On Pricing Derivatives under GARCH Models: A Dynamic Gerber-Shiu's Approach

Tak Kuen Siu*, Howell Tong† and Hailiang Yang‡

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Key words: Conditional Esscher transforms, Option pricing, GARCH Models, Infinitely divisible distributions, Dynamic utility framework

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*Tak Kuen Siu is an Assistant Professor in the Department of Mathematics, National University of Singapore, Singapore 119260, e-mail: tksiu2001@yahoo.com.hk

†Howell Tong, Hons. F.I.A., Ph.D., is the Chair Professor of the Department of Statistics and Actuarial Science, The University of Hong Kong, Pokfulam Road, Hong Kong, e-mail: htong@hku.hk. He is also a Chair Professor of the Department of Statistics, London School of Economics, U.K.

‡Hailiang Yang, A.S.A., Ph.D., is an Assistant Professor in the Department of Statistics and Actuarial Science, The University of Hong Kong, Pokfulam Road, Hong Kong, e-mail: hlyang@hkusua.hku.hk.

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Abstract

This paper proposes a method for pricing derivatives under the GARCH assumption for underlying assets in the context of a "dynamic" version of Gerber-Shiu's option-pricing model. Instead of adopting the notion of local risk-neutral valuation relationship (LRNVR) introduced by Duan (1995), we employ the concept of conditional Esscher transforms introduced by Bühlmann et al. (1996) to identify a martingale measure under the incomplete market setting. One advantage of our model is that it provides practitioners with an unified, direct and simple approach to deal with a wide class of parametric models for the innovation of the GARCH stock-price process. Another advantage is that a global preference-free risk-neutralized option price can be obtained in the case of conditionally normal stock innovation. An analytical pricing formula for a European call option can be obtained in this case. We can justify our pricing result within the dynamic framework of utility maximization problems which makes the economic intuition of our pricing result more transparent.

Key words: Conditional Esscher transforms, Option pricing, GARCH Models, Infinitely divisible distributions, Dynamic utility framework

§1. Introduction

Option pricing is one of the major areas in modern financial theory and practice. Since the introduction of the celebrated Black-Scholes option-pricing model, there is an explosive growth in the trading activities on derivatives in the worldwide financial markets. The main contribution of the seminal work of Black and Scholes (1973) and Merton

(1973) is the introduction of a preference-free option-pricing formula which does not involve an investor's risk preferences and subjective views. Due to its compact form and computational simplicity, the Black-Scholes formula enjoys a great popularity in the finance industries. One of the important economic insight underlying the preference-free option-pricing result is the concept of perfect replication of contingent claims by continuously adjusting a self-financing portfolio under the no-arbitrage principle. Cox, Ross and Rubinstein (1979) provided further insights in the concept of perfect replication by introducing the notion of risk-neutral valuation and establishing its relationship with the no-arbitrage principle in a transparent way under a discrete-time binomial setting. Harrison and Kreps (1979) and Harrison and Pliska (1981) established a solid mathematical foundation for the relationship between the no-arbitrage principle and the notion of risk-neutral valuation using the modern language of probability theory. They proposed the "Fundamental theorem for asset pricing" which states that the absence of arbitrage opportunities is equivalent to the existence of an equivalent martingale measure. If the securities market is complete, there is a unique martingale measure and hence the unique price of any contingent claim is given by its expected discounted payoff at expiry under the martingale measure. However, the assumption of market completeness is questionable in the real-world securities market. Under an incomplete market, there is more than one equivalent martingale measure and hence a range of no-arbitrage prices for a contingent claim. One crucial issue is to identify an equivalent martingale measure which gives an economically consistent and justifiable price for the contingent claim. Föllmer and Sondermann (1986), Föllmer and Schweizer (1991) and Schweizer (1996) identified a unique equivalent martingale measure by minimizing the variance of the hedging loss. In fact, the quadratic loss of the hedge position can be related to the concept of a quadratic utility (see Boyle and Wang (2001)). An interesting paper by Gerber and Shiu (1994) provided an elegant way to choose an equivalent martingale measure using the Esscher transformation, a time-honour tool in actuarial science for premium calculation, in an incomplete market setting. Their approach provides practitioners with the flexibility of choosing a wide variety of parametric models within the class of infinitely divisible distributions. The novelty of their approach is that the no-arbitrage price chosen by the Esscher transformation can be justified by maximizing the expected power utility of a representative agent. Their seminal work also provides an important insights in bridging the gap between the financial and insurance pricing problems in an incomplete market.

Bühlmann et al. (1996) generalized the classical notion of Esscher transform to stochastic processes.[1] They introduced the concept of conditional Esscher transforms in order to incorporate a richer theory of semi-martingale under the no-arbitrage condition in the context of Gerber-Shiu's option-pricing model.

Market incompleteness can arise through a variety of extensions to the standard Black-Scholes economy. One major stream of the extensions is the relaxation of the Black-Scholes assumption of Geometric Brownian Motion (GBM). Numerous models have been proposed to replace the stringent GBM assumption in the finance and actuarial science literature. In particular, the changing volatility (or heteroskedastic) models and the Lévy-type models (or its extension, the class of infinitely divisible distributions) are two major classes of general models which are of great interests from both theoretical and practical viewpoints. According to Cox (1981), the changing volatility models can be classified into the observation-driven GARCH models and parameter-driven stochastic volatility (SV) models. Recently, the ARCH-type models has gained its empirical success in modelling many "stylized" facts in financial time series, especially in modelling the changing variance and covariance structure of financial time series. It is a more realistic model for modelling the dynamics of underlying assets compared with GBM. However, it is the discrete-time and continuous-state nature of the ARCH-type (GARCH) models which makes the market incomplete and hence complicates the pricing issue. Duan (1995) introduced the notion of locally risk-neutral valuation relationship (LRNVR) which provides one way to choose a particular equivalent martingale pricing measure under the incomplete GARCH model with conditionally normal stock innovation. It has been shown empirically that Duan's GARCH option-pricing model can indeed explain some "stylized" systematic pricing biases of the Black-Scholes model (see Duan (1995)). One feature of the GARCH option-pricing model is that the LRNVR preserves the one-period ahead (local) conditional variance of the stock-price dynamic under changing the statistical probability measure to the risk-neutralized pricing measure. However, the global conditional variance process is not preserved by the change of probability measures via the LRNVR. It depends

[1] Siu, Tong and Yang (2001) provided another direction of generalizing the classical notion of Esscher transform, namely random Esscher transform, by assuming the Esscher parameter as a random variable. One application of random Esscher transform is to generate a family of random generalized "scenarios" for risk measurement.

on the constant unit risk premium of the underlying stock under the risk-neutralized probability measure. This makes the GARCH option price not preference-free. One may adopt other approaches to choose another risk-neutralized probability measure, for instance, the minimization of the variance hedge proposed by Föllmer and Sondermann (1986), Föllmer and Schweizer (1991) and Schweizer (1996) (see Härdle and Hafner (2000)). In any case, it seems to be difficult to obtain a preference-free option-pricing result from the existing approaches and a completely riskless hedging policy due to the impossibility of perfectly replicating the option's payoff at expiry. This is the case even when one considers the standard GARCH model with conditionally normal noise process.

In this paper, we propose an alternative approach for pricing derivatives under a general class of GARCH models in the context of a "dynamic" version of Gerber-Shiu's option-pricng model. We adopt the concept of conditional Esscher transforms introduced by Bühlmann et al. (1996) to identify an equivalent martingale pricing measure and hence the no-arbitrage price for a contingent claim or any security under the incomplete setting. Here, there are two major sources of market incompleteness, namely the discrete-time and continuous-state nature of the GARCH model and the non-normality of the GARCH innovation. One obvious advantage of the use of conditional Esscher transforms for picking an equivalent martingale measure is its capability in incorporating different infinitely divisible distributions for the GARCH innovations in an unified, direct and simple way. Indeed, our direction of generalization of the standard GARCH option-pricing model in Duan (1995) is different from the corresponding generalization in Duan (1999) which mainly considered the class of GARCH models with conditional fat-tailed distribution. One advantage of our generalization is the provision of a bridge between the changing volatility models and the Lévy-type models. Another advantage of our approach is that the choice of the no-arbitrage price by conditional Esscher transforms maximizes the conditional one-period expected power utility of a representative agent recursively. This constitutes a sound and intuitive economics-equilibrium argument to justify our choice of the no-arbitrage price. The concept of conditional Esscher transforms also provides an interesting linkage between the changing conditional variances of the GARCH model and the sequence of the conditional risk-averse parameters in the dynamic power utility framework. This gives an example to illustrate the applications of recursive or dynamic utility framework to option pricing. One interesting special case of our model is the case

of conditional normality for the GARCH innovation. In this case, we start with the same GARCH process as in Duan (1995) under the statistical probability measure. The main difference between our choice of no-arbitrage price and that in Duan (1995) is that our price for an European option is preference-free due to the fact that the conditional variance process does not involve the constant unit risk premium under the risk-neutralization using the conditional Esscher transform. We can also obtain an analytical pricing formula for an European call option using the techniques in Heston and Nandi (1997) under the conditional normality assumption. Last but not the least, our work further illustrates that Esscher transform, in particular its offspring conditional Esscher transform, is indeed a powerful tool for option valuation. We aim to highlight some implications to the interplay between mathematical finance and actuarial science by exploring the relationship between actuarial and financial pricing in an incomplete setting. See Embrechts (2000) for a detailed discussion about the interaction between financial and actuarial pricing. We organise this paper in the following way.

The next section presents the general setup of our model. We consider a discrete-time economy consisting of two primary assets, namely a risk-free bond and a risky stock in the context of Gerber-Shiu's option-pricing model. We assume that the distribution of the stock innovation is infinitely divisible with a finite moment generating function. First, we will present the probabilistic setup of our GARCH model. The construction of risk-neutralized probability measure using the concept of conditional Esscher transform and the martingale condition for the no-arbitrage price will be presented next. In particular, we focus on an arbitrary (possibly path-dependent) European option. Finally, we justify our choice of no-arbitrage price by the dynamic utility framework. Section three discusses some special cases of our model, namely the conditional normality for the stock innovation and the conditional shifted gamma distribution for the innovation. In the former special case, we illustrate how the unit risk premium is "absorbed" by the sequence of risk-aversa parameters under the changing volatility environment. An analytical pricing formula will be provided. In the latter special case, we can describe the dynamic of the logarithmic stock-price returns under the risk-neutralized probability measure. As in Gerber-Shiu's option-pricing model, the conditional distribution of the logarithmic returns of the underlying stock is still a shifted gamma distribution under the risk-neutralized measure. The final section concludes this paper and proposes some possible topics for further research.

§2. Option Pricing under GARCH Model via Conditional Esscher Transforms

We deal with the pricing problem of an European-type (possibly path-dependent) option written on the underlying stock S under the GARCH assumption for the dynamic of the underlying stock S. First, we consider a discrete-time financial model consisting of one risk-free bond B and one risky stock S in the context of Gerber-Shiu's option-pricing model. For the sake of generality, we do not impose any stringent parametric assumption on the noise process (innovation) of the underlying stock S. We only assume that the distribution of the noise process is infinitely divisible and that the moment generating function of the infinitely divisible distribution exists. The latter condition is a reasonable one since, otherwise, the conditional expectation of the one-period simple returns for the underlying asset is unbounded (see Duan (1999)). Examples for such a class of infinitely divisible distributions include the normal distribution, the gamma distribution, the Poisson distribution, the inverse-Gamma distribution and so forth. In this way, our model provides two major directions of generalization for the dynamics of the stock-price process, namely the relaxation of the time-independence of the stock returns and the normality assumption for the distribution of the stock innovation. Both directions of generalization provide sources for incompleteness of our market model. Bollerslev (1987), Baillie and Bollerslev (1989), Hsieh (1989), Baillie and DeGennaro (1990) and Wang et al. (1996) pointed out that the conditional normality assumption for the asset innovation fails to incorporate the leptokurtic behavior of asset returns. Another shortcoming of the conditional normality assumption is that it fails to incorporate the skewness in the conditional innovation of some financial time series, such as the exchange-rate time series (see Wang et al. (2000)). For the exchange-rate time series, it may be more appropriate to consider other conditional distributions for the innovation, such as conditional shifted gamma distributions and conditional shifted inverse Gaussian distributions, which can incorporate the positively skewed behavior of the exchange-rate time series. Duan (1999) extended the classical notion of LRNVR to generalized LRNVR (GLRNVR) in order to deal with the conditional non-normality of the GARCH innovation. The main focus of his approach is the use of functional transformations for a conditionally fat-tailed GARCH innovation in order to make it be a conditionally normal one. Here, we focus on a different distributional class of GARCH innovation. Due to the generality of our stock innovation,

our model can also be applied to deal with other financial time series which exhibit different potential skewed behavior. Instead of using the concept of local risk-neutral valuation relationship (LRNVR) introduced by Duan (1995), we employ the concept of conditional Esscher transforms introduced by Bühlmann et al. (1996) to construct a risk-neutralized pricing probability measure. In this way, we can give a no-arbitrage price for a derivative V under different infinitely divisible distributions beyond the normality assumption for the noise of the GARCH stock-price process. The concept of conditional Esscher transforms provides the degree of freedom for adjusting their corresponding parameters according to the changing conditional variance. It is well-suited for changing probability measures in the GARCH setting. The merit of conditional Esscher transforms is to provide an economically consistent way for choosing a martingale pricing measure under the GARCH setting. As in the Gerber-Shiu's option-pricing model, we can justify the pricing result by considering a dynamic utility maximization problem of a representative agent. In particular, we consider the maximization of the one-period conditional expected utility on the wealth of the representative agent given the market information up to and including the last period using a power utility function. The risk-averse parameter of the power utility can be adjusted dynamically according to the arrival of new market information. In fact, our model provides a transparent and intuitively appealing way to visualise the concept of risk-neutralization for pricing options under the GARCH assumption. We present the setup and the main idea of our model as follows.

Suppose $(\Omega, \mathcal{F}, P)$ is a given complete probability space, where P is the statistical or data-generating probability measure. Let $\mathcal{T}$ be the time index set $\{0, 1, 2, \ldots, T\}$ of our financial model such that all economic activities take place at each time point $t \in \mathcal{T}$. We equip our sample space $(\Omega, \mathcal{F})$ with a filtration $\Phi := \{\Phi_t\}_{t \in \mathcal{T}}$, that is, $\Phi_{t_1} \subseteq \Phi_{t_2}$ for any $t_1, t_2 \in \mathcal{T}$ with $t_1 \leq t_2$ and $\Phi_t \subseteq \mathcal{F}$ for all $t \in \mathcal{T}$. For each $t \in \mathcal{T}$, the sub-σ-algebra Φ_t of $\mathcal{F}$ is the information set representing all information up to and including time t for each $t \in \mathcal{T}$. For theoretical purposes, we impose the following assumptions for the filtration Φ.

Assumption 2.1:

1. $\Phi_0 = \{\phi, \Omega\}$ with ϕ being the empty set
2. Φ_1 is complete with respect to the measure P, that is, it contains all P-null sets

3. $\Phi_T = \mathcal{F}$

Let $\{\xi_t\}_{t\in\mathcal{T}}$ denote an Φ-adapted stochastic process on the probability space $(\Omega, \mathcal{F}, P)$ taking values on the real line R, with $\xi_0 = 0$, P-a.s., which represents the random fluctuations of the stock-price process. Denote by $\{h_t\}_{t\in\mathcal{T}}$ a Φ-predictable conditional variance process, that is h_t is Φ_{t-1}-measurable, for each $t \in \mathcal{T}\backslash\{0\}$. We assume that $\{\xi_t\}_{t\in\mathcal{T}}$ follows a GARCH process with orders p and q (GARCH(p, q)) under the statistical measure P. More precisely, under P,

1. For each $t \in \mathcal{T}\backslash\{0\}$, $\xi_t|\Phi_{t-1} \sim F(0, h_t)$, where $F(0, h_t)$ represents an infinitely divisible distribution with mean zero and conditional variance h_t.
2. For each $t \in \mathcal{T}\backslash\{0\}$,

$$h_t = \alpha_0 + \sum_{i=1}^{q} \alpha_i \xi_{t-i}^2 + \sum_{j=1}^{p} \beta_j h_{t-j} \ ,$$

   where $p \geq 1$, $q \geq 1$ and $\alpha_0 > 0$, $\alpha_i \geq 0$, $i \in \{1, 2, \ldots, q\}$, $\beta_j \geq 0$, $j \in \{1, 2, \ldots, p\}$ in order to ensure the positivity of h_t.

As in Duan (1995), in order to ensure covariance stationarity of the GARCH(p, q) model, we further impose the condition that

$$\sum_{i=1}^{q} \alpha_i + \sum_{j=1}^{p} \beta_j < 1$$

For detailed discussions of ARCH-type models, see Engle (1982), Bollerslev (1986), Taylor (1986), Tong (1990) and Gouriéroux (1997). Let r be the constant continuously compounded risk-free interest rate of the bond B and λ the constant unit risk premium representing a preference parameter. Then, we assume that, under the measure P, the dynamics of the bond-price process $\{B_t\}_{t\in\mathcal{T}}$ and the stock-price process $\{S_t\}_{t\in\mathcal{T}}$ satisfy:

$$\begin{aligned} B_t &= B_{t-1} e^r \ , \quad B_0 = 1 \ , \\ S_t &= S_{t-1} \exp(r + \lambda\sqrt{h_t} - \frac{1}{2} h_t + \xi_t) \ , \quad S_0 = s \ , \ t \in \mathcal{T}\backslash\{0\}. \end{aligned} \tag{2.1}$$

Here, the real-world stock-price dynamic under the measure P is the same as that in Duan (1995) except that our noise process is more general than the conditional normal noise process adopted in Duan (1995). Under the statistical measure P, the unit risk premium λ can be estimated from historical stock-price data using some statistical techiques, such as maximum likelihood estimation (MLE). The underlying assumption for the statistical estimation is that the stock-price process contains information about preferences. As noted in Heston and Nandi (1997), the return premium $\lambda\sqrt{h_t}$ in the real-world stock-price dynamic can serve the purpose of both incorporating risk preferences under the statistical measure P and precluding arbitrage opportunities. The latter follows from the fact that the logarithmic return $\ln(\frac{S_t}{S_{t-1}})$ of the stock S is the same as the risk-free interest rate r when the conditional variance h_t tends to zero.

For each $t \in \mathcal{T}\backslash\{0\}$, Y_t denotes the continuously compounded one-period rate of return $\ln(\frac{S_t}{S_{t-1}})$ of the stock S. Then, we can see that, under the measure P, the conditional distribution of Y_t given the information Φ_{t-1} is the infinitely divisible distribution $F(r + \lambda\sqrt{h_t} - \frac{1}{2}h_t, h_t)$ with conditional mean $r + \lambda\sqrt{h_t} - \frac{1}{2}h_t$ and conditional variance h_t. In the following, we construct conditional Esscher transforms for the GARCH process $\{Y_t\}_{t\in\mathcal{T}}$ associated with a sequence of conditional Esscher parameters $\{\theta_t\}_{t\in\mathcal{T}}$. See Bühlmann et al. (1996) for a detailed discussion about the concept of conditional Esscher transforms.

First, suppose that $\{\theta_t\}_{t\in\mathcal{T}\backslash\{0\}}$ is a Φ-predictable process, that is, the value of θ_t is known given Φ_{t-1} for each $t \in \mathcal{T}\backslash\{0\}$. Let $M_{Y_t|\Phi_{t-1}}(z)$ be the moment generating function of the conditional distribution Y_t given Φ_{t-1} under the statistical measure P, where $z \in R$. That is,

$$M_{Y_t|\Phi_{t-1}}(z) := E_P(e^{zY_t}|\Phi_{t-1}) = \int_{-\infty}^{\infty} e^{zy} dF(y|r + \lambda\sqrt{h_t} - \frac{1}{2}h_t, h_t) \ , \tag{2.2}$$

where $F(y|r + \lambda\sqrt{h_t} - \frac{1}{2}h_t, h_t) := P(Y_t \leq y|\Phi_{t-1})$ is the conditional distribution function of Y_t.

For each $t \in \mathcal{T}\backslash\{0\}$, we say that the moment generating function $M_{Y_t|\Phi_{t-1}}(z)$ exists at point z if $E_P(e^{zY_t}|\Phi_{t-1}) < \infty$. Assume that $M_{Y_t|\Phi_{t-1}}(\theta)$ exists, for all $t \in \mathcal{T}$. As in Bühlmann et al. (1996), we define a sequence $\{\Lambda_t\}_{t\in\mathcal{T}}$ with $\Lambda_0 = 1$ and

$$\Lambda_t = \prod_{k=1}^{t} \frac{e^{\theta_k Y_k}}{M_{Y_k|\Phi_{k-1}}(\theta_k)} \ , \ t \in \mathcal{T}\backslash\{0\}. \tag{2.3}$$

Then, it is easy to check that $\{\Lambda_t\}_{t \in \mathcal{T}}$ is a (Φ, P)-martingale. Write P_t for the restriction $P|\Phi_t$ of the statistical measure P on the σ-algebra Φ_t, for each $t \in \mathcal{T} \backslash \{0\}$, where $P_T = P$. We define a family of probability measures $\{P_{t,\Lambda_t}\}_{t \in \mathcal{T} \backslash \{0\}}$ such that

1. For each $t \in \mathcal{T} \backslash \{0\}$, the adjusted probability measure P_{t,Λ_t} equivalent to P_t on Φ_t is given by

$$dP_{t,\Lambda_t} = \Lambda_t dP_t. \tag{2.4}$$

2. $P_{t,\Lambda_t} = P_{t+1,\Lambda_{t+1}}|\Phi_t$, for each $t \in \mathcal{T} \backslash \{T\}$.

The first property is the Radon-Nikodym derivative of the adjusted measure P_{t,Λ_t} on Φ_t with respect to the restricted measure P_t while the second property is the consistent property for restricting the probability $P_{t+1,\Lambda_{t+1}}$ on the σ-algebra Φ_t. Now, by Bayes' rule, we have the following conditional probability given the information set Φ_{t-1} under P_{t,Λ_t}:

$$P_{t,\Lambda_t}(\{Y_t \in B\}|\Phi_{t-1}) = E_{P_t}\left(I\{Y_t \in B\} \frac{e^{\theta_t Y_t}}{E_{P_t}(e^{\theta_t Y_t}|\Phi_{t-1})} |\Phi_{t-1} \right) , \tag{2.5}$$

where B is a Borel subset of the real line and $I\{Y_t \in B\}$ represents the indicator function of the Φ_{t-1}-measurable event $I\{Y_t \in B\}$.

We call the conditional probability (2.5) the conditional Esscher transform and the associated parameter θ_t the conditional Esscher parameter given the information set Φ_{t-1} (see Bühlmann et al. (1996)). Note that we apply the Bayes's rule sequentially for changing one-period conditional probability measures in order to generate a family of conditional Esscher transforms and hence their corresponding one-period conditional distributions. Write $F(y; \theta_t|\Phi_{t-1})$ for the probability distribution of Y_t given Φ_{t-1} under the measure P_{t,Λ_t}. That is,

$$F(y; \theta_t|\Phi_{t-1}) := P_{t,\Lambda_t}(Y_t \leq y|\Phi_{t-1}). \tag{2.6}$$

From (2.5), $F(y; \theta_t|\Phi_{t-1})$ is given by:

$$dF(y; \theta_t|\Phi_{t-1}) = \frac{e^{\theta_t y} dF(y|r + \lambda\sqrt{h_t} - \frac{1}{2}h_t, h_t)}{M_{Y_t|\Phi_{t-1}}(\theta_t)} \tag{2.7}$$

Let $M_{Y_t|\Phi_{t-1}}(z;\theta_t)$ denote the moment generating function of the adjusted distribution function $F(y;\theta_t|\Phi_{t-1})$. That is,

$$M_{Y_t|\Phi_{t-1}}(z;\theta_t) := \int_{-\infty}^{\infty} e^{zy} dF(y;\theta_t|\Phi_{t-1}) \tag{2.8}$$

Then, we can see that $M_{Y_t|\Phi_{t-1}}(z;\theta_t)$ can be related to $M_{Y_t|\Phi_{t-1}}(z)$ as follows.

$$M_{Y_t|\Phi_{t-1}}(z;\theta_t) = \frac{M_{Y_t|\Phi_{t-1}}(z+\theta_t)}{M_{Y_t|\Phi_{t-1}}(\theta_t)}. \tag{2.9}$$

For pricing the derivative V, we construct an equivalent martingale measure $Q \sim P$ on $(\Omega, \mathcal{F})$ by adopting the concept of conditional Esscher transforms. First, we choose a sequence of conditional Esscher parameters $\{\theta_t^q\}_{t\in\mathcal{T}\backslash\{0\}}$ according to the following set of equations:

$$r = \ln\{M_{Y_t|\Phi_{t-1}}(1;\theta_t^q)\}\ ,\ t \in \mathcal{T}\backslash\{0\}. \tag{2.10}$$

Note that the sequence $\{\theta_t^q\}_{t\in\mathcal{T}\backslash\{0\}}$ of conditional Esscher parameters can be determined uniquely from the set of equations (2.10). Then, we can define a family of probability measures $\{P_{t,\Lambda_t^q}\}_{t\in\mathcal{T}\backslash\{0\}}$ associated with $\{\theta_t^q\}_{t\in\mathcal{T}\backslash\{0\}}$ as follows.

1. Generate a sequence $\{\Lambda_t^q\}_{t\in\mathcal{T}}$, with $\Lambda_0^q = 1$, P-a.s., by $\{\theta_t^q\}_{t\in\mathcal{T}\backslash\{0\}}$:

$$\Lambda_t^q = \prod_{k=1}^{t} \frac{e^{\theta_k^q Y_k}}{M_{Y_k|\Phi_{k-1}}(\theta_k^q)}. \tag{2.11}$$

2. For each $t \in \mathcal{T}\backslash\{0\}$, the probability measure P_{t,Λ_t^q} equivalent to P_t on Φ_t is given by

$$dP_{t,\Lambda_t^q} = \Lambda_t^q dP_t. \tag{2.12}$$

By definition, we can check that the family $\{P_{t,\Lambda_t^q}\}_{t\in\mathcal{T}\backslash\{0\}}$ satisfies the following consistency property:

$$P_{t,\Lambda_t^q} = P_{s,\Lambda_s^q}|\Phi_t\ ,\ s,t \in \mathcal{T} \text{with } t \le s. \tag{2.13}$$

Let $Q = P_{T,\Lambda_T^q}$ be a probability measure on $\mathcal{F} = \Phi_T$. Then, we have the following proposition for the discounted stock-price process $\{e^{-rt}S_t\}_{t\in\mathcal{T}}$:

Proposition 2.1: The discounted stock-price process $\{e^{-rt}S_t\}_{t\in\mathcal{T}}$ is a (Φ, Q)-martingale.

Proof: First, we show that for each $t \in \mathcal{T}\backslash\{0\}$,

$$S_{t-1} = E_{P_{t,\Lambda_t^q}}(e^{-r}S_t|\Phi_{t-1}). \tag{2.14}$$

Since the conditional distribution of Y_t given Φ_{t-1} under the measure P_{t,Λ_t^q} is $F(y;\theta_t^q|\Phi_{t-1})$, $M_{Y_t|\Phi_{t-1}}(z;\theta_t^q)$ is the corresponding moment generating function of Y_t given Φ_{t-1}. Then, by (2.10), we have

$$\begin{aligned} & E_{P_{t,\Lambda_t^q}}(e^{-r}S_t|\Phi_{t-1}) = S_{t-1}e^{-r}E_{P_{t,\Lambda_t^q}}(e^{Y_t}|\Phi_{t-1}) \\ = \quad & S_{t-1}e^{-r}M_{Y_t|\Phi_{t-1}}(1;\theta_t^q) = S_{t-1}. \end{aligned}$$

By the consistency property (2.13), for any $t_1, t_2, t_3 \in \mathcal{T}$ with $t_1 \leq t_2 \leq t_3$,

$$E_{P_{t_3,\Lambda_{t_3}^q}}(e^{-rt_2}S_{t_2}|\Phi_{t_1}) = E_{P_{t_2,\Lambda_{t_2}^q}}(e^{-rt_2}S_{t_2}|\Phi_{t_1}). \tag{2.15}$$

Then, we are going to show that, for any $u, t \in \mathcal{T}$ with $t < u$,

$$E_Q(e^{-ru}S_u|\Phi_t) = e^{-rt}S_t \ , \ P - a.s. \tag{2.16}$$

By using double expectation formulas, (2.14) and (2.15) iteratively, we get, for any $u, t \in \mathcal{T}$ with $t < u$,

$$\begin{aligned} & E_Q(e^{-ru}S_u|\Phi_t) = E_{P_{u,\Lambda_u^q}}(e^{-ru}S_u|\Phi_t) = E_{P_{u,\Lambda_u^q}}[E_{P_{u,\Lambda_u^q}}(e^{-ru}S_u|\Phi_{u-1})|\Phi_t] \\ = \quad & E_{P_{u,\Lambda_u^q}}(e^{-r(u-1)}S_{u-1}|\Phi_t) = E_{P_{u-1,\Lambda_{u-1}^q}}(e^{-r(u-1)}S_{u-1}|\Phi_t) \\ = \quad & \ldots = E_{P_{t+1,\Lambda_{t+1}^q}}(e^{-r(t+1)}S_{t+1}|\Phi_t) = e^{-rt}S_t \ , \ P - a.s. \end{aligned}$$

Hence, $\{e^{-rt}S_t\}_{t\in\mathcal{T}}$ is a (Φ, Q)-martingale.

□

By Proposition 2.1, there is an equivalent martingale measure $Q \sim P$ on $\mathcal{F}$ under which the discounted stock-price process is a martingale with respect to the filtration Φ.

By the fundamental theorem of asset pricing (see Harrison and Kreps (1979), Harrison and Pliska (1981), Dybyig and Ross (1987) and Delbaen and Schachermayer (1994)), Proposition 2.1 implies that there is no arbitrage opportunity in our market model. Under the no-arbitrage market model, we can define a no-arbitrage price for the derivative V at time $t \in \mathcal{T}$ by its discounted expected terminal payoff under Q as follows.

$$V_t = E_Q(e^{-r(T-t)} V_T | \Phi_t). \tag{2.17}$$

We call Q a conditional risk-neutralized Esscher pricing measure. The price (2.17) is only one particular choice for a no-arbitrage price. If the market is incomplete (as usual in a general GARCH setting), there is more than one equivalent martingale measure each with its corresponding no-arbitrage price. Hence, one crucial question is how to justify our particular choice of the price. Instead of using the Euler's condition in Duan (1995), we justify our pricing result by solving a dynamic utility maximization problem of a representative agent as in Gerber-Shiu's option-pricing model. We present the main idea as follows.

Let $\{\gamma_t\}_{t \in \mathcal{T}}$ denote an Φ-adapted stochastic process on $(\Omega, \mathcal{F})$. We define the following sequence of random utility functions $\{u_t\}_{t \in \mathcal{T}}$ on the real line associated with $\{\gamma_t\}_{t \in \mathcal{T}}$:

$$u_t(x) = \begin{cases} \frac{x^{1-\gamma_t}}{1-\gamma_t} & \text{if } \gamma_t \neq 1, \\ \ln x & \text{if } \gamma_t = 1. \end{cases}$$

For each fixed $x \in R$, $u_t(x)$ is Φ_t-measurable. $u_t : R \to R$ represents a power utility function with paramter γ_t, for each $t \in \mathcal{T}$. As in Gerber-Shiu's option-pricing model, we assume that a representative agent makes financial decisions according to the sequence of utility functions $\{u_t\}_{t \in \mathcal{T}}$. In particular, the representative agent adjusts or decides dynamically the risk-averse parameter γ_t based on the information Φ_t up to and including time t, for each $t \in \mathcal{T}$. Following Gerber and Shiu (1994), we impose the following assumptions about the representative agent.

Assumption 2.2:

1. The representative agent has m_t units of stock S over the time horizon $[t, t+1)$,

where m_t can be decided according to the information Φ_t up to and including time t, for each $t \in \mathcal{T} \backslash \{T\}$.

2. For each $t \in \mathcal{T}$, $\tilde{V}_t$ is the representative agent's price of the derivative V at time t with $\tilde{V}_T = V_T$, such that it is optimal for the representative agent not to buy or sell any unit of the derivative V at time t.

The second statement of Assumption 2.2 can be related to a dynamic version of the variational argument in actuarial science. See Gerber and Shiu (2000) for the applications of the variational arguement to the problem of dynamic asset allocation. For each $t \in \mathcal{T} \backslash \{T\}$, we define the following conditional expected utility function H_t on the representative agent's wealth at time $t+1$ given the information Φ_t up to and including time t:

$$H_t(\eta_t) = E_P\{u_t(m_t S_{t+1} + \eta_t[\tilde{V}_{t+1} - e^r \tilde{V}_t])|\Phi_t\}, \tag{2.18}$$

where η_t represents the number of units of the derivative V held by the representative agent over the time horizon $[t, t+1)$, which is the only choice variable for the maximization of H_t.

By adopting the approach used in Gerber and Shiu (1994) under our dynamic setting, we justify our pricing result in the following proposition.

Proposition 2.2: For each $t \in \mathcal{T} \backslash \{T\}$, $\tilde{V}_t = V_t$.

Proof: The idea of the proof is similar to that in Gerber and Shiu (1994). First, by translating the second assumption in Assumption 2.2 mathematically, we have that $H_t(\eta_t)$ attains its maximum value when $\eta_t = 0$, for each $t \in \mathcal{T} \backslash \{0\}$. This implies that

$$H_t'(0) = 0, \tag{2.19}$$

where $H_t'(\eta_t)$ is the derivative of $H_t(\eta_t)$ with respect to η_t.

From (2.19), the Φ-adapted price process $\{\tilde{V}_t\}_{t \in \mathcal{T}}$ of the representative agent for the derivative V satisfies the following recursive equations:

$$\tilde{V}_t = e^{-r} \frac{E_P(\tilde{V}_{t+1} {S_{t+1}}^{-\gamma_t} | \Phi_t)}{E_P({S_{t+1}}^{-\gamma_t} | \Phi_t)}\ ,\ t \in \mathcal{T} \backslash \{T\}. \tag{2.20}$$

In fact, for all derivative securities, their corresponding representative agent's price processes must satisfy the recursive relation (2.20). In particular, if we consider the trivial derivative instrument, namely the stock S, the recursion (2.20) becomes

$$S_t = e^{-r} \frac{E_P({S_{t+1}}^{1-\gamma_t}|\Phi_t)}{E_P({S_{t+1}}^{-\gamma_t}|\Phi_t)} \ , \ t \in \mathcal{T}\backslash\{T\}. \tag{2.21}$$

From (2.21), we get

$$\begin{aligned} S_t &= e^{-r} S_t \frac{E_P\{\exp[(1-\gamma_t)Y_{t+1}]|\Phi_t\}}{E_P[\exp(-\gamma_t Y_{t+1})|\Phi_t]} \\ &= e^{-r} S_t \frac{E_{P_{t+1}}\{\exp[(1-\gamma_t)Y_{t+1}]|\Phi_t\}}{E_{P_{t+1}}[\exp(-\gamma_t Y_{t+1})|\Phi_t]} \\ &= e^{-r} S_t \frac{M_{Y_{t+1}|\Phi_t}(1-\gamma_t)}{M_{Y_{t+1}|\Phi_t}(-\gamma_t)} \ , \ t \in \mathcal{T}\backslash\{T\}. \end{aligned} \tag{2.22}$$

Hence, by (2.9), (2.10) and (2.22), we have

$$\begin{aligned} M_{Y_{t+1}|\Phi_t}(1;\theta^q_{t+1}) &= e^r = \frac{M_{Y_{t+1}|\Phi_t}(1-\gamma_t)}{M_{Y_{t+1}|\Phi_t}(-\gamma_t)} \\ &= M_{Y_{t+1}|\Phi_t}(1;-\gamma_t) \ , \ t \in \mathcal{T}\backslash\{T\}. \end{aligned} \tag{2.23}$$

By the uniqueness of the conditional Esscher parameter, for each $t \in \mathcal{T}\backslash\{T\}$,

$$\gamma_t = -\theta^q_{t+1}. \tag{2.24}$$

Therefore, using (2.24) and Bayes' rule, the recursive equation (2.20) can be written as

$$\begin{aligned} \tilde{V}_t &= e^{-r} \frac{E_P(\tilde{V}_{t+1}{S_{t+1}}^{\theta^q_{t+1}}|\Phi_t)}{E_P({S_{t+1}}^{\theta^q_{t+1}}|\Phi_t)} = e^{-r} \frac{E_P[\tilde{V}_{t+1}\exp(\theta^q_{t+1}Y_{t+1})|\Phi_t]}{E_P[\exp(\theta^q_{t+1}Y_{t+1})|\Phi_t]} \\ &= e^{-r} \frac{E_{P_{t+1}}[\tilde{V}_{t+1}\exp(\theta^q_{t+1}Y_{t+1})|\Phi_t]}{E_{P_{t+1}}[\exp(\theta^q_{t+1}Y_{t+1})|\Phi_t]} = e^{-r} E_{P_{t+1,\Lambda^q_{t+1}}}(\tilde{V}_{t+1}|\Phi_t). \end{aligned} \tag{2.25}$$

As in the proof of Proposition 2.1, we use (2.25) and the consistency property (2.13) to get

$$\begin{aligned} \tilde{V}_t &= e^{-r} E_{P_{t+1,\Lambda^q_{t+1}}}(\tilde{V}_{t+1}|\Phi_t) = e^{-r(T-t)} E_{P_{T,\Lambda^q_T}}(\tilde{V}_T|\Phi_t) \\ &= e^{-r(T-t)} E_Q(V_T|\Phi_t) = V_t \ , \ t \in \mathcal{T}. \end{aligned} \tag{2.26}$$

□

Proposition 2.2 states that the representative agent's price process $\{\tilde{V}_t\}_{t\in\mathcal{T}}$ coincides with our price process $\{V_t\}_{t\in\mathcal{T}}$ in order to give the representative agent no incentive to buy or sell any fraction or multiple of the derivative V at any time point $t \in \mathcal{T}$. This justifies our pricing result under the market equilibrium. Hence, the price of the derivative V at time t is given by (2.17). An interesting feature for the choice of the conditional risk-neutralized Esscher pricing measure Q is that it satisfies the locally risk-neutral valuation relationship (LRNVR) proposed by Duan (1995) under some parametric distributional assumptions on the noise process, for instance, the conditional normality assumption for the noise process. In fact, the constant unit risk premium λ can be eliminated completely from the stock-price process and the conditional variance process under the conditional Esscher risk-neutraliztion. We will discuss the detail in the next section.

§3. Some Special Cases

In the following, we consider some special cases of our model by imposing some specific parametric assumptions on the conditional distribution of the noise ξ_t given Φ_{t-1}. First, we deal with the case of conditional normality for the noise process, that is, we start with the same stock-price and conditional-variance processes as those in Duan (1995) under the statistical measure P. Due to the discrete-time nature of GARCH models, the market is still incomplete even when we impose the conditional normality assumption. This implies that there is more than one equivalent martingale measure, and hence, a range of no-arbitrage prices for derivatives. There are different ways to choose an equilibrum price from the range of no-arbitrage prices. Duan (1995) adopted the LRNVR to specify one particular choice of no-arbitrage price which can be justified as an equilibrium price by the Euler's condition, whereas we use the concept of conditional Esscher transforms to fix up another no-arbitrage price and justify the no-arbitrage price as an equilibrium price which solves a dynamic utility maximization problem of a representative agent as in Gerber-Shiu's model. One interesting feature of our approach is that the constant unit risk premium λ has been "absorbed" by the conditional risk-neutralized Esscher parameters $\{\theta_t^q\}_{t\in\mathcal{T}\setminus\{0\}}$, which are dynamically updated by the changing real-world conditional variances, and hence, the constant unit risk premium λ does not appear in our

risk-neutralized stock-price and conditional-variance dynamics. This somehow suggests that our pricing model is preference-free in the case of conditional normality for the noise process. The conditional risk-neutralized pricing measure Q not only satisfies the locally risk-neutral valuation relationship, but also admits a relatively global risk-neutralization for the stock-price and conditional variance processes. By adopting the techniques in Heston and Nandi (1997), we can obtain an analytical pricing formula for a European call option. Then, we consider the case of conditional shifted gamma distribution for the stock innovation, that is, the conditional distribution of Y_t given Φ_{t-1} is a shifted gamma distribution with an adjusted (shifted) parameter equal to the negative of the conditional expectation of Y_t given Φ_{t-1} under the statistical measure P. As in Gerber-Shiu's option pricing model, the conditional distribution of Y_t given Φ_{t-1} is again a shifted gamma distribution with a risk-neutralized scale parameter under the Q-measure. However, the risk-neutralized conditional variance is no longer a GARCH process. We can only obtain a formula for updating the risk-neutralized conditional variance process according to the real-world conditional variance process. Finally, we would like to point out that the homoskedastic case of our GARCH option-pricing model coincides with the discrete-time version of Gerber-Shiu's option pricing model. The homoskedastic case is still very flexible to incorporate different infinitely divisible distributions for the stock-price returns as long as the moment generating function for the infinitely divisible distributions exists.

§3.1. Conditional Normality for the Noise Process

We consider the case that the conditional distribution of the noise ξ_t given Φ_{t-1} follows a normal distribution with conditional mean zero and conditional variance h_t under the statistical measure P. In this case, we can also obtain a formula for the conditional risk-neutralized Esscher parameter θ_t^q, for each $t \in \mathcal{T} \backslash \{0\}$. By using the same technique as in Heston and Nandi (1997), we can obtain an analytical pricing formula for a European call option.

Suppose that $\{\xi_t\}_{t \in \mathcal{T}}$ follows a GARCH (p, q) process with the conditional distribution $F(0, h_t)$ given Φ_{t-1} being a normal distribution with conditional mean zero and variance h_t under the statistical measure P. Then, the conditional distribution of Y_t given Φ_{t-1} under the measure P is a normal distribution with mean $r + \lambda\sqrt{h_t} - \frac{1}{2}h_t$ and variance

h_t. For each $t \in \mathcal{T}\backslash\{0\}$, the conditional risk-neutralized Esscher parameter θ_t^q is given by the following proposition.

Proposition 3.1: For each $t \in \mathcal{T}\backslash\{0\}$,

$$\theta_t^q = -\frac{\lambda}{\sqrt{h_t}} \ , \tag{3.1}$$

and hence

$$\gamma_{t-1} = \frac{\lambda}{\sqrt{h_t}}. \tag{3.2}$$

Proof: By (2.9) and (2.10), we have

$$\begin{aligned} r &= \ln\{M_{Y_t|\Phi_{t-1}}(1;\theta_t^q)\} = \ln\left\{\frac{M_{Y_t|\Phi_{t-1}}(1+\theta_t^q)}{M_{Y_t|\Phi_{t-1}}(\theta_t^q)}\right\} \\ &= \ln\left\{\frac{\exp[(1+\theta_t^q)(r+\lambda\sqrt{h_t}-\frac{1}{2}h_t)+\frac{1}{2}(1+\theta_t^q)^2 h_t]}{\exp[\theta_t^q(r+\lambda\sqrt{h_t}-\frac{1}{2}h_t)+\frac{1}{2}\theta_t^{q2}h_t]}\right\} \\ &= r+\lambda\sqrt{h_t}+\theta_t^q h_t \ , \ t \in \mathcal{T}\backslash\{0\}. \end{aligned}$$

This implies the results (3.1) and (3.2) follows immediately from (2.24).

□

As time goes by, the evolution of $\{\theta_t^q\}_{t\in\mathcal{T}\backslash\{0\}}$ can capture the changing nature of the conditional variance h_t. Then, by (2.9) and (3.1), the moment generating function $M_{Y_t|\Phi_{t-1}}(z;\theta_t^q)$ is given by

$$\begin{aligned} M_{Y_t|\Phi_{t-1}}(z;\theta_t^q) &= \frac{M_{Y_t|\Phi_{t-1}}(z+\theta_t^q)}{M_{Y_t|\Phi_{t-1}}(\theta_t^q)} \\ &= \frac{\exp[(z+\theta_t^q)(r+\lambda\sqrt{h_t}-\frac{1}{2}h_t)+\frac{1}{2}(z+\theta_t^q)^2 h_t]}{\exp[\theta_t^q(r+\lambda\sqrt{h_t}-\frac{1}{2}h_t)+\frac{1}{2}\theta_t^{q2}h_t]} \\ &= \exp[z(r+\lambda\sqrt{h_t}-\frac{1}{2}h_t+\theta_t^q h_t)+\frac{1}{2}z^2 h_t)] \\ &= \exp[z(r-\frac{1}{2}h_t)+\frac{1}{2}z^2 h_t)]. \end{aligned} \tag{3.3}$$

Hence, under Q, the conditional distribution of Y_t given Φ_{t-1} is a normal distribution with conditional mean $r-\frac{1}{2}h_t$ and conditional variance h_t, where h_t is given by

$$h_t = \alpha_0 + \sum_{i=1}^{q}\alpha_i\xi_{t-i}^2 + \sum_{j=1}^{p}\beta_j h_{t-j} \ , \ t \in \mathcal{T}\backslash\{0\}.$$

This reveals that the GARCH (p, q) structure for the conditional variance process $\{h_t\}_{t\in\mathcal{T}}$ and the conditional normality for the rate of return process $\{Y_t\}_{t\in\mathcal{T}}$ are preserved under changing probability measures from P to Q via the concept of conditional Esscher transforms. The conditional risk-neutralized Esscher pricing measure Q satisfies the locally risk-neutral valuation relationship (LRNVR) proposed in Duan (1995) in the case of conditional normality for the noise process. That is, Q satisfies the following three conditions:

1. Q is equivalent to the statistical measure P on $(\Omega, \mathcal{F})$.
2. Under Q, $\frac{S_t}{S_{t-1}}$ is lognormally distribution.
3. $E_Q(\frac{S_t}{S_{t-1}}|\Phi_{t-1}) = e^r$, P-a.s.
4. $Var_Q[\ln(\frac{S_t}{S_{t-1}})|\Phi_{t-1}] = Var_P[\ln(\frac{S_t}{S_{t-1}})|\Phi_{t-1}]$, P-a.s. That is, the conditional variance of the logarithmic return $\ln(\frac{S_t}{S_{t-1}})$ is invariant under the change of probability measures from P to Q almost surely with respect to P.

Since the constant unit risk premium λ is completely "absorbed" by the conditional risk-neutralized Esscher parameter θ_t^q, it does not appear in our risk-neutralized stock-price and conditional-variance dynamics. This somehow suggests that our pricing model is preference-free in the case of conditional normality for the noise process. For the valuation of path-independent options, it is convenient for us to write the terminal stock price as a function of the current stock price under the Q-measure as follows.

$$S_T = S_t \exp\{r(T-t) - \frac{1}{2}\sum_{s=t+1}^{T} h_s + \sum_{s=t+1}^{T} \xi_s\}. \tag{3.4}$$

Apparently, the stock-price dynamic (3.4) under Q is the same as that in Duan (1995). In fact, this is not the case since the conditional variance of the stock-price dynamic under Q does not involve a parameter representing the constant unit risk premium of the stock. It only depends on the GARCH parameters or coefficients which can be estimated easilty from empirical data. In the following, we derive the analytical pricing formula for a European call option by adopting the same technique as in Heston and Nandi (1997). First, we consider a European call option C written on the underlying stock S, with

strike price K and expiry at time T. Write $X_t = \ln S_t$ and $M_Q^X(t,T;z)$ the moment generating function of X_T given Φ_t under the Q-measure. We assume that the moment generating function $M_Q^X(t,T;z)$ is of the following log-linear form.

$$\begin{aligned} M_Q^X(t,T;z) &= \exp(zX_t + A(t,T;z) + \sum_{j=1}^{p} B_j(t,T;z)h_{t+2-j} + \\ &\quad \sum_{k=1}^{q-1} C_k(t,T;z)\xi_{t+1-k}^2). \end{aligned} \tag{3.5}$$

As in Heston and Nandi (1997), we derive a system of recursive formulas for the calculations of the coefficients $A(t,T;z)$, $B_j(t,T;z)$ and $C_k(t,T;z)$, for $j = 1,2,\ldots,p$ and $k = 1,2,\ldots,q-1$ by the method of undetermined coefficients and first step analysis. The following proposition summarizes the result.

Proposition 3.2:

1. At time $t = T-1$,

$$\begin{aligned} &A(T-1,t;z) = rz \\ &B_1(T-1,T;z) = \frac{1}{2}z(z-1) \\ &B_j(T-1,T;z) = 0\ ,\ j = 1,2,\ldots,p\ , \\ &C_k(T-1,T;z) = 0\ ,\ k = 1,2,\ldots,q-1\ . \end{aligned}$$

2. For $t = 1,2,\ldots,T-2$,

$$\begin{aligned} &A(t,T;z) = rz + A(t+1,T;z) + B_1(t+1,T;z)\alpha_0 - \\ &\frac{1}{2}\ln(1 - 2B_1(t+1,T;z)\alpha_1 - 2C_1(t+1,T;z)) \\ &+\frac{z^2}{1 - 2B_1(t+1,T;z) - 2C_1(t+1,T;z)}\ , \\ &B_1(t,T;z) = B_1(t+1,T;z)\beta_1 - \frac{1}{2}z + B_2(t+1,T;z)\ , \\ &B_j(t,T;z) = \beta_j B_1(t+1,T;z) + B_{j+1}(t+1,T;z)\ ,\ j = 1,2,\ldots,p\ , \\ &C_k(t,T;z) = B_1(t+1,T;z)\alpha_{k+1} + C_{k+1}(t+1,T,z)\ ,\ k = 1,2,\ldots,q-1\ , \\ &B_{p+1}(t,T;z) = C_q(t,T;z) = 0\ . \end{aligned}$$

Proof: The proof of Proposition 3.2 is adapted to the proof of Proposition 2 in Heston and Nandi (1997). The proof of the first result relies on the fact that the conditional distribution of X_T given Φ_{T-1} is a normal distribution with conditional mean $\ln S_{T-1} + r - \frac{1}{2}h_T$ and variance h_T. The proof of the second result rests on the use of first step analysis and the following property of a standard normal random variable Z, namely

$$E\{\exp[a_1(Z+a_2)^2]\} = \exp[-\frac{1}{2}\ln(1-2a_1) + \frac{a_1 a_2^2}{1-2a_1}], \tag{3.6}$$

where $a_1, a_2 \in R$.

By the first step analysis (or double expectation formula) and (3.5),

$$\begin{aligned}
M_Q^X(t,T;z) &= E_Q\{E_Q[\exp(zX_T)|\Phi_{t+1}]|\Phi_t\} \\
&= E_Q\{M_Q^X(t+1,T;z)|\Phi_t\} \\
&= E_Q\Big\{\exp\Big[zX_{t+1} + rz + A(t+1,T;z) + \sum_{j=1}^{p} B_j(t+1,T,z)h_{t+3-j} + \\
&\quad \sum_{k=1}^{q-1} C_k(t+1,T;z)\xi_{t+2-k}^2\Big]\Big|\Phi_t\Big\} \\
&= \exp\{zX_t + rzA(t+1,T;z) + B_1(t+1,T;z)\alpha_0 - \frac{1}{2}\ln[1-2B_1(t+1,T;z) \\
&\quad -2C_1(t+1,T;z)] + \frac{z^2}{1-2B_1(t+1,T;z)-2C_1(t+1,T;z)} + \\
&\quad (B_1(t+1,T;z)\beta_1 - \frac{1}{2} + B_2(t+1,T;z))h_{t+1}) + \\
&\quad \sum_{j=2}^{p}(\beta_j B_1(t+1,T;z) + B_{j+1}(t+1,T;z))h_{t+2-j} \\
&\quad \sum_{k=1}^{q-1}(B_1(t+1,T;z)\alpha_{k+1} + C_{k+1}(t+1,T;z))\xi_{t+1-k}^2\}.
\end{aligned} \tag{3.7}$$

By convention, we assume the following starting values for the coefficients:

$$B_{p+1}(t+1,T;z) = C_q(t+1,T;z) = 0.$$

The proof is completed by equating the coefficients of (3.5) and (3.7).

□

Let i denote $\sqrt{-1}$. Then, the characteristic function of X_T given Φ_t under Q is given

by:

$$M_Q^X(t,T;iz) = S_t^{iz}\exp(A(t,T;iz) + \sum_{j=1}^{p} B_j(t,T;iz)h_{t+2-j} + \sum_{k=1}^{q-1} C_k(t,T;iz)\xi_{t+1-k}^2) \ , \tag{3.8}$$

where the coefficients $A(t,T;iz)$, $B_j(t,T;iz)$ and $C_k(t,T;iz)$, for $j = 1,2,\ldots,p$ and $k = 1,2,\ldots,q-1$, can be determined by the recursive formulas in Proposition 2.4 with the real parameter z replaced by the purely imaginary parameter iz.

Let $Q_{X_T|\Phi_t}(x)$ be the conditional probability distribution function of X_T given Φ_t under the Q-measure. Define another conditional probability distribution function $Q^*_{X_T|\Phi_t}(x)$ by the following classical Esscher transform with the Esscher parameter being one:

$$dQ^*_{X_T|\Phi_t}(x) = \frac{e^x dQ_{X_T|\Phi_t}(x)}{M_Q^X(t,T;1)}. \tag{3.9}$$

Hence, the characteristic function of $Q^*_{X_T|\Phi_t}(x)$ is given by

$$M_{Q^*}^X(t,T;iz) = \frac{M_Q^X(t,T;iz+1)}{M_Q^X(t,T;1)}. \tag{3.10}$$

Then, we can obtain an analytical pricing formula for the call option C at time t in terms of the characteristic functions $M_{Q^*}^X(t,T;iz)$ and $M_Q^X(t,T;iz)$ which can be calculated using the recursive formulas in Proposition 2.4. We express the pricing result in the following proposition.

Proposition 3.3: Under the GARCH(p, q) model with conditionally normal noise process, the price C_t for the call option C at time t is given by:

$$C_t = \frac{1}{2}S_t + \frac{e^{-r(T-t)}}{\pi}\int_0^{\infty} Re\Big[\frac{K^{-iz}M_Q^X(t,T;iz+1)}{iz}\Big]dz - Ke^{-r(T-t)}\Big\{\frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} Re\Big[\frac{K^{-iz}M_Q^X(t,T;iz)}{iz}\Big]dz\Big\}. \tag{3.11}$$

Proof: The idea of this proof is similar to the proof of Proposition 3 in Heston and Nandi (1997). For the purpose of exposition, we only provide the main step of the proof

here. First, by Feller (1971), we get the following inversion formula for the characteristic function $M_Q^X(t,T;iz)$:

$$\int_{\ln K}^{\infty} dQ_{X_T|\Phi_t}(x) = \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} Re\Big[\frac{e^{-iz\ln K}M_Q^X(t,T;iz)}{iz}\Big]dz. \tag{3.12}$$

Similarly, the inversion formula for the characteristic function $M_{Q^*}^X(t,T;iz)$ is given by

$$\int_{\ln K}^{\infty} dQ^*{}_{X_T|\Phi_t}(x) \;\; = \;\; \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} Re\Big[\frac{e^{-iz\ln K}M_{Q^*}^X(t,T;iz)}{iz}\Big]dz. \tag{3.13}$$

By Proposition 2.1, we get

$$\begin{aligned}
M_Q^X(t,T;1) &:= E_Q[\exp(X_T)|\Phi_t] \\
&= E_Q(S_T|\Phi_t) \\
&= e^{r(T-t)}S_t \ , \ P-a.s.
\end{aligned} \tag{3.14}$$

From (3.9), (3.10), (3.12), (3.13) and (3.14), we can calculate the price of the call option at time t as follows.

$$\begin{aligned}
C_t &= e^{-r(T-t)}E_Q[\max(e^{X_T}-K,0)|\Phi_t] \\
&= e^{-r(T-t)}\int_{\ln K}^{\infty} e^x dQ_{X_T|\Phi_t}(x) - e^{-r(T-t)}K\int_{\ln K}^{\infty} dQ_{X_T|\Phi_t}(x) \\
&= e^{-r(T-t)}M_Q^X(t,T;1)\int_{\ln K}^{\infty}\frac{e^x}{M_Q^X(t,T;1)}dQ_{X_T|\Phi_t}(x) \\
&\quad -e^{-r(T-t)}K\int_{\ln K}^{\infty} dQ_{X_T|\Phi_t}(x) \\
&= S_t\int_{\ln K}^{\infty} dQ^*{}_{X_T|\Phi_t}(x) - e^{-r(T-t)}K\int_{\ln K}^{\infty} dQ_{X_T|\Phi_t}(x) \\
&= S_t\Big\{\frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} Re\Big[\frac{e^{-iz\ln K}M_Q^X(t,T;iz+1)}{izM_Q^X(t,T;1)}\Big]dz\Big\} \\
&\quad -e^{-r(T-t)}K\Big\{\frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} Re\Big[\frac{e^{-iz\ln K}M_Q^X(t,T;iz)}{iz}\Big]dz\Big\} \\
&= \frac{1}{2}S_t + \frac{e^{-r(T-t)}}{\pi}\int_0^{\infty} Re\Big[\frac{K^{-iz}M_Q^X(t,T;iz+1)}{iz}\Big]dz \\
&\quad -Ke^{-r(T-t)}\Big\{\frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} Re\Big[\frac{K^{-iz}M_Q^X(t,T;iz)}{iz}\Big]dz\Big\}.
\end{aligned} \tag{3.15}$$

□

For the GARCH(1, 1) case ($p = q = 1$), we obtain a pricing formula for the call option C which is a function of the current stock price S_t and the conditional variance h_{t+1}.

Corollary 2.4: Under the GARCH(1, 1) model with conditionally normal noise process, the pricing formula for the call option C is given by

$$\begin{aligned}
&C_t = S_t\Big\{\frac{1}{2} + \frac{e^{-r(T-t)}}{\pi}\int_0^{\infty} Re\Big[\frac{K^{-iz}}{iz}\exp\Big(iz\ln(\frac{S_T}{K}) + A(t,T;iz+1) + \\
&B_1(t,T;iz+1)h_{t+1}\Big)\Big]dz\Big\} - Ke^{-r(T-t)}\Big\{\frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} Re\Big[\frac{K^{-iz}}{iz}\exp\Big(iz\ln(\frac{S_T}{K}) \\
&+A(t,T;iz) + B_1(t,T;iz)h_{t+1}\Big)\Big]dz\Big\}\ , \qquad (3.16)
\end{aligned}$$

where the coefficients $A(t,T;iz)$ and $B_1(t,T;iz)$ can be calculated recursively by:

$$\begin{aligned}
A(t,T;iz) &= irz + A(t+1,T;iz) + B_1(t+1,T;iz)\alpha_0 - \\
&\quad \frac{1}{2}\ln[1-2B_1(t+1,T;iz)\alpha_1] - \frac{z^2}{1-2B_1(t+1,T;iz)\alpha_1}\ , \\
B_1(t,T;iz) &= B_1(t+1,T;iz)\beta_1 - \frac{1}{2}iz\ , \\
A(T-1,T;iz) &= irz\ ,\ B_1(t,T;iz) = -\frac{1}{2}z(z+i)\ ,
\end{aligned} \qquad (3.17)$$

and the coefficients $A(t,T;iz+1)$ and $B_1(t,T;iz+1)$ can be calculated by the following recursive formulas:

$$\begin{aligned}
A(t,T;iz+1) &= r(iz+1) + A(t+1,T;iz+1) + B_1(t+1,T;iz+1)\alpha_0 - \\
&\quad \frac{1}{2}\ln[1-2B_1(t+1,T;iz+1)\alpha_1] + \frac{1+2iz-z^2}{1-2B_1(t+1,T;iz+1)\alpha_1}\ , \\
B_1(t,T;iz+1) &= B_1(t+1,T;iz+1)\beta_1 - \frac{1}{2}(iz+1)\ ,\ t=0,1,\ldots,T-2\ , \\
A(T-1,T;iz+1) &= r(iz+1)\ ,\ B_1(T-1,T;iz+1) = \frac{1}{2}z(i-z). \qquad (3.18)
\end{aligned}$$

Proof: The proof of this corollary follows immediately from Proposition 3.2 and Proposition 3.3.

$\square$

§3.2. Conditional Shifted Gamma distribution for the Noise Process

In this subsection, we deal with the case of the conditional shifted gamma distribution for the noise process. First, we consider an Φ-adapted stochastic process $\{X_t\}_{t\in\mathcal{T}\backslash\{0\}}$ such that under the statistical measure P,

1. $\{X_t\}_{t\in\mathcal{T}\backslash\{0\}}$ are independent and identically distributed.
2. For each $t \in \mathcal{T}\backslash\{0\}$, $X_t \sim Ga(a,b)$, where $Ga(a,b)$ represents a gamma distribution with shape parameter a and scale parameter b. That is, the density function $f_{X_t}(x)$ of X_t is given by:

$$f_{X_t}(x) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx} \ , \ t \in \mathcal{T}\backslash\{0\}. \tag{3.19}$$

For each $t \in \mathcal{T}\backslash\{0\}$, we define a standardized gamma random variable

$$z_t := \frac{X_t - \frac{a}{b}}{\sqrt{\frac{a}{b^2}}}. \tag{3.20}$$

By definition, it is easy to see that under P,

1. $\{z_t\}_{t\in\mathcal{T}\backslash\{0\}}$ are independent and identically distributed.
2. For each $t \in \mathcal{T}\backslash\{0\}$, $z_t \sim SGa(0,1)$, where $SGa(0,1)$ represents a shifted gamma distribution with mean zero and variance one.

Then, we assume that the stock innovation ξ_t is given as follows:

$$\xi_t = z_t\sqrt{h_t} \ , \ t \in \mathcal{T}\backslash\{0\}. \tag{3.21}$$

Hence, under P, the conditional distribution of ξ_t given Φ_{t-1} is a shifted gamma distribution with mean zero and variance h_t. That is,

$$\xi_t|\Phi_{t-1} \sim SGa(0, h_t) \ , \ t \in \mathcal{T}\backslash\{0\}. \tag{3.22}$$

In this case, we have the following specifications for the logarithmic stock-price returns process $\{Y_t\}_{t\in\mathcal{T}\backslash\{0\}}$ and the conditional variance process $\{h_t\}_{t\in\mathcal{T}\backslash\{0\}}$.

1. $Y_t = r + \lambda\sqrt{h_t} - \frac{1}{2}h_t + \xi_t$, where the conditional distribution $F(0, h_t)$ of ξ_t given Φ_{t-1} is $SGa(0, h_t)$.
2. $h_t = \alpha_0 + \sum_{i=1}^{q}\alpha_i\xi_{t-i}^2 + \sum_{j=1}^{p}\beta_j h_{t-j}$.

We can write the dynamic of the logarithmic stock-price return process $\{Y_t\}_{t\in\mathcal{T}\backslash\{0\}}$ under the P-measure in terms of $\{X_t\}_{t\in\mathcal{T}\backslash\{0\}}$ as follows.

$$Y_t = r + \lambda\sqrt{h_t} - \frac{1}{2}h_t - \sqrt{ah_t} + b\sqrt{\frac{h_t}{a}}X_t \ , \ t \in \mathcal{T}\backslash\{0\}. \tag{3.23}$$

By noticing that $b\sqrt{\frac{h_t}{a}}X_t|\Phi_{t-1} \sim Gamma(a, \sqrt{\frac{a}{h_t}})$, the conditional distribution of Y_t given Φ_{t-1} is a shifted gamma distribution with shape parameter a, scale parameter $\sqrt{\frac{a}{h_t}}$ and shifted parameter $-r - \lambda\sqrt{h_t} + \frac{1}{2}h_t + \sqrt{ah_t}$. Hence, the moment generating function of Y_t given Φ_{t-1} is given by

$$M_{Y_t|\Phi_{t-1}}(\theta) = \Big(\frac{\sqrt{\frac{a}{h_t}}}{\sqrt{\frac{a}{h_t}} - \theta}\Big)^a \exp[(r + \lambda\sqrt{h_t} - \frac{1}{2}h_t - \sqrt{ah_t})\theta] \ , \ \theta < \sqrt{\frac{a}{h_t}}. \tag{3.24}$$

The moment generating function $M_{Y_t|\Phi_{t-1}}(z;\theta_t)$ of Y_t given Φ_{t-1} under the transformed probability measure P_{t,Λ_t} is

$$M_{Y_t|\Phi_{t-1}}(z;\theta_t) = \Big(\frac{\sqrt{\frac{a}{h_t}} - \theta_t}{\sqrt{\frac{a}{h_t}} - \theta_t - z}\Big)^a \exp[(r + \lambda\sqrt{h_t} - \frac{1}{2}h_t - \sqrt{ah_t})z] \ , \tag{3.25}$$

provided that $z < \sqrt{\frac{a}{h_t}} - \theta_t$.

By (2.10) and (3.25) , we get an explicit formula for the conditional risk-neutralized Esscher parameter as follows.

$$\theta_t^q = \sqrt{\frac{a}{h_t}} - \Big[1 - \exp(\frac{\lambda\sqrt{h_t} - \frac{1}{2}h_t - \sqrt{ah_t}}{a})\Big]^{-1}. \tag{3.26}$$

Write b_t for the scale parameter $\sqrt{\frac{a}{h_t}}$ of the shifted gamma distribution under the statistical measure P. Then, we define an Φ_{t-1}-measurable parameter b_t^q as follows.

$$\begin{aligned} b_t^q &:= b_t - \theta_t^q \\ &= \left[1 - \exp\left(\frac{\lambda\sqrt{h_t} - \frac{1}{2}h_t - \sqrt{ah_t}}{a}\right)\right]^{-1}, \ t \in \mathcal{T}\backslash\{0\}. \end{aligned} \tag{3.27}$$

From (3.23) and (3.25), we see that under Q,

$$Y_t|\Phi_{t-1} \sim SGa(a, b_t^q; -r - \lambda\sqrt{h_t} + \frac{1}{2}h_t + \sqrt{ah_t}) \ , \ t \in \mathcal{T}\backslash\{0\}. \tag{3.28}$$

In constrast with the conditional normality case, the relationship among the conditional risk-neutralized Esscher parameter θ_t^q, the constant unit risk premium λ and the conditional risk-averse parameter γ_t becomes more oburred in this case. However, as in Gerber-Shiu's option-pricing model, the conditional distribution of Y_t given Φ_{t-1} under the Q-measure is still a shifted gamma distribution. In fact, the conditional risk-neutralized Esscher transform only changes the real-world scale parameter b_t of the original shifted gamma distribution under the P-measure to a conditional risk-neutralized scale parameter b_t^q. Then, we can also write the dynamic of the logarithmic stock-price returns under the Q-measure in the following form:

$$Y_t = r + \lambda\sqrt{h_t} - \frac{1}{2}h_t - \sqrt{ah_t} + X_t^q \ , \ t \in \mathcal{T}\backslash\{0\} \ , \tag{3.29}$$

where $X_t^q \sim Ga(a, b_t^q)$.

From the dynamic (3.23), we see that the constant unit risk premium λ cannot be eliminated under the Q-measure as in the case of conditional normality for the stock innovation. Hence, unlike the case of conditional normality for the noise process, the price of the call option C is not preference-free in the case of conditional shifted gamma distribution for the stock innovation.

For each $t \in \mathcal{T}\backslash\{0\}$, the conditional variance h_t^q under the Q-measure may not follow a GARCH (p,q) process. That is, the GARCH property of the conditional variance process may not preserve under the change of probability measures from P to Q in the case of shifted gamma distribution for the noise process. However, we can update the conditional

risk-neutralized variance process $\{h_t^q\}_{t\in\mathcal{T}\backslash\{0\}}$ using the following formula:

$$\begin{aligned} h_t^q &= \frac{a}{b_t^q}, \\ &= a\Big[1-\exp\Big(\frac{\lambda\sqrt{h_t}-\frac{1}{2}h_t-\sqrt{ah_t}}{a}\Big)\Big]\ ,\ t\in\mathcal{T}\backslash\{0\}. \end{aligned} \tag{3.30}$$

Finally, it is interesting to point out that we can follow the same procedure in this subsection to deal with the cases of the conditional shifted inverse Gaussian distribution and the conditional shifted Poisson distribution for the noise process. In fact, the GARCH stock-price dynamic with conditional shifted Poisson distribution for the innovation is a relatively new class of GARCH model which may have potential applications to modelling stock-price behaviour.

§4. Conclusion and Further Research

We have proposed an option-pricing model under a general GARCH assumption for the underlying stock-price dynamic. Our model is flexible enough to incorporate different parametric assumptions for the GARCH innovation, namely a class of infinitely divisible distributions, in the context of Gerber-Shiu's option-pricing model. We have adopted the concept of conditional Esscher transforms to pick a particular pricing probability measure under the incomplete market induced by both the discrete-time and continuous-state nature of the GARCH process and the conditional non-normality of the GARCH innovation. Our pricing result can be justified by the dynamic utility framework which provides a sound and intuitively appealing arguement for market equilibrium in our discrete-time economy. We have obtained a global preference-free pricing result and a compact analytical pricing formula for an European option in the case of conditional normality of the GARCH innovation.

One possible direction of investigation is to explore the applications of our model to different types of underlying financial instruments which exhibit deviation of the conditional non-normality assumption of the GARCH innovation. We may also consider the extension of our model to deal with American-style options. For other extensions of our model, we may consider the option-pricing model under different types of GARCH specifications, for instances, exponential GARCH (EGARCH) and thresold GARCH (T-

GARCH). Härdle and Hafner (2000) and Tong (1990) may provide some clues for the development of our model in this direction. Applications of our model to price various kinds of reinsurance products which exhibit "derivative" feature are a topic of practical interest. Due to the flexibility of our GARCH innovation and the positively skewed behavior of the exchange-rate time series, we may consider the case of conditional shifted inverse-Gaussian GARCH innovation in order to price foreign exchange options. Since our model is only in its theoretical stage, empirical research will enlighten us on further exploring the applications of our general GARCH option-pricing model.

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