Periodic time series models: a structural approach

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SUMMARY

Time series with periodic autocorrelation structure are commonly modelled using periodic autoregressive (PARMA) processes. In most applications, the moving average terms are excluded for ease of estimation. We propose a new class, called periodic structural time series models (PSTSM). Parameter estimates for PSTSM are readily interpreted; the estimated coefficients correspond to variances of the measurement noise and of the error terms in unobserved components. We show that PSTSM have correlation structure equivalent to that of a periodic integrated moving average (PIMA) process. In a comparison of forecast performance for a set of quarterly macroeconomic series, PSTSM outperform periodic autoregressive (PAR) models both within and out of sample. We conclude that PSTSM are a natural framework for series with periodic autocorrelation structure both in terms of interpretability and forecasting accuracy

Some key words: periodic autoregressive moving average models, seasonality, structural time series model

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1 Introduction

Economists traditionally view seasonality as a redundant feature of a time series that needs to be removed before economic analysis. In recent years there has been increased interest in modelling seasonality and an understanding that economic analysis could be flawed if seasonality is ignored. Seasonality is usually viewed as an unobserved component with constant variance and zero sum over the seasonal period. Our interest lies in seasonal time series with periodicity in the second moments, that is in the autocovariance function. This gives rise to models for which the confidence interval of the forecasts is season dependent. Models with a periodic autocovariance function have been investigated within the autoregressive moving average (ARMA) framework. Periodic ARMA are specified in a similar way to non-periodic ARMA models but the former have parameters which change with season. Most applications exclude the MA part for ease of estimation. Periodic AR models (PAR) have been used successfully in economic time series. Novales and de Frutto (1997), Franses (1996), Osborn and Smith (1989) show that a large proportion of macroeconomic time series have periodic second moments.

An alternative class of time series models are the structural time series models (Harvey 1989). Following this methodology, the salient features of the data such as trend, seasonal and irregular are modelled directly as stochastic processes. The model is cast in state-space form, and the Kalman filter is used for estimation. Structural time series models (STSM) can be extended to incorporate seasonality in the second moments. We propose a new class of periodic structural time series models (PSTSM) and show that PSTSM are observationally equivalent to periodic integrated moving average (PIMA) models.

Periodic models are efficiently represented in a vector form with the time index measured in years and estimated using multivariate analysis (Gladyshev 1961). PAR models and their vector representation are described in §2 along with PMA models. §3 describes the extensions of STSM to the periodic case and the relation between PSTSM and PARMA models. §4 looks at the forecasting accuracy of PAR and PSTSM. We compare PAR and PSTSM on a data set of eleven quarterly macroeconomic variables from USA, Canada, Germany, and UK. PSTSM produce better forecasts, both within and out of sample, for the majority of the series concerned. The final section presents conclusions.

2 Periodic AR and MA models

Consider an observed time series $y_{s,n}$, where $s = 1, \ldots, S$ denotes the season and $n = 1, \ldots, N$ the year. A simple periodic AR(1), or PAR(1), has the form

$$y_{s,n} = \phi_{1,s} y_{s-1,n} + \epsilon_{s,n}, \qquad \{\epsilon_{s,n}\} \sim \mathrm{NID}(0, \sigma^2), \tag{1}$$

where $y_{i,n} = y_{S+i,n-1}$ when $i \leq 0$, and NID denotes normal and independently distributed. The variance and the autocovariance function of this process are periodic. The vector form of PAR is used in many studies, see for example Ghysels and Osborn (2001), Franses (1996), or Troutman (1979). For the case $S = 4$, equation (1) becomes

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -\phi_{1,2} & 1 & 0 & 0 \\ 0 & -\phi_{1,3} & 1 & 0 \\ 0 & 0 & -\phi_{1,4} & 1 \end{pmatrix} \begin{pmatrix} y_{1,n} \\ y_{2,n} \\ y_{3,n} \\ y_{4,n} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \phi_{1,1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_{1,n-1} \\ y_{2,n-1} \\ y_{3,n-1} \\ y_{4,n-1} \end{pmatrix} + \begin{pmatrix} \epsilon_{1,n} \\ \epsilon_{2,n} \\ \epsilon_{3,n} \\ \epsilon_{4,n} \end{pmatrix}$$

or

$$\begin{aligned} \Phi_0 Y_n &= \Phi_1 Y_{n-1} + E_n \\ \Rightarrow Y_n &= \Phi_0^{-1} \Phi_1 Y_{n-1} + \Phi_0^{-1} E_n. \end{aligned} \tag{2}$$

From equations (1) and (2), we see that a PAR(1) process can be written in a VAR(1) representation. In general, a PAR(p) process results in a VAR(P) representation, where $P = [\frac{p+S-1}{S}]$ and [ ] denotes the integer part. For example, for $S = 4$, a PAR(4) will still have a VAR(1) representation with

$$\begin{aligned} \Phi_0 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\phi_{1,2} & 1 & 0 & 0 \\ -\phi_{2,3} & -\phi_{1,3} & 1 & 0 \\ -\phi_{3,4} & -\phi_{2,4} & -\phi_{1,4} & 1 \end{pmatrix} \\ \Phi_1 &= \begin{pmatrix} \phi_{4,1} & \phi_{3,1} & \phi_{1,2} & \phi_{1,1} \\ 0 & \phi_{4,2} & \phi_{3,2} & \phi_{2,2} \\ 0 & 0 & \phi_{4,3} & \phi_{3,3} \\ 0 & 0 & 0 & \phi_{4,4} \end{pmatrix} \end{aligned}$$

The general VAR(P) representation is

$$\begin{aligned} Y_n &= \Phi_0^{-1} \Phi_1 Y_{n-1} + \ldots + \Phi_0^{-1} \Phi_P Y_{n-P} + \Phi_0^{-1} E_n \\ &= A_1 Y_{n-1} + \ldots + A_p Y_{n-P} + U_n, \end{aligned}$$

where $A_i = \Phi_0^{-1}\Phi_i$, for $(i = 1, \ldots, P)$, and $U_n = \Phi_0^{-1}E_n$.

A VAR(P) process is stationary if the roots of $|\Phi_0 - \Phi_1 z - \ldots - \Phi_P z^P| = 0$ are greater than one in absolute value (Hamilton 1994). For a stationary VAR process, the matrix of autocovariances at lag K, $\Gamma(K) = \mathrm{E}(\mathrm{Y_n}\mathrm{Y}'_{\mathrm{n-K}})$, satisfies the vector Yule-Walker equations

$$\Gamma(K) = A_1\Gamma(K-1) + \ldots + A_p\Gamma(K-P), \qquad K \geq P.$$

For a stationary VAR process, estimation is relatively straightforward (Whittle 1963; Jones and Brelsford 1967) and the standard t and F tests are asymptotically valid.

We define periodic MA models (PMA) in a similar way. A periodic MA(1) process is

$$y_{s,n} = \epsilon_{s,n} + \theta_{1,s}\epsilon_{s-1,n}, \qquad \{\epsilon_{s,n}\} \sim \mathrm{NID}(0, \sigma^2), \tag{3}$$

where $\epsilon_{i,n} = \epsilon_{S+i,n-1}$ when $i \leq 0$. As in the PAR case, we can write PMA models in the vector form. For $S = 4$, (3) becomes

$$\begin{pmatrix} y_{1,n} \\ y_{2,n} \\ y_{3,n} \\ y_{4,n} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \theta_{1,2} & 1 & 0 & 0 \\ 0 & \theta_{1,3} & 1 & 0 \\ 0 & 0 & \theta_{1,4} & 1 \end{pmatrix} \begin{pmatrix} \epsilon_{1,n} \\ \epsilon_{2,n} \\ \epsilon_{3,n} \\ \epsilon_{4,n} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & \theta_{1,1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \epsilon_{1,n-1} \\ \epsilon_{2,n-1} \\ \epsilon_{3,n-1} \\ \epsilon_{4,n-1} \end{pmatrix}$$

or

$$Y_n = \Theta_1 E_n + \Theta_2 E_{n-1}. \tag{4}$$

As in the PAR case, a PMA(q) process will result in a VMA(Q) representation, where $Q = [\frac{q+S-1}{S}]$. The autocovariance function for the model defined by (4) is

$$\begin{aligned} \Gamma(0) &= \sigma^2(\Theta_1\Theta_1' + \Theta_2\Theta_2'), \\ \Gamma(1) &= \sigma^2\Theta_2\Theta_1', \\ \Gamma(k) &= 0, \qquad \text{for } k \geq 2. \end{aligned}$$

As in the univariate case, we can standardise the autocovariance matrix to get the autocorrelation matrix $P(k) = D_0^{-1}\Gamma(k)D_0^{-1}$ where $D_0^2 = \mathrm{diag}\{\Gamma_{11}(0), \ldots, \Gamma_{\mathrm{SS}}(0)\}$ and $\Gamma_{ii}(k)$ is the i^{th} diagonal element of $\Gamma(k)$. The autocorrelation matrices for (3) in the quarterly case are

$$P(0) = \begin{pmatrix} 1 & \frac{\theta_{1,2}}{\sqrt{1+\theta_{1,1}^2}\sqrt{1+\theta_{1,2}^2}} & 0 & 0 \\ \frac{\theta_{1,2}}{\sqrt{1+\theta_{1,1}^2}\sqrt{1+\theta_{1,2}^2}} & 1 & \frac{\theta_{1,3}}{\sqrt{1+\theta_{1,2}^2}\sqrt{1+\theta_{1,3}^2}} & 0 \\ 0 & \frac{\theta_{1,3}}{\sqrt{1+\theta_{1,2}^2}\sqrt{1+\theta_{1,3}^2}} & 1 & \frac{\theta_{1,4}}{\sqrt{1+\theta_{1,3}^2}\sqrt{1+\theta_{1,4}^2}} \\ 0 & 0 & \frac{\theta_{1,4}}{\sqrt{1+\theta_{1,3}^2}\sqrt{1+\theta_{1,4}^2}} & 1 \end{pmatrix} \tag{5}$$

$$P(1) = \begin{pmatrix} 0 & 0 & 0 & \frac{\theta_{1,1}}{\sqrt{1+\theta_{1,1}^2}\sqrt{1+\theta_{1,4}^2}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{6}$$

The identification of a PAR(p) or a PMA(q) model is not straightforward. In practice, a model selection criterion such as AIC or BIC are used to choose the appropriate order of the model (Franses 1996).

3 Periodic structural times series models

A structural time series model represents the observed series as a sum of unobserved components. Each component is represented explicitly as a stochastic process. We propose a new class of periodic structural time series models. By examining the reduced form we show that PSTSM are observationally equivalent to PIMA models.

3.1 Periodic local level model

Consider a time series y_t for $t = 1, \ldots, SN$ with S and N defined in §2. A standard decomposition into a trend μ_t and an irregular component ϵ_t is the local level model

$$\begin{array}{rcll} y_t & = & \mu_t + \epsilon_t, & \{\epsilon_t\} \sim \text{NID}(0, \sigma_\epsilon^2), \\ \mu_t & = & \mu_{t-1} + \eta_t, & \{\eta_t\} \sim \text{NID}(0, \sigma_\eta^2), \end{array} \tag{7}$$

where $\{\epsilon_t\}$ and $\{\eta_t\}$ are mutually independent. The stationary form of (7) is

$$\triangle y_t = \eta_t + \triangle \epsilon_t.$$

Allowing the variances in (7) to be season dependent gives the following model

$$\begin{array}{rcll} y_{s,n} & = & \mu_{s,n} + \epsilon_{s,n}, & \{\epsilon_{s,n}\} \sim \text{NID}(0, \sigma_{\epsilon,s}^2), \\ \mu_{s,n} & = & \mu_{s-1,n} + \eta_{s,n}, & \{\eta_{s,n}\} \sim \text{NID}(0, \sigma_{\eta,s}^2), \end{array} \tag{8}$$

where $\mu_{i,n} = \mu_{S+i,n-i}$ for $i \leq 0$. Model (8) yields a stationary form with seasonal autocovariance function.

The relationship between PMA and our representation for periodic structural models is established using the vector representation. We rewrite (7) using the notation of §2

$$Y_n = \mu_n + \epsilon_n, \tag{9}$$

where $\mathrm{Var}(\epsilon_\mathrm{n}) = \Sigma_\epsilon$ is restricted to be diagonal; in the non-periodic case $\Sigma_\epsilon = \sigma_\epsilon^2 I_s$. The vector representation for the trend is

$$\begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ -1 & 1 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \dots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & -1 & 1 \end{pmatrix} \begin{pmatrix} \mu_{1,n} \\ \mu_{2,n} \\ \mu_{3,n} \\ \vdots \\ \mu_{S,n} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} \mu_{1,n-1} \\ \mu_{2,n-1} \\ \vdots \\ \mu_{S,n-1} \end{pmatrix} + \begin{pmatrix} \eta_{1,n} \\ \eta_{2,n} \\ \vdots \\ \eta_{S,n} \end{pmatrix}$$

or

$$\begin{aligned} D\mu_n &= \Phi\mu_{n-1} + \eta_n \qquad (10) \\ \mu_n &= D^{-1}\Phi\mu_{n-1} + D^{-1}\eta_n. \end{aligned}$$

Rearranging yields

$$\begin{pmatrix} \mu_{1,n} \\ \mu_{2,n} \\ \vdots \\ \mu_{S,n} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} \begin{pmatrix} \mu_{1,n-1} \\ \mu_{2,n-1} \\ \vdots \\ \mu_{S,n-1} \end{pmatrix} + \begin{pmatrix} 1 & 0 & \dots & 0 \\ 1 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 1 & 1 & \dots & 1 \end{pmatrix} \begin{pmatrix} \eta_{1,n} \\ \eta_{2,n} \\ \vdots \\ \eta_{S,n} \end{pmatrix}$$

or

$$\mu_n = T\mu_{n-1} + R\eta_n,$$

with $\eta_n \sim NID(0, \Sigma_\eta)$ where Σ_η is restricted to be diagonal.

T has one eigenvalue equal to one so $\{\mu_n\}$ is non-stationary (Hamilton 1994). Applying the difference operator to all elements of μ_n and noting that T is idempotent ($T^2 = T$) we have

$$\begin{aligned} \triangle\mu_n &= \mu_n - \mu_{n-1} \\ &= T\mu_{n-1} + R\eta_n - T\mu_{n-2} - R\eta_{n-1} \\ &= T^2\mu_{n-2} + TR\eta_{n-1} + R\eta_n - T\mu_{n-2} - R\eta_{n-1} \\ &= R\eta_n + (TR - R)\eta_{n-1} \\ &= R\eta_n + W\eta_{n-1}, \end{aligned}$$

where $W = TR - R$. We can then write the vector stationary form of (9) as

$$\triangle Y_n = R\eta_n + W\eta_{n-1} + \triangle\epsilon_n.$$

Thus, $\{\triangle Y_n\}$ is a vector MA(1) with autocovariance matrices

$$\begin{aligned}\Gamma_{LLM}(0) &= R\Sigma_\eta R' + W\Sigma_\eta W' + 2\Sigma_\epsilon, \\ \Gamma_{LLM}(1) &= W\Sigma_\eta R' - \Sigma_\epsilon, \\ \Gamma_{LLM}(k) &= 0, \qquad \text{for } k \geq 2.\end{aligned}$$

The non-zero off diagonal elements of the autocorrelation matrices in the quarterly case are

$$\begin{aligned}P_{12}(0) &= -\frac{\sigma^2_{\epsilon_1}}{\sqrt{\sigma^2_{\epsilon_1}+\sigma^2_{\epsilon_2}+\sigma^2_{\eta1}}\sqrt{\sigma^2_{\epsilon_1}+\sigma^2_{\epsilon_4}+\sigma^2_{\eta2}}}, \\ P_{23}(0) &= -\frac{\sigma^2_{\epsilon_2}}{\sqrt{\sigma^2_{\epsilon_1}+\sigma^2_{\epsilon_4}+\sigma^2_{\eta2}}\sqrt{\sigma^2_{\epsilon_2}+\sigma^2_{\epsilon_3}+\sigma^2_{\eta3}}}, \\ P_{34}(0) &= -\frac{\sigma^2_{\epsilon_3}}{\sqrt{\sigma^2_{\epsilon_2}+\sigma^2_{\epsilon_3}+\sigma^2_{\eta3}}\sqrt{\sigma^2_{\epsilon_3}+\sigma^2_{\epsilon_4}+\sigma^2_{\eta4}}}, \\ P_{14}(1) &= -\frac{\sigma^2_{\epsilon_4}}{\sqrt{\sigma^2_{\epsilon_1}+\sigma^2_{\epsilon_4}+\sigma^2_{\eta1}}\sqrt{\sigma^2_{\epsilon_3}+\sigma^2_{\epsilon_4}+\sigma^2_{\eta4}}},\end{aligned} \tag{11}$$

By construction $P_{11}(0) = P_{22}(0) = P_{33}(0) = P_{44}(0) = 1$. Equating the autocorrelations matrices in equations (5) and (11) gives expressions for $\theta_{1,s}$ for $s = 1, \ldots, S$. Using the PSTSM (8), the admissible region for $\theta_{1,s}$ is (-1,0) for $s = 1, \ldots, S$.

The issue of identifiability is of particular importance in the context of structural models; it is easy to set up models which are non-identifiable. We assume normality so the identifiability of the model depends on the form of the autocovariance matrix. Hotta (1989) shows that a sufficient condition for identifiability of an unobserved components ARIMA model is that all the M components have $p_m + d_m \geq q_m + 1$ where p_m, and q_m is the order of the AR and MA polynomials and d_m is the order of difference for each unobserved component. Since the periodic structural time series models have reduced form which is a periodic IMA process, it is possible to check whether the conditions for identifiability are satisfied for the PSTSM by looking at the equivalent unobserved component PIMA model. Alternatively following Harvey (1989, p.207), we can check the identifiability of a periodic structural time series model without invoking the general result. Using model (8), we denote $\gamma_s(\tau)$ the autocovariance function of $\triangle y_t$ for season s at lag τ. We have

$$\begin{aligned}\gamma_s(0) &= \sigma^2_{\epsilon,s} + \sigma^2_{\epsilon,s-1} + \sigma^2_{\eta,s}, \\ \gamma_s(1) &= \sigma^2_{\epsilon,s},\end{aligned} \tag{12}$$

for $s = 1, \ldots, S$. This indicates that the model is identifiable; (12) provide $2S$ linearly independent equations which give unique solutions for the $2S$ quantities $\sigma^2_{\epsilon_1}, \ldots, \sigma^2_{\epsilon_S}$ and $\sigma^2_{\eta_1}, \ldots, \sigma^2_{\eta_S}$. A similar argument establishes identifiability of other periodic structural time series models.

3.2 Periodic local linear trend model

We may add a slope component in our model to give the periodic local linear trend model

$$\begin{array}{rclclclll} y_{s,n} & = & \mu_{s,n} & + & & & \epsilon_{s,n}, & \{\epsilon_{s,n}\} \sim \mathrm{NID}(0, \sigma^2_{\epsilon,\mathrm{s}}), \\ \mu_{s,n} & = & \mu_{s-1,n} & + & \beta_{s-1,n} & + & \eta_{s,n}, & \{\eta_{s,n}\} \sim \mathrm{NID}(0, \sigma^2_{\eta,\mathrm{s}}), \\ \beta_{s,n} & = & \beta_{s-1,n} & + & & & \zeta_{s,n}, & \{\zeta_{s,n}\} \sim \mathrm{NID}(0, \sigma^2_{\zeta,\mathrm{s}}), \end{array} \tag{13}$$

where $\beta_{i,n} = \beta_{S+i,n-i}$ for $i \leq 0$. The vector representation is then

$$\begin{aligned} Y_n &= \mu_n + \epsilon_n, \\ \mu_n &= T\mu_{n-1} + R\beta_n + R\eta_n, \\ \beta_n &= T\beta_{n-1} + R\zeta_n, \end{aligned} \tag{14}$$

with $\zeta_n \sim NID(0, \Sigma_\zeta)$. We have

$$\triangle\beta_n = R\zeta_n + W\zeta_{n-1},$$

so that

$$\begin{aligned} \triangle\mu_n &= T\mu_{n-1} - T\mu_{n-2} + R^2\zeta_n + RW\zeta_{n-1} + R\eta_n - R\eta_{n-1} \\ &= T\mu_{n-2} + TR\beta_{n-1} + TR\eta_{n-1} - T\mu_{n-2} + \\ &\quad R^2\zeta_n + RW\zeta_{n-1} + R\eta_n - R\eta_{n-1} \\ &= TR\beta_{n-1} + R\eta_n + W\eta_{n-1} + R^2\zeta_n + RW\zeta_{n-1}. \end{aligned}$$

It follows that

$$\begin{aligned} \triangle^2\mu_n &= TR^2\zeta_{n-1} + TRW\zeta_{n-2} + R\eta_n - R\eta_{n-1} + W\eta_{n-1} - \\ &\quad W\eta_{n-2} + R^2\zeta_n - R^2\zeta_{n-1} + RW\zeta_{n-1} - RW\zeta_{n-2} \\ &= R\eta_n + (W - R)\eta_{n-1} - W\eta_{n-2} + \\ &\quad R^2\zeta_n + (TR^2 - R^2 + RW)\zeta_{n-1} + (TRW - RW)\zeta_{n-2} \\ &= R\eta_n + (W - R)\eta_{n-1} - W\eta_{n-2} + \\ &\quad R^2\zeta_n + (WR + RW)\zeta_{n-1} + W^2\zeta_{n-2}. \end{aligned}$$

The stationary form is then

$$\triangle^2 Y_n \;=\; \triangle^2 \mu_n + \epsilon_n - 2\epsilon_{n-1} + \epsilon_{n-2}.$$

Thus, $\triangle^2 Y_n$ is a vector MA(2) with autocovariance function

$$\begin{aligned}
\Gamma_{LLT}(0) &= 2R\Sigma_\eta R' + 2W\Sigma_\eta W' - W\Sigma_\eta R' - R\Sigma_\eta W' + \\
&\quad R^2\Sigma_\zeta R'^2 + WR\Sigma_\zeta R'W' + WR\Sigma_\zeta W'R' + \\
&\quad RW\Sigma_\zeta R'W' + RW\Sigma_\zeta W'R' + W^2\Sigma_\zeta W^{2'} + 6\Sigma_\epsilon, \\
\Gamma_{LLT}(1) &= 2W\Sigma_\eta R' - R\Sigma_\eta R' - W\Sigma_\eta W' + \\
&\quad WR\Sigma_\zeta R^{2'} + RW\Sigma_\zeta R^{2'} + W^2\Sigma_\zeta R'W' + W^2\Sigma_\zeta W'R' - 4\Sigma_\epsilon, \\
\Gamma_{LLT}(2) &= -W\Sigma_\eta R' + W^2\Sigma_\zeta R^{2'} + \Sigma_\epsilon, \\
\Gamma_{LLT}(k) &= 0, \qquad \text{for } k \geq 3.
\end{aligned}$$

$\Gamma_{LLT}(2)$ is an upper triangular matrix which corresponds to $\{\triangle_S^2 y_t\}$ being a PMA(2S).

3.3 Periodic basic structural model

Adding a seasonal component γ_n to our model results in a periodic version of the basic structural model (BSM), (Harvey 1989). The vector representation is

$$Y_n = \mu_n + \gamma_n + \epsilon_n. \tag{15}$$

We examine two possible representations of the seasonal component below.

Harrison-Stevens seasonality model

For the seasonal component, it is natural to use the Harrison and Stevens (1976) seasonal model

$$\gamma_n = \gamma_{n-1} + \omega_n, \tag{16}$$

where ω_n is a zero-mean process with variance

$$\mathrm{Var}(\omega_n) = \Omega = \mathrm{D} - \frac{1}{\mathrm{i'_s D i_s}} \mathrm{D i_s i'_s D}, \tag{17}$$

where $D = \mathrm{diag}\{\sigma^2_{\omega,1}, \ldots, \sigma^2_{\omega,\mathrm{S}}\}$ and $i_s = [1, 1, \ldots, 1]'$ is an $S \times 1$. The variance-covariance matrix Ω enforces the constraint that $i'_s \mathrm{Var}(\omega_t) = 0$. This implies that $\mathrm{Var}(\mathrm{i'_s}\omega_\mathrm{t}) = 0$ which, for $i'_s\gamma_0 = 0$, gives $i'_s\gamma_t = 0$. Hence, the model enforces the

constraint that seasonality adds up to zero within a year. $\{\triangle\gamma_n\}$ is a vector MA(1) process with autocovariance

$$\begin{aligned}\Gamma_{\triangle_\gamma}(0) &= \Omega,\\ \Gamma_{\triangle_\gamma}(1) &= V,\\ \Gamma_{\triangle_\gamma}(k) &= 0, \qquad \text{for } k \geq 2.\end{aligned}$$

The elements of V are defined as follows

$$V_{i,j} = \begin{cases} \Omega_{i,j}, & \text{if } i < j,\\ 0, & \text{if } i \geq j.\end{cases}$$

A HS seasonal component in the BSM (15), results in $\{\triangle^2 y_n\}$ being a stationary vector MA(2) as is the case of model (13). Similarly, adding a HS seasonal component in the periodic local level model (8) still results in a vector MA(1) process. However, there are fewer non-zero elements in the autocorrelation matrices than in the non-seasonal case. In fact the autocorrelation matrices of the local level model with HS seasonality correspond to $\{\triangle_s y_t\}$ being a PMA(S). Equating the two autocorrelation matrices we get exact expressions for the $S \times S$ unknown $\theta_{i,s}$.

Dummy seasonality

An alternative model for γ_n is the dummy seasonality model. In this case, the vector representation for the seasonal component is

$$\begin{pmatrix} 1 & 0 & \dots & 0\\ 1 & 1 & \ddots & 0\\ \vdots & \vdots & \ddots & 0\\ 1 & 1 & \dots & 1\end{pmatrix}\begin{pmatrix}\gamma_{1,n}\\ \gamma_{2,n}\\ \vdots\\ \gamma_{S,n}\end{pmatrix} = -\begin{pmatrix} 0 & 1 & \dots & 1\\ 0 & 0 & \ddots & \vdots\\ \vdots & \vdots & \ddots & 1\\ 0 & 0 & \dots & 0\end{pmatrix}\begin{pmatrix}\gamma_{1,n-1}\\ \gamma_{2,n-1}\\ \vdots\\ \gamma_{S,n-1}\end{pmatrix} + \begin{pmatrix}\omega_{1,n}\\ \omega_{2,n}\\ \vdots\\ \omega_{S,n}\end{pmatrix}$$

or in vector notation

$$\begin{aligned} R\gamma_n &= -W\gamma_{n-1} + \omega_n,\\ \gamma_n &= -R^{-1}W\gamma_{n-1} + R^{-1}\omega_n, \end{aligned} \tag{18}$$

with $\omega_n \sim NID(0, \Sigma_\omega)$ where Σ_ω is restricted to be diagonal. In the time invariant case $\Sigma_\omega = \sigma_\omega^2 I$. (18) becomes

$$\begin{pmatrix} \gamma_{1,n} \\ \gamma_{2,n} \\ \gamma_{3,n} \\ \vdots \\ \gamma_{S,n} \end{pmatrix} = -\begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 0 & -1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \dots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & -1 \end{pmatrix} \begin{pmatrix} \gamma_{1,n-1} \\ \gamma_{2,n-1} \\ \gamma_{3,n-1} \\ \vdots \\ \gamma_{S,n-1} \end{pmatrix} + \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ -1 & 1 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \dots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & -1 & 1 \end{pmatrix} \begin{pmatrix} \omega_{1,n} \\ \omega_{2,n} \\ \omega_{3,n} \\ \vdots \\ \omega_{4,n} \end{pmatrix}$$

or

$$\gamma_n = -J\gamma_{n-1} + D\omega_n.$$

Since J has $S-1$ eigenvalues equal to -1, γ_n is not stationary. Applying the difference operator to all elements of γ_n and noting that $J^2 = -J$ we have

$$\begin{aligned} \triangle\gamma_n &= \gamma_n - \gamma_{n-1} \\ &= -J\gamma_{n-1} + D\omega_n + J\gamma_{n-2} - D\omega_{n-1} \\ &= J^2\gamma_{n-2} - JD\omega_{n-1} + D\omega_n + J\gamma_{n-2} - D\omega_{n-1} \\ &= D\omega_n - (J + I_s)D\omega_{n-1} \\ &= D\omega_n - \Phi\omega_{n-1}, \end{aligned}$$

since $(J+I_s)D = \Phi$ where Φ is defined in (10). Thus, $\{\triangle\gamma_n\}$ is a vector MA(1) process with autocovariance function

$$\begin{aligned} \Gamma_{\triangle_\gamma}(0) &= D\Sigma_\omega D' + \Phi\Sigma_\omega\Phi', \\ \Gamma_{\triangle_\gamma}(1) &= -\Phi\Sigma_\omega D', \\ \Gamma_{\triangle_\gamma}(k) &= 0, \qquad \text{for } k \geq 2. \end{aligned}$$

As in the case of Harrison-Stevens, adding a dummy seasonal component in the local level model (7), still gives a vector MA(1) process.

4 Applications

We propose a strategy to determine the appropriate periodic structural time series model. Franses (1996) suggests the use of model selection criteria such as AIC or BIC for determining the order of PAR. We adapt a similar approach for PSTSM. As a first step we select the appropriate time-invariant STSM by using the diagnostic tests suggested by Harvey (1989). We use the standardized one-step ahead prediction errors

$e_t = \frac{\upsilon_t}{\sqrt{F_t}}$, where υ_t is the one-step ahead prediction error and F_t is its variance. If the model is correctly specified and all the parameters known, then $\{e_t\} \sim N(0,1)$. We use common tests for normality and serial correlation such as those suggested by Bowman and Shenton (1975) and Ljung and Box (1978). The likelihood for periodic structural time series is calculated using the Kalman filter. The efficiency of the Kalman filter makes likelihood ratio tests a convenient tool for inference. The algorithm used for estimation and signal extraction is implemented in Ox (Doornik 1998) using SSfPack (S. J. Koopman and Doornik 1998). We test for seasonal heteroscedasticity in every season separately using a LR test and then use the AIC to choose the appropriate combination of periodic variances to formulate a PSTSM. Finally the chosen PSTSM model is checked for normality and independence of the residuals using the tests discussed above.

We compare the performance of PAR models with the PSTSM on a data set of quarterly macroeconomic variables. They include several macroeconomic indicators from UK, USA, Canada, and Germany. All data in Table 1 come from Franses (1996). In that monograph, the series are scrutinized for periodicity in the AR parameters using a battery of tests. The author arrived in an optimal order of periodic AR using the Schwarz information criterion and an F-test for the significance of $\phi_{p+1,s}$ parameters. Alternatively, the Akaike information criterion can be used. We use the same p as Franses (1996) who also considered PAR models which include deterministic seasonal trends. PAR models for all series were cast in state-space form and estimated in Ox (Doornik 1998) by maximum likelihood using the Kalman filter. We give results for the chosen PAR models with and without trend. Novales and de Frutto (1997) report that the forecasting performance of PAR models improves considerably by imposing non-periodic coefficients across some seasons so we included the results of a constrained version not considered by Franses (1996). Column C in Table 1 shows the seasons that have varying coefficients in the constrained PAR model. For all series except the unemployment figures for Canada, the constrained PAR has lower AIC than the unconstrained version.

For the PSTSM we followed the model selection strategy described at the beginning of this section. All series are well fitted by a Harrison and Stevens (1976) seasonal model with periodic variances. Column S shows the seasons with season-specific variance. In all cases, the periodic STSM outperform the PAR models in terms of goodness of fit. We are also interested in the out-of-sample performance of our forecasts. The best model from Table 1 was fitted without the last three years and forecasts were generated for the last 12 observations. Table 2 shows the percentage root mean square error for the out-of-sample forecasts. The results suggest that, in 8 out of 10 series, IMA provides

a better representation of the correlation structure and thus PSTSM outperform PAR models. We conclude that, for these series, PSTSM produce considerable gains in accuracy over PAR models.

5 Conclusions

Periodic processes where the coefficients change with the season represent a real feature of economic time series. The most widely used framework is the periodic autoregression (PAR). We advocate representation of periodic processes in the structural time series framework. We established that, periodic structural models have a vector integrated moving average representation. Although vector moving average and vector integrated moving average models have been investigated as part of VARIMA class of models, they are little used in practice. We have shown that a class of models with VIMA correlation structure provide parsimonious models for univariate series with periodic second moments. Moreover, in practical applications, the structural framework provides greater insight into the nature of the series than PAR models. PSTSM relate the seasonality in the autocovariance function to specific unobserved components. In comparison, PAR models parameters are not readily interpretable. We compare the forecasting accuracy of PSTSM with PAR models. PSTSM produced better forecasts both within and out of sample for the majority of the series concerned. We conclude that structural time series models are a natural framework for modelling periodic processes both in terms of interpretability and forecasting accuracy.

Table 1: Comparison of PAR and STSM

Variable	p	$AIC_{PAR}{}^{a}$	$AIC_{PAR}{}^{b}$	$AIC_{PAR}{}^{e}$	C	S	AIC_{PSTSM}
USA Ind. production	2	-6.73	-6.71	-6.73^{a}	2,4	4	-9.89^{c}
CAN Unemployment	4	9.57	9.65	9.64^{a}	3	$-^{f}$	17.33^{d}
DEU GNP,	2	-6.69	-6.68	-6.73^{b}	2,4	2	-12.23^{d}
UK GDP	2	-6.35	-6.34	-6.47^{a}	2,3	3	-8.86^{e}
UK consumption	1	-6.82	-6.80	-6.95^{a}	2	3	-9.36^{e}
UK cons. nondur.	1	-7.47	-7.43	-7.61^{a}	2	3=4	-10.47^{d}
UK Exports	2	-4.88	-4.86	-4.99^{b}	2	3	-7.36^{e}
UK Imports	1	-4.95	-4.98	-5.08^{b}	2,4	3	-9.30^{d}
UK pub. investment	2	-2.53	-2.62	-2.73^{b}	1	2	-4.94^{e}
UK workforce	2	-9.43	-9.4	-9.61^{a}	1	1,2,3,4	-9.62^{e}

[a] Model without trend

[b] Model with Trend

[c] Constrained PAR

[d] Fixed Level+Stochastic Slope

[e] Stochastic Level+ Fixed Slope

[f] No heteroscedasticity

Table 2: Out of Sample Comparison of PAR and STSM (percentage RMSE)

Variable	PAR	PSTSM
USA, Industrial production	0.19	0.28
CAN Unemployment	3.3	7.14
DEU, Real GNP	0.15	0.14
UK GDP	0.14	0.11
UK total consumption	0.13	0.08
UK consumer nondurables	0.09	0.06
UK Exports	0.29	0.27
UK Imports	0.30	0.22
UK public investment	3.10	2.87
UK workforce	0.03	0.03

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