Cycle properties of Square Matrices

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Abstract

We find a simple condition that a square matrix provides a mapping that has an optimal property for cyclic permutations. The maximum length of a cycle with the optimum property gives an index of symmetry for the matrix.
Keywords: Cyclic Monotone Relations

1 Linear Cyclic Monotone Transformations

In [2] we looked at cycle properties of transformations, and gave simple examples using linear transformations. This paper shows how to obtain the most general possible results for the linear case, and justifies the examples given in [2]. We use fairly well known characterisations of skew symmetric matrices to obtain our results. The presentation of the proofs could have been made with direct products, but the approach used here with complex numbers is slightly less cumbersome.

Suppose that $x_j \in R^k$ for $j = 1, \ldots r$. Then we can interpret $x = (x_1, \ldots, x_r)$ as an *r-cycle* in R^k by making a convention that $x_{r+1} = x_1$ producing a cycle $\{x_1, x_2, \ldots, x_r, x_{r+1}\}$.

When x is an r-cycle and C is a $k \times k$ square matrix define

$$V_r(C; x) = \sum_{j=1}^{r} x_j' C(x_j - x_{j+1}). \tag{1}$$

Following [3], we call a linear transformation

$$y = Cx$$

r-cyclic monotone if $V_r(C;x) \geq 0$ for every r-cycle x, and *cyclic monotone* if it is r-cyclic monotone for all r. Since one can choose $x_r = x_{r-1}$, if C is r-cyclic monotone, it is also $(r-1)$-cyclic monotone.

Let $S = (C + C')/2$ and $A = (C - C')/2$ be the symmetric and skew-symmetric parts of C. Then $V_r(C;x)$ may be written as $V_r(S;x) + V_r(A;x)$, where

$$V_r(S;x) = \sum_{j=1}^{r} (x_j - x_{j+1})' S (x_j - x_{j+1})/2, \tag{2}$$

and

$$V_r(A;x) = -\sum_{j=1}^{r} x_j' A x_{j+1} = \frac{1}{2}\sum_{j=1}^{r} [x_{j+1}' A x_j - x_j' A x_{j+1}]. \tag{3}$$

Note that $V_2(A;x) = 0$, so that $V_2(C;x) \geq 0$ if and only if $V_2(S;x) \geq 0$. ie if and only if S is positive semi-definite. If C is 2-cyclic monotone, we say that C is *monotone*, and this is the same as the condition that S is positive semi-definite.

We are interested in r-cyclic monotonicity for $r > 2$, and will assume in all that follows the rather stronger condition that S is positive definite.

2 Geometry of $V_r(I + E; x)$

Using the assumption that S is semi positive-definite it is clear from (2) that if C is symmetric, then C is cyclic monotone. However, if C is not symmetric we shall see below that there is a maximum value of r for which C is r-cyclic monotone. We shall call this largest r the *symmetry index* of C, or SI(C).

Instead of looking at the cycle properties of $C = S + A$, we look equivalently at cycle properties of $I + E$, where E retains the skew-symmetric property, and can be written $S^{-\frac{1}{2}} A S^{-\frac{1}{2}}$.

It is possible to give an intuitive interpretation of $V_r(I + E; x)$. We work with even k below, but small modifications cover k odd. The skew symmetric matrix E can be written in the canonical form

$$E = \sum_{i \leq k/2} \lambda_i (u_i v_i' - v_i u_i'), \tag{4}$$

where $\{u_i\}, \{v_i\}$ are an orthonormal set of vectors, and λ_i's are non-negative. Using (2) gives

$$V_r(I;x) = \sum_{i \leq k/2} \sum_{j=1}^{r} \frac{1}{2} [\mid u_i' x_j - u_i' x_{j+1} \mid^2 + \mid v_i' x_j - v_i' x_{j+1} \mid^2]. \tag{5}$$

From (3)

$$V_r(E;x) = \sum_{i\le k/2} \lambda_i \sum_{j=1}^{r} \frac{1}{2} \sum_{j=1}^{r} [x_j' u_i v_i' (x_{j-1} - x_{j+1}) - x_j' v_i u_i' (x_{j-1} - x_{j+1})]. \quad (6)$$

.

From (4), (5) and (6) $V_r(I+E;x)$ can be written as the sum of the $k/2$ pieces from the first summation. The i'th piece can be interpreted using the polygon in two dimensions which has vertices with coordinates $(u_i' x_j, v_i' x_j)$ for $j = 1, \ldots, r$. The interpretation of the i'th piece is that it is half the sum of the squared lengths on the sides of the polygon minus a multiple λ_i of the (signed) area of the polygon.

One would expect that if $V_r(I+E;x)$ were non-negative for all x, then each of the pieces would be non-negative. One would require that λ_i was not too large. The critical case would be for the largest λ_i when the polygon with vertices with coordinates $(u_i' x_j, v_i' x_j)$ was regular, and so had large area compared with the lengths of its sides. We demonstrate these results in Section 3.

This interpretation is particularly simple if $k = 2$. For instance, taking $e > 0$ gives an E that may be written

$$E = e \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and $V_r(E;x)$ is e times the area of the polygon with vertices $x_1, x_2, \ldots, x_r$.

3 Extreme configurations

In all that follows, it is assumed that S is of full rank.

Writing $\omega = \cos 2\pi/r + i \sin 2\pi/r$ where i is the imaginary quantity, and $\eta_{vj} = \omega^{(v-1)(j-1)}/\sqrt{r}$ it is easy to show (with the bar meaning complex conjugate and $j = r+1$, $j = 0$ interpreted as $j = 1$, $j = r$) that for u, v integers in the range 1 to r,

$$-i \sum_{v=1}^{r} s_v \eta_{vl} \bar{\eta}_{vj} = \begin{cases} 0 & l \ne j+1 \text{ or } j-1 \\ 1 & l = j+1 \\ -1 & l = j-1 \end{cases} ,$$

and also that

$$\sum_{v=1}^{r} c_v \eta_{vl} \bar{\eta}_{vj} = \begin{cases} 0 & l \ne j,\ j+1 \text{ or } j-1 \\ -\frac{1}{2} & l = j+1 \text{ or } l = j-1 \\ 1 & l = j \end{cases} ,$$

where $s_v = \sin \frac{2(v-1)\pi}{r}$ and $c_v = 1 - \cos \frac{2(v-1)\pi}{r}$. The skew symmetric matrix E has a spectral decomposition which can be written see, for instance, [1]

$$\sum_{u=1}^{k} i\lambda_u h_u h_u^* \tag{7}$$

where the * gives complex conjugate transpose, and λ_u is real. We shall arrange the eigenvectors so that $\lambda_{u+1} \geq \lambda_u$ for $u = 1, \ldots, (k-1)$, and note that then λ_u is equal to $-\lambda_{k-u+1}$ and $h_{k-u+1} = \bar{h}_u$.

Using (7),

$$\begin{aligned}
V_r(I;w) &= \sum_{j=1}^{r} w_j'(w_j - w_{j+1}) \\
&= \sum_{j=1}^{r}[w_j'w_j - \frac{1}{2}w_j'w_{j+1} - \frac{1}{2}w_j'w_{j-1}] \\
&= \sum_{j=1}^{r} w_j' \sum_{l=1}^{r} w_l \sum_{v=1}^{r} c_v \eta_{vl}\bar{\eta}_{vj} \\
&= \sum_{v=1}^{r} c_v \sum_{j=1}^{r}\sum_{l=1}^{r} \bar{\eta}_{vj} w_j' w_l \eta_{vl} \\
&= \sum_{v=1}^{r} c_v \sum_{j=1}^{r}\sum_{l=1}^{r} \bar{\eta}_{vj} w_j' \sum_{u=1}^{k} h_u h_u^* w_l \eta_{vl} \\
&= \sum_{u=1}^{k}\sum_{v=1}^{r} c_v \sum_{j=1}^{r} \bar{\eta}_{vj} w_j' h_u \sum_{l=1}^{r} \eta_{vl} h_u^* w_l \\
&= \sum_{u=1}^{k}\sum_{v=1}^{r} c_v \mid \sum_{j=1}^{r} h_u^* w_j \eta_{vj} \mid^2 .
\end{aligned}$$

A similar approach to $V_r(E;w)$ leads to

$$\begin{aligned}
V_r(I+E;w) &= \sum_{u=1}^{k}\sum_{v=1}^{r}(c_v + \lambda_u s_v) \mid \sum_j h_u^* w_j \eta_{vj} \mid^2 \\
&= \sum_{u=1}^{k}\sum_{v=2}^{r}[1 + \cot(v-1)\pi/r\lambda_u] \mid \sum_j h_u^* w_j \eta_{vj} c_v^{\frac{1}{2}} \mid^2 .
\end{aligned}$$

So, $V_r(I+E;w)$ is non-negative for all w if the largest λ_u is no greater than $\tan \pi/r$. That this condition is also necessary can be seen by taking for $j = 1, \ldots, r$

$$w_j = c_{v_0}^{-\frac{1}{2}}[h_{u_0}\bar{\eta}_{v_0 j} + \bar{h}_{u_0}\eta_{v_0 j}]$$

which are, as required, real vectors. For this choice, since we can also write

$$w_j = c_{v_0}^{-\frac{1}{2}}[h_{u_0}\bar{\eta}_{v_0 j} + h_{k+1-u_0}\bar{\eta}_{r+2-v_0,j}]$$

it follows that

$$V_r(I+E;w) = 2[1+\cot(v_0-1)\pi/r\lambda_{u_0}].$$

The extreme configuration $w_j = \tilde{w}_j = c_2^{-\frac{1}{2}}[h_1\bar{\eta}_{2j} + \bar{h}_1\eta_{2j}]$ which gives rise to $V_r(I+E;w) = 2[1+\lambda_1 \cot \pi/r]$ has the regular polygonal structure anticipated at the end of the Section 2. Since all the $\tilde{w}_j$ are linear combinations of the real and imaginary parts of h_1 they lie in a 2-dimensional subspace.

These points $\tilde{w}_j$ form a regular polygon, since the matrix

$$Q = (\lambda_1 \cot \pi/rI - E)(\lambda_1 \cot \pi/rI + E)^{-1}$$

is orthogonal and has the property $Q\tilde{w}_j = \tilde{w}_{j+1}$ for $j = 1, \ldots, r$.

4 Properties of the Index of Symmetry

The index of symmetry $\mathrm{SI}(C) = S + A$ is the greatest integer r for which the largest eigenvalue of $S^{-\frac{1}{2}}AS^{-\frac{1}{2}}$ is no greater than $\tan \pi/r$.

It is easily seen that if D is an orthogonal matrix, then $\mathrm{SI}(DCD') = \mathrm{SI}(C)$. Since replacing C by C' changes the sign of A and leaves S unchanged, $\mathrm{SI}(C') = \mathrm{SI}(C)$. It is clear from (1), on replacing x_i by $C^{-1}x_i$, that for invertible C, $\mathrm{SI}(C) = \mathrm{SI}(C'^{-1}) = \mathrm{SI}(C^{-1})$.

The index of symmetry is therefore unchanged under the operations of transpose, inversion, and change of basis.

References

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