Muckenhoupt's (A_p)-condition and existence of optimal martingale measure

Dmitry Kramkov
joint work with Kim Weston (preprint on arXiv)

Carnegie Mellon University

Outline

Setup: optimal investment

Motivation: utility-based valuation

Idea: the case of power utilities with $a \in (0, 1)$

Local martingale property for $\widehat{Y}$

Main results

Connection with BMO

Summary

Financial market

The uncertainty is modeled by $(\Omega, \mathcal{F}_T, \mathbf{F} = (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$;

- $T < \infty$ is the maturity.

There are

- bank account with $r = 0$;
- d stocks with prices $S = (S_t)$.

Denote

$$\mathcal{Q} \triangleq \{\mathbb{Q} \sim \mathbb{P} : \; S \; \text{is} \; \mathbb{Q} - \text{local martingale}\}.$$

Assumption (No Arbitrage)

$$\mathcal{Q} \neq \emptyset.$$

Optimal investment

$x_0 > 0$: initial capital,

$U = (U(x))_{x>0}$: strictly concave, $\mathbf{C}^1$-utility function with Inada's conditions:

$$U'(0) = \infty, \quad U'(\infty) = 0.$$

Problem: find the optimal strategy

$$\widehat{X} = \operatorname*{arg\,max}_{X \in \mathcal{X},\, X_0 = x_0} \mathbb{E}\left[U(X_T)\right],$$

where

$$\mathcal{X} \triangleq \{X \geq 0 : \; X = X_0 + \int H dS\}.$$

Duality characterization

$$\widehat{X} \text{ is optimal} \quad \Leftrightarrow \quad U'(\widehat{X}_T) = \widehat{Y}_T,$$

where the process $\widehat{Y}$ is such that

1. $X\widehat{Y}$ is a supermartingale, $\forall X \in \mathcal{X}$,
2. $\widehat{X}\widehat{Y}$ is a UI martingale.

In complete case, where there is only one $\mathbb{Q} \in \mathcal{Q}$,

$$\widehat{Y}_T = y_0 \frac{d\mathbb{Q}}{d\mathbb{P}} \text{ with } y_0 = \widehat{Y}_0.$$

Goal: in incomplete case, find conditions $\Longrightarrow$

$$\widehat{Y}_T = y_0 \frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}} \text{ for some } \widehat{\mathbb{Q}} \in \mathcal{Q}.$$

Motivation: utility-based valuation

Take a non-traded contingent claim Ψ.
In complete case, where $\mathcal{Q}$ contains just one $\mathbb{Q}$,

$$\text{Arbitrage-Free Price} = \mathbb{E}^{\mathbb{Q}}[\Psi] \quad \text{(by replication)}.$$

In incomplete case, $p \in \mathbf{R}$ is called a *Utility-Based Price* for Ψ if the investor will not trade Ψ at p:

$$\mathbb{E}\left[U(X_T + q(\Psi - p))\right] \leq \mathbb{E}\left[U(\widehat{X}_T)\right], \quad q \in \mathbf{R}, X \in \mathcal{X}(x_0).$$

Such UBP p always exists. However, see (Hugonnier et al, 05),

$$p \text{ is unique } \forall \text{ bounded } \Psi \Leftrightarrow \widehat{Y}_T = y_0 \frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}} \text{ for some } \widehat{\mathbb{Q}} \in \mathcal{Q}.$$

Key fact: dual minimizer

Denote

$$\mathcal{Z} \triangleq \{Z = (Z_t) : \; Z_t = \frac{d\mathbb{Q}_t}{d\mathbb{P}_t}, \quad \mathbb{Q} \in \mathcal{Q}\},$$
$$V(y) \triangleq \max_{x>0}(U(x) - xy), \quad y > 0.$$

Key fact: $\widehat{Y}$ is the dual *minimizer*:

$$\mathbb{E}\left[V(\widehat{Y}_T)\right] = \inf_{Z\in\mathcal{Z}} \mathbb{E}\left[V(y_0 Z_T)\right].$$

(We want to know when inf is attained, that is, $\widehat{Y} \in y_0\mathcal{Z}$.)

Power utility function with $a \in (0,1)$

If U is the power utility,

$$U(x) = \frac{x^{1-a}}{1-a}, \quad x > 0,$$

with the risk-aversion $a \in (0,1)$, then

$$V(y) \triangleq \max_{x>0}(U(x) - xy) = (p-1)\left(\frac{1}{y}\right)^{\frac{1}{p-1}},$$

where

$$p = \frac{1}{1-a} > 1.$$

Hence,

$$\mathbb{E}\left[\left(\frac{y_0}{\widehat{Y}_T}\right)^{\frac{1}{p-1}}\right] \leq \mathbb{E}\left[\left(\frac{1}{Z_T}\right)^{\frac{1}{p-1}}\right], \quad Z \in \mathcal{Z}.$$

Power utility function with $a \in (0,1)$

This inequality also admits a conditional version:

$$\mathbb{E}\left[\left(\frac{\widehat{Y}_\tau}{\widehat{Y}_T}\right)^{\frac{1}{p-1}}\middle|\mathcal{F}_\tau\right] \leq \mathbb{E}\left[\left(\frac{Z_\tau}{Z_T}\right)^{\frac{1}{p-1}}\middle|\mathcal{F}_\tau\right], \quad Z \in \mathcal{Z},$$

for every stopping time τ. This leads to (A_p)-inequality.

Remark

- Similar optimality of $\widehat{Y}$ has been observed for quadratic, power and exponential utilities on real line in (Delbaen et al, 97), (Grandis and Krawczyk, 98), and (Delbaen et al, 02). These papers use appropriate version of the Reverse Hölder inequality.
- Reverse Hölder is dual to (A_p). However, it *requires* UI martingale property.

Mukenhoupt's (A_p)-condition

Definition

Let $p > 1$. A process $R > 0$ satisfies (A_p) if there is a constant $c > 0$ such that for every stopping time τ

$$\mathbb{E}\left[\left(\frac{R_\tau}{R_T}\right)^{\frac{1}{p-1}}\middle|\mathcal{F}_\tau\right] \leq c.$$

Known fact: if $R > 0$ satisfies (A_p) and $\mathbb{E}[R_T] < \infty$, then R is of class $(\mathbf{D})$:

$$\{R_\tau : \tau \text{ is a stopping time}\} \text{ is UI.}$$

In particular, if R is a local martingale, then R is a UI martingale.

Power utility function with $a \in (0, 1)$

Thus for $U(x) = \frac{x^{1-a}}{1-a}$ with $a \in (0, 1)$ we have that

$$\exists\, Z \in \mathcal{Z} \text{ satisfying } (A_p) \text{ with } p = \frac{1}{1-a} \quad \Rightarrow \quad \widehat{Y} \text{ satisfies } (A_p)$$

$$\Rightarrow \quad \widehat{Y} \text{ is of class } (\mathbf{D}).$$

Since,

$$\widehat{Y}_T = y_0 \frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}} \text{ for some } \widehat{\mathbb{Q}} \in \mathcal{Q} \quad \Leftrightarrow \quad \widehat{Y} \in y_0 \mathcal{Z}$$

$$\Leftrightarrow \quad \widehat{Y} \text{ is a UI martingale,}$$

we only have to show that

$$\widehat{Y} \text{ is a local martingale.}$$

Local martingale property for $\widehat{Y}$

In general, $\widehat{Y}$ is not a local martingale; see example for logarithmic utility in single-period model in (K. Schachermayer, 1999).

Definition

A semimartingale R is *σ-bounded* if there is a predictable process $h > 0$ such that the stochastic integral $\int hdR$ is bounded.

Assumption

For all wealth processes $X > 0$ and $X' \geq 0$, the process X'/X is σ-bounded.

Proposition

If Assumption holds, then $\widehat{Y}$ is a local martingale.

Local martingale property for $\widehat{Y}$

Assumption holds in the following cases, see (K. Sîrbu, 2006):

1. S is continuous (easy).
2. $\exists$ a *finite-dimensional* local martingale M such that every bounded *purely discontinuous* martingale N admits the integral representation:

$$N = N_0 + \int H dM.$$

3. $\exists$ a *complete* financial market on $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$.

Main result

Theorem

Suppose that there are bounds $a \in (0,1)$ *and* $b \geq a$ *for the relative risk-aversion:*

$$a \leq -\frac{xU''(x)}{U(x)} \leq b, \quad x > 0,$$

and there is $Z \in \mathcal{Z}$ *(a density of a martingale measure) satisfying* (A_p) *with*

$$p = \frac{1}{1-a}.$$

Then $\widehat{Y}$ *satisfies* $(A_{p'})$ *with*

$$p' = 1 + \frac{b}{1-a} \qquad (\geq \frac{1}{1-a}).$$

If, in addition, σ*-boundedness Assumption holds, then* $\widehat{Y}_T / y_0 \in \mathcal{Z}$*.*

Key lemma

Lemma

Suppose there is a constant $a \in (0,1)$ *such that*

$$a \leq -\frac{xU''(x)}{U(x)}, \quad x > 0,$$

and there is $Z \in \mathcal{Z}$ *satisfying* (A_p) *with*

$$p = \frac{1}{1-a}.$$

Then there is a constant $c > 0$ *such that for every stopping time* τ*,*

$$U'(c\widehat{X}_\tau) \leq \widehat{Y}_\tau.$$

Example: bounded market price of risk

Suppose that the maturity T is bounded and

$$dS = S\sigma(\lambda dt + dB),$$

where B is a Brownian motion, $\sigma = (\sigma_t)$ is the volatility, and $\lambda = (\lambda_t)$ is the *market price of risk*.
Recall that the *minimal* martingale measure $\widetilde{\mathbb{Q}}$ has the density

$$\widetilde{Z} = \mathcal{E}(-\int \lambda dB) = \exp(-\int \lambda dB - \frac{1}{2}\int \lambda^2 ds).$$

We have that

$$|\lambda| \leq \mathrm{const} \quad \Rightarrow \quad \widetilde{Z} \text{ satisfies } (A_p), \forall p > 1.$$

In this case, we get the result ($\widehat{Y} \in y_0\mathcal{Z}$) as soon as the relative risk-aversion of U is bounded away from 0 and ∞.

Sharpness of the (A_p) condition

Theorem

Let constants a and p be such that

$$0 < a < 1 \text{ and } p > \frac{1}{1-a}.$$

Then there exists a financial market with a continuous stock price S such that

1. *There is $\mathbb{Q} \in \mathcal{Q}$, whose density process Z satisfies (A_p).*
2. *In the optimal investment problem with the power utility function*

$$U(x) = \frac{x^{1-a}}{1-a}, \quad x > 0,$$

the dual minimizer $\widehat{Y}$ with $\widehat{Y}_0 = 1$ is well-defined but does not belong to $\mathcal{Z}$ ($\widehat{Y}$ is a strict local martingale).

Connection with BMO

Assume now (to be able to refer to (Kazamaki, 94)) that all martingales on $(\Omega, \mathcal{F}_T, \mathbf{F} = (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ are continuous.

- For a continuous martingale M with $M_0 = 0$,

$$\|M\|_{\mathrm{BMO}} \triangleq \sup_{\tau} \|\mathbb{E}\left[|M_T - M_\tau|^2 \middle| \mathcal{F}_\tau \right]^{1/2}\|_\infty,$$

  where the supremum is taken with respect to all stopping times τ.

- For a martingale $R > 0$ with $R_0 = 1$ we have the equivalences:

$$R \text{ satisfies } (A_p) \text{ for some } p > 1$$
$$\Updownarrow$$
$$R = \mathcal{E}(M) \triangleq \exp\left(M - \frac{1}{2}\langle M\rangle\right) \text{ for some } M \in \mathrm{BMO}.$$

Connection with BMO

Hereafter, $\mathbb{Q}$ and $\widehat{\mathbb{Q}}$ are the measures from the main theorem.

Corollary

Under the conditions of the main theorem, we have that

$$\widehat{Y} = y_0\mathcal{E}(M) \text{ with } M \in \mathrm{BMO}.$$

Moreover, if $X \in \mathcal{X}$ has the form $X = X_0\mathcal{E}(L)$ with $L \in \mathrm{BMO}(\mathbb{Q})$, then $L \in \mathrm{BMO}(\widehat{\mathbb{Q}})$. In particular, X is a UI martingale under $\widehat{\mathbb{Q}}$.

Remark

The second assertion is important as, see (Hugonnier et al, 05),

$$\text{UBP for (unbounded) } \Psi \text{ is unique} \quad \Leftrightarrow \quad \exists X \in \mathcal{X} \text{ such that}$$
$$|\Psi| \leq X_T \text{ and } X \text{ is a UI martingale under } \widehat{\mathbb{Q}}.$$

Connection with BMO

Corollary

Suppose there is a constant $b > 0$ *such that*

$$1 \leq -\frac{xU''(x)}{U(x)} \leq b, \quad x > 0,$$

and there is $Z \in \mathcal{Z}$ *such that* $Z = \mathcal{E}(L)$ *with* $L \in \mathrm{BMO}$. *Then*

$$\widehat{Y} = y_0 \mathcal{E}(M) \ \text{ with } \ M \in \mathrm{BMO};$$

in particular, $\widehat{Y} \in y_0 \mathcal{Z}$.

Remark

The lower bound $1 \leq -\frac{xU''(x)}{U(x)}$ is sharp. In particular, the result does not hold for

$$U(x) = \frac{x^{1-a}}{1-a} \ \text{ with } \ 0 < a < 1.$$

Summary

- In various studies involving utility-based arguments, it is important to know whether the dual minimizer corresponds to a martingale measure.
- Our main condition consists in the verification of (A_p) for density process of some martingale measure with

$$p = \frac{1}{1-a}, \quad \text{where} \quad -\frac{xU''(x)}{U'(x)} \geq a \in (0,1).$$

- The above inequality for p is sharp even for power utilities and continuous S.