Single season heteroscedasticity in time series

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Abstract

We consider seasonal time series in which one season has variance that is different from all the others. This behaviour is evident in production series where the variability of the index of production is higher for the month with the lowest seasonal level. We show that when only one month has different variability from others there are constraints on the seasonal models that can be used. Most studies use a dummy or a trigonometric seasonality representation. We show that both the dummy and the trigonometric models are not effective in modelling seasonal series with this type of variability. We conclude that only the seasonal model derived by Harrison and Stevens (1976) (HS) and the periodic irregular model can track appropriately this behavior. We argue that HS is more appropriate for economic time series. Thus, modelling the seasonal variability enhances our understanding for the properties of the series. We present a likelihood ratio test for the presence of periodic variance in one season.

Keywords: heteroscedasticity, seasonality, structural time-series modelling, periodic models, likelihood ratio
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1 Introduction

Series with autocovariance structure that varies with season arise in hydrology, see for example, Hipel and McLeod (1994), Troutman (1979), Pagano (1978), Jones and Brelsford (1967). The fact that many economic times series have one season that exhibits a higher volatility than other seasons is often overlooked (Osborn and Smith 1989). This behaviour is evident in production series where the variability of the index of production is higher for the month with the lowest seasonal level. For example, the seasonal component for August in most European countries has the lowest level within a year due to summer holiday factory shut-downs. August also shows higher variability than other months. Miron (2001) shows that this is consistent with backward L-shaped marginal cost curves. Modelling this type of behaviour correctly is of importance for forecasting and seasonal adjustment. We show that, when one month only has different variability from others there are constraints on the seasonal models we can use.

The two most common methods for seasonal adjustment, X-12-ARIMA (Findley, Monsell, Otto, Bell, and Pugh 1998) and TRAMO-SEATS (Gomez and Maravall 1996), have substantial restrictions in modelling periodic variances. In the case of X-12-ARIMA, periodic variance is dealt by applying different seasonal moving averages to each season. This ad-hoc method has proved flexible in fitting models but provides little help in detecting seasonal heteroscedasticity. There is no attempt to understand the structure and the relationship between different seasons, a common criticism for the overall philosophy of the Census X-11 method. TRAMO-SEATS uses an ARIMA model based decomposition of the time series which has currently no way of modelling periodic variances. An alternative approach is to use a structural time series model and estimate it using the Kalman filter (Harvey 1989).

This paper explores ways of modelling periodic variances in a single month using a structural model. Section 2 reviews well-known common models for seasonality and examines whether they can be used to model periodic variance in one month alone. Burridge and Wallis (1988) include periodic variances in a structural model using dummy seasonality, but we show that both the dummy seasonality (DS) and the trigonometric seasonality(TS) are not effective in modelling seasonal series with this type of variability. We conclude that only the seasonal model derived by Harrison and Stevens (1976) and the periodic irregular model can track appropriately this behaviour. A likelihood ratio test for seasonal heteroscedasticity is given in section 3, while

section 4 provides real data examples. The final section presents conclusions.

2 Periodic Variance and Seasonal models

2.1 Dummy seasonality model(DS)

A widely used framework for seasonality is the model commonly known as dummy seasonality. Consider an observed time series y_t for $t = 1, \ldots, n$ and let γ_t denote the seasonal effect at time t. The simplest way to model seasonality is to assume that the seasonal effects sum to zero within a year, that is $\gamma_t = -\sum_{j=1}^{s-1} \gamma_{t-j}$. We allow seasonality to evolve over time by adding a noise term ω_t. This gives the relationship

$$\gamma_t = -\sum_{j=1}^{s-1} \gamma_{t-j} + \omega_t \tag{1}$$

Subtracting γ_{t-s} from both sides of (1), we get

$$\gamma_t - \gamma_{t-s} = -\sum_{j=1}^{s} \gamma_{t-j} + \omega_t \Longrightarrow \Delta_s \gamma_t = \omega_t - \omega_{t-1} \tag{2}$$

since $S(L)\gamma_{t-1} = \omega_{t-1}$ and $\omega_t \sim \mathrm{NID}(0, \sigma_\omega^2)$. Thus γ_t follows and ARIMA $(0,0,1)(0,1,0)_s$.

We take the following simple model (commonly known as the local level model) consisting of a trend μ_t and a dummy seasonal component:

$$\begin{aligned} y_t &= \mu_t + \gamma_t + \epsilon_t \qquad \{\epsilon_t\} \sim \mathrm{NID}(0, \sigma_\epsilon^2) \\ \mu_t &= \mu_{t-1} + \eta_t \qquad \{\eta_t\} \sim \mathrm{NID}(0, \sigma_\eta^2) \end{aligned} \tag{3}$$

with γ_t as in (1), and $\{\epsilon_t\}, \{\eta_t\}, \{\omega_t\}$ mutually uncorrelated. The stationary form of (3) is:

$$\triangle_s y_t = S(L)\eta_t + \triangle_s \gamma_t + \triangle_s \epsilon_t$$

In order to model seasonal heteroscedasticity we take $\omega_t \sim \mathrm{NID}(0, \sigma_{\mathrm{r}}^2)$ for $r = t(\bmod s)$. Using (2), the variance of $\triangle_s \gamma_t$ is:

$$\mathrm{Var}(\triangle_{\mathrm{s}}\gamma_{\mathrm{t}}) = \mathrm{Var}(\omega_{\mathrm{t}} - \omega_{\mathrm{t-1}}) = \begin{cases} \sigma_1^2 + \sigma_s^2 & \text{for } t = 1, s+1, 2s+1, \ldots \\ \sigma_r^2 + \sigma_{r-1}^2 & \text{otherwise} \end{cases}$$

Following the same argument and using (2) we see that:

$$\mathrm{Cov}(\triangle_{\mathrm{s}}\gamma_{\mathrm{t}}, \triangle_{\mathrm{s}}\gamma_{\mathrm{t-1}}) = \begin{cases} -\sigma_s^2 & \text{for } t = 1, s+1, 2s+1, \ldots \\ -\sigma_{r-1}^2 & \text{otherwise} \end{cases}$$

and

$$\mathrm{Cov}(\triangle_{\mathrm{s}}\gamma_{\mathrm{t}}, \triangle_{\mathrm{s}}\gamma_{\mathrm{t-j}}) = 0 \qquad \text{for j} > 1$$

The dummy seasonality model was used by Burridge and Wallis (1988) to account for periodic variances. They propose this framework to model cases where the final estimates of the seasonal component exhibit seasonal variation. We show that this framework is not appropriate for cases where the seasonal component of only one month exhibits different variation from all other months. We show that it is impossible to model a single season's variance to be different from all the other seasons using the dummy seasonal model. Without loss of generality, let the variability in the s^{th} season be different. Then :

$$\mathrm{Var}(\triangle_{\mathrm{s}}\gamma_{\mathrm{t}}) = \begin{cases} \sigma_s^2 + \sigma_{s-1}^2 = \lambda_1 & \text{for } t = s, 2s, 3s, \ldots \\ \sigma_1^2 + \sigma_s^2 = \lambda_2 & \text{for } t = 1, s+1, 2s+1, \ldots \\ \sigma_r^2 + \sigma_{r-1}^2 = \lambda_3 & \text{otherwise} \end{cases} \tag{4}$$

Since we have only one season exhibiting different variance from all the others, $\lambda_1 \neq \lambda_2$, $\lambda_1 \neq \lambda_3$ and $\lambda_2 = \lambda_3$. But from the third line of (4) we have:

$$\sigma_{s-1}^2 = \lambda_3 - \sigma_{s-2}^2 = \lambda_3 - (\lambda_3 - \sigma_{s-3}^2)$$
$$\Longrightarrow \sigma_{s-1}^2 = \sigma_{s-3}^2 = \ldots = \begin{cases} \sigma_1^2 & s \text{ even} \\ \sigma_2^2 & s \text{ odd} \end{cases}$$

When s is odd we have from the first line of (4)

$$\begin{aligned} & \sigma_{s-1}^2 && = \lambda_1 - \sigma_s^2 \\ \Longleftrightarrow \quad & \sigma_2^2 && = \lambda_1 - \sigma_s^2 = \lambda_1 - \lambda_2 + \sigma_1^2 = \lambda_1 - \lambda_2 + \lambda_3 - \sigma_2^2 \\ \Longleftrightarrow \quad & \sigma_2^2 && = \frac{\lambda_1 - \lambda_2 + \lambda_3}{2} \end{aligned}$$

and

$$\begin{aligned} & \sigma_1^2 && = \lambda_2 - \sigma_s^2 = \lambda_2 - \lambda_1 + \sigma_2^2 = \lambda_1 - \lambda_2 + \lambda_3 - \sigma_1^2 \\ \Longleftrightarrow \quad & \sigma_1^2 && = \frac{\lambda_1 - \lambda_2 + \lambda_3}{2} \end{aligned}$$

so for the case when s is odd, we also have $\sigma_{s-1}^2 = \sigma_1^2$. Then from the first two lines of (4) we have $\lambda_1 = \lambda_2$ which contradicts our assumption that $\lambda_1 \neq \lambda_2$. Even if we assign different variance for the innovation error ω_t to one season only, the subsequent season will have the same variability which will be different from the variability of all other months. Therefore, in the DS case we can have different variances for at least two seasons and there is no way to represent a different variance for a single season.

2.2 Trigonometric seasonal model (TS)

The TS case exhibits a different type of problem in dealing with heteroscedasticity. In the trigonometric case, the seasonal effect is the combination of [s/2] cycles that is $\gamma_t = \sum_{j=1}^{[s/2]} \gamma_{j,t}$ where [s/2] is the integer part of s/2 and :

$$\begin{pmatrix} \gamma_{j,t} \\ \gamma_{j,t}^* \end{pmatrix} = \begin{pmatrix} \cos\lambda_j & \sin\lambda_j \\ -\sin\lambda_j & \cos\lambda_j \end{pmatrix} \begin{pmatrix} \gamma_{j,t-1} \\ \gamma_{j,t-1}^* \end{pmatrix} + \begin{pmatrix} \omega_{j,t} \\ \omega_{j,t}^* \end{pmatrix} \tag{5}$$

where $j = 1, ..., [s/2]$, t=1,...,n and $\lambda_j = 2\pi j/s$ is the frequency, in radians. In the homoscedastic case $\omega_{j,t}$ and $\omega_{j,t}^*$ are independent $\mathrm{N}(0, \sigma_\omega^2)$ variables. The component $\gamma_{j,t}^*$ appear as a matter of construction.

For simplicity consider the quarterly case where we have 2 seasonal frequencies $\pi/2$ and π. Then (5) becomes:

$$\gamma_{1,t} = \gamma_{1,t-1}^* + \omega_{1,t} \tag{6}$$
$$\gamma_{1,t}^* = -\gamma_{1,t-1} + \omega_{1,t}^* \tag{7}$$
$$\gamma_{2,t} = -\gamma_{2,t-1} + \omega_{2,t} \Longrightarrow \gamma_{2,t} + \gamma_{2,t-1} = \omega_{2,t} \tag{8}$$

From (6) and (7) we get:

$$\gamma_{1,t} = -\gamma_{1,t-2} + \omega_{1,t} + \omega_{1,t-1}^* \Longrightarrow \gamma_{1,t} + \gamma_{1,t-2} = \omega_{1,t} + \omega_{1,t-1}^* \tag{9}$$

From (8) and (9) we get:

$$S(L)\gamma_{1,t} = \omega_{1,t} + \omega_{1,t-1} + \omega_{1,t-1}^* + \omega_{1,t-2}^* \tag{10}$$
$$S(L)\gamma_{2,t} = \omega_{2,t} + \omega_{2,t-2} \tag{11}$$

Thus

$$S(L)\gamma_t = \sum_{j=1}^{2} S(L)\gamma_{j,t} = \omega_{1,t} + \omega_{1,t-1} + \omega^*_{1,t-1} + \omega^*_{1,t-2} + \omega_{2,t} + \omega_{2,t-2} \qquad (12)$$

From (12) we can see that $S(L)\gamma_t$ follows and MA(2) process in the quarterly case. In the general trigonometric seasonality case, $S(L)\gamma_t$ results in a MA(s-2) process.

The trigonometric seasonal model (5) will not exhibit seasonal heteroscedasticity even if we assign different variances for different frequencies. For example, taking σ^2_1, $\sigma^2_{1^*}$ and σ^2_2 to be the variances in the respective cycle and using (12) we see that:

$$\begin{aligned} \mathrm{Var}(\Delta_\mathrm{s}\gamma_\mathrm{t}) &= \mathrm{Var}(\Delta \mathrm{S(L)}\gamma_\mathrm{t}) \\ &= \mathrm{Var}(\omega_{1,\mathrm{t}} + \omega^*_{1,\mathrm{t}-1} - \omega_{1,\mathrm{t}-2} - \omega^*_{1,\mathrm{t}-3} + \\ &\quad + \omega_{2,t} - \omega_{2,t-1} + \omega_{2,t-2} - \omega_{2,t-3}) \\ &= 2\sigma^2_1 + 2\sigma^2_{1^*} + 4\sigma^2_2 \end{aligned}$$

Therefore all seasons exhibit the same variance.

Modelling heteroscedasticity using a trigonometric cyclical component is probably more appropriate in business cycle analysis rather than in seasonal analysis. Our interest lies on whether a particular month has different variance rather than whether a cycle with a particular periodicity has higher variance than cycles in other periodicities.

2.3 Harrison and Stevens seasonal model (HS)

An alternative seasonal specification is the Harrison and Stevens (1976) seasonal model. This representation has a time-varying observation equation, in which the seasonal factors are explicitly modelled as a multivariate random walk. The state space model for the seasonal factors in the constant variance case is :

$$\begin{aligned} \gamma_t &= \boldsymbol{x}'_t \boldsymbol{\delta}_t \\ \boldsymbol{\delta}_t &= \boldsymbol{\delta}_{t-1} + \boldsymbol{\omega}_t, \qquad \boldsymbol{\omega}_t \sim \mathrm{NID}(\mathbf{0}, \boldsymbol{\Omega}) \end{aligned} \qquad (13)$$

where

$$\boldsymbol{\Omega} = \sigma_\omega^2[\boldsymbol{I_s} - \tfrac{\mathbf{1}}{\boldsymbol{s}}\boldsymbol{i_s i_s'}] = \sigma_\omega^2 \begin{pmatrix} 1-\frac{1}{s} & -\frac{1}{s} & \dots & -\frac{1}{s} \\ -\frac{1}{s} & 1-\frac{1}{s} & \dots & -\frac{1}{s} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{s} & \dots & \dots & 1-\frac{1}{s} \end{pmatrix} \tag{14}$$

and $\boldsymbol{\delta}$ is an $s \times 1$ vector containing the seasonal effects, $\mathbf{x_t'} = [D_{1t}, \dots, D_{st}]$, with $D_{jt} = 1$ in season j and 0 otherwise, and $\mathbf{i_s} = [1, 1, \dots, 1]'$ is an $s \times 1$ vector. From (14) we get that $\mathbf{i'}_s\mathrm{Var}(\boldsymbol{\omega_t}) = 0$ which implies that $\mathrm{Var}(\mathbf{i'}_s\boldsymbol{\omega_t}) = 0$, and by taking $\mathbf{i'}_s\boldsymbol{\omega_0} = 0$, $\mathbf{i'}_s\boldsymbol{\omega_t} = 0$ is satisfied . Then from equation (13) we have that $\mathbf{i'}_s\boldsymbol{\delta_t} = \mathbf{i'}_s\boldsymbol{\delta_{t-1}}$, which for $\mathbf{i'}_s\boldsymbol{\delta_0} = 0$ gives $\mathbf{i'}_s\boldsymbol{\delta_t} = 0$

By repeated substitution in (13) (Proietti 1998)

$$\begin{aligned} S(L)\gamma_t &= \sum_{\boldsymbol{j=0}}^{\boldsymbol{s-1}} \boldsymbol{x'_{t-j}\delta_{t-j}} \\ &= \sum_{\boldsymbol{j=0}}^{\boldsymbol{s-1}}\sum_{\boldsymbol{k=j}}^{\boldsymbol{s-2}} \boldsymbol{x'_{t-j}\omega_{t-j-k}} + \left(\sum_{\boldsymbol{j=0}}^{\boldsymbol{s-1}} \boldsymbol{x'_{t-j}}\right)\boldsymbol{\delta_{t-s+1}} \\ &= \boldsymbol{x_t\omega_t + (x_t + x_{t-1})'\omega_{t-1} + \ldots + (x_t + \ldots + x_{t-s+2})'\omega_{t-s+2}} \end{aligned}$$

since $\mathbf{i'}_s\boldsymbol{\delta_t} = \left(\sum_{\boldsymbol{j=0}}^{\boldsymbol{s-1}} \boldsymbol{x'_{t-j}}\right)\boldsymbol{\delta_{t-s+1}} = 0$. Hence,

$$\begin{aligned} \mathrm{Var(S(L)\gamma_t)} &= \mathbf{x_t'}\mathrm{Var}(\boldsymbol{\omega_t})\mathbf{x_t} + \ldots + (\mathbf{x_t} + \ldots + \mathbf{x_{t-s+2}})'\mathrm{Var}(\boldsymbol{\omega_{t-s+2}})(\mathbf{x_t} + \ldots + \mathbf{x_{t-s+2}}) \\ &= \sum_{k=0}^{s-2}\sum_{j=0}^{k}\sum_{\ell=0}^{k} \mathbf{x'_{t-j}}\boldsymbol{\Omega}\mathbf{x_{t-\ell}} \end{aligned}$$

In general the autocovariance function of $S(L)\gamma_t$ is:

$$c(\tau) = \begin{cases} \sum_{k=\tau}^{s-2}\sum_{j=0}^{k}\sum_{\ell=\tau}^{k} \mathbf{x'_{t-j}}\boldsymbol{\Omega}\mathbf{x_{t-\ell}} & \text{for } \tau \leq s-2 \\ 0 & \text{for } \tau > s-2 \end{cases}$$

which shows that $S(L)\gamma_t \sim MA(s-2)$.

The appropriate covariance matrix for the case of seasonal heteroscedasticity is given by Proietti (1998):

$$\mathrm{Var}(\boldsymbol{\omega_t}) = \mathbf{V} \quad = \quad [\mathbf{D} - \frac{\mathbf{1}}{\mathbf{i_s D i_s'}}\mathbf{D i_s i_s' D}] =$$

$$= \frac{1}{\sum_{i=1}^{s}\sigma_i^2}\begin{pmatrix} \sum_{i\neq 1}^{s}\sigma_1^2\sigma_i^2 & -\sigma_1^2\sigma_2^2 & \dots & -\sigma_1^2\sigma_s^2 \\ -\sigma_1^2\sigma_2^2 & \sum_{i\neq 2}^{s}\sigma_2^2\sigma_i^2 & \dots & -\sigma_2^2\sigma_s^2 \\ \vdots & \vdots & \ddots & \vdots \\ -\sigma_1^2\sigma_s^2 & \dots & \dots & \sum_{i\neq s}^{s}\sigma_s^2\sigma_i^2 \end{pmatrix} \tag{15}$$

where $D = \mathrm{diag}\{\sigma_1^2, \dots, \sigma_{\mathrm{s}}^2\}$. The multivariate variance-covariance matrix enforces the constraint that $S(L)\gamma_t$ is stationary. Since HS is a multivariate random walk the variance at each time-point is season specific and there is no element of spreading heteroscedasticity into different seasons as there was in the DS model. It should be noted that in the heteroscedastic case σ_r^2 is no longer the variance of the seasonal error in season r. The variance at season r is the r^{th} diagonal element of the matrix $\mathbf{V}$.

An interesting relationship between the variances of two seasonal effects errors arises in the heteroscedastic case of the HS model when only one season exhibits different variability. Let the r^{th} diagonal element of D be σ_r^2 while all other diagonal are σ^2. From (15) we see that the relationship between the variance in season r and the variances in any other season is the ratio of the diagonal elements in the r^{th} row over the diagonal element in any other row:

$$\frac{\mathbf{V}_{rr}}{\mathbf{V}_{jj}} = \frac{(s-1)\sigma_r^2\sigma^2}{(s-2)\sigma^4 + \sigma_r^2\sigma^2} = \frac{(s-1)\sigma_r^2}{(s-2)\sigma^2 + \sigma_r^2} = \frac{(s-1)}{(s-2)q+1} \tag{16}$$

where $q = \frac{\sigma^2}{\sigma_r^2} \geq 0$ and $j \neq r$. We can easily see that $\frac{\mathbf{V}_{rr}}{\mathbf{V}_{jj}}$ is a decreasing function with respect to q with a maximum at $s-1$ when $q = 0$. Consequently, the variance of the seasonal effect in season r can be at most $s-1$ times higher that the variance of the other seasons.

2.4 Periodic Irregular model

An alternative approach is to superimpose periodic heteroscedastic measurement noise on homoscedastic seasonality. It is similar to the deseasonalised model (Hipel and McLeod 1994) used in hydrological time series where the seasonal component γ_t is deterministic and ϵ_t has variance $\sigma_{\epsilon,r}^2$ varying with season, $r = 1, \dots, s$. In our model, γ_t is allowed to be stochastic with constant variance σ_ω^2. We can chose any of the three seasonal models for γ_t, with very similar results. In our applications we chose the Harrison-Stevens representation.

In order to illustrate the differences between HS heteroscedastic model and the periodic irregular models, we derive the autocovariance functions of the stationary forms for the local level and local linear trend model in the quarterly case. Denote the autocovariance function at lag τ in the r^{th} season ($r = 1, \ldots, s$) as $\delta_r(\tau)$. Then $\delta_r(\tau)$ is a function of season r as well as of the lag τ.

For HS heteroscedastic seasonal model (13) the autocovariance function of $\Delta_4 y_t$ for (3) is :

$$\begin{array}{lcllllll}
\delta_r(0) & = & 4\sigma_\eta^2 & +2\sigma_\epsilon^2 & +2c_r(0) & -2c_r(1) & \\
\delta_r(1) & = & 3\sigma_\eta^2 & & -c_r(0) & +2c_r(1) & -c_r(2) \\
\delta_r(2) & = & 2\sigma_\eta^2 & & & -c_r(1) & +c_r(2) \\
\delta_r(3) & = & \sigma_\eta^2 & & & & -c_r(2) \\
\delta_r(4) & = & -\sigma_\epsilon^2 & & & & \\
\delta_r(\tau) & = & 0 & \text{for} & \tau > 4 & &
\end{array} \tag{17}$$

where $c_r(\tau)$ with $r = t(\text{mod } s)$ is the autocovariance function of $S(L)\gamma_t$ at lag τ for season s. The autocovariance function of $\Delta_4 y_t$ for the periodic irregular model is :

$$\begin{array}{lcllllll}
\delta_r(0) & = & 4\sigma_\eta^2 & +2\sigma_{\epsilon,r}^2 & +2c(0) & -2c(1) & \\
\delta_r(1) & = & 3\sigma_\eta^2 & & -c(0) & +2c(1) & -c(2) \\
\delta_r(2) & = & 2\sigma_\eta^2 & & & -c(1) & +c(2) \\
\delta_r(3) & = & \sigma_\eta^2 & & & & -c(2) \\
\delta_r(4) & = & -\sigma_{\epsilon,r}^2 & & & & \\
\delta_r(\tau) & = & 0 & \text{for} & \tau > 4 & &
\end{array} \tag{18}$$

where $c(\tau)$ is the autocovariance function of $S(L)\gamma_t$ at lag τ, and γ_t follows a HS homoscedastic model. From (17), the autocovariance function of $\Delta_4 y_t$ for the HS heteroscedastic depends on season r for all lags up to $s-1$. Contrast this with (18), which shows that the periodic irregular has seasonal autocovariance only at lags 0 and s.

By adding a slope in the local level model we obtain the following model:

$$\begin{array}{lcll}
y_t & = & \mu_t + \gamma_t + \epsilon_t & \{\epsilon_t\} \sim \text{NID}(0, \sigma_\epsilon^2) \\
\mu_t & = & \mu_{t-1} + \beta_{t-1} + \eta_t & \{\eta_t\} \sim \text{NID}(0, \sigma_\eta^2) \\
\beta_t & = & \beta_{t-1} + \zeta_t & \{\zeta_t\} \sim \text{NID}(0, \sigma_\zeta^2)
\end{array} \tag{19}$$

with $\{\epsilon_t\}$, $\{\eta_t\}$ and $\{\zeta_t\}$ mutually uncorrelated. With seasonal factor γ_t following any of the seasonal models discussed in the previous sections, the stationary form of this model is:

$$\triangle\triangle_s y_t = S(L)\zeta_t + \triangle_s \eta_t + \triangle^2 S(L)\gamma_t + \triangle\triangle_s \epsilon_t$$

Table 1: Lags with periodic autocovariance

	Local level	Local linear trend
HS heteroscedastic	$0, 1, \ldots, s-1$	$0, 1, \ldots, s$
Per. Irregular	$0, s$	$0, 1, s-1, s, s+1$

where $S(L)\gamma_t$ is a $MA(s-2)$ for Harrison and Stevens seasonal model. In Table (1) we show the lags for which the autocovariance function of the stationary form of (3) and (19) is periodic for the Harrison and Stevens seasonal model and the periodic irregular model.

The periodicity in the autocovariance function implies that the relation between the heteroscedastic season and all other months is different in the periodic irregular model to the HS heteroscedastic model. In the periodic irregular model the relation between seasons is the same within the year. In the HS model, the relation between the heteroscedastic season and all other seasons differs from the relation between any two other seasons. If increased variability in a particular month is superimposed on the series, the periodic irregular model is more appropriate than any other model with seasonal heteroscedasticity. This is very common in hydrological series where high variability in one month is usually caused by extreme weather conditions, for example floods would usually occur in the same month but do not necessarily happen every year. On the other hand, in many economic time series high variability in one month is endogenous in the data generating process, for example factory owners react to lower production in one month and adapt the production in subsequent months to bring the overall output to the desired level. By comparing the periodic irregular model with HS we can infer whether heteroscedasticity is endogenous or superimposed on the data generating process.

In theory we are able to identify the appropriate model for a particular series by comparing the periodicity of the sample autocovariance with the theoretical autocovariance. There are several methods for testing for periodicities in the autocorrelation function, see for example (Vecchia and Ballerini 1991) and (Hurd and Gerr 1991). However, the power of these tests is very small for samples less than 30 years so they are impractical for many economic time series. Thus, we do not attempt to differentiate between seasonal models using the periodicities in the theoretical autocovariance. We rather use post-fit diagnostics and knowledge about specific series to chose the appropriate model.

3 Test for seasonal heteroscedasticity

Despite the fact that seasonal heteroscedasticity is relatively common, there are few methods to test for its presence. Existing tests are based of the likelihood ratio, Wald or Lagrange multiplier principles (Engle 1984). In practice, some version of Goldfeld and Quandt (1965) or White (1980) test for heteroscedasticity, adjusted for seasonal series is used. In fact these are mispecification tests rather than tests of a specific hypothesis.

In this section we suggest a likelihood ratio test for testing the hypothesis that one month exhibits different variance than the others, under the null hypothesis that all seasons have the same variance. With a parameter vector ψ, we denote the likelihood function of the null model as L_0 while the likelihood function of the alternative model that one month has different variance we denote as L_1. The likelihood ratio test statistics is then:

$$LR = 2(\log L_1 - \log L_0)$$

which is asymptotically distributed as χ_1^2 under the null hypothesis.

We use a Monte Carlo experiment to see the power of the test for several levels of heteroscedasticity. In the first experiment, 10000 replications of quarterly data with different length were simulated from a local level model with seasonality with irregular component

$$\epsilon_t \sim \begin{cases} \mathrm{N}(0, \sigma_1^2) & \text{for } s = 1 \\ \mathrm{N}(0, 1) & \text{otherwise} \end{cases}$$

The results are shown in Table 2.

In the second experiment a local level model with Harrison-Stevens seasonality with variance

$$\mathrm{Var}(\omega_\mathbf{t}) = \mathbf{V} = [\mathbf{D} - \frac{\mathbf{1}}{\mathbf{i_s D i_s'}} \mathbf{D i_s i_s' D}]$$

where

$$\mathbf{D} = \begin{pmatrix} \sigma_1^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The results are shown in Table (3).

From the results we see that when the null is true, that is for $\sigma_1^2 = 1$, the rejection frequency is very close to that of the χ_1^2 distribution. We can also see that the test performs better in the HS heteroscedastic case for relatively high values of q. On the other hand, for small values of q the test of the periodic irregular performs better.

Table 2: Percentage of rejecting the null hypothesis of no periodic variance

σ_1^2	N=40	N=80	N=120	N=240	N=480
0.1	9.67	23.3	38.6	69.9	95.0
0.5	5.14	9.41	13.3	22.2	39.0
1	4.03	5.03	4.68	4.49	5.00
2	6.54	9.17	12.7	40.9	68.8
5	44.9	74.4	89.2	99.5	100
10	79.5	97.5	99.8	100	100
100	99.8	100	100	100	100

10000 replications of a local level model + seasonality with periodic irregular

Table 3: Percentage of rejecting the null hypothesis of no periodic variance

σ_1^2	N=40	N=80	N=120	N=240	N=480
0.1	29.1	64.4	83.9	99.3	100
0.5	9.06	14.7	18.2	29.6	54.1
1	6.2	6.9	5.8	5.6	5.05
2	8.61	14.1	17.4	28.7	50.9
5	18.3	37.1	51.5	80.7	97.9
10	25.1	51.4	69.2	93.8	99.8
100	35.9	68.5	84.7	98.8	100

10000 replications of a local level model + Harrison-Stevens seasonality

Table 4: Italian Index of Production: parameter estimates

	Homoscedastic	Periodic irregular	Heteroscedastic
$\sigma^2_{\epsilon_1}$	$3.0\text{x}10^{-5}$	$3.7\text{x}10^{-5}$	$2.1\text{x}10^{-5}$
$\sigma^2_{\epsilon_2}$		0.007	
σ^2_{η}	$4.7\text{x}10^{-5}$	$5.0\text{x}10^{-5}$	$4.6\text{x}10^{-5}$
σ^2_{ζ}	$4.9\text{x}10^{-9}$	$1.0\text{x}10^{-10}$	$2.7\text{x}10^{-13}$
$\sigma^2_{\omega_1}$	$6.2\text{x}10^{-6}$	$6.1\text{x}10^{-7}$	$1.6\text{x}10^{-6}$
$\sigma^2_{\omega_2}$			$4.3\text{x}10^{-5}$
LR		75	114

4 Applications

We consider the problem of modelling index of production for some European countries which exhibit higher variability in one month. As mentioned before, economic theory predicts that the lowest season will exhibit higher variability. We consider the index of production for France, Italy and Spain. Figure (1) shows the monthly Index of Production for Italy from January 1960 to December 1997 and for France and Spain from January 1960 to January 2003. In all three countries the lowest month is August which coincides with the holiday season and the shut-down of several industrial units in most countries. The local linear trend model (19) was found more appropriate for these series with γ_t following the HS model. The models were estimated under the null of homoscedasticity and the alternative of seasonal heteroscedasticity and periodic variance of the irregular component. The algorithm used for estimation and signal extraction is the Kalman filter and smoother that was implemented in Ox (Doornik 1998). The parameter estimates for the three specifications are presented in Tables (4), (6), (8). $\sigma^2_{\epsilon_2}$ and $\sigma^2_{\omega_2}$ refer to the variance of the respective components for August. The likelihood ratio test for both heteroscedastic models is significant for Italy and France while for Spain only the HS heteroscedastic model is significant. Using the AIC criterion, we see that the best performance is observed for the HS heteroscedastic model rather than the periodic irregular model. Even though the ratio of the seasonal variances is very high there was no problem for distinguishing heteroscedastic from homoscedastic cases in the HS model. Intuitively, the better fit with the HS heteroscedastic model means that the high variability in August is a feature of the seasonal component which is balanced across all seasons.

We turn into an example to illustrate the periodic irregular model. Figure

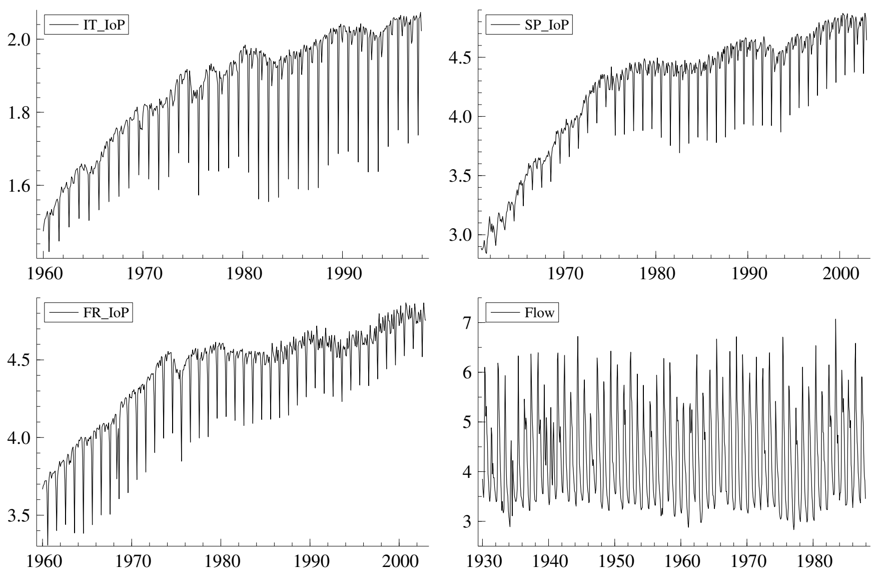

Figure 1: Series

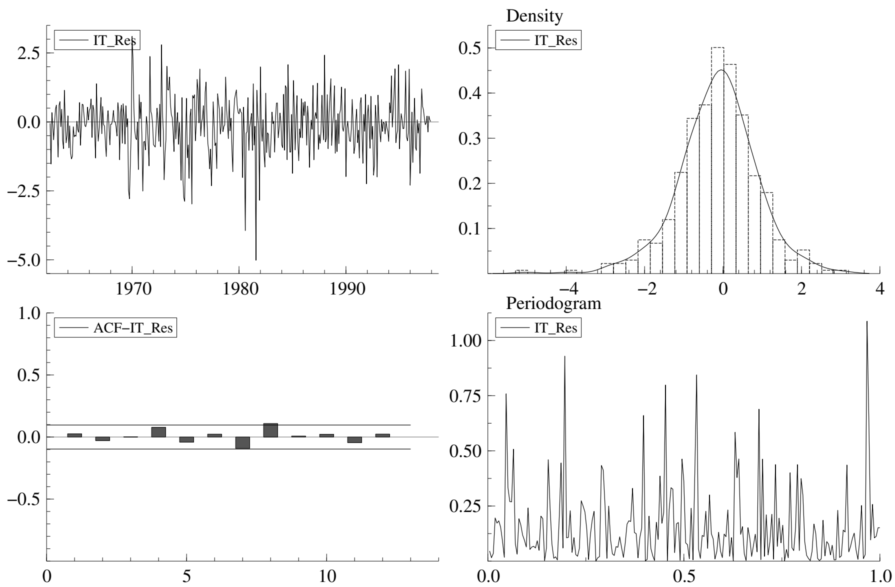

Figure 2: Italy IoP: Residuals

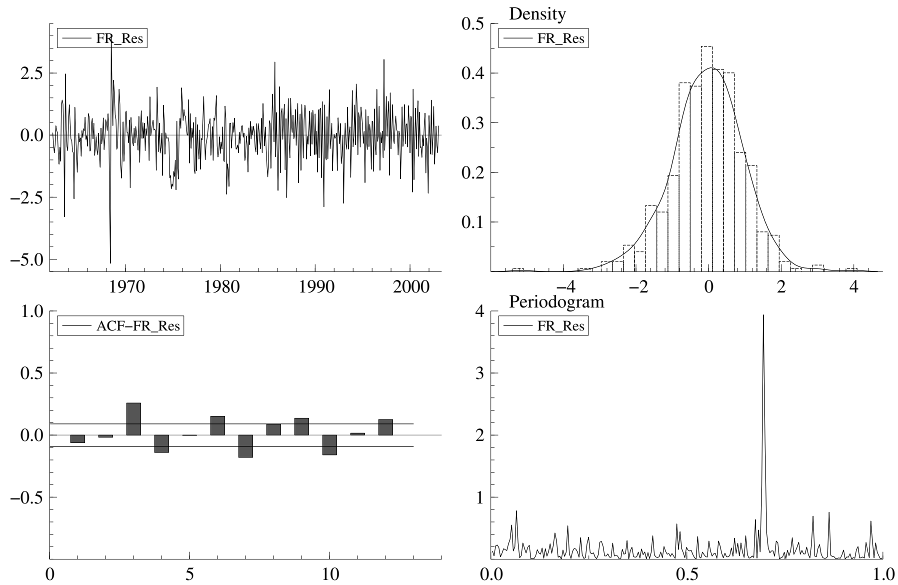

Figure 3: France IoP: Residuals

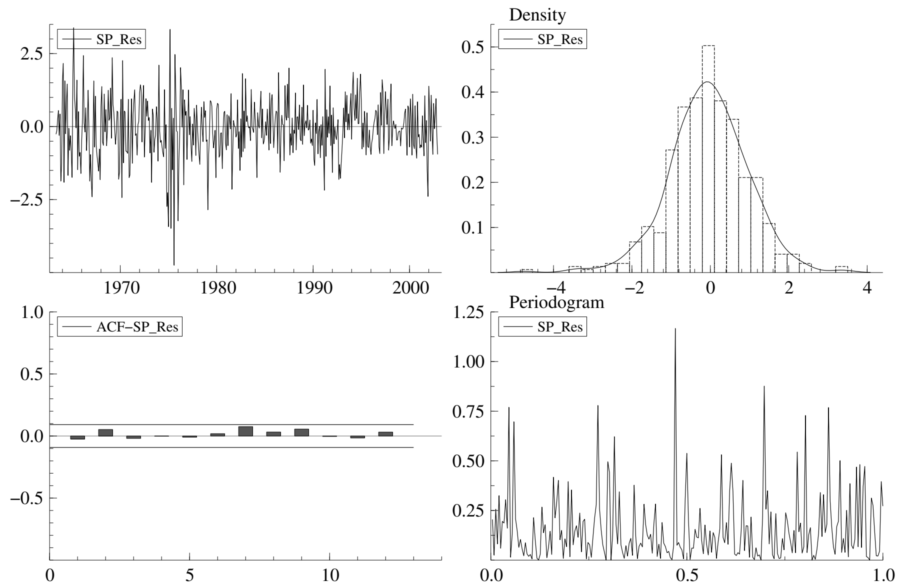

Figure 4: Spain IoP: Residuals

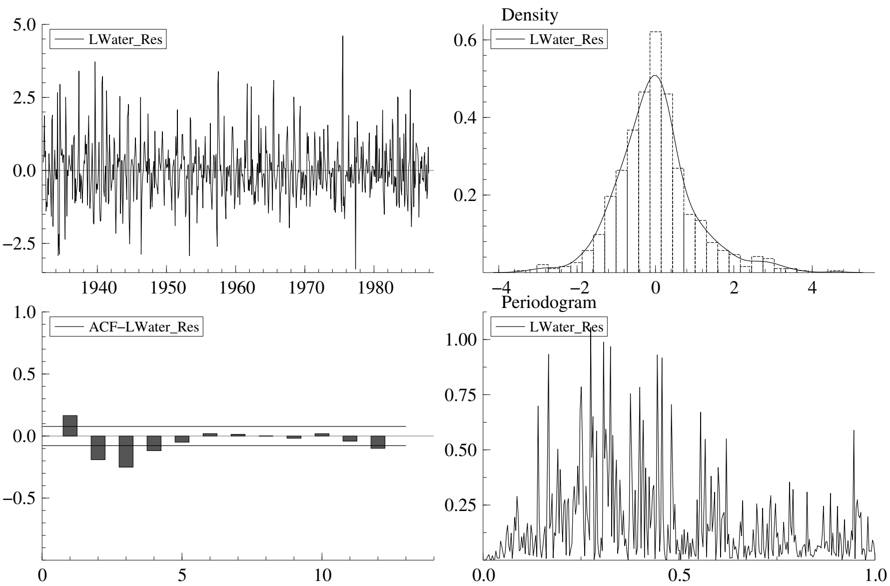

Figure 5: Whiterocks river flow: Residuals

Table 5: Italian Index of Production: diagnostics

	$r_1{}^a$	$r_{12}{}^b$	$\mathrm{Q}(12)^c$	S^d	K^e	N^f	AIC
Homoscedastic	-0.002	0.03	15.3	-0.8	7.7	450	-13.9
Periodic Irregular	0.007	0.08	21.0	0.0	4.2	25	-14.3
HS	0.026	0.02	15.0	-0.4	5.1	97	-14.9

[a] Residual autocorrelation at lag 1
[b] Residual autocorrelation at lag 12
[c] Ljung-Box statistic based on 12 residual autocorrelations
[d] Test for residual skewness
[e] Test for residual kurtosis
[f] Bowman-Shenton test for non-normality

Table 6: French Index of Production: parameter estimates

	Homoscedastic	Periodic irregular	Heteroscedastic
$\sigma^2_{\epsilon_1}$	0.0002	0.0002	0.0002
$\sigma^2_{\epsilon_2}$		0.001	
σ^2_{η}	0.0001	0.0001	0.0001
σ^2_{ζ}	2.6x10^{-8}	2.7x10^{-8}	2.7x10^{-8}
$\sigma^2_{\omega_1}$	1.8x10^{-5}	1.2x10^{-5}	5.6x10^{-6}
$\sigma^2_{\omega_2}$			0.0003
LR		17	49

A correction for an outlier in 1968.5 is included

Table 7: French Index of Production: diagnostics

	r_1	r_{12}	Q(12)	S	K	N	AIC
Homoscedastic	-0.05	0.08	83.4	-0.4	5.4	134	-11.5
Periodic Irregular	-0.05	0.11	94.4	-0.4	4.9	91.8	-11.6
HS	-0.06	0.12	109	-0.3	4.9	91.0	-11.7

Table 8: Spanish Index of Production: parameter estimates

	Homoscedastic	Periodic irregular	Heteroscedastic
$\sigma^2_{\epsilon_1}$	0.0002	0.0002	0.0002
$\sigma^2_{\epsilon_2}$		0.016	
σ^2_{η}	0.0001	0.0001	0.0001
σ^2_{ζ}	1.6x10^{-7}	1.5x10^{-7}	1.6x10^{-7}
$\sigma^2_{\omega_1}$	3.0x10^{-5}	1.3x10^{-5}	1.1x10^{-6}
$\sigma^2_{\omega_2}$			0.0002
LR		0	77

Table 9: Spanish Index of Production: diagnostics

	r_1	r_{12}	Q(12)	S	K	N	AIC
Homoscedastic	-0.05	0.004	15.7	-0.6	6.2	240	-11.1
Periodic Irregular	-0.04	0.06	6.7	-0.1	3.9	18	-11.1
HS	-0.02	0.03	7.7	-0.3	4.7	64	-11.4

Table 10: River flow in Whiterocks river: parameter estimates

	Homoscedastic	Periodic irregular	Heteroscedastic
$\sigma^2_{\epsilon_1}$	0.05	0.03	0.15
$\sigma^2_{\epsilon_2}$		0.22	
σ^2_{η}	0.03	0.03	0.03
$\sigma^2_{\omega_1}$	3.3x10^{-5}	3.0x10^{-5}	2.5x10^{-5}
$\sigma^2_{\omega_2}$			0.05
LR		74.1	1.13

Table 11: River flow in Whiterocks river: diagnostics

	r_1	r_{12}	Q(12)	S	K	N	AIC
Homoscedastic	0.17	-0.14	96.3	0.6	5.9	298	-2.2
Periodic Irregular	0.17	-0.10	107.0	0.6	5.1	168	-2.4
HS	0.18	-0.14	101	0.5	5.7	234	-2.2

1 also displays the monthly river flow in feet in logarithms for the Whiterocks river (Utah). The peak in river flow occurs between May and July. Variation in rainfall upstream would cause higher variability in the river flow in these months and especially in June as we see from the graph. The local level model (3) was found to describe better the series. The parameter estimates for the three models are presented in Table 10. The homoscedastic hypothesis is rejected only when the periodic irregular model is used. Moreover, the periodic irregular fits the data better that the seasonal heteroscedastic model. This means that the higher variability in one month is superimposed in the series. Tables (11), (5), (7), and (9) show some diagnostics for all models for the two series. Graphs (2),(3),(4), and (5) show some residual graphs for the models with the lowest AIC.

5 Conclusions

This paper discusses models for single season heteroscedasticity. Calculating the correct variance is essential for confidence interval estimation which is important for forecasting as well as descriptive purposes. Understanding that some months exhibit different seasonal variability from others is also essential for seasonal adjustment or for detecting relations between the seasonal and the business cycles. Economic time series often have one month with different variability from others. By comparing the different properties of some well known seasonal models, we concluded that seasonal heteroscedasticity using HS type of seasonality and the periodic irregular model are appropriate for modelling series where one season exhibits higher variance than others. We also presented a likelihood ratio test for the case where one month exhibits different variance than others. Monte Carlo analysis showed that the power of this test is very good for medium size sample (more than 20 years). We applied our outlined methodology to some real series. From the application, it seems that the HS heteroscedasticity model is more appropriate for economic time series. In these series, the higher variability in one month is a feature of the seasonal component. Economic agents have good knowledge of seasonality and are able to counteract the higher variability in one month by adjusting the production in all other months. On the other hand, in hydrology, there is no inherent mechanism to balance the higher variability in one month across all seasons. The higher variability is a result of an exogenous effect (rainfall in the case of the flow series) which is not balanced across the year. Modelling the seasonal variability has enhanced our understanding for the properties of the series.

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