Multivariate distribution models with generalized hyperbolic margins

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Abstract

Multivariate generalized hyperbolic distributions represent an attractive family of distributions (with exponentially decreasing tails) for multivariate data modelling. However, in a limited data environment, robust and fast estimation procedures are rare. In this paper we propose an alternative class of multivariate distributions (with exponentially decreasing tails) belonging to affine-linear transformed random vectors with stochastically independent and generalized hyperbolic marginals. The latter distributions possess good estimation properties and have attractive dependence structures which we explore in detail. In particular, dependencies of extreme events (tail dependence) can be modelled within this class of multivariate distributions. In addition we introduce the necessary estimation and random-number generation procedures. Various advantages and disadvantages of both types of distributions are discussed and illustrated via a simulation study.

Key words: Multivariate distributions, Generalized hyperbolic distributions, Affine-linear transforms, Copula, Tail dependence, Estimation, Random number generation

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1 Introduction

Data analysis with generalized hyperbolic distributions has become quite popular in various areas of theoretical and applied statistics. Originally, Barndorff-Nielsen (1977) utilized this class of distributions to model grain size distributions of wind-blown sand (cf. also Barndorff-Nielsen and Blæsild (1981) and Olbricht (1991)). In econometrical finance the latter family of distributions has been used for multidimensional asset-return modelling. In this context, the generalized hyperbolic distribution replaces the Gaussian distribution, which is not able to describe the fat tails and the (distributional) skewness of most financial asset-return data. References are Eberlein and Keller (1995), Prause (1999), Bauer (2000), Bingham and Kiesel (2001), and Eberlein (2001).

Multivariate generalized hyperbolic distributions (in short: MGH distributions) were introduced and investigated by Barndorff-Nielsen (1978) and Blæsild and Jensen (1981). These distributions have attractive analytical and statistical properties whereas robust and fast parameter estimations turn out to be difficult in higher dimensions. Furthermore, MGH distribution functions possess no parameter constellation for which they are the product of their marginal distribution functions. However, many applications require the multivariate distribution function to model both: Marginal dependence and independence. Because of these and other shortcomings (see also Section 5) we introduce and explore a new class of multivariate distributions, the so called multivariate affine generalized hyperbolic distributions (in short: MAGH distributions). This class of distributions has an appealing stochastic representation and, in contrast to the MGH distributions, the estimation and simulation algorithms are easier. Moreover, our simulation study reveals that the goodness-of-fit of the MAGH distribution is comparable to that of the MGH distribution. The one-dimensional marginals of an MAGH distribution are even more flexible due to more flexibility in the parameter choice.

We show that the dependence structure of an MAGH distribution is appropriate for data modelling. In particular, the correlation matrix as an important dependence measure is more intuitive and easier to handle for an MAGH distribution than for an MGH distribution. Further, zero correlation implies independent margins of the MAGH distribution whereas the margins of an MGH distribution do not have this property. Moreover, in contrast to the MGH distributions, the MAGH distributions can model dependencies of extreme events (so called tail dependence).

On the other hand, MGH distributions embrace a large subclass of distributions which belong to the family of elliptically or spherically contoured distributions. The family of elliptically contoured distributions inherits many useful properties from the multivariate normal distribution, like closure properties for linear regression and for passing to marginal distributions (see Cambanis, Huang, and Simons (1981)). Moreover, various tests for statistical inference under the assumption of a normal distribution are also applicable in the elliptical world (see Fang, Kotz, and Ng (1990)). Therefore an appropriate choice of the latter distributions depends very much on the kind of application.

Sections 2 and 3 introduce the MGH and MAGH distributions and characterize the subclass of elliptically contoured distributions. In Section 4 we investigate the dependence structure of both distributions by utilizing the theory of copulae. Section 5 discusses the

advantages and disadvantages of the MGH and MAGH distributions. The corresponding estimation and random-number generation algorithms are established in Sections 6 and 7. Finally, in Section 8, we present a detailed simulation study to illustrate the elaborated results. The Appendix contains various results and proof mainly relating to the tail behavior of MGH and MAGH distributions.

2 Multivariate generalized hyperbolic model (MGH)

In the first place, a subclass of MGH distributions, namely the hyperbolic distributions, has been introduced via so-called variance-mean mixtures of inverse Gaussian distributions. This subclass suffers from not having hyperbolic distributed marginals, i.e., the subclass is not closed with respect to passing to marginal distributions. Therefore and because of other theoretical aspects, Barndorff-Nielsen (1977) extended this class to the family of MGH distributions. Many different parametric representations of MGH density functions are provided in the literature, see e.g. Blæsild and Jensen (1981). The following density representation is appropriate in our context.

Definition 1 (MGH distribution) *An n-dimensional random vector X is said to have a* multivariate generalized hyperbolic *(MGH) distribution with location vector $\mu \in \mathbb{R}^n$ and scaling matrix $\Sigma \in \mathbb{R}^{n\times n}$ if it has the stochastic representation*

$$X \stackrel{d}{=} A'Y + \mu \tag{1}$$

for some lower triangular matrix $A' \in \mathbb{R}^{n\times n}$ such that $A'A = \Sigma$ is positive-definite and Y has a density function of the form

$$f_Y(y) = c\frac{K_{\lambda-n/2}(\alpha\sqrt{1+y'y})}{(1+y'y)^{n/4-\lambda/2}}e^{\alpha\beta' y}, \quad y \in \mathbb{R}^n. \tag{2}$$

The normalizing constant c is given by

$$c = \frac{\alpha^{n/2}\,(1-\beta'\beta)^{\lambda/2}}{(2\pi)^{n/2}K_\lambda(\alpha\sqrt{1-\beta'\beta})}, \tag{3}$$

where K_ν denotes the modified Bessel-function of the third kind with index ν (cf. Magnus, Oberhettinger, and Soni (1966), pp. 65) and the parameter domain is $\|\beta\|_2 < 1$, $\alpha > 0$ and $\lambda \in \mathbb{R}$ ($\|\cdot\|_2$ denotes the Euclidian norm). The family of n-dimensional generalized hyperbolic distributions is denoted by $MGH_n(\mu, \Sigma, \omega)$, where $\omega := (\lambda, \alpha, \beta)$.

An important property of the above parameterization of the MGH density function is its invariance under affine-linear transformations. For $\lambda = (n+1)/2$ we obtain the *multivariate hyperbolic* density and for $\lambda = -1/2$ the *multivariate normal inverse Gaussian* density. Hence $\lambda = 1$ leads to hyperbolically distributed one-dimensional margins. It can be shown that MGH distributions with $\lambda = 1$ are closed with respect to passing to marginal distributions and under affine-linear transformations. The latter subclass turns out to be important for practical applications (see e.g. Section 8.1).

An MGH distribution belongs to the class of elliptically contoured distributions if and only if $\beta = (0, \ldots, 0)'$. In this case the density function of X can be represented as

$$f_X(x) = |\Sigma|^{-1/2} g((x-\mu)'\Sigma^{-1}(x-\mu)), \quad x \in I\!R^n, \tag{4}$$

for some density generator function $g : I\!R_+ \to I\!R_+$. Consequently, the random vector Y in (1) is spherically distributed. The density generator g in (4) is given by $g(u) = cK_{\lambda-n/2}(\alpha\sqrt{1+u})/(1+u)^{n/4-\lambda/2}$, $u \in I\!R$, with some normalizing constant c. For a detailed treatment of elliptically contoured distributions, see Fang, Kotz, and Ng (1990) or Cambanis, Huang, and Simons (1981).

Remark. Usually the following representation of an MGH density is given in the literature:

$$\bar{f}_X(x) = \bar{c}\frac{K_{\bar{\lambda}-n/2}(\bar{\alpha}\sqrt{\bar{\delta}^2 + (x-\bar{\mu})'\bar{\Sigma}^{-1}(x-\bar{\mu})})}{\left(\bar{\alpha}^{-1}\sqrt{\bar{\delta}^2 + (x-\bar{\mu})'\bar{\Sigma}^{-1}(x-\bar{\mu})}\right)^{n/2-\bar{\lambda}}} e^{\bar{\beta}'(x-\bar{\mu})}, \quad x \in I\!R^n, \tag{5}$$

with some normalizing constant $\bar{c}$. The domain of variation[2] of the parameter vector $\bar{\omega} = (\bar{\lambda}, \bar{\alpha}, \bar{\delta}, \bar{\beta})$ is as follows: $\bar{\lambda}, \bar{\alpha} \in I\!R$, $\bar{\beta}, \bar{\mu} \in I\!R^n$, $\bar{\delta} \in I\!R_+$, $\bar{\beta}'\bar{\Sigma}\bar{\beta} < \bar{\alpha}^2$ and $\bar{\Sigma} \in I\!R^{n\times n}$ being a positive-definite matrix with determinant $|\bar{\Sigma}| = 1$. The one-to-one mapping between the parameter vector ω corresponding to (2) and $\bar{\omega}$ corresponding to (5) is given by: $\lambda = \bar{\lambda}$, $\mu = \bar{\mu}$, $\alpha = \bar{\alpha}\bar{\delta}$, $\beta = 1/\bar{\alpha} \cdot \bar{A}\bar{\beta}$, $\bar{A}'\bar{A} = \bar{\Sigma}$, and $\Sigma = \bar{\delta}^2\bar{\Sigma}$.

3 Multivariate affine generalized hyperbolic model (MAGH)

A disadvantage of multivariate generalized hyperbolic distributions (and of many other families of multivariate distributions) is that the margins X_i of $X = (X_1, \ldots, X_n)'$ are not mutually independent for any choice of the scaling matrix Σ. In other words, they do not allow the modelling of phenomena where random variables result as the sum of independent random variables. This shortcoming is serious since the independence may be an undisputable property of the problem for which the stochastic model is sought. Furthermore, in case of asymmetry (i.e., $\beta \neq 0$) the covariance matrix is in a complex relationship with the matrix Σ which is shown in the next section.

Therefore we propose an alternative concept. Instead of a multivariate generalized hyperbolic distribution, a distribution is considered which is composed of n independent margins with univariate generalized hyperbolic distributions with zero location and unit scaling. Such a canonical random vector is then subject to an affine-linear transformation. As a consequence, the transformation matrix can be modelled proportionally to the square root of the covariance-matrix inverse even in the asymmetric case. This property holds, for example, for multivariate normal distributions.

[2] This representation omits the limiting distributions obtained at the boundary of the parameter space; see e.g. Blæsild and Jensen (1981)

Definition 2 (MAGH distribution) *An n-dimensional random vector X is said to be* multivariate affine generalized hyperbolic *(MAGH) distributed with location vector $\mu \in \mathbb{R}^n$ and scaling matrix $\Sigma \in \mathbb{R}^{n\times n}$ if it has the following stochastic representation:*

$$X \stackrel{d}{=} A'Y + \mu \tag{6}$$

for some lower triangular matrix $A \in \mathbb{R}^{n\times n}$ such that $A'A = \Sigma$ is positive-definite and the random vector $Y = (Y_1, \ldots, Y_n)'$ consists of mutually independent random variables $Y_i \in MGH_1(0, 1, \omega_i),\ i = 1, \ldots, n$. In particular the one-dimensional margins of Y are generalized hyperbolic distributed. The family of n-dimensional affine generalized hyperbolic distributions is denoted by $MAGH_n(\mu, \Sigma, \omega)$, where $\omega := (\omega_1, \ldots, \omega_n)$ and $\omega_i := (\lambda_i, \alpha_i, \beta_i)',\ i = 1, \ldots, n$.

Observe that an MAGH distribution has independent margins in case the scaling matrix Σ equals the identity matrix I. However, no MAGH distribution belongs to the class of elliptically contoured distributions for dimension $n \geq 2$ which is illustrated by the density contour-plots in Figure 2.

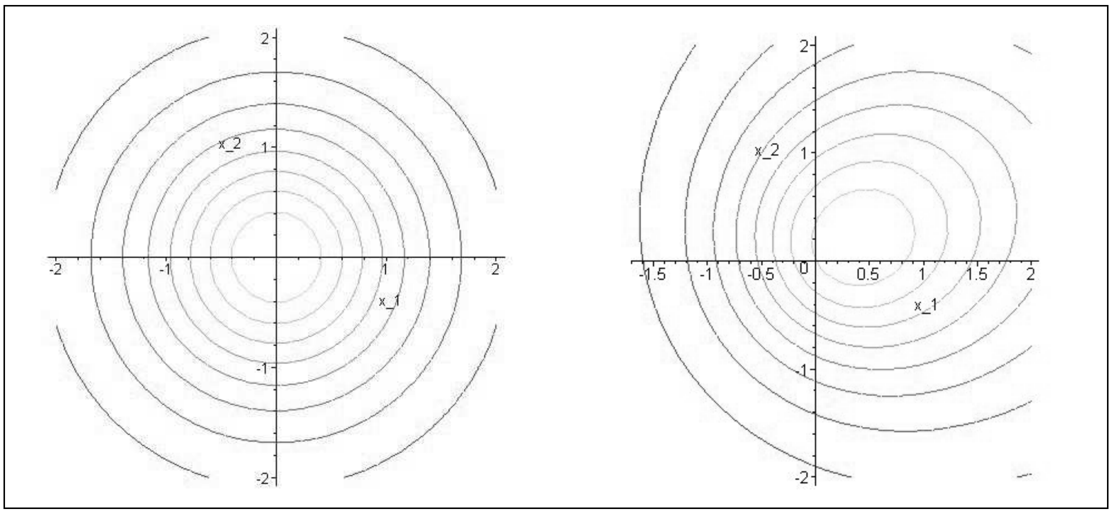


Fig. 1. Contour-plots of the bivariate density function of an $MGH_2(0, I, \omega)$ distribution with parameters $\lambda = 1,\ \alpha = 1$ and $\beta = (0, 0)'$ (left figure), $\beta = (0.5, 0.25)'$ (right figure)

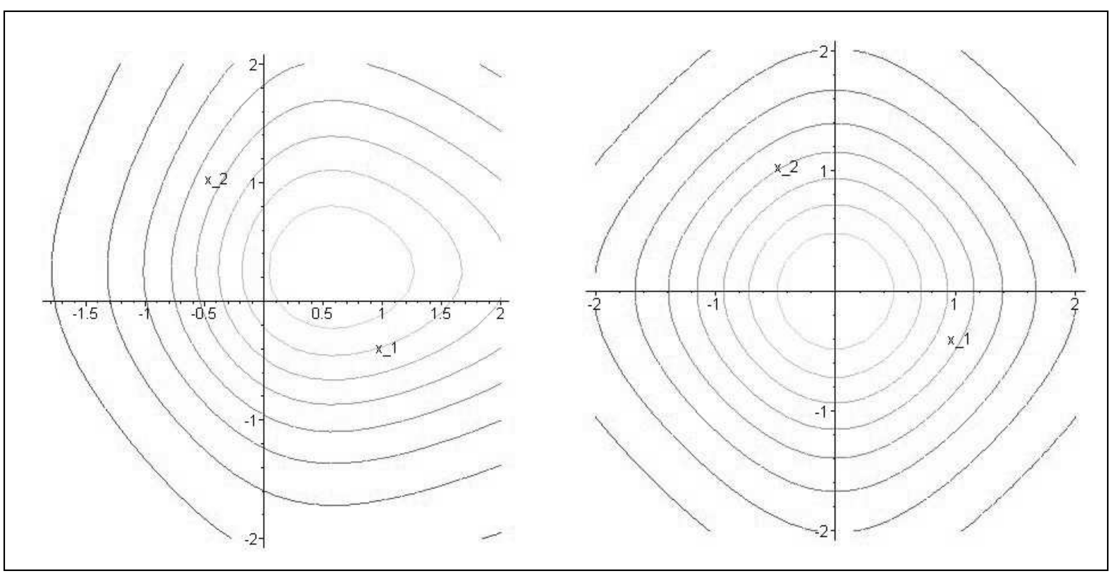


Fig. 2. Contour-plots of the bivariate density function of an $MAGH_2(0, I, \omega)$ distribution with parameters $\lambda = (1, 1)',\ \alpha = (1, 1)'$ and $\beta = (0, 0)'$ (left figure), $\beta = (0.5, 0.25)'$ (right figure).

General affine transformations. The consideration of the lower triangular matrix A' in the stochastic representations (1) and (6) is essential, since any other decomposition of the scaling matrix Σ would lead to a different class of distributions. This phenomenon and a possible extension are discussed below.

Only the elliptically contoured subclass of the MGH distributions is invariant with respect to different decompositions $A'A = \Sigma$. In particular, all decompositions of the scaling matrix Σ lead to the same distribution since they enter the characteristic function via the form $\Sigma = A'A$. Equation (4) also justifies the latter property. However, in the asymmetric or general affine case this equivalence does not hold anymore. In this case, for example, the matrix A can be sought via a singular value decomposition

$$A = UWV' \tag{7}$$

where W is a diagonal matrix having the square roots of eigenvalues of $\Sigma = A'A$ on its diagonal and where the matrix V consists of the corresponding eigenvectors of Σ. The matrices W and V are directly determined from Σ whereas the matrix U might be some arbitrary matrix with orthonormal columns (rotation and flip). However, the most common case, of course, is $U = I$. Here the matrix A is directly computed from Σ utilizing its eigenvalues and eigenvectors. Consequently, every margin of Y is distributed according to a linear combination of the margins of X determined by the principal components (PC) (i.e., the eigenvectors) of the covariance matrix Σ:

$$Y = A'^{-1}X = W^{-1}V'X. \tag{8}$$

4 Dependence structures of MGH and MAGH distributions

In this section we investigate and compare the dependence structures of the MGH distribution and the MAGH distribution. We consider the corresponding copulae and various dependence measures, like the covariance and correlation coefficients, Kendall's tau and tail dependence. It is precisely the copula which encodes all information on the dependence structure unencumbered by the information on marginal distributions and the copula couples the marginal distributions to give the joint distribution. In particular if $X = (X_1, \ldots, X_n)'$ has joint distribution F with continuous marginals $F_1, \ldots, F_n$, then the distribution function of the transformed vector $(F_1(X_1), \ldots, F_n(X_n))'$ is a copula C, and

$$F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)). \tag{9}$$

The MGH and MAGH copulae. According to Definitions 1 and 2, the MGH and MAGH distributions are represented by affine-linear transformations of random vectors following a standardized MGH distribution and MAGH distribution, respectively. Leaving the affine-linear transformation aside, we are interested in the dependence structure (copula) of the underlying random vector. In particular we set the scaling matrix $\Sigma = I$ and $\mu = 0$. Note that the copula of an MGH or MAGH distribution does not even depend on the location vector, i.e., μ is not a copula parameter.

Theorem 3 *Let $Y \in MGH_n(0, I, \omega)$. Then the copula density function of Y is given by*

$$c(u_1, \ldots, u_n) = c\frac{K_{\lambda-n/2}(\alpha\sqrt{1+y'y})}{(1+y'y)^{n/4-\lambda/2}} \prod_{i=1}^{n} \frac{(1+y_i^2)^{1/4-\lambda/2}}{K_{\lambda-1/2}(\alpha\sqrt{1+y_i^2})} \frac{\exp(\alpha\beta' y)}{\exp(\prod_{i=1}^{n} \alpha\beta_i y_i)}\Bigg|_{y_i=F_i^{-1}(u_i)}$$

for $u_i \in [0,1]$, $i = 1, \ldots, n$, and some normalizing constant c. Here F_i refers to the distribution function of the one-dimensional margin Y_i, $i = 1, \ldots, n$.

Let $Y \in MAGH_n(0, I, \omega)$. Then the corresponding copula equals the independence copula, i.e, the copula density function is given by

$$c(u_1, \ldots, u_n) \equiv 1, \quad u_i \in [0,1],\ i = 1, \ldots, n. \tag{10}$$

Proof. Suppose $Y \in MGH_n(0, I, \omega)$. Then Y has a continuous and strictly positive density function f. Utilizing formula (9) we obtain the copula density function by

$$c(u_1, \ldots, u_n) = \frac{f(y_1, \ldots, y_n)}{\prod_{i=1}^{n} f_i(y_i)}\Bigg|_{y_i=F_i^{-1}(u_i)}. \tag{11}$$

Inserting the density function (2) into (11) yields the first assertion. Assume now that $Y \in MAGH_n(0, I, \omega)$. Then Y has independent margins if and only if Y possesses the independence copula $C(u_1, \ldots, u_n) = u_1 \cdot \ldots \cdot u_n$ according to Theorem 2.10.14 in Nelsen (1999). □

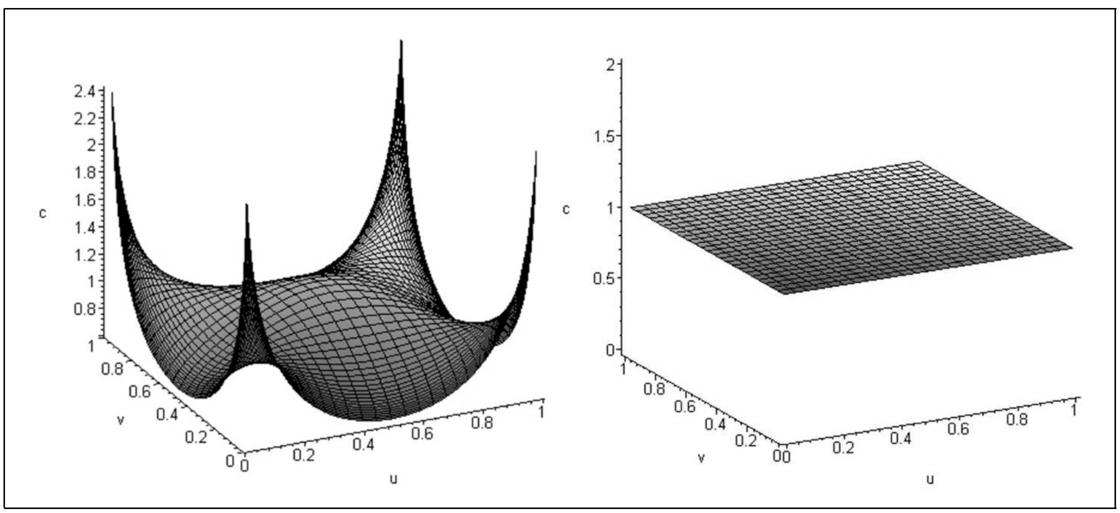


Fig. 3. Copula-density function $c(u, v)$ of an $MGH_2(0, I, \omega)$ distribution (left figure) with parameters $\lambda = 1$, $\alpha = 1$ and $\beta = (0, 0)'$ and that of an $MAGH_2(0, I, \omega)$ distribution (right figure) with arbitrary parameter constellation.

Figure 3 clearly reveals that the independence copula of an MAGH distribution is quite different to the copula function of an MGH distribution. In particular, the MGH copula shows dependence in the limiting corners like in the far right-upper and lower-left quadrant. This property is investigated in more detail later, when we discuss the concept of tail dependence.

According to Theorem 2.10.12 in Nelsen (1999), any copula function is bounded from below (above) by the lower (upper) Fréchet bound, i.e.,

$$W(u_1,\dots,u_n) \le C(u_1,\dots,u_n) \le M(u_1,\dots,u_n), \tag{12}$$

where $W(u_1,\dots,u_n) = \max(u_1+\ldots+u_n-n+1,0)$ and $M(u_1,\dots,u_n) = \min(u_1,\dots,u_n)$. Let $\Pi(u_1,\dots,u_n) = u_1\cdot\ldots\cdot u_n$ be the product or independence copula. Note that M and Π are always copula functions. However, W is only a copula for dimension $n=2$.

Theorem 4 (Limiting cases) *Let*

$$\Sigma^{(m)} := \begin{pmatrix} \sigma_{11}^{(m)} & \sigma_{12}^{(m)} \\ \sigma_{12}^{(m)} & \sigma_{22}^{(m)} \end{pmatrix} \in I\!R^{2\times 2},\ m \in I\!N,$$

be a sequence of symmetric positive-definite matrices and $\rho^{(m)} := \sigma_{12}^{(m)}/\sqrt{\sigma_{11}^{(m)}\sigma_{22}^{(m)}}$. *Suppose that* $X^{(m)} \in MGH_2(\mu,\Sigma^{(m)},\omega)$ *or* $X^{(m)} \in MAGH_2(\mu,\Sigma^{(m)},\omega)$ *for every* $m \in I\!N$, *and let* $C^{(m)}$ *denote the corresponding copula. Then*

i) $C^{(m)} \to M$ *pointwise if* $\rho^{(m)} \to 1,\ \sigma_{ij}^{(m)} \to \sigma_{ij} \neq 0,\ i,j=1,2,$ *as* $m\to\infty$,
ii) $C^{(m)} \to W$ *pointwise if* $\rho^{(m)} \to -1,\ \sigma_{ij}^{(m)} \to \sigma_{ij} \neq 0,\ i,j=1,2,$ *as* $m\to\infty$,
iii) if $X^{(m)} \in MGH_2(\mu,\Sigma^{(m)},\omega)$, *then* $C^{(m)} \neq \Pi$ *for each parameter constellation, and*
iv) if $X^{(m)} \in MAGH_2(\mu,\Sigma^{(m)},\omega)$, *then* $C^{(m)} = \Pi$ *if and only if* $\Sigma^{(m)} = I$.

Proof. i) For each $m \in I\!N$ the random vector $X^{(m)}$ possesses the stochastic representation $X^{(m)} \overset{d}{=} (A^{(m)})'Y + \mu$, with

$$(A^{(m)})' = \begin{pmatrix} \sqrt{\sigma_{11}^{(m)}} & 0 \\ \sigma_{12}^{(m)}/\sqrt{\sigma_{11}^{(m)}} & \sqrt{\sigma_{22}^{(m)}}\sqrt{1-(\rho^{(m)})^2} \end{pmatrix}$$

being the Cholesky decomposition of $\Sigma^{(m)}$. Note that the correlation coefficient of $X^{(m)} = (X_1^{(m)},X_2^{(m)})'$ fulfills $\rho^{(m)} = \sigma_{12}^{(m)}/\sqrt{\sigma_{11}^{(m)}\sigma_{22}^{(m)}} \in (-1,1)$ for all $m \in I\!N$ since $\Sigma^{(m)}$ is positive-definite. If now $\rho^{(m)} \to 1,\ \sigma_{ij}^{(m)} \to \sigma_{ij} \neq 0,\ i,j=1,2,$ as $m\to\infty$ then

$$X^{(m)} \xrightarrow{d} A'Y + \mu =: X \text{ as } m\to\infty \qquad \text{with } A' = \begin{pmatrix} \sqrt{\sigma_{11}} & 0 \\ \sigma_{12}/\sqrt{\sigma_{11}} & 0 \end{pmatrix}.$$

Consequently for $X = (X_1,X_2)'$ and $\mu = (\mu_1,\mu_2)'$ we obtain $X_1 = (X_2-\mu_2)\sigma_{11}/\sigma_{12} + \mu_1 = g(X_2)$ for some strictly increasing mapping $g : I\!R \to I\!R$ because $\sigma_{12} > 0$. Due to Theorem 2.4.3 in Nelsen (1999) the copula C of X is invariant under strictly increasing transformations of the margins because X possesses continuous marginal distribution functions. Therefore we conclude

$$C \equiv C_{X_1,X_2} \equiv C_{X_1,X_1} \equiv M.$$

Hence $C^{(m)} \to C \equiv M$ pointwise as $m\to\infty$ because $X^{(m)} \xrightarrow{d} X$ as $m\to\infty$ and the corresponding distribution functions are continuous.

ii) The second assertion can be shown analogously, using the fact that

$$C \equiv C_{X_1,X_2} \equiv C_{X_1,g(-X_1)} \equiv C_{X_1,-X_1} \equiv W$$

for some strictly increasing mapping $g : \mathbb{R} \to \mathbb{R}$.

iii) Suppose $X^{(m)} \in MGH_2(\mu, \Sigma^{(m)}, \omega)$. Then $X^{(m)}$ possesses the product copula if and only if it has independent margins (see Theorem 2.4.2 in Nelsen (1999)). According to Definition 1, $X^{(m)}$ does not have independent margins if $\Sigma^{(m)}$ is not a diagonal matrix. Thus it suffices to consider $\Sigma^{(m)} = I$. Further we can put $\beta = 0$ since β has no influence on the factorization of the density function of an MGH distribution (see formula (2)). However, in that case $X^{(m)}$ belongs to the family of elliptically contoured distributions. Therefore Theorem 4.11 in Fang, Kotz, and Ng (1990) implies that $X^{(m)}$ possesses independent margins if and only if $X^{(m)}$ is has a bivariate normal distribution. Since normal distributions and MGH distributions are disjoint classes of distributions, the assertion follows.

iv) According to Definition 2, MAGH distributions have independent margins if and only if $\Sigma = I$. □

Remark. The results of Theorem 4 can be extended to n-dimensional MGH and MAGH distributions. However, for $n \geq 3$ the lower Fréchet bound is not a copula function anymore; see Theorem 2.10.13 in Nelsen (1999) for an interpretation of the lower Fréchet bound in that case.

The covariance and correlation matrix. Among the large number of dependence measures for multivariate random vectors the covariance and the correlation matrix are still the most favorite ones in most practical applications. Many multivariate models in finance are based on these linear dependence measures which are appropriate under the assumption of a normal distribution. However, as we have already mentioned, several empirical investigations reject the hypothesis of a multivariate normal distribution to be suited for financial data-modelling. Consequently, new distribution models like the present MGH and MAGH model have been introduced where dependence measures describing only linear dependence between bivariate random vectors should be considered with care. Often, "scale-invariant" dependence measures like Kendall's tau seem to be more appropriate; for more details we refer to Embrechts, McNeil, and Straumann (1999). According to Lindskog, McNeil, and Schmock (2003), the correlation matrix represents a reasonable dependence measure within the class of elliptically contoured distributions.

Theorem 5 (Mean and covariance for MGH distributions)
Let $X \in MGH_n(\mu, \Sigma, \omega)$ and define $R_{\lambda,i}(x) := \frac{K_{\lambda+i}(x)}{x^i K_\lambda(x)}$. Then the mean vector and the covariance matrix of X are given by

$$E[X] = \mu + \alpha R_{\lambda,1}\left(\sqrt{\alpha^2(1-\beta'\beta)}\right) A'\beta$$

and

$$Cov[X] = R_{\lambda,1}\left(\sqrt{\alpha^2(1-\beta'\beta)}\right) \Sigma$$

$$+\left[R_{\lambda,2}\left(\sqrt{\alpha^2(1-\beta'\beta)}\right)-R^2_{\lambda,1}\left(\sqrt{\alpha^2(1-\beta'\beta)}\right)\right]\ \frac{A'\beta\beta'A}{1-\beta'\beta}$$

For the symmetric case $\beta=(0,\dots,0)'$ and $\lambda=1$, the mean vector and the covariance matrix of X simplify to $E[X]=0$ and $Cov[X]=K_2(\alpha)/(\alpha K_1(\alpha))\cdot\Sigma$.

Proof. Let $X\in MGH_n(\bar\mu,\bar\Sigma,\bar\omega)$ with parameter representation as in (5). Then X is distributed according to a variance-mean mixture of a multivariate normal distribution, i.e., $X|(Z=z)\sim N(\bar\mu+z\bar\Sigma\bar\beta,z\bar\Sigma)$, where the mixing random variable Z is distributed according to a generalized inverse Gaussian distribution $GIG(\bar\lambda,\bar\delta,\sqrt{\bar\delta^2(\bar\alpha^2-\bar\beta'\bar\Sigma\bar\beta)})$ (see e.g. Barndorff-Nielsen, Kent, and Sørensen (1982)). Hence,

$$E[X]=\bar\mu+\bar\Sigma\bar\beta E_{GIG}[Z]$$

and

$$E[XX']=\bar\mu\bar\mu'+(\bar\Sigma+\bar\mu\bar\beta'\bar\Sigma+\bar\Sigma\bar\beta\bar\mu')E_{GIG}[Z]+\bar\Sigma\bar\beta\bar\beta'\bar\Sigma E_{GIG}[Z^2].$$

Therefore, the covariance matrix of X is given by

$$Cov[X]=\bar\Sigma E_{GIG}[Z]+\bar\Sigma\bar\beta\bar\beta'\bar\Sigma Var_{GIG}[Z].$$

According to Eberlein and Prause (1999), mean and variance of the mixing random variable $Z\sim GIG(\bar\lambda,\,\bar\delta,\sqrt{\bar\delta^2(\bar\alpha^2-\bar\beta'\bar\Sigma\bar\beta)})$ are given by

$$E_{GIG}[Z]=R_{\bar\lambda,1}(\sqrt{\bar\delta^2(\bar\alpha^2-\bar\beta'\bar\Sigma\bar\beta)})$$

and

$$Var_{GIG}[Z]=\bar\delta^4\cdot[R_{\bar\lambda,2}(\sqrt{\bar\delta^2(\bar\alpha^2-\bar\beta'\bar\Sigma\bar\beta)})-R^2_{\bar\lambda,1}(\sqrt{\bar\delta^2(\bar\alpha^2-\bar\beta'\bar\Sigma\bar\beta)})].$$

Utilizing the parameter mapping $\bar\delta^2(\bar\alpha^2-\bar\beta'\bar\Sigma\bar\beta)=\alpha^2(1-\beta'\beta)$, $\bar\Sigma\bar\beta\bar\delta^2=\Sigma\alpha A^{-1}\beta$ and $\bar\delta^4\bar\Sigma\bar\beta\bar\beta'\bar\Sigma=A'\beta\beta'A$ yields the assertion. □

Theorem 6 (Mean and covariance for MAGH distributions)
Let $X\in MAGH_n(\mu,\Sigma,\omega)$. Then the mean vector and the covariance matrix are given by

$$E[X]=A'e_Y+\mu\quad\textit{and}\quad Cov[X]=A'CA,$$

respectively, where $e_Y=(E[Y_1],\dots,E[Y_n])'$ with $E[Y_i]=R_{\lambda_i,1}(\sqrt{\alpha_i^2(1-\beta_i^2)})\alpha_i\beta_i$ and $C=diag(c_{11},\dots,c_{nn})$ with

$$c_{ii}=R_{\lambda_i,1}(\sqrt{\alpha_i^2(1-\beta_i^2)})+[R_{\lambda_i,2}(\sqrt{\alpha_i^2(1-\beta_i^2)})-R^2_{\lambda_i,1}(\sqrt{\alpha_i^2(1-\beta_i^2)})]\frac{\beta_i^2}{1-\beta_i^2}.$$

The covariance matrix $Cov[X]$ is proportional to Σ if $\alpha=\alpha_i$, $\beta=\beta_i$ and $\lambda=\lambda_i$ for all $i=1,\dots,n$.

Proof. The assertion follows immediately from Theorem 5 and equation (6). □

Kendall's tau. The correlation coefficient is a measure of linear dependence between two random variables and therefore it is not invariant under monotone increasing transformations. However, not only does "scale-invariance" present an undisputable requirement for

a proper dependence measure in general (cf. Joe (1997), Chapter 5), but also in practice "scale-invariant" dependence measures play an increasing role in dependence modelling. Kendall's tau is the most famous one and therefore we determine it for MGH and MAGH distributions.

Definition 7 (Kendall's tau) *Let $X = (X_1, X_2)'$ and $\bar{X} = (\bar{X}_1, \bar{X}_2)'$ be independent bivariate random vectors with common continuous distribution function F and copula C. Kendall's tau is defined by*

$$\begin{aligned}\tau &= \mathbb{P}((X_1 - \bar{X}_1)(X_2 - \bar{X}_2) > 0) - \mathbb{P}((X_1 - \bar{X}_1)(X_2 - \bar{X}_2) < 0) \\ &= 4 \int_{[0,1]^2} C(u,v)\, dC(u,v) - 1. \end{aligned} \tag{13}$$

Theorem 8 *Let $\rho \in (-1, 1)$ be the correlation coefficient of X_1 and X_2.*

i) If $X \in MGH_2(\mu, \Sigma, \omega)$ with $\beta = 0$, then

$$\tau = \frac{2}{\pi} \arcsin(\rho). \tag{14}$$

ii) If $X \in MAGH_2(\mu, \Sigma, \omega)$ with stochastic representation $X \stackrel{d}{=} A'Y + \mu,\ A'A = \Sigma$, then for $\rho \neq 0$

$$\tau = \frac{4}{|c|} \int_{\mathbb{R}^2} f_{Y_1}(x_1) \left(\frac{x_2 - x_1}{c}\right) \cdot \int_{-\infty}^{x_1} F_{Y_2}\left(\frac{x_2 - z}{c}\right) f_{Y_1}(z)\, dz d(x_1, x_2) - 1, \tag{15}$$

where $c := sgn(\rho)\sqrt{1/\rho^2 - 1}$. Further $\tau = 0$ for $\rho = 0$.

The *proof* is given in the Appendix.

Remark. A closed form expression of Kendall's tau for MAGH distributions (given by (15)) cannot be expected. However, since the density functions of Y_1 and Y_2 are explicitly given, formula (15) yields a tractable numerical solution.

Tail dependence. A strong emphasis in this paper is put on the dependence structure of MGH and MAGH distributions. In this context we establish the following result about the dependence structure of extreme events (tail dependence or extremal dependence) related to the latter types of distributions. The importance of tail dependent especially in financial risk management is emphasized in Hauksson, Dacorogna, Domenig, Mueller, and Samorodnitsky (2001) and Embrechts, Lindskog, and McNeil (2003). The following definition (according to Joe (1997), p. 33) represents one of many possible definitions of tail dependence.

Definition 9 (Tail dependence) *Let $X = (X_1, X_2)'$ be a 2-dimensional random vector. We say that X is upper tail-dependent if*

$$\lambda_U := \lim_{u \to 1^-} \mathbb{P}(X_1 > F_1^{-1}(u) \mid X_2 > F_2^{-1}(u)) > 0, \tag{16}$$

where the limit is assumed to exist and F_1^{-1}, F_2^{-1} *denote the generalized inverse distribution functions of* X_1, X_2, *respectively. Consequently, we say that* X *is upper tail-independent if* λ_U *equals* 0. *Similarly,* X *is said to be lower tail-dependent (lower tail-independent) if* $\lambda_L > 0$ $(\lambda_L = 0)$, *where* $\lambda_L := \lim_{u \to 0^+} \mathbb{P}(X_1 \le F_1^{-1}(u) \mid X_2 \le F_2^{-1}(u))$ *if existent.*

Figure 3 reveals that bivariate standardized MGH distributions show more evidence of dependence in the upper-right and lower-left quadrant of its distribution function than MAGH distributions. However, the following theorem shows that MGH distributions are always tail independent whereas non-standardized MAGH distributions can even model tail dependence. For the sake of simplicity we restrict ourselves to the symmetric case $\beta = 0$.

Theorem 10 *Let* $\rho \in (-1, 1)$ *be the correlation coefficient of the bivariate distributions considered below. Suppose* $\beta = 0$. *Then*
i) the $MGH_2(\mu, \Sigma, \omega)$ *distributions are upper and lower tail-independent,*
ii) the $MAGH_2(\mu, \Sigma, \omega)$ *distributions are upper and lower tail-independent if* $\alpha_2 < \alpha_1\sqrt{1/\rho^2 - 1}$ *or* $\rho \le 0$, *and*
iii) the $MAGH_2(\mu, \Sigma, \omega)$ *distributions are upper and lower tail-dependent if* $\alpha_2 > \alpha_1\sqrt{1/\rho^2 - 1}$ *and* $\rho > 0$.

The *proof* is given in the Appendix.

Remark. Additionally to Theorem 10 it can be shown that an $MGH_2(\mu, \Sigma, \omega)$ distribution (with $\beta = 0$) is tail independent if $\alpha_2 = \alpha_1\sqrt{1/\rho^2 - 1}$ and $\lambda_2 < \lambda_1$.

5 MGH versus MAGH: Advantages and disadvantages

In this section we list and compare some advantages and disadvantages of MGH distributions and MAGH distributions. We start with the *distributional flexibility* to fit real data. An outstanding property of MAGH distributions is that, after an affine-linear transformation, all one-dimensional margins can be fitted separately via different generalized hyperbolic distributions. In contrast to this, the one-dimensional margins of MGH distributions are not that flexible since the parameters α and λ relate to the entire multivariate distribution and determine the strong structural behavior (see Definition 1). However, this structure causes a large subclass of MGH distributions to belong to the family of elliptically contoured distributions which inherit many useful statistical and analytical properties from multivariate normal distributions.

Regarding the *dependence structure*, the MAGH distributions may have independent margins for any parameter constellations $\omega = (\omega_1, \ldots, \omega_n)$ (see Theorem 4). In particular, they also support models which are based on a linear combination of independent factors. In contrast, the MGH distributions are not capable of modelling independent margins. They even yield "extremal" dependencies for bivariate distributions having correlation zero. Moreover, the correlation matrix of MAGH distributions is proportional to the scaling matrix Σ within a large subclass of asymmetric MAGH distributions (see Theorem 6), whereas Σ is hardly to interpret for skewed MGH distributions. Further, the copula

of MAGH distributions, being the dependence structure of an affine-linear transformed random vector with independent components, is quite illustrative and possesses many appealing modelling properties. On the other hand, the copula structure of MGH distributions may suffers from inflexibility. Regarding the dependence of extreme events, the MAGH distributions can model tail dependence whereas MGH distributions are always tail independent. Therefore, MAGH distributions are suitable especially within the field of risk management.

Sections 6 and 8 reveal that in contrast to MGH distributions, *parameter estimation* for MAGH distributions is considerably simpler and more robust. Even in an asymmetric environment, the parameters of MAGH distributions can be identified in a two stage procedure which has a considerable computational advantage in higher dimensions. The same procedure can be applied for elliptically contoured MGH distributions ($\beta = 0$). The random vector *generation* algorithms for MGH and MAGH distributions turn out to be equally efficient and fast, irrespective of the dimension.

The simulations in Section 8 show that both distributions fit simulated and real data well. Thus, summarizing the above advantages and disadvantages, the MAGH distributions have much to recommend them regarding their parameter estimation, dependence structure, and random vector generation. However, it depends also on the kind of application and the user's taste which model to prefer.

6 Parameter estimation

6.1 MGH distributions

Minimizing the cross entropy. A general method to measure the similarity between two distribution or density functions f^* and f, respectively, is given by the *Kullback entropy* (see Kullback (1959)) which is defined as

$$H_K(f, f^*) := \int_{\mathbb{R}^n} f^*(x) \log \frac{f^*(x)}{f(x)} dx \tag{17}$$

$$= \int_{\mathbb{R}^n} f^*(x) \log f^*(x) dx - \int_{\mathbb{R}^n} f^*(x) \log f(x) dx. \tag{18}$$

The Kullback entropy is always nonnegative and is zero if the densities f and f^* are identical. In our context, this relationship is useful to measure the similarity between the "true" density f^* and its approximation f. Therefore, minimizing the Kullback entropy by varying f is one way to find a good approximation of the "true" density f^*. Note that the first term in (17) is constant and can be dropped. This leads to the so-called *cross entropy* given by

$$H(f, f^*) := -\int_{\mathbb{R}^n} f^*(x) \log f(x) dx. \tag{19}$$

Obviously the integral in (19) cannot be evaluated without knowledge of f^*. Hence, the corresponding distribution is approximated by its empirical counterpart, i.e., by the empirical distribution function

$$F_e^*(x) = \frac{1}{m} \sum_{k=1}^{m} \mathbf{1}_{\{X_k \leq x\}}, \tag{20}$$

where $X_k,\ k = 1, \ldots, m$, denotes a random sample with distribution function F^*. It is well known that this procedure leads to an unbiased estimator of $F^*(x)$ (see, for example, Parzen (1962)).

An approximation of the integral in (19) is now given by

$$H(f, f^*) \approx -\int_{I\!R^n} \log f(x) dF_e^*(x) = -\frac{1}{m} \sum_{k=1}^{m} \log f(X_k). \tag{21}$$

Thus, minimizing the cross entropy is approximately equivalent to maximizing the log-likelihood function.

For computational reasons it may be advantageous to work with the standardized vector $y = B(x - \mu)$ where $B := A'^{-1}$. The density transformation theorem yields that the density function f_X of $X \stackrel{d}{=} A'Y + \mu \in MGH(\mu, \Sigma, \omega)$ is given by

$$f_X(x) = f_{A'Y+\mu}(x) = |B| f_Y(y), \tag{22}$$

with $|B| > 0$ being the determinant of B. The following expression indicates the parameterization of the negative log-likelihood function L while utilizing the above standardization:

$$L(X, \eta) = -\sum_{k=1}^{m} \log(|B|) - \sum_{k=1}^{m} \log(f_Y(\omega, B(X_k - \mu))) \tag{23}$$

with parameters $\eta = (\mu, B, \omega)$ and $\omega = (\lambda, \alpha, \beta)$.

While the location vector μ can take arbitrary values in $I\!R^n$, B and ω are subject to various constraints. The matrix B must be triangular with positive diagonal such that $A'A$ is positive-definite. The parameter α is supposed to be positive and the vector β must fulfill $||\beta||_2 \leq 1$.

Many approaches are possible regarding the latter optimization problem, inter alia we mention two methods:

- constrained nonlinear optimization methods, and
- unconstrained nonlinear optimization methods after suitable parameter transformations.

We prefer the unconstrained approach due to robustness and efficiency reasons. The following parameter transformations are appropriate.

The matrix B can be sought of the form $B = UD$ where D is some diagonal matrix having strictly positive elements and U is a triangular matrix having only ones on its diagonal. In order to enforce the strict positivity of the diagonal elements $d_{ii}, i = 1, \ldots, n$, of D, the following transformations are applied:

$$\begin{aligned} b_{ii} &= d_{ii} = e^{\nu_i}, \quad i = 1, \ldots, n, \\ b_{ij} &= d_{ii} u_{ij} = e^{\nu_i} u_{ij}, i = 1, \ldots, n-1; \quad j = 2, \ldots, n; \quad j > i \end{aligned}$$

with unknown parameters ν_i and u_{ij}. The parameter α is estimated via the same exponential map. For the vector β we utilize the smooth transformation

$$\beta = \gamma \frac{1}{1 + \exp(-\|\gamma\|_2)} \cdot \frac{1}{\|\gamma\|_2}. \tag{24}$$

After the transformations we are now confronted with the new optimization problem

$$\min_{\bar{\eta} \in \Omega} L(X, \bar{\eta})$$

where $\Omega = I\!R^{(3+(n-1)/2)n+2}$ and $\bar{\eta}$ denotes the transformed unconstrained parameters. Since in general the objective function is non-convex, a global numerical optimization method is required. Optimization methods can be subdivided into deterministic and probabilistic methods. As far as deterministic methods are concerned, certain assumptions on the objective function have to be imposed (like Lipschitz conditions) in order to guarantee a termination of the optimization routine in finitely many steps. Probabilistic methods, in contrast, have better convergence properties but only with a certain likelihood. Here the assumptions on the objective function are rather weak.

The algorithm we use belongs to the probabilistic ones and consists of two characteristic phases:

1. In a *global phase* the objective function is evaluated at a random number of points being part of the search space.
2. In a *local phase* the samples are transformed to become candidates for local optimization routines.

Thus, the global phase is responsible for the convergence result of the optimization routine while the local phase determines the efficiency of the optimization. In the ideal (but hardly attainable) case, only a few local optimizations are necessary as there are attractors in the range of the objective function (e.g. regions where the minimum is reached by gradient descent). We mentioned already that the probabilistic algorithms find a global solution only with a given probability. Rinnooy Kan and Timmer (1987) show the convergence of probabilistic algorithms to the global minimum under certain conditions. This motivates a usage of the Multi-Level Single-Linkage procedure (see Rinnooy Kan and Timmer (1987)). Here the global optimization method generates random samples from the search space by identifying the samples belonging to the same objective function attractor and eliminating multiple ones. For the remaining samples in the local phase a conjugate gradient method (see Fletcher (1987)) is started. Further, a Bayesian stopping rule is applied in order to assess the probability that all attractors have been explored. For an account on the Bayesian stopping rule we refer the reader to Boender and Rinnooy Kan (1987).

The parameters of the MAGH distribution function can be easily identified in a two-stage procedure comprising the following steps:

(1) Compute the sample covariance matrix S of the random vector X. Transforme X to the vector $Y = BX$ with independent margins. The matrix B is received via Cholesky decomposition $S^{-1} = B'B$.
(2) Identify the parameters λ_i, α_i and β_i which belong to the univariate marginal distributions of Y. The location vector μ can be received via $B^{-1}e$ with e_i being the location parameter of Y_i (see Theorem 5). The scaling matrix Σ equals $B^{-1'}DB^{-1}$ where the diagonal matrix D is determined by the scaling parameters of Y_i on its diagonal.

The latter procedure simplifies the complexity of the numerical optimization a lot.

Note that the parameters of MAGH distributions can be estimated analogously to the procedure described for MGH distributions. However, instead of the multivariate density (2), a product of n univariate densities

$$f_1(y) = c\frac{K_{\lambda-1/2}(\alpha\sqrt{1+y^2})}{(1+y^2)^{1/4-\lambda/2}}e^{\alpha\beta y} \tag{25}$$

with normalizing constant

$$c = \frac{\alpha^{1/2}\,(1-\beta^2)^{\lambda/2}}{(2\pi)^{1/2}K_\lambda(\alpha\sqrt{1-\beta^2})} \tag{26}$$

is considered. It is important for applications that the univariate densities are not necessarily identically parameterized. This means that the margins may have different parameters $\lambda_i, \alpha_i, \beta_i,\ i = 1, \ldots, n$. In other words, there is a considerable freedom of choosing the parameters λ, α and β. Therefore, in addition to a parameterization similar to that for MGH distributions (i.e., the same λ and α for all one-dimensional margins and different β_i) two further extreme alternatives are possible:

- Minimum parameterization: Equal parameters λ, α and β for all one-dimensional margins.
- Maximum parameterization: Individual parameters λ_i, α_i and β_i for all one-dimensional margins.

The appropriate parameterization depends on the kind of application and the available data volume.

The optimization procedure presented in this section is a special case of an identification-algorithm for conditional distributions explored in Stützle and Hrycej (2001, 2002a, 2002b).

7 Sampling from MGH and MAGH distributions

Additionally to the estimation procedures described in Section 6, in this section we provide an efficient and self-contained random vector generation for the families of MGH and MAGH distributions. Fast sampling from an $MGH_n(\mu, \Sigma, \omega)$ distribution is possible via the following variance-mean mixture representation.

Let the random variable Z be distributed according to a generalized inverse Gaussian distribution with parameters λ, χ and ψ. In particular the latter family is referred to as the $GIG(\lambda, \chi, \psi)$ distributions. Then, $X \in MGH_n(\mu, \Sigma, \omega)$ is conditionally normal distributed with mixing random variable Z, i.e., $X|(Z = z) \sim N_n(\mu + z\tilde{\beta}, z\Delta)$, where $\Delta \in \mathbb{R}^{n \times n}$ is a symmetric positive-definite matrix with determinant $|\Delta| = 1$ and $\mu, \tilde{\beta} \in \mathbb{R}^n$. The parameters Σ and $\omega = (\lambda, \alpha, \beta)$ are given by

$$\alpha = \sqrt{(\psi + \tilde{\beta}'\Delta\tilde{\beta})\chi},\ \beta = 1/\sqrt{\psi + \tilde{\beta}'\Delta\tilde{\beta}}L\tilde{\beta},\ \Sigma = \chi \cdot \Delta$$

with Cholesky decomposition $L'L = \Delta$. The inverse map is given by

$$\chi = |\Sigma|^{1/n},\ \psi = \alpha^2/|\Sigma|^{1/n} \cdot (1 - \beta'\beta),\ \Delta = \Sigma/|\Sigma|^{1/n},\ \text{and}\ \tilde{\beta} = \alpha \cdot (A)^{-1}\beta$$

with Cholesky decomposition $A'A = \Sigma$.

The sampling algorithm is now of the following form: A pseudo random number is sampled from a random variable Z having a generalized inverse Gaussian distribution with parameters λ, χ, and ψ. Then an n-dimensional random vector X being conditionally normal distributed with mean vector $\mu + Z\Delta\tilde{\beta}$ ("drift") and covariance matrix $Z\Delta$ (determinant $|\Delta| = 1$) is generated.

The density function of the generalized inverse Gaussian distribution $GIG(\lambda, \chi, \psi)$, is given by

$$f_Z(x) = c \cdot x^{\lambda-1} \exp\Big(-\frac{\chi}{2x} - \frac{\psi x}{2}\Big), \quad x > 0, \tag{27}$$

with normalizing constant $c = (\psi/\chi)^{\lambda/2}/(2K_\lambda(\sqrt{\psi\chi}))$. The range of the parameters is given by

i) $\chi > 0,\ \psi \geq 0$ if $\lambda < 0$, or
ii) $\chi > 0,\ \psi > 0$ if $\lambda = 0$, or
iii) $\chi \geq 0,\ \psi > 0$ if $\lambda > 0$.

The following efficient algorithm is formulated for multivariate generalized hyperbolic distributions $MGH_n(\mu, \Sigma, \omega)$ with parameter $\lambda = 1$. Section 8.1 justifies the restriction to that class of distributions. In this context the $GIG(\lambda, \chi, \psi)$ distribution is referred to as inverse Gaussian distribution. However, the algorithm can be easily extended to general λ. Our empirical study shows that the algorithm outperforms the efficiency of the sampling algorithm proposed by Atkinson (1982) which suites to a larger class of distributions (see also Prause (1999), Section 4.6). Moreover, the algorithm avoids tedious minimization routines and time-consuming evaluations of the Bessel function K_λ. The generation utilizes

a rejection method (see Ross (1997), pp. 565) with a three part rejection-envelop. We define the envelop $d : I\!R_+ \to I\!R_+$ by

$$d(x) := \begin{cases} d_1(x) = ca_1 \exp(b_1 x), & \text{if } 0 < x < x_1, \\ d_2(x) = ca_2, & \text{if } x_1 \leq x < x_2, \\ d_3(x) = ca_3 \exp(-b_3 x), & \text{if } x_2 \leq x < \infty, \end{cases} \tag{28}$$

with $a_i > 0,\ i = 1, \ldots, 3,\ b_i > 0,\ i = 1, 3$, and $x_1 \leq x_2 \leq x_3$ to be defined later. Let $z_i,\ i = 1, 3$ denote the inflection points and $z_2 = \sqrt{\chi/\psi}$ denote the mode of the unimodal density f_Z. Further we require

$$d_1(z_1) = f_Z(z_1), \quad d_2(z_2) = f_Z(z_2), \quad d_3(z_3) = f_Z(z_3). \tag{29}$$

The points $x_1 > 0$ and $x_2 > 0$ correspond to the intersection points of d_1, d_2 and d_2, d_3, respectively, i.e.,

$$d_1(x_1) = d_2(x_1), \qquad d_2(x_2) = d_3(x_2). \tag{30}$$

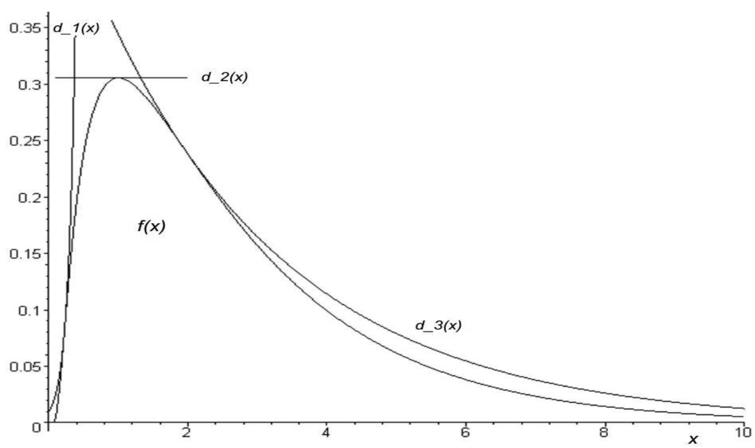


Fig. 4. Three part envelop d for the inverse Gaussian density function f_Z with parameters $\chi = 1$, $\psi = 1$.

Primarily, the rejection method requires the generation of random numbers with density $s \cdot d(x)$ where the scaling factor s has to be computed in order to obtain a density function $s \cdot d(x), x > 0$. This scaling factor is derived below.

Pseudo algorithm for generating an inverse Gaussian random number:

(1) Compute the zeros z_1, z_2 for $\psi^2 z^4 - 2\chi\psi z^2 - 4\chi z + \chi^2 = 0$.
(2) Set $b_1 = (\chi/z_1^2 - \psi)/2$ and $a_1 = \exp(-\chi/z_1)$.
(3) Set $a_2 = \exp(-\sqrt{\chi\psi})$.
(4) If $(\psi - \chi/z_2^2)/2 > 0$ then
    Set $b_3 = (\psi - \chi/z_2^2)/2$ and $a_3 = \exp(-\chi/z_2)$
  Else Set $b_3 = \psi/2$ and $a_3 = 1$.
(5) Set $x_1 = \ln(a_2/a_1)/b_1$ and $x_2 = -\ln(a_2/a_3)/b_3$.

(6) Set

$$s = \Big(\frac{a_1}{b_1}\exp(b_1 x_1) - \frac{a_1}{b_1} + (x_2 - x_1)a_2 + \frac{a_3}{b_3}\exp(-b_3 x_2)\Big).$$

(7) Set

$$k_1 = \frac{1}{s}\Big(\frac{a_1}{b_1}\exp(b_1 x_1) - \frac{a_1}{b_1}\Big) \quad \text{and} \quad k_2 = k_1 + \frac{1}{s}(x_2 - x_1)a_2.$$

(8) Generate independent and uniformly distributed random numbers U and V on the interval $[0, 1]$.

(9) If $U \leq k_1$ goto step 10.
ElseIf $k_1 < U \leq k_2$ goto step 11.
Else goto step 12.

(10) Set

$$x = \frac{1}{b_1}\ln\Big(\frac{b_1}{a_1}sU + 1\Big).$$

If

$$V \leq \frac{f_Z(x)}{d_1(x)} = \frac{1}{a_1}\exp\Big(-\big(\frac{\chi x^{-1} + \psi x}{2} + b_1 x\big)\Big)$$

Then Return x
Else goto step 8.

(11) Set

$$x = \frac{sU}{a_2} - \frac{a_1}{b_1 a_2}\Big(\exp(b_1 x_1) - 1\Big) + x_1.$$

If

$$V \leq \frac{f_Z(x)}{d_2(x)} = \frac{1}{a_2}\exp\Big(-\big(\frac{\chi x^{-1} + \psi x}{2}\big)\Big)$$

Then Return x
Else goto step 8.

(12) Set

$$x = -\frac{1}{b_3}\ln\Big[-\frac{b_3}{a_3}\Big\{sU - \frac{a_1}{b_1}(e^{b_1 x_1} - 1) - (x_2 - x_1)a_2 - \frac{a_3}{b_3}e^{-b_3 x_2}\Big\}\Big].$$

If

$$V \leq \frac{f_Z(x)}{d_3(x)} = \frac{1}{a_3}\exp\Big(-\big(\frac{\chi x^{-1} + \psi x}{2}\big) + b_3 x\Big)$$

Then Return x
Else goto step 8.

Remark. In order to generate a sequence of inverse Gaussian random numbers repeat step 8.

So far we have generated random numbers from an univariate inverse Gaussian distribution. We turn now to the generation of multivariate generalized hyperbolic random vectors. For this we exploit the above introduced mixture representation.

Pseudo algorithm for generating an MGH vector:

(1) Set $\Delta = L'L$ via Cholesky decomposition.
(2) Generate an inverse Gaussian random number Z with parameters χ and ψ.
(3) Generate a standard normal random vector N.

(4) Return $X = \mu + Z\Delta\tilde{\beta} + \sqrt{Z}L'N$.

Pseudo algorithm for generating an MAGH vector:

(1) Set $\Sigma = A'A$ via Cholesky decomposition.
(2) Generate a random vector Y with independent $MGH_1(0, 1, \omega_i)$, $i = 1, \ldots, n$, distributed components (see above).
(3) Return $X = \mu + A'Y$.

Table 1 presents values of empirical efficiency of the MGH random vector generator for various parameter constellations. In our framework, efficiency is defined by the following ratio

$$\text{Efficiency} = \frac{\sharp \text{ of generated samples}}{\sharp \text{ of algorithm-passes including rejections}} .$$

χ/ψ	0.1	0.5	1	2	5	10
0.1	0.94	0.913	0.904	0.886	0.877	0.872
0.5	0.916	0.884	0.877	0.865	0.859	0.852
1	0.901	0.877	0.867	0.863	0.857	0.851
2	0.889	0.866	0.857	0.856	0.856	0.845
5	0.876	0.858	0.854	0.852	0.847	0.849
10	0.866	0.86	0.851	0.853	0.842	0.847

Table 1
Empirical efficiency of the MGH random number generator for $\lambda = 1$ and $10,000$ generated samples.

8 Simulation and empirical study

A series of computational experiments with simulated data is performed in this section. The experiments disclose that

- MGH density functions with arbitrary parameter λ seem to have very close counterparts in the MGH-subclass with $\lambda = 1$,
- The MAGH model can closely approximate the MGH model with similar parameter values.

8.1 Arbitrary MGH distributions versus MGH distributions with parameter $\lambda = 1$ (MH)

Figure 5 shows the identification results of four univariate distributions with random samples drawn from the following reference distributions. The identification takes place

either under the assumption of an arbitrary generalized hyperbolic distribution or of a generalized hyperbolic distribution with fixed $\lambda = 1$ (in short: MH distribution). In both pictures on the left half of the figure, samples were drawn from the generalized hyperbolic distribution with fixed parameter $\lambda = 1$. They illustrate the phenomenon that although the identification procedure for MGH densities frequently produces $\lambda \neq 1$, the approximation of the density function remains good. The pictures on the right side illustrate the opposite case, namely, a good approximation of the MGH density function ($\lambda \neq 1$) via the MH density function ($\lambda = 1$).

Consider now an $MGH_2(\mu, \Sigma, \omega)$ distribution with $\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}$. The mutual tradeoff between λ, α and the scaling parameters $S_1 := \sqrt{\sigma_{11}}$ and $S_2 := \sqrt{\sigma_{22}}$ for the latter distribution function is shown in Table 2. While all samples were drawn from an MH distribution, MGH identifications usually lead to an overestimate of λ which is traded off by lower values of α, S_1 and S_2. In contrast to that, the parameter identifications for MH distributions are close to the reference values. However, the differences between the cross entropies are hardly discernible, showing that both parameter combinations correspond to densities which are close to each other.

The following conclusions can be drawn:

- The fit of both distributions, MGH and MH distribution, measured by cross entropy and visual closeness of the plots, is satisfying. This implies the existence of multiple parameter constellations for MGH distribution functions which lead to quite similar density functions. Similar results have been observed for the corresponding tail functions.
- Generalized hyperbolic densities seem to have very close counterparts in the class of MH distributions (even for large λ). Therefore, the class of MH distributions will be sufficiently rich for our considerations.

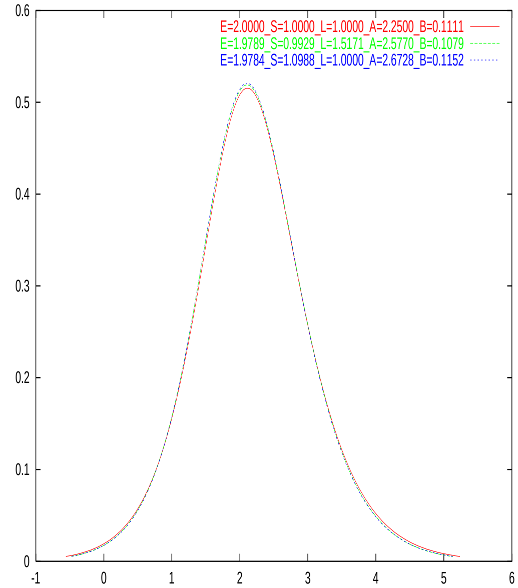
In view of the above results, only MH distributions and the corresponding MAH distributions (multivariate affine generalized hyperbolic distributions with $\lambda = 1$) will be considered in the next section.

8.2 MH distributions versus MAH distributions

The estimation of parameters is compared for the following three classes of bivariate distributions:

(1) MH distributions.
(2) MAH distributions with minimal parameter configuration (same value of α and β for each margin).
(3) MAH distributions with maximal parameter configuration (different values of α_i and β_i for each margin).

All types of distributions in Table 3 have been identified from data which were sampled from an MH distribution. The two stage algorithm introduced in Section 6.2 has been used for the identification of the MAHmax model (determining first the sample correlation ma-



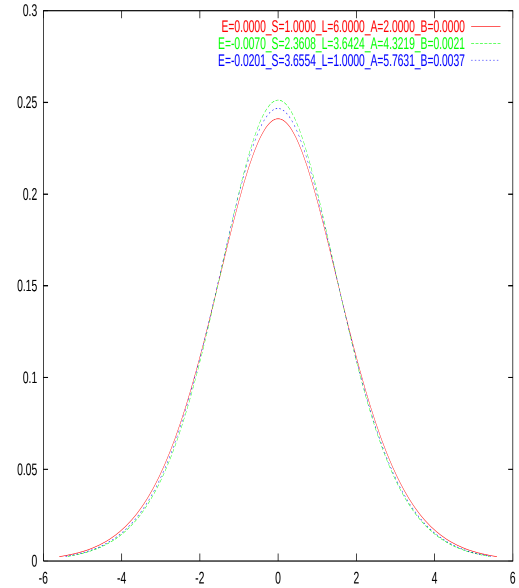
$(\mu, \sqrt{\Sigma}, \lambda, \alpha, \beta)' =$

$(2.00, 1.00, 1.00, 2.25, 0.11)'$ (solid line)

$(m(\hat{\mu}), m(\hat{\sqrt{\Sigma}}), m(\hat{\lambda}), m(\hat{\alpha}), m(\hat{\beta}))' =$

$(1.98, 0.99, 1.51, 2.58, 0.11)'$ (bright dotted line)

$(1.98, 1.10, 1.00, 2.67, 0.12)'$ (dark dotted line)



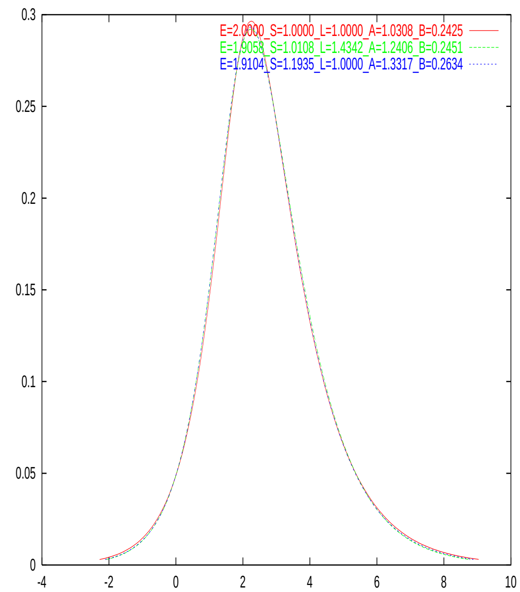
$(\mu, \sqrt{\Sigma}, \lambda, \alpha, \beta)' =$

$(0.00, 1.00, 6.00, 2.00, 0.00)'$ (solid line)

$(m(\hat{\mu}), m(\hat{\sqrt{\Sigma}}), m(\hat{\lambda}), m(\hat{\alpha}), m(\hat{\beta}))' =$

$(0.01, 2.36, 3.64, 4.32, 0.00)'$ (bright dotted line)

$(0.02, 3.66, 1.00, 5.76, 0.00)'$ (dark dotted line)



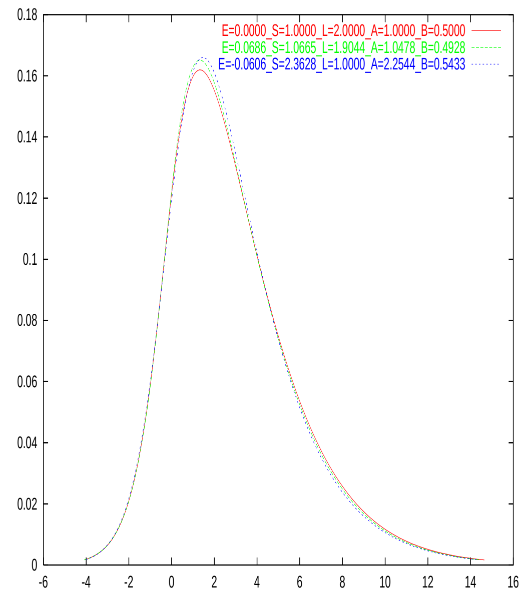
$(\mu, \sqrt{\Sigma}, \lambda, \alpha, \beta)' =$

$(2.00, 1.00, 1.00, 1.03, 0.24)'$ (solid line)

$(m(\hat{\mu}), m(\hat{\sqrt{\Sigma}}), m(\hat{\lambda}), m(\hat{\alpha}), m(\hat{\beta}))' =$

$(1.91, 1.01, 1.43, 1.24, 0.24)'$ (bright dotted line)

$(1.91, 1.19, 1.00, 1.33, 0.26)'$ (dark dotted line)



$(\mu, \sqrt{\Sigma}, \lambda, \alpha, \beta)' =$

$(0.00, 1.00, 2.00, 1.00, 0.50)'$ (solid line)

$(m(\hat{\mu}), m(\hat{\sqrt{\Sigma}}), m(\hat{\lambda}), m(\hat{\alpha}), m(\hat{\beta}))' =$

$(0.07, 1.07, 1.90, 1.05, 0.49)'$ (bright dotted line)

$(0.06, 2.36, 1.00, 2.25, 0.54)'$ (dark dotted line)

Fig. 5. Univariate MGH and MH distributions. Reference densities (solid lines) and identified MGH and MH densities (bright and dark dotted lines) averaged from 100 random samples for various parameter constellations. The values $m(\cdot)$ denote the sample mean of the respective parameters.

trix, then transforming the variables, and finally identifying the univariate distributions). The identification results are provided in Table 3.

The following conclusions can be drawn:

- For all three models, most parameters show an acceptable fit regarding the sample bias and the sample standard deviation. The relative variability of the estimates increases with decreasing α (fatter tailed distributions). Such fatter tailed distributions seem to be more ill-posed with respect to the estimation of individual parameters.
- The differences between the parameter estimates obtained either for the MH, the MAHmin, or the MAHmax distribution are negligible (although the data are drawn from an MH distribution).
- The fit in terms of the cross entropy does not differ significantly between the various models. As expected, the MAHmax estimates are closer to the MH reference distribution than the MAHmin estimates are in terms of the cross entropy (Note that in one case they are even better than the MH estimates). The fitting capability of the MAHmax model comes at the expense of a larger variability and sometimes larger bias ("overlearning effect").

9 Application to financial data

The MGH and MAGH distributions have been fitted to various asset-return data. In particular, the following distributions have been used:

(1) MGH/MH,
(2) MAGH/MAH with minimum parameterization (denoted by (min)), that is, with all margins equally parameterized,
(3) MAGH/MAH with maximum parameterization (denoted by (max)), that is, with each margin individually parameterized.

For some of these distributions we have also estimated the symmetric pendant (denoted by (sym)), i.e., $\beta = 0$. Further, we illustrate estimations following the affine transformation method provided at the end of Section 3 (denoted by (PC)).

The results are presented in Table 4. The dependence parameter Dep.Par. refers to the sub-diagonal elements of the normed matrix Σ, i.e., Dep.Par.$:= \sigma_{ij}/\sqrt{\sigma_{ii}\sigma_{jj}} = \sigma_{ij}/(S_i S_j)$.

The maximum parameterized MAGH distribution frequently reach the best fit in terms of the cross entropy. Considerable variations between λ and α among the various models can be observed. This may lead to a similar variation of the scaling parameters which frequently behave contrary to the variation of the shape parameters.

Due to the total or approximate symmetry of some distribution models, the dependence parameters can be roughly interpreted as "correlation coefficients" for MGH and MAGH-min distributions. They are even close to the corresponding sample correlation coefficient (column "Corr."). Note that such an interpretation is not possible for the MAGHmax model, due to the different parameterization of the one-dimensional margins.

Conclusion

Summarizing the results we have investigated an interesting new class of multidimensional distributions with exponentially decreasing tails to analyze high-dimensional data, namely the multivariate affine generalized hyperbolic distributions. These distributions are attractive regarding the estimation of unknown parameters and random vector generation. We illustrated that this class of distributions possesses an appealing dependence structure and we derived several dependence properties. Finally, an extensive simulation study showed the flexibility and robustness of the introduced model. Thus, the use of multivariate affine generalized hyperbolic distributions is recommended for multidimensional data modelling in statistical and financial applications.

Appendix

Proof (Theorem 8). i) Suppose $X \in MGH_2(\mu, \Sigma, \omega)$ with parameter $\beta = 0$. Then X belongs to the family of elliptically contoured distributions and the assertion follows by Theorem 2 in Lindskog, McNeil, and Schmock (2003).

ii) Suppose $X \in MAGH_2(\mu, \Sigma, \omega)$ with stochastic representation $X \stackrel{d}{=} A'Y + \mu,\ A'A = \Sigma$ and copula C. In particular

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} \quad \text{and} \quad A' = \begin{pmatrix} \sqrt{\sigma_{11}} & 0 \\ \sigma_{12}/\sqrt{\sigma_{11}} & \sqrt{\sigma_{22}}\sqrt{1-\rho^2} \end{pmatrix}$$

with ρ being the correlation coefficient of X_1 and X_2. For $\rho = 0$, the assertion follows by Theorem 5.1.9 in Nelsen (1999) due to the independence of X_1 and X_2. For the remaining case we can assume $\rho > 0$ as for $\rho < 0$ the assertion is shown similarly. According to Theorem 5.1.3 in Nelsen (1999), Kendall's tau is a copula property what justifies to put $\mu = 0$. Further, Kendall's tau is invariant under strictly increasing transformations of the margins (see Theorem 5.1.9 in Nelsen (1999)) and therefore we may set

$$A' = \begin{pmatrix} 1 & 0 \\ 1 & c \end{pmatrix} \quad \text{with } c := \sqrt{1/\rho^2 - 1}.$$

Consequently, the distribution function of $X = (X_1, X_2)'$ has the form

$$F_X(x_1, x_2) = \mathbb{P}(Y_1 \le x_1, Y_1 + cY_2 \le x_2) = \int_{-\infty}^{x_1} F_{Y_2}\left(\frac{x_2 - z}{c}\right) f_{Y_1}(z)\, dz$$

and the corresponding density function is

$$f_X(x_1, x_2) = \frac{1}{|c|} f_{Y_2}\left(\frac{x_2 - x_1}{c}\right) f_{Y_1}(x_1).$$

Thus, using the fact that

$$\tau = 4 \int_{[0,1]^2} C(u,v)\, dC(u,v) - 1 = 4 \int_{I\!R^2} F_X(x_1,x_2) f_X(x_1,x_2)\, d(x_1,x_2) - 1,$$

formula (15) is shown. □

In order to prove Theorem 10 we first investigate the tail behavior of the univariate symmetric MGH and MAGH distributions. The tail of the distribution function F, as always, is denoted by $\overline{F} := 1 - F$.

Definition 11 (Semi-heavy tails) *A continuous (symmetric) function $g : I\!R \to (0,\infty)$ is called semi-heavy tailed (or exponentially tailed) if it satisfied*

$$g(x) \sim c|x|^{\nu} \exp(-\eta|x|) \quad \textit{as } x \to \pm\infty \tag{31}$$

with $\nu \in I\!R$, $\eta > 0$ and some positive constant c. The class of (symmetric) semi-heavy tailed functions is denoted by $L_{\nu,\eta}$.

Lemma 12 *Let f be a density function such that $f \in L_{\nu,\eta}$, $\nu \in I\!R$, $\eta > 0$. Then the corresponding distribution function F possesses the same asymptotic behavior as its density, i.e., $F(x) \sim \bar{c}|x|^{\nu} \exp(-\eta|x|)$ as $x \to -\infty$ and $\bar{F}(x) \sim \bar{c}x^{\nu}\exp(-\eta x)$ as $x \to \infty$ for some positive constant $\bar{c}$; write $F \in L_{\nu,\eta}$.*

Proof. Consider e.g. the tail function $\bar{F}$. Applying partial integration we obtain

$$\begin{aligned}\bar{F}(x) &= \int_x^{\infty} f(u)\, du \sim c \int_x^{\infty} u^{\nu} \exp(-\eta u)\, du \\ &= c\eta x^{\nu} \exp(-\eta x) + \eta c\nu \int_x^{\infty} u^{\nu-1} \exp(-\eta u)\, du.\end{aligned}$$

Thus, the proof is complete if we show that

$$\int_x^{\infty} u^{\nu-1} \exp(-\eta u)\, du \Big/ x^{\nu} \exp(-\eta x) = o(1) \quad \text{as } x \to \infty.$$

Rewriting the latter quotient yields

$$0 \le \frac{1}{x} \int_x^{\infty} \Big(\frac{u}{x}\Big)^{\nu-1} \exp(-\eta(u-x))\, du = \frac{1}{x} \int_0^{\infty} \Big(\frac{u+x}{x}\Big)^{\nu-1} \exp(-\eta u)\, du.$$

The assertion is now immediate because

$$\Big(\frac{u}{x} + 1\Big)^{\nu-1} \le (u+1)^{\nu-1} \text{ for } \nu \ge 1 \text{ and } \Big(\frac{u}{x} + 1\Big)^{\nu-1} \le 1 \text{ for } \nu < 1$$

and the corresponding integrals exist. □

The next lemma is quite useful; it states that the tail of the convolution of two semi-heavy tailed distributions is determined by the heavier tail.

Lemma 13 *Let F_1 and F_2 be distribution functions with $F_1 \in L_{\nu_1,\eta_1}$ and $F_2 \in L_{\nu_2,\eta_2}$ where $0 < \eta_2 < \eta_1$, $\nu_1, \nu_2 \in \mathbb{R}$. Then $F_1 * F_2 \in L_{\nu_2,\eta_2}$ and, moreover,*

$$\lim_{t\to\infty} \overline{F_1 * F_2}(t)/\overline{F}_2(t) = m_1 := \int_{-\infty}^{\infty} e^{\eta_2 u} dF_1(u). \tag{32}$$

Proof. For some fixed $s > 1$, we have

$$\begin{aligned}
\overline{F_1 * F_2}(t) &= \int_{-\infty}^{\infty} \overline{F}_2(t-u)dF_1(u) \\
&= \int_{-\infty}^{t/s} \overline{F}_2(t-u)dF_1(u) - \int_{-\infty}^{t-t/s} \overline{F}_2(u)dF_1(t-u) \\
&= \int_{-\infty}^{t/s} \overline{F}_2(t-u)dF_1(u) - \Big[F_1(t/s) - 1 - \int_{-\infty}^{t-t/s} F_2(u)dF_1(t-u)\Big] \\
&= \int_{-\infty}^{t/s} \overline{F}_2(t-u)dF_1(u) + \int_{-\infty}^{t-t/s} \overline{F}_1(t-u)dF_2(u) + \overline{F}_2(t-t/s)\overline{F}_1(t/s),
\end{aligned}$$

where the last equality follows by partial integration. Thus, dominated convergence yields

$$\begin{aligned}
\lim_{t\to\infty} \overline{F_1 * F_2}(t)/\overline{F}_2(t) &= \lim_{t\to\infty} \int_{-\infty}^{t/s} \overline{F}_2(t-u)/\overline{F}_2(t)dF_1(u) \\
&= \int_{-\infty}^{\infty} e^{\eta_2 u} dF_1(u) =: m_1 < \infty
\end{aligned}$$

because

$$0 \le \int_{-\infty}^{t-t/s} \overline{F}_1(t-u)/\overline{F}_2(t)dF_2(u) \le \frac{\overline{F}_1(t/s)}{\overline{F}_2(t)} \cdot \overline{F}_2(t-t/s) \to 0 \quad \text{as } t \to \infty.$$

A consequence of the symmetric tails of F_1 and F_2 is that $\lim_{t\to-\infty} F_1 * F_2(t)/F_2(t) = m_1$. Hence, $F_1 * F_2 \in L_{\nu_2,\eta_2}$ is proven. □

According to Barndorff-Nielsen and Blæsild (1981), the univariate MGH distributions have semi-heavy tails, in particular

$$MGH_1(0,1,\omega) \sim c|x|^{\lambda-1} \exp((\mp\alpha + \alpha\beta)x) \text{ as } x \to \pm\infty \tag{33}$$

with some positive constant c. Hence, in the symmetric case $\beta = 0$ we obtain $MGH_1(0,1,\omega) \in L_{\nu,\eta}$ with $\nu = \lambda - 1$ and $\eta = \alpha$. Now we are ready to prove Theorem 10.

Proof (Theorem 10). We only show upper tail-dependence and upper tail-independence, respectively, as the lower pendant is obtained similarly. Recall that tail dependence is a copula property and therefore we may put $\mu = 0$.

i) Let $X \in MGH_2(0, \Sigma, \omega)$ with $\beta = 0$. In that case X belongs to the family of elliptically contoured distributions. According to Theorem 6.8 in Schmidt (2002) the assertion follows because of the exponentially-tailed density generator.

ii) Let $X \in MAGH_2(0, \Sigma, \omega)$ with stochastic representation $X \stackrel{d}{=} A'Y$ and Cholesky matrix $A' = \begin{pmatrix} a_{11} & 0 \\ a_{12} & a_{22} \end{pmatrix}$. Note that $a_{11}, a_{22} > 0$. Then

$$\begin{aligned}
&\mathbb{P}(X_2 > F_{X_2}^{-1}(v) \mid X_1 > F_{X_1}^{-1}(v)) \\
&= \frac{\mathbb{P}(a_{11}Y_1 > F_{X_1}^{-1}(v), a_{12}Y_1 + a_{22}Y_2 > F_{X_2}^{-1}(v))}{\mathbb{P}(a_{11}Y_1 > F_{X_1}^{-1}(v))} \\
&= \frac{1}{1-v}\mathbb{P}(Y_1 > F_{Y_1}^{-1}(v), a_{12}Y_1 + a_{22}Y_2 > F_{X_2}^{-1}(v)) \\
&= \frac{1}{1-v}\int_{-\infty}^{\infty} \mathbb{P}(y > F_{Y_1}^{-1}(v), a_{12}y + a_{22}Y_2 > F_{X_2}^{-1}(v)) f_{Y_1}(y)\, dy \\
&= \frac{1}{1-v}\int_{F_{Y_1}^{-1}(v)}^{\infty} \mathbb{P}(Y_2 > (F_{X_2}^{-1}(v) - a_{12}y)/a_{22}) f_{Y_1}(y)\, dy =: \ I
\end{aligned}$$

because Y_1 and Y_2 are independent random variables. If $\rho \le 0$ then $a_{12} \le 0$ and upper tail-independence immediately follows by dominated convergence.

Consider now $\rho > 0$ and therefore $a_{12} > 0$. Let $\alpha_2 < \alpha_1\sqrt{1/\rho^2 - 1}$. Then $\alpha_2/a_{22} < \alpha_1/a_{12}$. For all $\varepsilon \in (0, 1]$ we conclude with $u := 1 - v$ that

$$\begin{aligned}
I \ &\le \varepsilon + \frac{1}{u}\int_{F_{Y_1}^{-1}(1-u)}^{F_{Y_1}^{-1}(1-u\varepsilon)} \mathbb{P}(Y_2 > (F_{X_2}^{-1}(1-u) - a_{12}y)/a_{22}) f_{Y_1}(y)\, dy \\
&\le \varepsilon + (1-\varepsilon)\mathbb{P}(Y_2 > (F_{X_2}^{-1}(1-u) - a_{12}F_{Y_1}^{-1}(1-u\varepsilon))/a_{22}). \qquad (34)
\end{aligned}$$

Due to (33) we know that $F_{a_{12}Y_1} \in L_{\nu_1,\eta_1}$ with $\nu_1 = \lambda_1 - 1$ and $\eta_1 = \alpha_1/a_{12}$. Thus, Lemma 13 gives $F_{X_2} \in L_{\nu_2,\eta_2}$ with $\nu_2 = \lambda_2 - 1$ and $\eta_2 = \alpha_2/a_{22}$ as $0 < \eta_2 < \eta_1$. Then the probability in (34) converges to zero as $u \to 0^+$ if $F_{X_2}^{-1}(1-u) - a_{12}F_{Y_1}^{-1}(1-u\varepsilon) = F_{a_{12}Y_1+a_{22}Y_2}^{-1}(1-u) - F_{a_{12}Y_1}^{-1}(1-u\varepsilon) \to \infty$ as $u \to 0^+$. Put $x_u := F_{a_{12}Y_1}^{-1}(1-u\varepsilon)$ and $y_u := F_{a_{12}Y_1+a_{22}Y_2}^{-1}(1-u)$. Then

$$u = \frac{1}{\varepsilon}\overline{F}_{a_{12}Y_1}(x_u) = \overline{F}_{X_2}(y_u) \sim \frac{c_1}{\varepsilon}x_u^{\nu_1}\exp(-\eta_1 x_u) \sim c_2 y_u^{\nu_2}\exp(-\eta_2 y_u) \qquad (35)$$

as $u \to 0^+$ and therefore $x_u, y_u \to \infty$. The asymptotic behavior (35) implies $y_u - x_u \to \infty$

as $u \to 0^+$ because $0 < \eta_2 < \eta_1$. Hence, upper tail-independence is shown.

iii) Now suppose $\rho > 0$ and $\alpha_2 > \alpha_1\sqrt{1/\rho^2 - 1}$ which yields $a_{12} > 0$ and $\alpha_2/a_{22} > \alpha_1/a_{12}$. According to Lemma 13 we have $F_{a_{12}Y_1} \in L_{\nu_1,\eta_1}$ and $F_{X_2} \in L_{\nu_1,\eta_1}$ with $\nu_1 = \lambda_1 - 1$ and $\eta_1 = \alpha_1/a_{12}$. Notice that with $u := 1 - v$

$$I \geq \mathbb{P}(Y_2 > (F_{X_2}^{-1}(1-u) - a_{12}F_{Y_1}^{-1}(1-u\varepsilon))/a_{22}) \tag{36}$$

Further

$$u = \overline{F}_{a_{12}Y_1}(x_u) = \overline{F}_{X_2}(y_u) \sim c_1 x_u^{\nu_1}\exp(-\eta_1 x_u) \sim c_2 y_u^{\nu_1}\exp(-\eta_1 y_u).$$

Hence

$$\frac{c_2}{c_1}\Big(\frac{y_u}{x_u}\Big)^{\nu_1}\exp(-\eta_1(y_u - x_u)) \to 1 \text{ as } u \to 0^+.$$

Suppose that $\limsup_{u\to 0^+}(y_u - x_u) = \infty$. If $\nu_1 \geq 0$, then the fact

$$\begin{aligned}&\liminf_{u\to 0^+}\frac{c_2}{c_1}\Big(\frac{y_u}{x_u}\Big)^{\nu_1}\exp(-\eta_1(y_u - x_u))\\ &\leq \liminf_{u\to 0^+}\frac{c_2}{c_1}(2(y_u - x_u))^{\nu_1}\exp\Big(-\eta_1(y_u - x_u)\Big) = 0 \text{ as } u \to 0^+\end{aligned}$$

would lead to a contradiction. On the other hand, if $\nu_1 < 0$ then

$$\liminf_{u\to 0^+}\frac{c_2}{c_1}\Big(\frac{y_u}{x_u}\Big)^{\nu_1}\exp(-\eta_1(y_u - x_u)) \leq \liminf_{u\to 0^+}\frac{c_2}{c_1}\exp\Big(-\eta_1(y_u - x_u)\Big) = 0 \text{ as } u \to 0^+$$

would also lead to a contradiction. Therefore we conclude that $\liminf_{u\to 0^+}\mathbb{P}(Y_2 > (F_{X_2}^{-1}(1-u) - a_{12}F_{Y_1}^{-1}(1-u\varepsilon))/a_{22}) > 0$ as Y_2 is supported on $\mathbb{R}$. Finally, the limit $\lim_{v\to 1-}\mathbb{P}(X_2 > F_{X_2}^{-1}(v) \mid X_1 > F_{X_1}^{-1}(v))$ exists due to the convexity of $\overline{F}_{X_1}$ and $\overline{F}_{X_2}$ for large arguments. □

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Table 2
Bivariate MGH and MH distribution. For each parameter λ, α, $S_1 = \sqrt{\sigma_{11}}$, and $S_2 = \sqrt{\sigma_{22}}$ the table lists the reference value, the sample mean $m(\cdot)$, and the sample standard deviation $\sigma(\cdot)$ of various parameter estimations derived from 100 samples of sample-size 1000 each. In the last column, the sample mean and the sample standard deviation of the corresponding cross entropy H are provided.

	λ	$m(\hat{\lambda})$	$\sigma(\hat{\lambda})$	α	$m(\hat{\alpha})$	$\sigma(\hat{\alpha})$	S_1	$m(\hat{S}_1)$	$\sigma(\hat{S}_1)$	S_2	$m(\hat{S}_2)$	$\sigma(\hat{S}_2)$	$m(\hat{H})$	$\sigma(\hat{H})$
MGH	1.0000	1.1061	0.0571	0.3200	0.2724	0.0867	0.3200	0.2574	0.0735	0.3200	0.2595	0.0746	3.4525	0.0438
MH	1.0000	1.0000	0.0000	0.3200	0.3286	0.1228	0.3200	0.3229	0.1078	0.3200	0.3243	0.1119	3.4177	0.3450
MGH	1.0000	1.3745	0.1651	1.1456	0.8758	0.2085	1.0000	0.7096	0.1341	1.0000	0.7141	0.1343	3.8683	0.0455
MH	1.0000	1.0000	0.0000	1.1456	1.1609	0.2922	1.0000	0.9954	0.1912	1.0000	1.0037	0.1931	3.8683	0.0454
MGH	1.0000	2.1779	0.5777	2.2400	1.8217	0.5151	1.0000	0.6897	0.1603	1.0000	0.6931	0.1621	2.5389	0.0411
MH	1.0000	1.0000	0.0000	2.2400	2.4543	0.6451	1.0000	1.0475	0.1859	1.0000	1.0522	0.1862	2.5390	0.0411

Table 3
Bivariate MH and MAH distribution. Reference value, sample bias and sample standard deviation (in brackets) are provided for each parameter estimate derived from 100 samples of sample-size 1000 each. Denote Dep.Par.:= $\sigma_{12}/(S_1 S_2)$. The last column lists the sample mean and the sample standard deviation of the cross entropy H.

Distribution	μ_1	μ_2	S_1	S_2	α_1	α_2	β_1	β_2	Dep.Par.	Cross Ent. H
	0.000 value:	0.000 value:	1.000 value:	1.000 value:	1.000 value:	1.000 value:	0.000 value:	0.000 value:	0.000 value:	
MAHmin	0.004 (0.085)	0.002 (0.087)	0.019 (0.208)	0.025 (0.213)	0.040 (0.265)	0.040 (0.265)	0.002 (0.032)	0.002 (0.032)	0.005 (0.037)	3.769 (0.039)
MAHmax	0.005 (0.111)	0.014 (0.112)	0.034 (0.323)	0.060 (0.300)	0.064 (0.398)	0.102 (0.385)	0.006 (0.045)	0.003 (0.044)	0.005 (0.048)	3.768 (0.039)
MH	0.003 (0.095)	0.007 (0.094)	0.023 (0.163)	0.029 (0.168)	0.042 (0.210)	0.042 (0.210)	0.005 (0.040)	0.001 (0.039)	0.003 (0.036)	3.753 (0.038)
	0.000 value:	0.000 value:	0.320 value:	0.320 value:	0.320 value:	0.320 value:	0.000 value:	0.000 value:	0.000 value:	
MAHmin	0.022 (0.830)	0.051 (0.644)	0.001 (0.126)	0.003 (0.131)	0.008 (0.137)	0.008 (0.137)	0.004 (0.115)	0.004 (0.115)	0.019 (0.109)	3.457 (0.112)
MAHmax	0.060 (0.464)	0.015 (0.200)	0.000 (0.205)	0.031 (0.194)	0.011 (0.214)	0.017 (0.203)	0.020 (0.111)	0.008 (0.122)	0.004 (0.214)	3.382 (0.142)
MH	0.071 (0.697)	0.100 (1.015)	0.003 (0.108)	0.004 (0.112)	0.009 (0.123)	0.009 (0.123)	0.004 (0.035)	0.007 (0.067)	0.012 (0.105)	3.418 (0.345)
	0.000 value:	0.000 value:	1.155 value:	1.155 value:	2.236 value:	2.236 value:	0.000 value:	0.000 value:	0.500 value:	
MAHmin	0.001 (0.120)	0.001 (0.075)	0.139 (0.238)	0.143 (0.191)	0.077 (0.626)	0.077 (0.626)	0.003 (0.042)	0.003 (0.042)	0.055 (0.027)	2.687 (0.040)
MAHmax	0.002 (0.110)	0.002 (0.127)	0.085 (0.294)	0.056 (0.320)	0.318 (1.083)	0.270 (0.969)	0.004 (0.064)	0.001 (0.053)	0.004 (0.145)	2.686 (0.040)
MH	0.003 (0.118)	0.002 (0.083)	0.159 (0.224)	0.128 (0.177)	0.128 (0.603)	0.128 (0.603)	0.003 (0.058)	0.000 (0.048)	0.054 (0.026)	2.680 (0.040)
	2.000 value:	4.000 value:	1.155 value:	1.155 value:	1.258 value:	1.258 value:	0.229 value:	0.512 value:	0.500 value:	
MAHmin	0.147 (0.201)	0.759 (0.129)	0.098 (0.253)	0.006 (0.234)	0.224 (0.279)	0.224 (0.279)	0.145 (0.034)	0.138 (0.034)	0.050 (0.029)	4.210 (0.050)
MAHmax	1.470 (0.173)	0.498 (0.171)	0.504 (0.511)	0.521 (0.192)	0.134 (0.483)	0.400 (0.303)	0.154 (0.039)	0.050 (0.014)	0.144 (0.123)	4.175 (0.048)
MH	0.201 (0.117)	0.260 (0.082)	0.031 (0.214)	0.186 (0.174)	0.182 (0.240)	0.182 (0.240)	0.128 (0.038)	0.042 (0.013)	0.007 (0.027)	4.128 (0.045)

Table 4
Estimations from two-dimensional asset-return data (DAX-CAC, DAX-DOW, NIKKEI-CAC).

Data	Model	Location μ	Scaling S	Dep.Par.	Corr.	λ	α	β	CrossEnt
DaxCac	MAGHmin	0.0012 0.0008	0.0062 0.0060	0.689	0.709	0.889	0.675	-0.034	-6.183
DaxCac	MAHmin	0.0011 0.0008	0.0062 0.0059	0.688	0.709	1.000	0.699	-0.029	-6.183
DaxCac	MAGHmin sym.	0.0004 0.0003	0.0056 0.0054	0.689	0.709	1.002	0.631	0.000	-6.182
DaxCac	MAGHmaxPC	0.0012 0.0005	0.0103 0.0046	0.941	0.709	0.812 0.732	0.192 1.363	0.037 0.000	-6.208
DaxCac	MAHmaxPC	0.0015 0.0007	0.0095 0.0042	0.898	0.709	1.000 1.000	0.261 1.315	0.039 0.000	-6.208
DaxCac	MAGHmaxPC sym.	0.0002 0.0001	0.0107 0.0048	0.999	0.709	0.635 0.645	0.050 1.427	0.000 0.000	-6.275
DaxCac	MAGHmax	0.0010 0.0001	0.0086 0.0082	0.957	0.709	0.873 0.800	0.247 1.490	-0.039 0.035	-6.201
DaxCac	MAHmax	-0.9333 -0.6267	0.0071 0.0071	0.999	0.709	1.000 1.000	0.050 1.302	0.600 0.037	-6.389
DaxCac	MAGHmax sym.	0.0001 0.0000	0.0085 0.0085	1.000	0.709	0.635 0.639	0.050 1.463	0.000 0.000	-6.265
DaxCac	MGH	0.0010 0.0005	0.0045 0.0044	0.672	0.709	1.134	0.535	-0.038 -0.017	-6.213
DaxCac	MH	0.0011 0.0006	0.0059 0.0058	0.673	0.709	1.000	0.689	-0.044 -0.019	-6.213
DaxCac	MGH sym.	0.0005 0.0003	0.0043 0.0042	0.672	0.709	1.143	0.504	0.000 0.000	-6.212
DaxDow	MAGHmin	0.0025 0.0013	0.0137 0.0097	0.505	0.498	0.891	1.285	-0.067	-5.743
DaxDow	MAHmin	0.0021 0.0011	0.0124 0.0088	0.503	0.498	1.000	1.169	-0.056	-5.743
DaxDow	MAGHmin sym.	0.0002 0.0001	0.0137 0.0097	0.500	0.498	0.759	1.208	0.000	-5.741
DaxDow	MAGHmaxPC	0.0009 0.0006	0.0145 0.0092	0.287	0.498	0.694 0.654	1.014 1.623	-0.001 0.000	-5.746
DaxDow	MAHmaxPC	0.0012 0.0002	0.0122 0.0082	0.373	0.498	1.000 1.000	0.868 1.561	0.032 0.000	-5.746
DaxDow	MAGHmaxPC sym.	0.0000 0.0001	0.0147 0.0093	0.277	0.498	0.650 0.657	1.013 1.634	0.000 0.000	-5.746
DaxDow	MAGHmax	0.0016 0.0010	0.0137 0.0114	0.477	0.498	1.160 0.751	1.194 1.825	-0.060 -0.014	-5.732
DaxDow	MAHmax	0.0017 0.0010	0.0146 0.0107	0.420	0.498	1.000 1.000	1.276 1.786	-0.063 -0.013	-5.732
DaxDow	MAGHmax sym.	0.0000 0.0002	0.0172 0.0119	0.397	0.498	0.650 0.653	1.423 1.878	0.000 0.000	-5.732
DaxDow	MGH	0.0020 0.0003	0.0135 0.0097	0.489	0.498	1.087	1.359	-0.077 -0.010	-5.757
DaxDow	MH	0.0021 0.0004	0.0137 0.0098	0.489	0.498	1.000	1.346	-0.079 -0.015	-5.757
DaxDow	MGH sym.	0.0001 0.0001	0.0130 0.0093	0.488	0.498	1.097	1.289	0.000 0.000	-5.755
NikkeiCac	MAGHmin	0.0000 0.0004	0.0028 0.0027	0.196	0.236	0.870	0.272	-0.011	-5.884
NikkeiCac	MAHmin	-0.0000 0.0004	0.0024 0.0023	0.199	0.236	1.000	0.245	-0.008	-5.884
NikkeiCac	MAGHmin sym.	-0.0001 0.0003	0.0028 0.0027	0.197	0.236	0.880	0.278	0.000	-5.883
NikkeiCac	MAGHmaxPC	0.0008 0.0007	0.0084 0.0069	0.170	0.236	0.646 0.902	0.700 0.957	0.009 0.000	-5.839
NikkeiCac	MAHmaxPC	0.0008 0.0008	0.0067 0.0058	0.352	0.236	1.000 1.000	0.564 0.859	0.008 0.000	-5.838
NikkeiCac	MAGHmaxPC sym.	0.0000 0.0003	0.0090 0.0076	0.278	0.236	0.652 0.639	0.714 1.020	0.000 0.000	-5.838
NikkeiCac	MAGHmax	0.0059 0.0018	0.0016 0.0067	0.987	0.236	0.635 0.696	0.050 0.732	-0.491 -0.015	-5.964
NikkeiCac	MAHmax	0.0044 0.0015	0.0012 0.0049	0.976	0.236	1.000 1.000	0.050 0.587	-0.322 -0.015	-6.123
NikkeiCac	MAGHmax sym.	0.0000 0.0003	0.0016 0.0065	0.986	0.236	0.636 0.689	0.050 0.709	0.000 0.000	-5.997
NikkeiCac	MGH	0.0004 0.0005	0.0032 0.0031	0.217	0.236	1.117	0.355	-0.026 -0.014	-5.885
NikkeiCac	MH	0.0004 0.0005	0.0043 0.0041	0.216	0.236	1.000	0.459	-0.028 -0.017	-5.885
NikkeiCac	MGH sym.	-0.0000 0.0003	0.0032 0.0031	0.217	0.236	1.118	0.349	0.000 0.000	-5.885

Table 5
Estimations from two- and four-dimensional asset-return data (NEMAX-NASDAQ, DAX-DOW-NIKKEI-CAC, DAX-NEMAX-DOW-NASDAQ).

Data	Model	Location μ	Scaling S	λ	α	β	CrossEnt
NemaxNasdaq	MAGHmin	0.0029 0.0023	0.0057 0.0044	1.562	0.337	-0.052	-4.610
NemaxNasdaq	MAHmin	0.0021 0.0018	0.0169 0.0129	1.000	0.902	-0.045	-4.609
NemaxNasdaq	MAGHmin sym.	-0.0002 0.0005	0.0011 0.0008	1.705	0.072	0.000	-4.760
NemaxNasdaq	MAGHmaxPC	-0.0000 -0.0001	0.0247 0.0151	1.143 1.111	1.255 1.542	0.007 0.000	-4.596
NemaxNasdaq	MAHmaxPC	-0.0000 -0.0000	0.0266 0.0160	1.000 1.000	1.329 1.599	0.007 0.000	-4.596
NemaxNasdaq	MAGHmaxPC sym.	-0.0006 -0.0001	0.0291 0.0172	0.797 0.837	1.424 1.669	0.000 0.000	-4.596
NemaxNasdaq	MAGHmax	-0.0013 0.0004	0.0133 0.0233	1.913 0.858	0.321 2.183	0.010 -0.020	-4.592
NemaxNasdaq	MAHmax	-0.0013 0.0005	0.0252 0.0230	1.000 1.000	1.242 2.219	0.013 -0.021	-4.592
NemaxNasdaq	MAGHmax sym.	-0.0013 -0.0003	0.0100 0.0188	2.209 1.512	0.050 1.915	0.000 0.000	-4.938
NemaxNasdaq	MGH	0.0007 0.0016	0.0040 0.0031	1.907	0.263	-0.005 -0.037	-4.616
NemaxNasdaq	MH	0.0008 0.0019	0.0214 0.0164	1.000	1.205	-0.006 -0.051	-4.614
NemaxNasdaq	MGH sym.	-0.0003 0.0003	0.0055 0.0042	1.937	0.363	0.000 0.000	-4.614
DaxDowNikkeiCac	MAGHmin	0.0009 0.0007 0.0002 0.0006	0.0051 0.0038 0.0051 0.0049	0.888	0.539	-0.020	-12.405
DaxDowNikkeiCac	MAHmin	0.0008 0.0006 0.0001 0.0005	0.0044 0.0033 0.0044 0.0042	1.000	0.481	-0.017	-12.405
DaxDowNikkeiCac	MAGHmin sym.	0.0004 0.0004 -0.0001 0.0003	0.0050 0.0037 0.0050 0.0048	0.894	0.525	0.000	-12.405
DaxDowNikkeiCac	MAGHmaxPC	0.0016 0.0015 0.0005 0.0010	0.0099 0.0047 0.0022 0.0045	0.92 0.90 1.05 0.83	0.31 0.17 0.52 1.33	0.032 0.000 0.000 0.000	-12.403
DaxDowNikkeiCac	MAHmaxPC	-1.5272 -0.7592 -0.8804 -1.4177	0.0093 0.0062 0.0016 0.0041	1.00 1.00 1.00 1.00	0.05 0.05 0.79 1.29	0.039 0.000 0.000 0.000	-12.827
DaxDowNikkeiCac	MAGHmaxPC sym.	0.0009 0.0007 0.0002 0.0008	0.0104 0.0045 0.0022 0.0048	0.63 0.88 1.01 0.63	0.05 0.16 0.46 1.40	0.000 0.000 0.000 0.000	-12.451
DaxDowNikkeiCac	MAGHmax	0.0010 0.0013 0.0049 0.0003	0.0085 0.0040 0.0007 0.0077	0.87 1.09 0.63 0.72	0.24 0.61 0.05 1.37	-0.039 -0.051 -0.464 0.054	-12.505
DaxDowNikkeiCac	MAHmax	-0.9333 -0.2867 -0.2327 -0.6268	0.0073 0.0033 0.0007 0.0070	1.00 1.00 1.00 1.00	0.05 0.47 0.05 1.30	0.600 0.007 -0.100 0.046	-12.869
DaxDowNikkeiCac	MAGHmax sym.	0.0001 0.0004 0.0000 0.0000	0.0085 0.0030 0.0008 0.0082	0.63 0.98 0.63 0.637	0.05 0.39 0.05 1.42	0.000 0.000 0.000 0.000	-12.575
DaxDowNikkeiCac	MGH	0.0013 0.0009 0.0004 0.0007	0.0046 0.0036 0.0049 0.0045	2.000	0.726	-0.032 -0.023 -0.021 -0.019	-12.441
DaxDowNikkeiCac	MH	0.0012 0.0009 0.0004 0.0007	0.0076 0.0059 0.0081 0.0075	1.000	0.939	-0.035 -0.026 -0.023 -0.023	-12.453
DaxDowNikkeiCac	MGH sym.	0.0005 0.0005 -0.0001 0.0003	0.0043 0.0033 0.0045 0.0042	2.000	0.664	0.000 0.000 0.000 0.000	-12.441
DaxNemDowNasd	MAGHmin	0.0006 0.0006 0.0006 0.0009	0.0104 0.0177 0.0076 0.0136	1.136	1.000	-0.015	-10.903
DaxNemDowNasd	MAHmin	0.0006 0.0005 0.0006 0.0009	0.0115 0.0197 0.0084 0.0151	1.000	1.087	-0.015	-10.903
DaxNemDowNasd	MAGHmin sym.	-0.0001 -0.0004 0.0002 0.0004	0.0106 0.0180 0.0077 0.0138	1.127	1.019	0.000	-10.903
DaxNemDowNasd	MAGHmaxPC	0.0011 0.0008 -0.0001 -0.0000	0.0357 0.0203 0.0081 0.0037	1.63 0.84 1.85 0.90	1.08 1.89 0.26 2.18	-0.030 0.000 0.000 0.000	-10.868
DaxNemDowNasd	MAHmaxPC	0.0012 0.0005 -0.0002 -0.0002	0.0358 0.0201 0.0105 0.0062	1.00 1.00 1.00 1.00	1.47 1.84 1.14 2.13	-0.035 0.000 0.000 0.000	-10.868
DaxNemDowNasd	MAGHmaxPC sym.	-0.0000 -0.0005 0.0001 -0.0002	0.0377 0.0207 0.0092 0.0038	1.40 0.65 2.18 0.81	1.29 1.95 0.05 2.14	0.000 0.000 0.000 0.000	-11.217
DaxNemDowNasd	MAGHmax	0.0016 0.0006 0.0010 0.0027	0.0145 0.0023 0.0127 0.0072	1.16 1.98 0.64 1.02	1.19 0.05 1.79 0.70	-0.060 0.010 -0.018 -0.066	-11.279
DaxNemDowNasd	MAHmax	0.0017 0.0006 0.0010 0.0027	0.0162 0.0060 0.0119 0.0073	1.00 1.00 1.00 1.00	1.27 0.37 1.77 0.72	-0.063 0.029 -0.013 -0.067	-10.939
DaxNemDowNasd	MAGHmax sym.	0.0000 -0.0012 0.0002 0.0011	0.0172 0.0006 0.0118 0.0003	0.65 1.89 0.64 2.73	1.42 0.05 1.84 0.06	0.000 0.000 0.000 0.000	-11.583
DaxNemDowNasd	MGH	0.0014 0.0021 0.0011 0.0023	0.0038 0.0064 0.0028 0.0049	1.990	0.434	-0.014 -0.026 -0.003 -0.053	-10.967
DaxNemDowNasd	MH	0.0015 0.0022 0.0010 0.0024	0.0132 0.0220 0.0095 0.0169	1.000	1.258	-0.019 -0.030 -0.001 -0.064	-10.964
DaxNemDowNasd	MGH sym.	0.0001 -0.0001 0.0003 0.0005	0.0027 0.0044 0.0019 0.0034	1.987	0.296	0.000 0.000 0.000 0.000	-10.968