Filtering and smoothing of state vector for diffuse state space models

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Abstract

This paper presents exact recursions for calculating the mean and mean square error matrix of the state vector given the observations for the multivariate linear Gaussian state space model in the case where the initial state vector is (partially) diffuse.

Keywords: Diffuse initialisation; Kalman filter; Smoothing; State space.

1 Introduction

In this paper we develop efficient recursions for filtering and smoothing of the state vector for the multivariate linear Gaussian state space time series model given by

$$\begin{aligned} y_t &= Z_t\alpha_t + \varepsilon_t, & \varepsilon_t &\sim \mathrm{N}(0, H_t), \\ \alpha_{t+1} &= T_t\alpha_t + R_t\eta_t, & \eta_t &\sim \mathrm{N}(0, Q_t), \qquad t = 1, \ldots, n, \end{aligned} \tag{1}$$

where y_t is the $N \times 1$ vector of observations and α_t is the $m \times 1$ unobserved state vector, in a situation where the initial state vector α_1 is, at least partially, diffuse. The matrices Z_t, T_t, R_t, H_t and Q_t are assumed to be known. The $N \times 1$ disturbance vector ε_t and the $r \times 1$ disturbance vector η_t are assumed to be serially independent and independent of each other at all time points. Similar results to those in this paper can also be derived for the more general model of de Jong (1991).

The initial state vector α_1 is specified as

$$\alpha_1 = a + A\delta + R_0\eta_0, \qquad \eta_0 \sim \mathrm{N}(0, Q_0), \tag{2}$$

where δ is a $q \times 1$ vector of unknown quantities, the $m \times 1$ vector a is known, the $m \times q$ matrix A and the $m \times (m-q)$ matrix R_0 are selection matrices, that is, they consist of columns of the identity matrix I_m; they are defined so that when taken together, their columns constitute all the columns of I_m so $A'R_0 = 0$ and $\delta = A'\alpha_1$. The matrix Q_0 is assumed to be positive definite and known. In most cases vector a will be treated as a zero vector unless some elements of the initial state vector are known constants. The vector δ is random and we assume that

$$\delta \sim \mathrm{N}(0, \kappa I_q), \tag{3}$$

where $\kappa > 0$ is given. Therefore, the initial conditions for the state vector become $\mathrm{E}(\alpha_1) = a$ and $\mathrm{Var}(\alpha_1) = P$ where

$$P = \kappa P_\infty + P_*, \tag{4}$$

and where we let $\kappa \to \infty$ where appropriate. Here $P_\infty = AA'$ and $P_* = R_0Q_0R_0'$; given the matrix structure of A, it follows that P_∞ is an $m \times m$ diagonal matrix with q elements equal to one and the other elements equal to zero. Without loss of generality, when a diagonal element of P_∞ is non-zero we take the corresponding element of a to be zero. A vector δ with distribution $\mathrm{N}(0, \kappa I_q)$ as $\kappa \to \infty$ is said to be diffuse.

The Kalman filter is a forward recursion for calculating $a_{t+1} = \mathrm{E}(\alpha_{t+1}|Y_t)$ with $Y_t = \{y_1, \ldots, y_t\}$ together with the mean square error matrix $P_{t+1} = \mathrm{Var}(\alpha_{t+1}|Y_t)$ for $t = 1, \ldots, n$. The disturbance smoother is a backward recursion for calculating $\hat{\varepsilon}_t = \mathrm{E}\left(\varepsilon_t|Y_n\right)$ and $\hat{\eta}_t = \mathrm{E}\left(\eta_t|Y_n\right)$, together with their corresponding mean square error matrices while the state smoother is a backward recursion for calculating $\hat{\alpha}_t = \mathrm{E}\left(\alpha_t|Y_n\right)$ together with $V_t = \mathrm{Var}\left(\alpha_t|Y_n\right)$ for $t = n, \ldots, 1$. Well-known Kalman filter and smoothing recursions are available for model (1) and (2) with given $\kappa > 0$. The objective in this paper is to derive limiting versions of these recursions when $\kappa \to \infty$.

A simple technique for obtaining an approximation to the exact limiting results is to replace κ in (4) by a large value such as 10^7 and use the standard recursions; see Harvey and Phillips (1979). However, the numerical solution obtained in this way will contain numerical inaccuracies and is theoretically unsatisfactory. Two main approaches have appeared in the literature on the exact treatment of a diffuse initial vector for filtering and smoothing:

- An exact solution as $\kappa \to \infty$ was first considered in detail by Ansley and Kohn (1985). They provided intricate equations for state filtering and smoothing. Ansley and Kohn (1990) presented a more simple treatment for mainly univariate models. The solution consists of modifications to the Kalman filter which remain until the influence of κ has disappeared from the updating equations. The transition to the usual Kalman filter at

some time point $t = d$ is automatic; there is no choice of doing a transition or not. The usual state smoothing recursions can be applied for $t = n, \ldots, d+1$. The modifications for state smoothing are only required for the initial period $t = d, \ldots, 1$.

- An alternative and simpler approach for diffuse likelihood evaluation and state filtering and smoothing was given by de Jong (1991). This solution requires augmenting the vector recursions of the Kalman filter and smoother to matrix recursions and adding an extra matrix recursion. Regression calculations applied to the auxiliary part of the Kalman filter and smoother provide the solution. A transition to the usual Kalman filter after some time point $t = d$ is optional but it is normally advisable to collapse at the earliest time possible. However, as is the case for the Ansley and Kohn solution, modifications for initial state smoothing are required when a collapse has taken place. Details of these modifications are given by Chu-Chun-Lin and de Jong (1993).

Other contributions on exact treatments of diffuse initialisation have appeared in the literature but these do not specifically deal with the problem of both filtering and smoothing of the state vector. Bell and Hillmer (1991) consider likelihood evaluation of ARIMA models; Snyder and Saligari (1996) present a strategy for initialising the Kalman filter using a square root formulation but their technique does not provide a solution for initial state smoothing ($t = d, \ldots, 1$); Koopman (1997) presents a more transparent treatment of diffuse filtering and likelihood evaluation using the Ansley and Kohn approach in which the collapse to the Kalman filter is automatic. In this paper we provide a solution to diffuse state smoothing using the approach of Koopman (1997). The paper is organised as follows. Sections 2 and 3 provide exact initial recursions for diffuse state filtering and smoothing, respectively. Section 4 presents useful matrix formulations of the filtering and smoothing recursions. Proofs are given in Appendices.

2 State filtering

2.1 Non-diffuse state filtering

The Kalman filter for model (1) for given $\kappa > 0$ is

$$a_{t+1} = T_t a_t + K_t v_t, \qquad P_{t+1} = T_t P_t L_t' + R_t Q_t R_t', \qquad t = 1, \ldots, n, \tag{5}$$

where

$$\begin{aligned} v_t &= y_t - Z_t a_t, & F_t &= Z_t P_t Z_t' + H_t, \\ K_t &= T_t P_t Z_t' F_t^{-1} & L_t &= T_t - K_t Z_t, \end{aligned} \tag{6}$$

with $a_1 = a$ and $P_1 = P$. The proof can be found in Anderson and Moore (1979, Chapter 3) and Durbin and Koopman (2001, §4.2.1).

2.2 Diffuse state filtering

The mean square error matrix P_t is decomposed in a similar way to matrix P in (4), that is

$$P_t = \kappa P_{\infty,t} + P_{*,t} + O(\kappa^{-1}), \qquad t = 1, \ldots, n, \tag{7}$$

where $P_{\infty,t}$ and $P_{*,t}$ do not depend on κ. It is shown by Ansley and Kohn (1985) and Koopman (1997) that the influence of the term $P_{\infty,t}$ will disappear after a limited number of updates d of the exact initial Kalman filter. Therefore, the state filtering equations (5) apply without change for $t = d + 1, \ldots, n$ when $\kappa \to \infty$. Note that when all state elements follow stationary processes or are known, $P_\infty = 0$ and $d = 0$.

Our approach relies on the expansion of the inverse of matrix

$$F_t = \kappa F_{\infty,t} + F_{*,t} + O(\kappa^{-1}), \qquad t = 1, \ldots, n,$$

which appears in the definition of K_t in (6). The expression of F_t is analogous to the decomposition of (7). The solution that we shall give for both filtering and smoothing apply to all univariate series. For multivariate series we give explicit solutions for the special cases $F_{\infty,t}$ nonsingular and $F_{\infty,t} = 0$. These two cases apply to nearly all time series occurring in practice. Although an explicit solution can be given as in Koopman (1997), we have found that for the rare cases where $F_{\infty,t}$ is a nonzero singular matrix, it is more efficient to convert the multivariate series to a univariate series in the way described in Koopman and Durbin (2000) and to apply the recursions of this paper to the resulting univariate series.

In Appendix A we show that for $t = 1, \ldots, d$ the exact initial state filtering equations when $F_{\infty,t}$ is nonsingular are given by

$$\begin{aligned} a_{t+1} &= T_t a_t + K_{\infty,t} v_t, \\ P_{\infty,t+1} &= T_t P_{\infty,t} L'_{\infty,t}, \\ P_{*,t+1} &= T_t P_{*,t} L'_{\infty,t} - K_{\infty,t} F_{\infty,t} K'_{*,t} + R_t Q_t R'_t, \end{aligned} \tag{8}$$

where

$$\begin{aligned} v_t &= y_t - Z_t a_t, & F_{\infty,t} &= Z_t P_{\infty,t} Z'_t, & F_{*,t} &= Z_t P_{*,t} Z'_t + H_t, \\ K_{\infty,t} &= T_t P_{\infty,t} Z'_t F^{-1}_{\infty,t}, & L_{\infty,t} &= T_t - K_{\infty,t} Z_t, & K_{*,t} &= (T_t P_{*,t} Z'_t - K_{\infty,t} F_{*,t}) F^{-1}_{\infty,t}, \end{aligned}$$

with the initialisations $a_1 = a$, $P_{*,1} = P_*$ and $P_{\infty,1} = P_\infty$. The recursions (8) are referred to as the exact initial Kalman filter following Koopman (1997).

In the case $F_{\infty,t} = 0$, we have

$$\begin{aligned} a_{t+1} &= T_t a_t + K_{*,t} v_t, \\ P_{\infty,t+1} &= T_t P_{\infty,t} T'_t, \\ P_{*,t+1} &= T_t P_{*,t} L'_{*,t} + R_t Q_t R'_t, \end{aligned} \tag{9}$$

where

$$\begin{aligned} v_t &= y_t - Z_t a_t, & F_{*,t} &= Z_t P_{*,t} Z'_t + H_t, \\ K_{*,t} &= T_t P_{*,t} Z'_t F^{-1}_{*,t}, & L_{*,t} &= T_t - K_{*,t} Z_t, \end{aligned} \tag{10}$$

for $t = 1, \ldots, d$.

3 State smoothing

3.1 Non-diffuse state smoothing

For given κ, the conditional mean of the state α_t given all the observations Y_n, that is $\hat{\alpha}_t = \mathrm{E}(\alpha_t|Y_n)$, together with its mean square error matrix V_t can be evaluated recursively by well-known smoothing recursions provided by Anderson and Moore (1979). A computationally more efficient set of recursions is developed by de Jong (1988), de Jong (1989) and Kohn and Ansley (1989). For model (1) and for given $\kappa > 0$ these are

$$\begin{aligned} r_{t-1} &= Z_t'F_t^{-1}v_t + L_t'r_t, & N_{t-1} &= Z_t'F_t^{-1}Z_t + L_t'N_tL_t, \\ \hat{\alpha}_t &= a_t + P_tr_{t-1}, & V_t &= P_t - P_tN_{t-1}P_t, \qquad t = n, n-1, \ldots, 1, \end{aligned} \tag{11}$$

with $r_n = 0$ and $N_n = 0$. Storage space is required for the quantities v_t, F_t, K_t, a_t and P_t, which are calculated by the Kalman filter, for $t = 1, \ldots, n$.

3.2 Diffuse state smoothing

The state smoothing recursions (11) apply to the period $t = n, n-1, \ldots, d+1$ for which contribution to P_t of terms depending on κ vanishes as $\kappa \to \infty$. We show in Appendix B that for the initial time period $t = d, d-1, \ldots, 1$, the exact initial state smoothing equations when $F_{\infty,t}$ is nonsingular are given by

$$\begin{aligned} \hat{\alpha}_t &= a_t + P_{*,t}r_{t-1}^{(0)} + P_{\infty,t}r_{t-1}^{(1)}, \\ V_t &= P_{*,t} - P_{*,t}N_{t-1}^{(0)}P_{*,t} - < P_{\infty,t}N_{t-1}^{(1)}P_{*,t} > - P_{\infty,t}N_{t-1}^{(2)}P_{\infty,t}, \end{aligned} \tag{12}$$

where for any square matrix A, $< A > = A + A'$, together with the backwards recursions

$$\begin{aligned} r_{t-1}^{(0)} &= L_{\infty,t}'r_t^{(0)}, & r_{t-1}^{(1)} &= Z_t'(F_{\infty,t}^{-1}v_t - K_{*,t}'r_t^{(0)}) + L_{\infty,t}'r_t^{(1)}, \\ N_{t-1}^{(0)} &= L_{\infty,t}'N_t^{(0)}L_{\infty,t}, & N_{t-1}^{(1)} &= Z_t'F_{\infty,t}^{-1}Z_t + L_{\infty,t}'N_t^{(1)}L_{\infty,t} - < L_{\infty,t}'N_t^{(0)}K_{*,t}Z_t >, \\ & & N_{t-1}^{(2)} &= Z_t'F_{\#,t}Z_t + L_{\infty,t}'N_t^{(2)}L_{\infty,t} - < L_{\infty,t}'N_t^{(1)}K_{*,t}Z_t >, \end{aligned} \tag{13}$$

where

$$F_{\#,t} = K_{*,t}'N_t^{(0)}K_{*,t} - F_{\infty,t}^{-1}F_{*,t}F_{\infty,t}^{-1},$$

and with initialisations $r_d^{(0)} = r_d$, $r_d^{(1)} = 0$ and $N_d^{(0)} = N_d$, $N_d^{(1)} = N_d^{(2)} = 0$.

In the case $F_{\infty,t} = 0$, we have

$$\begin{aligned} r_{t-1}^{(0)} &= Z_t'F_{*,t}^{-1}v_t + L_{*,t}'r_t^{(0)}, & r_{t-1}^{(1)} &= T_t'r_t^{(1)}, \\ N_{t-1}^{(0)} &= Z_t'F_{*,t}^{-1}Z_t + L_{*,t}'N_t^{(0)}L_{*,t}, & N_{t-1}^{(1)} &= T_t'N_t^{(1)}L_{*,t}, \\ & & N_{t-1}^{(2)} &= T_t'N_t^{(2)}T_t. \end{aligned} \tag{14}$$

Storage space is required for the quantities v_t, $F_{*,t}$, $F_{\infty,t}$, $K_{*,t}$, $K_{\infty,t}$, a_t, $P_{*,t}$ and $P_{\infty,t}$ which are calculated by the exact initial Kalman filter.

4 Matrix formulations

A convenient representation of the exact initial state filtering equations for the case $F_{\infty,t}$ nonsingular is given by

$$a_{t+1} = T_t a_t + K_{\infty,t} v_t, \qquad P_{t+1}^{\dagger} = T_t P_t^{\dagger} L_t^{\dagger\prime} + (R_t Q_t R_t', 0), \qquad t = 1, \ldots, d, \tag{15}$$

with the initialisation $a_1 = a$ and $P_1^{\dagger} = P^{\dagger} = (P_*, P_\infty)$. Elements of a corresponding to diffuse elements of α_1 can be put equal to zero. The matrices $P_t^{\dagger}$ and $L_t^{\dagger}$ are defined as

$$P_t^{\dagger} = (P_{*,t}, P_{\infty,t}), \qquad L_t^{\dagger} = \begin{bmatrix} L_{\infty,t} & -K_{*,t} Z_t \\ 0 & L_{\infty,t} \end{bmatrix}. \tag{16}$$

It is noteworthy that, apart from redefinitions of matrices, the initial recursions (15) have the same structure as the standard recursions (5).

A similar representation to (15) for the case $F_{\infty,t} = 0$ can be obtained by replacing $K_{\infty,t}$ in (15) by $K_{*,t}$ from (10) and the matrix $L_t^{\dagger}$ by

$$L_t^* = \begin{bmatrix} L_{*,t} & 0 \\ 0 & T_t \end{bmatrix}.$$

Define

$$Z_t^{\dagger} = \begin{bmatrix} Z_t & 0 \\ 0 & Z_t \end{bmatrix}, \qquad v_t^{\dagger} = \begin{pmatrix} v_t \\ 0 \end{pmatrix}, \qquad F_t^{\dagger} = \begin{bmatrix} 0 & F_{\infty,t}^{-1} \\ F_{\infty,t}^{-1} & -F_{\infty,t}^{-1} F_{*,t} F_{\infty,t}^{-1} \end{bmatrix},$$

$$r_t^{\dagger} = \begin{pmatrix} r_t^{(0)} \\ r_t^{(1)} \end{pmatrix}, \qquad N_t^{\dagger} = \begin{bmatrix} N_t^{(0)} & N_t^{(1)} \\ N_t^{(1)} & N_t^{(2)} \end{bmatrix}.$$

A convenient representation of the exact initial state smoothing equations for the case $F_{\infty,t}$ nonsingular is given by

$$\begin{aligned} r_{t-1}^{\dagger} &= Z_t^{\dagger\prime} F_t^{\dagger} v_t^{\dagger} + L_t^{\dagger\prime} r_t^{\dagger}, & N_{t-1}^{\dagger} &= Z_t^{\dagger\prime} F_t^{\dagger} Z_t^{\dagger} + L_t^{\dagger\prime} N_t^{\dagger} L_t^{\dagger}, \\ \hat{\alpha}_t &= a_t + P_t^{\dagger} r_{t-1}^{\dagger}, & V_t &= P_{*,t} - P_t^{\dagger} N_{t-1}^{\dagger} P_t^{\dagger\prime}, \qquad t = d, \ldots, 1, \end{aligned} \tag{17}$$

with $r_d^{\dagger} = (r_d', 0)'$ and $N_d^{\dagger} = \mathrm{diag}\,(N_d, 0)$, where the partioned matrices $L_t^{\dagger}$ and $P_t^{\dagger}$ are as in (16). As with the filter (15), these recursions have the same structure as the standard recursions (11) apart from redefinition of matrices. These results are obtained after some minor manipulation of the equations in §§2.2 and 3.2.

When $F_{\infty,t} = 0$, the smoothing equations (17) apply with $F_t^{\dagger}$ and $L_t^{\dagger}$ replaced by

$$F_t^* = \begin{bmatrix} F_{*,t}^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \qquad L_t^* = \begin{bmatrix} L_{*,t} & 0 \\ 0 & T_t \end{bmatrix},$$

respectively.

Acknowledgements

The first author was working at the Center for Economic Research of Tilburg University, The Netherlands as a Research Fellow of the Royal Netherlands Academy of Arts and Sciences during the main part of this project. The financial support of the Academy is gratefully acknowledged.

Appendix A Diffuse filtering

In this Appendix we derive the diffuse filtering results. The decomposition (7) leads to the similar decompositions for $F_t = Z_t P_t Z_t' + H_t$ and $M_t = T_t P_t Z_t'$, that is

$$F_t \;=\; \kappa F_{\infty,t} + F_{*,t} + O(\kappa^{-1}), \qquad M_t \;=\; \kappa M_{\infty,t} + M_{*,t} + O(\kappa^{-1}), \tag{18}$$

where

$$\begin{aligned} F_{\infty,t} &= Z_t P_{\infty,t} Z_t', & F_{*,t} &= Z_t P_{*,t} Z_t' + H_t, \\ M_{\infty,t} &= T_t P_{\infty,t} Z_t', & M_{*,t} &= T_t P_{*,t} Z_t', \end{aligned} \tag{19}$$

for $t = 1, \ldots, d$. The derivation of the exact initial Kalman filter for the case $F_{\infty,t}$ nonsingular is based on the expansion for $F_t^{-1} = [\kappa F_{\infty,t} + F_{*,t} + O(\kappa^{-1})]^{-1}$ as a power series in κ^{-1}, that is

$$F_t^{-1} = F_t^{(0)} + \kappa^{-1} F_t^{(1)} + \kappa^{-2} F_t^{(2)} + O\left(\kappa^{-3}\right), \tag{20}$$

for large κ. Since $I_p = F_t F_t^{-1}$ we have

$$I_p \;=\; (\kappa F_{\infty,t} + F_{*,t} + \kappa^{-1} F_{a,t} + \kappa^{-2} F_{b,t} + \ldots)(F_t^{(0)} + \kappa^{-1} F_t^{(1)} + \kappa^{-2} F_t^{(2)} + \ldots).$$

On equating coefficients of κ^j for $j = 0, -1, -2, \ldots$ we obtain

$$\begin{aligned} F_{\infty,t} F_t^{(0)} &= 0, \\ F_{*,t} F_t^{(0)} + F_{\infty,t} F_t^{(1)} &= I_p, \\ F_{a,t} F_t^{(0)} + F_{*,t} F_t^{(1)} + F_{\infty,t} F_t^{(2)} &= 0, \quad \text{etc.} \end{aligned} \tag{21}$$

We need to solve equations (21) for $F_t^{(0)}$, $F_t^{(1)}$ and $F_t^{(2)}$; further terms will not be required.

For $F_{\infty,t}$ nonsingular, we have from (21),

$$F_t^{(0)} = 0, \qquad F_t^{(1)} = F_{\infty,t}^{-1}, \qquad F_t^{(2)} = -F_{\infty,t}^{-1} F_{*,t} F_{\infty,t}^{-1}. \tag{22}$$

The matrix $K_t = M_t F_t^{-1}$ depends on the inverse matrix F_t^{-1} so it also can be expressed as a power series in κ^{-1}. We have

$$\begin{aligned} K_t &= [\kappa M_{\infty,t} + M_{*,t} + O(\kappa^{-1})](\kappa^{-1} F_t^{(1)} + \kappa^{-2} F_t^{(2)} + \ldots) \\ &= K_{\infty,t} + \kappa^{-1} K_{*,t} + O(\kappa^{-2}), \end{aligned}$$

where

$$\begin{aligned}
K_{\infty,t} &= M_{\infty,t}F_t^{(1)} \\
&= M_{\infty,t}F_{\infty,t}^{-1}, \\
K_{*,t} &= M_{*,t}F_t^{(1)} + M_{\infty,t}F_t^{(2)} \\
&= M_{*,t}F_{\infty,t}^{-1} - M_{\infty,t}F_{\infty,t}^{-1}F_{*,t}F_{\infty,t}^{-1} \\
&= (M_{*,t} - K_{\infty,t}F_{*,t})F_{\infty,t}^{-1}.
\end{aligned} \tag{23}$$

The updating equation (5) for a_{t+1} can now be expressed as

$$\begin{aligned}
a_{t+1} &= T_t a_t + K_t v_t \\
&= T_t a_t + [K_{\infty,t} + \kappa^{-1}K_{*,t} + O\left(\kappa^{-2}\right)]v_t,
\end{aligned}$$

which becomes as $\kappa \to \infty$,

$$a_{t+1} = T_t a_t + K_{\infty,t}v_t. \tag{24}$$

The updating equation (5) for $P_{t+1} = \kappa P_{\infty,t+1} + P_{*,t+1}$ is

$$\begin{aligned}
P_{t+1} &= T_t P_t L_t' + R_t Q_t R_t' \\
&= T_t[\kappa P_{\infty,t} + P_{*,t} + O(\kappa^{-1})]\left\{T_t - \left[K_{\infty,t} + \kappa^{-1}K_{*,t} + O\left(\kappa^{-2}\right)\right] Z_t\right\}' + R_t Q_t R_t'.
\end{aligned}$$

Consequently, the updates for $P_{\infty,t+1}$ and $P_{*,t+1}$ as $\kappa \to \infty$ are given by

$$\begin{aligned}
P_{\infty,t+1} &= T_t P_{\infty,t}(T_t - K_{\infty,t}Z_t)', \\
P_{*,t+1} &= T_t P_{*,t}(T_t - K_{\infty,t}Z_t)' - T_t P_{\infty,t}Z_t'K_{*,t}' + R_t Q_t R_t'.
\end{aligned} \tag{25}$$

The matrix P_{t+1} also depends on terms in κ^{-1}, κ^{-2}, etc. but these terms will not be multiplied by κ or higher powers of κ within the Kalman filter recursions. Thus the updating equations for P_{t+1} do not need to take account of these terms when $\kappa \to \infty$.

For the case $F_{\infty,t} = 0$, we have $P_{\infty,t}Z_t = 0$, $M_{\infty,t} = 0$ and

$$F_t^{-1} = [F_{*,t} + O(\kappa^{-1})]^{-1} = F_{*,t}^{-1} + \kappa^{-1}F_{a,t} + O(\kappa^{-2}).$$

It follows that

$$\begin{aligned}
K_t &= [M_{*,t} + \kappa^{-1}M_{a,t} + O(\kappa^{-2})][F_{*,t}^{-1} + \kappa^{-1}F_{a,t} + O(\kappa^{-2})] \\
&= K_{*,t} + \kappa^{-1}K_{a,t} + O(\kappa^{-2})
\end{aligned}$$

where

$$\begin{aligned}
K_{*,t} &= M_{*,t}F_{*,t}^{-1}, \\
K_{a,t} &= M_{*,t}F_{a,t} + M_{a,t}F_{*,t}^{-1}.
\end{aligned} \tag{26}$$

We will show below that no explicit definitions for $F_{a,t}$, $M_{a,t}$ and $K_{a,t}$ are required. The filtering equations are given by

$$
\begin{aligned}
a_{t+1} &= T_t a_t + [K_{*,t} + \kappa^{-1} K_{a,t} + O(\kappa^{-2})] v_t, \\
P_{t+1} &= T_t [\kappa P_{\infty,t} + P_{*,t} + O(\kappa^{-1})] \{T_t - [K_{*,t} + \kappa^{-1} K_{a,t} + O(\kappa^{-2})] Z_t\}' + R_t Q_t R_t'.
\end{aligned}
$$

When $\kappa \to \infty$, we obtain

$$
\begin{aligned}
a_{t+1} &= T_t a_t + K_{*,t} v_t, \\
P_{\infty,t+1} &= T_t P_{\infty,t} T_t', \\
P_{*,t+1} &= T_t P_{*,t} (T_t - K_{*,t} Z_t)' + R_t Q_t R_t', \qquad (27)
\end{aligned}
$$

since $P_{\infty,t} Z_t' = 0$.

Appendix B Diffuse smoothing

B1 Derivation of smoothed mean of state

To obtain the limiting recursions for the smoothing equation $\hat{\alpha}_t = a_t + P_t r_{t-1}$ given in (11) for $t = d, \ldots, 1$, we consider the recursion $r_{t-1} = Z_t' F_t^{-1} v_t + L_t' r_t$. Since r_{t-1} depends linearly on F_t^{-1} and K_t which can both be expressed as power series in κ^{-1} we write

$$
r_{t-1} = r_{t-1}^{(0)} + \kappa^{-1} r_{t-1}^{(1)} + O(\kappa^{-2}), \qquad t = d, \ldots, 1. \qquad (28)
$$

Substituting the relevant expansions into the recursion for r_{t-1} in (11) we have for the case $F_{\infty,t}$ nonsingular,

$$
\begin{aligned}
r_{t-1}^{(0)} + \kappa^{-1} r_{t-1}^{(1)} + \ldots &= Z_t' \left(\kappa^{-1} F_t^{(1)} + \kappa^{-2} F_t^{(2)} + \ldots\right) v_t \\
&\quad + \left\{T_t - \left[K_{\infty,t} + \kappa^{-1} K_{*,t} + O\left(\kappa^{-2}\right)\right] Z_t\right\}' \left(r_t^{(0)} + \kappa^{-1} r_t^{(1)} + \ldots\right),
\end{aligned}
$$

leading to recursions for $r_t^{(0)}$ and $r_t^{(1)}$,

$$
\begin{aligned}
r_{t-1}^{(0)} &= L_{\infty,t}' r_t^{(0)}, \\
r_{t-1}^{(1)} &= Z_t' \left(F_{\infty,t}^{-1} v_t - K_{*,t}' r_t^{(0)}\right) + L_{\infty,t}' r_t^{(1)}, \qquad (29)
\end{aligned}
$$

for $t = d, \ldots, 1$ with $r_d^{(0)} = r_d$ and $r_d^{(1)} = 0$. Note that $L_{\infty,t} = T_t - K_{\infty,t} Z_t$.

The smoothed state vector is

$$
\begin{aligned}
\hat{\alpha}_t &= a_t + P_t r_{t-1} \\
&= a_t + P_{*,t} \left(r_{t-1}^{(0)} + \kappa^{-1} r_{t-1}^{(1)}\right) + \kappa P_{\infty,t} \left(r_{t-1}^{(0)} + \kappa^{-1} r_{t-1}^{(1)}\right) + O\left(\kappa^{-1}\right) \\
&= a_t + P_{*,t} r_{t-1}^{(0)} + \kappa P_{\infty,t} r_{t-1}^{(0)} + P_{\infty,t} r_{t-1}^{(1)} + O\left(\kappa^{-1}\right). \qquad (30)
\end{aligned}
$$

It is obvious that for this expression to make sense we must have $P_{\infty,t} r_{t-1}^{(0)} = 0$ for all t. Thus we need to show that $\mathrm{Var}(\alpha_t | Y_n)$ is finite as $\kappa \to \infty$. Analogously to the arguments in de Jong (1991) we can express α_t as a linear function of $\delta, \eta_0, \eta_1, \ldots, \eta_{t-1}$. But $\mathrm{Var}(\delta | Y_d)$ is finite by definition of d so $V_t = \mathrm{Var}(\alpha_t | Y_n)$ must be finite as $\kappa \to \infty$ since $d < n$. Also, $Q_j = \mathrm{Var}(\eta_j)$ is finite so $\mathrm{Var}(\eta_j | Y_n)$ is finite for $j = 0, \ldots, t-1$. It follows that $\mathrm{Var}(\alpha_t | Y_n)$ is finite for all t as $\kappa \to \infty$ so from (30) $P_{\infty,t} r_{t-1}^{(0)} = 0$. Letting $\kappa \to \infty$ we obtain the expression for $\hat{\alpha}_t$ in (12).

For the case $F_{\infty,t} = 0$, the recursion for r_t becomes

$$
\begin{aligned}
r_{t-1}^{(0)} + \kappa^{-1} r_{t-1}^{(1)} + \ldots &= Z_t' \left(F_{*,t}^{-1} + \kappa^{-1} F_{a,t} + \ldots \right) v_t \\
&\quad + \left\{ T_t - \left[K_{*,t} + \kappa^{-1} K_{a,t} O\left(\kappa^{-2}\right) \right] Z_t \right\}' \left(r_t^{(0)} + \kappa^{-1} r_t^{(1)} + \ldots \right),
\end{aligned}
$$

leading to recursions for $r_t^{(0)}$ and $r_t^{(1)}$,

$$
\begin{aligned}
r_{t-1}^{(0)} &= Z_t' F_{*,t}^{-1} v_t + L_{*,t}' r_t^{(0)}, \\
r_{t-1}^{(1)} &= Z_t' \left(F_{a,t} v_t - K_{a,t}' r_t^{(0)} \right) + L_{*,t}' r_t^{(1)},
\end{aligned}
\qquad (31)
$$

for $t = d, \ldots, 1$ with matrix $L_{*,t}$ given by (10). Since $r_{t-1}^{(1)}$ is premultiplied by matrix $P_{\infty,t}$ in (12) and since $P_{\infty,t} Z_t' = 0$, the recursion for $r_t^{(1)}$ reduces to the one given in (14).

B2 Derivation of smoothed variance of state

To obtain exact finite expressions for V_t and N_{t-1} in (11) when $F_{\infty,t}$ is nonsingular and $\kappa \to \infty$, for $t = d, \ldots, 1$, we need to take three-term expansions instead of the two-term expressions previously employed:

$$
N_t = N_t^{(0)} + \kappa^{-1} N_t^{(1)} + \kappa^{-2} N_t^{(2)} + O(\kappa^{-3}). \qquad (32)
$$

Ignoring residual terms and on substituting in the expression $N_{t-1} = Z_t' F_t^{-1} Z_t + L_t' N_t L_t$, we obtain the recursion for N_{t-1} as

$$
\begin{aligned}
& N_{t-1}^{(0)} + \kappa^{-1} N_{t-1}^{(1)} + \kappa^{-2} N_{t-1}^{(2)} + \ldots \\
= \; & Z_t' \left(\kappa^{-1} F_t^{(1)} + \kappa^{-2} F_t^{(2)} + \ldots \right) Z_t \\
& + \left\{ T_t - \left[K_{\infty,t} + \kappa^{-1} K_{*,t} + \kappa^{-2} K_{\#,t} + O\left(\kappa^{-3}\right) \right] Z_t \right\}' \left(N_t^{(0)} + \kappa^{-1} N_t^{(1)} + \kappa^{-2} N_t^{(2)} + \ldots \right) \\
& \times \left\{ T_t - \left[K_{\infty,t} + \kappa^{-1} K_{*,t} + \kappa^{-2} K_{\#,t} + O\left(\kappa^{-3}\right) \right] Z_t \right\},
\end{aligned}
$$

for the case $F_{\infty,t}$ is nonsingular. We will argue below that no explicit definition for $K_{\#,t}$ is needed. This leads to the set of recursions

$$
\begin{aligned}
N_{t-1}^{(0)} &= L_{\infty,t}' N_t^{(0)} L_{\infty,t}, \\
N_{t-1}^{(1)} &= Z_t' F_{\infty,t}^{-1} Z_t + L_{\infty,t}' N_t^{(1)} L_{\infty,t} - < Z_t' K_{*,t}' N_t^{(0)} L_{\infty,t} >, \\
N_{t-1}^{(2)} &= -Z_t' F_{*,t} F_{\infty,t}^{-1} F_{*,t} Z_t + L_{\infty,t}' N_t^{(2)} L_{\infty,t} - < Z_t' K_{*,t}' N_t^{(1)} L_{\infty,t} > \\
&\quad - < Z_t' K_{\#,t}' N_t^{(0)} L_{\infty,t} > + Z_t' K_{*,t}' N_t^{(0)} K_{*,t} Z_t,
\end{aligned}
\qquad (33)
$$

with $N_d^{(0)} = N_d$ and $N_d^{(1)} = N_d^{(2)} = 0$. Note that $< A >= A + A'$ for any square matrix A.

Substituting the power series in κ^{-1}, κ^{-2}, etc. and the expression $P_t = P_{*,t} + \kappa P_{\infty,t}$ into the relation $V_t = P_t - P_t N_{t-1} P_t$ we obtain

$$\begin{aligned} V_t &= P_{*,t} + \kappa P_{\infty,t} \\ &\quad - (P_{*,t} + \kappa P_{\infty,t}) \left(N_{t-1}^{(0)} + \kappa^{-1} N_{t-1}^{(1)} + \kappa^{-2} N_{t-1}^{(2)} + \dots \right) (P_{*,t} + \kappa P_{\infty,t}) \\ &= -\kappa^2 P_{\infty,t} N_{t-1}^{(0)} P_{\infty,t} \\ &\quad + \kappa \left(P_{\infty,t} - P_{\infty,t} N_{t-1}^{(0)} P_{*,t} - P_{*,t} N_{t-1}^{(0)} P_{\infty,t} - P_{\infty,t} N_{t-1}^{(1)} P_{\infty,t} \right) \\ &\quad + P_{*,t} - P_{*,t} N_{t-1}^{(0)} P_{*,t} - P_{*,t} N_{t-1}^{(1)} P_{\infty,t} - P_{\infty,t} N_{t-1}^{(1)} P_{*,t} - P_{\infty,t} N_{t-1}^{(2)} P_{\infty,t} \\ &\quad + O\left(\kappa^{-1}\right). \end{aligned} \tag{34}$$

It was argued in the previous section that $V_t = \mathrm{Var}(\alpha_t | Y_n)$ is finite for $t = 1, \dots, n$. Thus the two matrix terms associated with κ and κ^2 in (34) must be zero. Letting $\kappa \to \infty$, the smoothed state variance matrix is given by

$$V_t = P_{*,t} - P_{*,t} N_{t-1}^{(0)} P_{*,t} - < P_{*,t} N_{t-1}^{(1)} P_{\infty,t} > - P_{\infty,t} N_{t-1}^{(2)} P_{\infty,t}. \tag{35}$$

Since we have argued that

$$P_{\infty,t} N_{t-1}^{(0)} P_{\infty,t} = 0, \quad \text{where} \quad N_{t-1}^{(0)} = L'_{\infty,t} N_t^{(0)} L_{\infty,t},$$

it follows that $P_{\infty,t} L_{\infty,t} N_t^{(0)} = 0$, when $F_{\infty,t}$ is nonsingular. Therefore, we can proceed in effect as if the recursion for $N_{t-1}^{(2)}$ in (33) is

$$N_{t-1}^{(2)} = -Z_t' F_{*,t} F_{\infty,t}^{-1} F_{*,t} Z_t + L'_{\infty,t} N_t^{(2)} L'_{\infty,t} - < Z_t' K'_{*,t} N_t^{(1)} L_{\infty,t} > + Z_t' K'_{*,t} N_t^{(0)} K_{*,t} Z_t.$$

It is convenient that $K_{\#,t}$ drops out from our calculations for $N_t^{(2)}$.

For the case $F_{\infty,t} = 0$, the recursion for N_t becomes

$$\begin{aligned} & N_{t-1}^{(0)} + \kappa^{-1} N_{t-1}^{(1)} + \kappa^{-2} N_{t-1}^{(2)} + \dots \\ = \; & Z_t' \left(F_{*,t}^{-1} + \kappa^{-1} F_{a,t} + \dots \right) Z_t \\ & + \left\{ T_t - \left[K_{*,t} + \kappa^{-1} K_{a,t} + \kappa^{-2} K_{b,t} + O(\kappa^{-3}) \right] Z_t \right\}' \left(N_t^{(0)} + \kappa^{-1} N_t^{(1)} + \kappa^{-2} N_t^{(2)} + \dots \right) \\ & \times \left\{ T_t - \left[K_{*,t} + \kappa^{-1} K_{a,t} + \kappa^{-2} K_{b,t} + O(\kappa^{-3}) \right] Z_t \right\}, \end{aligned}$$

leading to the recursions

$$\begin{aligned} N_{t-1}^{(0)} &= Z_t' F_{*,t}^{-1} Z_t + L'_{*,t} N_t^{(0)} L_{*,t}, \\ N_{t-1}^{(1)} &= Z_t' F_{a,t} Z_t + L'_{*,t} N_t^{(1)} L_{*,t} - < Z_t' K'_{a,t} N_t^{(0)} L_{*,t} >, \\ N_{t-1}^{(2)} &= Z_t' F_{b,t} Z_t + L'_{*,t} N_t^{(2)} L_{*,t} - < Z_t' K'_{a,t} N_t^{(1)} L_{*,t} > \\ &\quad - < Z_t' K'_{b,t} N_t^{(0)} L_{*,t} > + Z_t' K'_{a,t} N_t^{(0)} K_{a,t} Z_t, \end{aligned} \tag{36}$$

for $t = d, \ldots, 1$ with matrix $L_{*,t}$ given by (10). Similar to the arguments below (35), we have

$$P_{\infty,t} N_{t-1}^{(0)} P_{\infty,t} = 0, \quad \text{and} \quad P_{\infty,t} N_{t-1}^{(0)} P_{\infty,t} = P_{\infty,t} L_{*,t}' N_t^{(0)} L_{*,t} P_{\infty,t} = 0,$$

since $P_{\infty,t} Z_t' = 0$. Further, matrix $N_{t-1}^{(1)}$ is pre- or post-multiplied by matrix $P_{\infty,t}$ in (12) and since $P_{\infty,t} Z_t' = 0$ and $P_{\infty,t} L_{*,t}' N_t^{(0)} = 0$, the recursion for $N_t^{(1)}$ reduces to the one given in (14). Similar reductions are obtained for the recursion for $N_t^{(2)}$ because it is pre- and post-multiplied by matrix $P_{\infty,t}$ in (12). We do not need to give explicit definitions for the matrices $F_{a,t}$, $F_{b,t}$, $K_{a,t}$ and $K_{b,t}$ due to these reductions.

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