Model-independent bounds for Asian options

A dynamic programming approach

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Model-independent bounds for option prices

- Aim: make statements about the price of options given very mild modelling assumptions
- Incorporate market information by supposing the prices of vanilla call options are known
- Typically want to know the largest/smallest price of an exotic option (Lookback option, Barrier option, Variance option, Asian option,...) given observed call prices, but with (essentially) no other assumptions on behaviour of underlying

Financial Setting

- Option priced on an asset $(S_t)_{t\in[0,T]}$, option payoff X_T
- Dynamics of S unspecified, but suppose paths are continuous, and we see prices of call options at all strikes K and at maturity time T
- Assume for simplicity that all prices are discounted — this won't affect our main results
- Under risk-neutral measure, S should be a (local-)martingale, and we can recover the law of S_T at time T, μ say, from call prices $C(K)$

Existing Literature

Rich literature on these problems:

- Starting with Hobson ('98) connection with Skorokhod Embedding problem $\rightarrow$ explicit optimal solutions for many different payoff functions (Brown, C., Dupire, Henry-Labordère, Hobson, Klimmek, Obłój, Rogers, Spoida, Touzi, Wang,...)
- More recently, model-independent duality has been proved by Dolinsky-Soner ('14):

$$\sup_{\mathbb{Q}: S_T \sim \mu} \mathbb{E}^{\mathbb{Q}}[X_T] = \text{price of cheapest super-replication strategy}$$

  Here the super-replication strategy will use both calls and dynamic trading in underlying, and is model-independent. The sup is taken over measures $\mathbb{Q}$ for which S is a martingale. (See also Hou-Obłój and Beiglböck-C.-Huesmann-Perkowski-Prömel)
- The problem of finding the martingale S which maximises the expectation above is commonly called the Martingale Optimal Transport problem (MOT)

Explicit solutions and ‘convexity’

To date, explicit solutions to MOT have largely been constructed using the connection to the Skorokhod Embedding problem (SEP):

- Since S is a (continuous) martingale, it is the (continuous) time change of a Brownian motion, $S_t = B_{\tau_t}$. When the option payoff function X_T is independent of the time-scale (e.g. maximum), then choosing a model for S with $S_T \sim \mu$ is equivalent to finding a stopping time τ_T such that $B_{\tau_T} \sim \mu$ (the SEP).
- Finding a given model which maximises $\mathbb{E}^{\mathbb{Q}}[X_T]$ corresponds to finding a solution to the SEP with a certain optimality property
- Needs payoff to be invariant under time changes
- Historically, an optimal solution was produced using ad-hoc methods. In Beiglböck-C.-Huesmann, this was formalised in a monotonicity principle
- The monotonicity principle captures a certain type of convexity — essentially all known optimal solutions to the SEP exploit this convexity

- There exists $\Psi_\mu(\cdot)$ increasing such that:

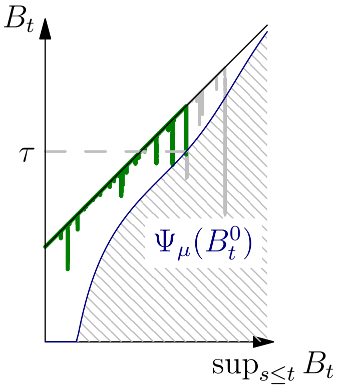
$$\tau = \inf\{t \geq 0 : B_t \leq \Psi_\mu(\sup_{s\leq t} B_s)\}$$

- Maximises

$$\mathbb{E}\left[F(\sup_{s\leq\tau} B_s)\right]$$

over all (well-behaved) embeddings, for any increasing function F.

- Model-independent bounds for lookback options with increasing F. General F unknown ...



- Azéma-Yor ('79)
- Hobson ('98)

Model-independent bounds for Asian options

- In this talk, want to consider Asian options, $X_T = F\left(\int_0^T S_r \,\mathrm{d}r\right)$ for arbitrary F
- The SEP methodology will not be effective: $A_T = \int_0^T S_r \,\mathrm{d}r$ is very dependent on the choice of the time-change.
- In the case of convex F, Jensen gives an easy solution: essentially, jump immediately to final law (see also Stebegg ('14))
- Our methods are not specific to Asian options: should(!) generalise

Dynamic programming approach

- One of the difficulties inherent in constructing solutions to the Martingale Optimal Transport problem (MOT) is that 'local' optimal behaviour behaviour is driven by the 'global' requirement that $S_T \sim \mu$
- Key idea: 'localise' the condition on the terminal law
- At any given time the 'state' of our process should be enhanced to include also the conditional terminal law:

$$\xi_t(A) = \mathbb{P}(S_T \in A | \mathcal{F}_t), \quad \xi_0 = \mu$$

  Note: implies $\xi_t(A)$ is a martingale for any A
- Model ξ_t rather than S_t — S_t can be recovered by $S_t = \int x \, \xi_t(\mathrm{d}x)$

Measure valued martingales

- Introduce the set of integrable probability measures:

$$\mathcal{P}^1 := \{\mu \in \mathcal{M}(\mathbb{R}_+) : \mu(\mathbb{R}_+) = 1, \int |x|\,\mu(\mathrm{d}x) < \infty\},$$

and the set of singular probability measures:

$$\mathcal{P}^s = \{\mu \in \mathcal{M}(\mathbb{R}_+) : \mu = \delta_y, y \in \mathbb{R}_+\}$$

- We say an adapted process $\xi_t \in \mathcal{P}^1$ is a measure-valued martingale (MVM) if for any $f \in C_b(\mathbb{R}_+)$, $\xi_r(f)$ is a martingale
- An MVM (on $[0, T]$) is terminating if $\xi_T \in \mathcal{P}^s$
- See Horowitz ('85); Walsh ('86); Dawson ('91); Eldan ('13).

Examples of measure valued martingale

The previous result tells us that we can construct an MVM from any (suitably) stopped Brownian motion:

- Exit from an interval: let $a < 0 < b$, $H := \inf\{t \geq 0 : B_t \notin (a, b)\}$, then
$$\xi_t := \frac{b - B_{t \wedge H}}{b - a}\delta_a + \frac{B_{t \wedge H} - a}{b - a}\delta_b$$
- 'Bass' solution: Let μ have distribution function F_μ, Φ the d.f. of an $N(0, 1)$, $h := F_\mu^{-1} \circ \Phi$, so $h(B_1) \sim \mu$. Then:
$$\xi_t(A) := \mathbb{P}(h(B_t) \in A | \mathcal{F}_t)$$
is an MVM with $\xi_0 = \mu$. Note that if $\mu = N(0, 1)$, we get the easy case where $\xi_t = N(B_t, 1 - t)$ for $t \in [0, 1]$

Measure valued martingales

Suppose we are given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_r)_{r\in[0-,T]}, \mathbb{P})$ satisfying the usual conditions, such that $\mathcal{F}_{0-}$ is trivial.

Lemma

1. *If* $(\xi_r)_{r\in[0,T]}$ *is a terminating measure-valued martingale with* $\xi_{0-} = \mu$, *then* $S_r := \int x\, \xi_r(\mathrm{d}x)$ *is a non-negative UI martingale with* $S_T \sim \mu$
2. *If* $(S_r)_{r\in[0,T]}$ *is a non-negative UI martingale with* $S_T \sim \mu$, *then* $\xi_r(A) := \mathbb{P}(S_T \in A|\mathcal{F}_r)$ *is a terminating MVM with* $\xi_{0-} = \mu$

[MOT] and [MVM] problem formulation

Given an integrable probability measure μ on $\mathbb{R}_+$ and a sufficiently nice function $F : \mathbb{R}_+ \to \mathbb{R}_+$,

[MOT] find a probability space $(\Omega, \mathcal{H}, (\mathcal{H}_t)_{t\in[0,T]}, \mathbb{P})$ and a càdlàg UI martingale $(S_t)_{t\in[0,T]}$ on this space with $S_T \sim \mu$ which maximises $\mathbb{E}\left[F(A_T)\right]$ over the class of such probability spaces and processes.

[MVM] find a probability space $(\Omega, \mathcal{G}, (\mathcal{G}_r)_{r\in[0-,\infty)}, \mathbb{P})$, a progressively measurable process $\lambda_r \in [0,1]$, and a terminating measure-valued $(\mathcal{G}_r)_{r\in[0-,\infty)}$-martingale $(\xi_r)_{r\in[0-,\infty]}$ with $\xi_{0-} = \mu$ and $\int x\,\xi_r(\mathrm{d}x)$ continuous a.s., which maximises $\mathbb{E}\left[F(A_T)\right]$ with A_T given by:

$$T_r = \int_0^r \lambda_s\,\mathrm{d}s, \quad A_T = \int_0^T \left\{ \int x\,\xi_{T_s^{-1}}(\mathrm{d}x) \right\} \mathrm{d}s$$

Equivalence of [MOT] and [MVM]

Lemma

[MOT] and [MVM] are equivalent. Moreover, if F is bounded above, then the value remains the same in [MVM] if we restrict to MVMs in a Brownian filtration which are (pathwise) continuous and almost surely terminate in finite time.

Here, (pathwise) continuity of MVMs is in the topology derived from the 1-Wasserstein metric on $\mathcal{P}^1$:

$$d_{\mathcal{W}_1}(\lambda, \mu) := \sup \left\{ \int \varphi(x) \, (\lambda - \mu)(\mathrm{d}x) : \varphi \text{ is 1-Lipschitz} \right\}.$$

Dynamic Programming Principle

Formulate dynamically: suppose that at time r, we have 'real' time $T_r = t$, current law $\xi_r = \xi \in \mathcal{P}^1$, running average $A_{T_r} = a$, and we wish to find:

$$U(r, t, \xi, a) = \sup \mathbb{E}\left[F(A_T) \middle| T_r = t, \xi_r = \xi, A_{T_r} = a\right],$$

where the supremum is taken over all time-change determining processes $(\lambda_u)_{u \in [r,\infty)}$ and continuous, finitely-terminating models $(\xi_u)_{u \in [r,\infty)}$.

Lemma

Suppose F is a non-negative, Lipschitz function. The function $U : \mathbb{R}_+ \times [0, T] \times \mathcal{P}^1 \times \mathbb{R}_+$ is continuous (here the topology on $\mathcal{P}^1$ is the topology derived from the Wasserstein-1 metric), and independent of r.

$\Longrightarrow$ Can approximate ξ by finite atomic measures

Approximation by atomic measures

When approximating by finitely supported measures, we have some nice structure to exploit:

- Suppose initially ξ supported on $0 \leq x_0 \leq x_1 \leq \cdots \leq x_N$, then at any later time ξ supported on some subset $x_{\alpha_0} \leq x_{\alpha_1} \leq \cdots \leq x_{\alpha_m}$, where $\alpha = \{\alpha_0, \alpha_1, \ldots, \alpha_m\} \subset \{0, 1, \ldots, N\}$
- Consider $\xi_r^i = \xi_r(\{x_i\})$ — each ξ^i is a martingale with values on $[0,1]$, constrained by $\sum_i \xi^i = 1$. I.e. ξ takes values on a simplex
- Consider a sequence of problems, where we run until the first time one of the ξ_i's hits zero: problem reduces to a smaller simplex
- Recalling that we can assume a Brownian filtration, $\mathrm{d}\xi_r^i = w_r^i \mathrm{d}W_r$, w_r^i part of control to be chosen
- Control also incorporates 'speed' how fast ξ_t evolves relative to 'real' time

Value function

Fix α, ξ with $|\alpha| = k+1$ then we write

$$\begin{aligned}
\boldsymbol{\xi}^\alpha &= (\xi^{\alpha_0}, \xi^{\alpha_1}, \ldots, \xi^{\alpha_k}) \in \Delta^{k+1} := \{\mathbf{z} \in \mathbb{R}_+^{k+1} : \sum z_i = 1\} \\
\mathbf{x}^\alpha &= (x_{\alpha_0}, x_{\alpha_1}, \ldots, x_{\alpha_k}) \\
\mathbb{S}^{k+1} &= \{\mathbf{z} \in \mathbb{R}^{k+1} : ||z|| = 1\}
\end{aligned}$$

Then the value function of interest is

$$V_\alpha(u, t, \boldsymbol{\xi}^\alpha, a) = U(u, t, \sum \xi^{\alpha_i} \delta_{x_{\alpha_i}}, a)$$

Main Result

Theorem

Suppose $F(a)$ is continuous, non-negative. Then V_α is independent of u and the smallest non-negative solution (in the viscosity sense) to

$$\max\left\{\frac{\partial V_\alpha}{\partial t} + \mathbf{x}^\alpha \cdot \boldsymbol{\xi}^\alpha \frac{\partial V_\alpha}{\partial a}, \sup_{\mathbf{w}\in\mathbb{S}^{k+1}} \left[\mathrm{tr}(\mathbf{w}\mathbf{w}^T D^2_{\boldsymbol{\xi}} V_\alpha)\right]\right\} = 0$$

for $\xi \in (\Delta^{k+1})^\circ$, and $t < T$, with the boundary conditions

$$\begin{aligned} V_\alpha(u, T, \boldsymbol{\xi}^\alpha, a) &= F(a) \\ V_\alpha(u, t, \boldsymbol{\xi}^\alpha, a) &= V_{\alpha'}(u, t, \boldsymbol{\xi}^{\alpha'}, a), \qquad \text{when } \boldsymbol{\xi}^\alpha \in \partial\Delta^{k+1} \end{aligned}$$

Here α' is the subset of α corresponding to non-zero entries of $\boldsymbol{\xi}^\alpha$, and $\boldsymbol{\xi}^{\alpha'}$ is the vector of non-zero values of $\boldsymbol{\xi}^\alpha$.

Example: Convex F

Lemma

Suppose the function F is convex and Lipschitz. Then for all $\xi \in \mathcal{P}^1(\mathbb{R}_+)$:

$$U(t, \xi, a) = \int F\left(a + (T - t)x\right)\, \xi(\mathrm{d}x).$$

Moreover, an optimal model is given by:

$$\begin{aligned} S_{0-} &= \int x\, \xi(\mathrm{d}x) \\ S_t &= S_T, \qquad\qquad t \geq 0, \end{aligned}$$

where $S_T \sim \xi$.

- Proof: check that U verifies our PDE
- Result due to Stebegg ('14): also provides a model-independent super-hedging strategy

Example: Non-convex F

Consider

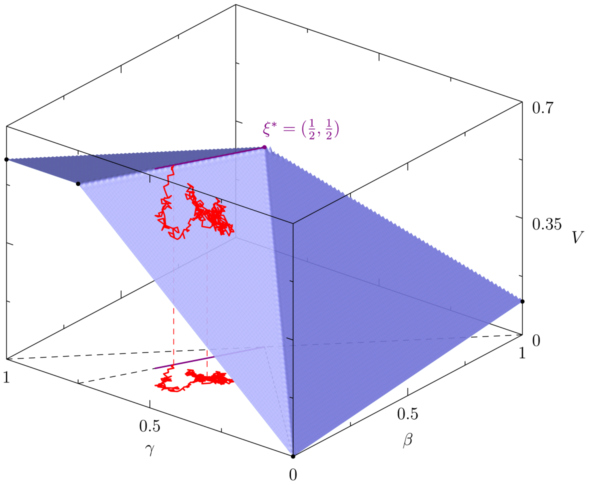
$$F(A_T) = (A_T - K_1)_+ - (A_T - K_2)_+, \qquad K_1 < K_2$$

- Conjecture: at time 0, run to final distribution for 'small' final values, or to K_2 for 'large' final values
- Simplify to three point case:

$$\xi_0 = (1 - \beta - \gamma)\delta_{-1} + \beta\delta_0 + \gamma\delta_1$$

- Explicit value function corresponding to conjectured solution can be computed, PDE can be verified for this solution $\implies$ optimality
- Note that general duality results (Dolinsky-Soner, ...) give existence of a model-independent super-hedging strategy

Example: non-convex F



Value function at $t = 0$ when $a = 0$, $T = 1$, $K_1 = -0.1$, $K_2 = 0.5$.

Open questions

- General starting law?
- Multi-marginal setup?
- Non-atomic formulation of the PDE?
- Other option types: Lookback options, Variance options etc.
- Higher dimensional problems — e.g. basket options.

Conclusions

- Formulated the model-independent pricing problem for Asian options in terms of a measure-valued martingale
- In this formulation, we can apply standard dynamic programming arguments, no convexity assumption required
- By discretising, can formulate as a PDE $\mapsto$ characterisation of value function
- Solve simple problems explicitly via verification