Composite likelihood goodness-of-fit testing for binary factor models under simple random and complex survey sampling

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Outline

- Brief introduction to latent variable models for categorical variables.
- Model framework.
- Estimation and inference framework: Pairwise Likelihood (PL)
- Limited goodness-of-fit tests under SRS and complex sample designs

Latent variables and measurement

Using statistical models to understand constructs better: a question of **measurement**

- Many theories in behavioral and social sciences are formulated in terms of theoretical constructs that are not directly observed

  attitudes, opinions, abilities, motivations, etc.

- The measurement of a construct is achieved through one or more observable **indicators** (questionnaire **items**, tests).
- The purpose of a measurement model is to describe how well the observed indicators serve as a measurement instrument for the constructs, also known as **latent variables**.
- **Measurement models** often suggest ways in which the observed measurements can be improved.

Motivation of our work

- Improve the estimation in cases of intractable integrals and complex models.
- Provide an inferential framework for model testing and model selection.

Notation

- $\mathbf{y}$: $p-$dimensional vector of the observed variables (binary, ordinal, continuous, mixed).
- $\mathbf{y}^\star$: $p-$dimensional vector of corresponding underlying continuous variables.
- The connection between y_i and $y_i^\star$ is

$$y_i = c_i \Longleftrightarrow \tau_{c_i-1}^{(y_i)} < y_i^\star < \tau_{c_i}^{(y_i)}, \tag{1}$$

$$-\infty = \tau_0^{(y_i)} < \tau_1^{(y_i)} < \ldots < \tau_{m_i-1}^{(y_i)} < \tau_{m_i}^{(y_i)} = +\infty.$$

- c : the c-th response category of variable y_i, $c = 1, \ldots, m_i$, $\tau_{i,c}$: the c-th threshold of variable y_i,
- In practice, $y_i^\star \sim N(0, 1)$
- y_i is continuous: $y_i = y_i^\star$.

Structural Equation Model

Following Muthén (1984):

$$\mathbf{y}^{\star} = \boldsymbol{\nu} + \Lambda\boldsymbol{\eta} + \boldsymbol{\epsilon}$$
$$\boldsymbol{\eta} = \boldsymbol{\alpha} + \mathrm{B}\boldsymbol{\eta} + \Gamma\mathbf{x} + \boldsymbol{\zeta}$$

$\boldsymbol{\eta}$: vector of latent variables, $q-$dimensional,
$\mathbf{x}$: vector of covariates,
$\boldsymbol{\epsilon}$ and $\boldsymbol{\zeta}$: vectors of error terms, and
$\boldsymbol{\nu}$ and $\boldsymbol{\alpha}$: vectors of intercepts.
Standard assumptions:

- $\boldsymbol{\eta}$, $\boldsymbol{\epsilon}$, $\boldsymbol{\zeta}$ follow multivariate normal distribution,
- $Cov\left(\boldsymbol{\eta}, \boldsymbol{\epsilon}\right) = Cov\left(\boldsymbol{\eta}, \boldsymbol{\zeta}\right) = Cov\left(\boldsymbol{\epsilon}, \boldsymbol{\zeta}\right) = \mathbf{0}$,
- $I - \mathrm{B}$ is non-singular, I the identity matrix.

Structural Equation Model

Based on the model:

$$\boldsymbol{\mu} \equiv E\left(\mathbf{y}^{\star}|\mathbf{x}\right) = \boldsymbol{\nu} + \Lambda\left(I - \mathrm{B}\right)^{-1}\left(\boldsymbol{\alpha} + \Gamma\mathbf{x}\right)$$

$$\Sigma \equiv Cov\left(\mathbf{y}^{\star}|\mathbf{x}\right) = \Lambda\left(I - \mathrm{B}\right)^{-1}\Psi\left[\left(I - \mathrm{B}\right)^{-1}\right]'\Lambda' + \Theta$$

Let $\boldsymbol{\theta}$ be the parameter vector of the model.

$$\boldsymbol{\theta}' = \left(\mathrm{vec}\left(\Lambda\right)', \mathrm{vec}\left(B\right)', \mathrm{vec}\left(\Gamma\right)', \mathrm{vech}\left(\Psi\right)', \mathrm{vech}\left(\Theta\right)', \boldsymbol{\alpha}', \boldsymbol{\nu}', \boldsymbol{\tau}'\right)$$

.

Likelihood Function

- Under the model, the probability of a response pattern r is:

$$\pi_r(\boldsymbol{\theta}) = \pi\left(y_1 = c_1, \ldots, y_p = c_p; \boldsymbol{\theta}\right) = \int \ldots \int \phi_p(\mathbf{y}^\star; \Sigma_{\mathbf{y}^\star}) d\mathbf{y}^\star \ , \tag{2}$$

  where $\phi_p(\mathbf{y}^\star; \Sigma_{\mathbf{y}^\star})$ is a p-dimensional normal density with zero mean, and correlation matrix $\Sigma_{\mathbf{y}^\star}$.
- The maximization of log-likelihood over the parameter vector $\boldsymbol{\theta}$ requires the evaluation of the p-dimensional integral which cannot be written in a closed form.
- Maximum likelihood infeasible for large number of observed variables.

Pairwise likelihood for SEM

Basic assumption:

$$\begin{pmatrix} y_i^\star \\ y_j^\star \end{pmatrix} \Bigg| \mathbf{x} \sim N_2\left(\begin{pmatrix} \mu_i \\ \mu_j \end{pmatrix}, \begin{pmatrix} \sigma_{ii} & \\ \sigma_{ji} & \sigma_{jj} \end{pmatrix}\right)$$

The pl for N independent observations[1]:

$$pl\left(\boldsymbol{\theta};\mathbf{y}|\mathbf{x}\right) = \sum_{n=1}^{N}\sum_{i<i'} \ln L\left(\boldsymbol{\theta};\left(y_{in},y_{i'n}\right)|\mathbf{x}\right) .$$

The specific form of $\ln L\left(\boldsymbol{\theta};\left(y_{in},y_{i'n}\right)|\mathbf{x}\right)$ depends on the type of the observed variables (binary/ ordinal, continuous).

[1] Myrsini Katsikatsou et al. "Pairwise likelihood estimation for factor analysis models with ordinal data". In: *Computational Statistics & Data Analysis* 56.12 (2012), pp. 4243–4258.

Pairwise Likelihood Estimation for Binary Responses (1) - no covariates

- For a pair of variables y_i and y_j. The basic pairwise log-likelihood takes the form

$$\sum_{i<j} \sum_{c_i=0}^{1} \sum_{c_j=0}^{1} n_{c_i c_j}^{(y_i y_j)} \ln \pi_{c_i c_j}^{(y_i y_j)}(\boldsymbol{\theta}) \tag{3}$$

where $n_{c_i c_j}$ is the observed frequency of sample units with $y_i = c_i$ and $y_j = c_j$.

- To accommodate complex sampling, the PL becomes:

$$pl(\boldsymbol{\theta}; \mathbf{y}) = \sum_{i<j} \sum_{c_i=0}^{1} \sum_{c_j=0}^{1} p_{c_i c_j}^{(y_i y_j)} \ln \pi_{c_i c_j}^{(y_i y_j)}(\boldsymbol{\theta}) \ , \tag{4}$$

where $p_{c_i c_j} = \sum_{h \in s} w_h I(y_i^{(h)} = c_i, y_j^{(h)} = c_j) / \sum_{h \in s} w_h$.

Pairwise Likelihood Estimation for Binary Responses (2)

The score function

$$\nabla pl(\boldsymbol{\theta}; \mathbf{y}) = \sum_{i<j} \sum_{c_i=0}^{1} \sum_{c_j=0}^{1} p_{c_i c_j}^{(y_i y_j)} (\pi_{c_i c_j}^{(y_i y_j)}(\boldsymbol{\theta}))^{-1} \frac{\partial \pi_{c_i c_j}^{(y_i y_j)}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}. \tag{5}$$

Using Taylor expansion, we may write

$$\hat{\boldsymbol{\theta}}_{PL} = \boldsymbol{\theta} + H(\boldsymbol{\theta})^{-1} \nabla pl(\boldsymbol{\theta}; \mathbf{y}) + o_p(N^{-1/2}) \tag{6}$$

where $H(\boldsymbol{\theta})$ is the sensitivity matrix, $H(\boldsymbol{\theta}) = E\left\{-\nabla^2 pl(\boldsymbol{\theta}; \mathbf{y})\right\}$. It follows that

$$\sqrt{N}\left(\hat{\boldsymbol{\theta}}_{PL} - \boldsymbol{\theta}\right) \xrightarrow{d} N_t\left(0, H(\boldsymbol{\theta}) J^{-1}(\boldsymbol{\theta}) H(\boldsymbol{\theta})\right) \ ,$$

where t is the dimension of $\boldsymbol{\theta}$, and $J(\boldsymbol{\theta})$ is the variability matrix, $J(\boldsymbol{\theta}) = Var\left\{\sqrt{N} \nabla pl(\boldsymbol{\theta}; \mathbf{y})\right\}$.

Overview of results so far

1. For factor analysis models with categorical data (Katsikatsou et al., 2012)
    - PL estimates and standard errors present a close-to-zero bias and mean squared error (MSE).
    - PL performs very similarly to three-stage least squares methods and maximum likelihood as implemented in the GLLVM approach.
2. Goodness-of-fit (Katsikatsou and Moustaki, 2016)
    - Pairwise Likelihood Ratio Test (PLRT) for overall fit
    - Pairwise Likelihood Ratio Test for comparing models (e.g. equality constraints)
    - Model selection criteria: PL versions of AIC and BIC
    - The PLRT statistic performs in accordance with the asymptotic results at 5% and 1% significance levels for $N = 500, 1000$ but not satisfactorily for $N = 200$.
    - Both adjusted AIC and BIC criteria perform very well with a minimum rate of success 82.9%.

Software

In the **R package** `lavaan`

PL is available for fitting and testing factor analysis models or SEMs where

- all observed variables are binary or ordinal, and
- the standard parametrization for the underlying variables is used (zero means and unit variances)
- Multigroup analysis is also possible.
- Handling MAR and Non ignorable missigness.

Limited Information Test Statistics for PL estimators

Overall goodness-of-fit tests, simple hypothesis

- Let us denote with $\mathbf{p}$ the $2^p \times 1$ vector of sample proportions corresponding to the vector of population proportions $\boldsymbol{\pi}$. Assuming i.i.d, it is known that:

$$\sqrt{N}(\mathbf{p} - \boldsymbol{\pi}) \xrightarrow{d} N(0, \Sigma), \tag{7}$$

- where $\Sigma = D(\boldsymbol{\pi}) - \boldsymbol{\pi}\boldsymbol{\pi}'$ and N is the sample size.
- Under complex sampling design, the vector $\mathbf{p}$ becomes the weighted vector of proportions $\mathbf{p}$ with elements $\sum_{h \in s} w_h I(\mathbf{y}^{(h)} = \mathbf{y}_r) / \sum_{h \in s} w_h$.
- Under suitable conditions (e.g. Fuller, 2009, sect. 1.3.2) we still have a central limit theorem, where the covariance matrix Σ need now not take a multinomial form.

Fit on the Lower order margins

- Let $\dot{\boldsymbol{\pi}}_1 = (P(y_1 = 1), P(y_2 = 1), \ldots, P(y_p = 1))'$ be the $p \times 1$ vector that contains all univariate probabilities of a positive response to an item.
- Let $\dot{\boldsymbol{\pi}}_2$ be the $\binom{p}{2} \times 1$ vector of bivariate probabilities with elements, $\dot{\pi}_{ij} = P(y_i = 1, y_j = 1), j < i$.
- Let $\boldsymbol{\pi}_2$ be the vector that contains both these univariate and bivariate probabilities with dimension $s = p + \binom{p}{2} = p(p+1)/2$.
- We also define an $s \times 2^p$ indicator matrix T_2 of rank s such that $\boldsymbol{\pi}_2 = T_2 \boldsymbol{\pi}$.

Limited information goodness-of-fit tests

Reiser (1996, 2008), Bartholomew and Leung (2002), Maydey-Olivares and Joe (2005, 2006) Cagnone and Mignani (2007).

The test statistics developed are based on marginal distributions rather than on the whole response pattern.

1. $H_o : \boldsymbol{\pi}_2 = \boldsymbol{\pi}_2(\boldsymbol{\theta})$ for some $\boldsymbol{\theta}$ versus $H_1 : \boldsymbol{\pi}_2 \neq \boldsymbol{\pi}_2(\boldsymbol{\theta})$ for any $\boldsymbol{\theta}$.
2. Construct test statistics based upon the residual vector $\hat{\mathrm{e}}_2 = \mathbf{p}_2 - \boldsymbol{\pi}_2(\hat{\boldsymbol{\theta}}_{PL})$ derived from the bivariate marginal distributions of $\mathbf{y}$ and with $\boldsymbol{\theta}_{PL}$.
3. We first derive the asymptotic distribution of $\hat{\mathrm{e}}_2$.

Distribution of residuals (1)

- Following earlier notation, we can write $s \times 1$ vectors: $\boldsymbol{\pi}_2(\boldsymbol{\theta}) = T_2\boldsymbol{\pi}(\boldsymbol{\theta})$ and $\mathbf{p}_2 = T_2\mathbf{p}$.
- It follows that:

$$\sqrt{n}(\mathbf{p}_2 - \boldsymbol{\pi}_2(\boldsymbol{\theta})) \xrightarrow{d} N(0, \Sigma_2), \tag{8}$$

  where $\Sigma_2 = T_2 \Sigma T_2'$.
- Because T_2 is of full rank s, Σ_2 is also of full rank s.

Distribution of residuals (2)

Noting that $\boldsymbol{\pi}_2(\boldsymbol{\theta}) = T_2\boldsymbol{\pi}(\boldsymbol{\theta})$, a Taylor series expansion gives:

$$\boldsymbol{\pi}_2(\hat{\boldsymbol{\theta}}_{PL}) = \boldsymbol{\pi}_2(\boldsymbol{\theta}) + T_2\Delta(\hat{\boldsymbol{\theta}}_{PL} - \boldsymbol{\theta}) + o_p(N^{-1/2}), \tag{9}$$

where $\Delta = \frac{\partial \boldsymbol{\pi}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}$
Hence, using

$$\hat{\boldsymbol{\theta}}_{PL} - \boldsymbol{\theta} = H(\boldsymbol{\theta})^{-1}\nabla pl(\boldsymbol{\theta}; \mathbf{y}) + o_p(N^{-1/2})$$

we have

$$\hat{\mathrm{e}}_2 = \mathbf{p}_2 - \boldsymbol{\pi}_2(\hat{\boldsymbol{\theta}}_{PL}) = \mathbf{p}_2 - \boldsymbol{\pi}_2(\boldsymbol{\theta}) - T_2\Delta H(\boldsymbol{\theta})^{-1}\nabla pl(\boldsymbol{\theta}; \mathbf{y}) + o_p(N^{-1/2}). \tag{10}$$

Finally we need to express $\nabla pl(\boldsymbol{\theta}; \mathbf{y})$ in terms of $\mathbf{p}_2 - \boldsymbol{\pi}_2(\boldsymbol{\theta})$

Distribution of residuals (3)

Hence, there is a $t \times s$ matrix $B(\boldsymbol{\theta})$ such that

$$\nabla pl(\boldsymbol{\theta}; \mathbf{y}) = B(\boldsymbol{\theta})(\mathbf{p}_2 - \boldsymbol{\pi}_2(\boldsymbol{\theta})) \tag{11}$$

Hence, from (10)

$$\hat{\mathbf{e}}_2 = (I - T_2 \Delta H(\boldsymbol{\theta})^{-1} B(\boldsymbol{\theta}))(\mathbf{p}_2 - \boldsymbol{\pi}_2(\boldsymbol{\theta})) + o_p(N^{-1/2}) \tag{12}$$

So from (8), we have under H_0 that:

$$\sqrt{N}\hat{\mathbf{e}}_2 \xrightarrow{d} N(0, \Omega). \tag{13}$$

where $\Omega = (I - T_2 \Delta H(\boldsymbol{\theta})^{-1} B(\boldsymbol{\theta}))\Sigma_2(I - T_2 \Delta H(\boldsymbol{\theta})^{-1} B(\boldsymbol{\theta}))'$.

Distribution of residuals (4)

To estimate the asymptotic covariance matrix of $\hat{\mathrm{e}}_2$, we evaluate $\frac{\partial \boldsymbol{\pi}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}$ at the PL estimate $\hat{\boldsymbol{\theta}}_{PL}$ to obtain $\hat{\Delta}$ and set:

$$\hat{\Omega} = (I - T_2\hat{\Delta}\hat{H}(\hat{\boldsymbol{\theta}}_{PL})^{-1}B(\hat{\boldsymbol{\theta}}_{PL}))\hat{\Sigma}_2(I - T_2\hat{\Delta}\hat{H}(\hat{\boldsymbol{\theta}}_{PL})^{-1}B(\hat{\boldsymbol{\theta}}_{PL}))',$$

where $\hat{\Sigma}_2 = T_2\hat{\Sigma}T_2'$.

- In the case of iid observations with a multinomial covariance matrix, we may set $\hat{\Sigma} = D(\boldsymbol{\pi}(\hat{\boldsymbol{\theta}})) - \boldsymbol{\pi}(\hat{\boldsymbol{\theta}})\boldsymbol{\pi}(\hat{\boldsymbol{\theta}})'$.
- In the case of a complex sample design we need to derive a consistent estimator for Σ

Proposed test statistics

Wald test type statistics

A Wald test statistic is given by:

- $$L_2 = N(\mathbf{p}_2 - \boldsymbol{\pi}_2(\hat{\boldsymbol{\theta}}_{PL}))'\hat{\Omega}^{+}(\mathbf{p_2} - \boldsymbol{\pi_2}(\hat{\boldsymbol{\theta}}_{\mathbf{PL}})), \tag{14}$$
  - $\hat{\Omega}^{+}$ is the Moore-Penrose inverse of $\hat{\Omega}$.
  - Under H_0, this test statistic is asymptotically distributed as χ^2 with degrees of freedom equal to the rank of $\hat{\Omega}^{+}$, which is between $s - t$ and s.
- An alternative Wald test: $\hat{\boldsymbol{\Xi}}_2 = \mathrm{diag}(\hat{\boldsymbol{\Omega}}_2)^{-1}$ is used instead of the pseudoinverse of $\boldsymbol{\Omega}_2$. We refer to this *Diagonal Wald test*, (Wald v2). Its distribution needs to be determined using moment-matching procedures. We employ a three moment adjustment.
- The estimation of Ω_2 can be computationally involved in some cases (large models).
- The rank of Ω_2 cannot be determined a priori instead one needs to inspect the eigen values of $\hat{\Omega}_2$.

Variance-covariance free Wald test, Wald v3

Maydeu-Olivares and Joe (2005, 2006) suggested using a weight matrix $\mathbf{\Xi}_2$ such that $\mathbf{\Omega}_2$ is a generalized inverse of $\mathbf{\Xi}_2$, i.e. $\mathbf{\Xi}_2 = \mathbf{\Xi}_2\mathbf{\Omega}_2\mathbf{\Xi}_2$.
The test statistic proposed:

$$X^2 = N\hat{\mathbf{e}}_2^\top \hat{\mathbf{\Xi}}_2 \hat{\mathbf{e}}_2 = N\hat{\mathbf{e}}_2^\top \hat{\mathbf{\Delta}}_2^\perp \left((\hat{\mathbf{\Delta}}_2^\perp)^\top \hat{\mathbf{\Sigma}}_2 \hat{\mathbf{\Delta}}_2^\perp \right)^{-1} (\hat{\mathbf{\Delta}}_2^\perp)^\top \hat{\mathbf{e}}_2$$

- where $\mathbf{\Delta}_2^\perp$ is an $S \times (S-t)$ orthogonal complement to $\mathbf{\Delta}_2$, i.e. it satisfies $(\mathbf{\Delta}_2^\perp)^\top \mathbf{\Delta}_2 = \mathbf{0}$.
- It converges in distribution to a χ^2_{S-t} variate as $N \to \infty$.

Pearson Chi-square Test Statistic

- Let D_2 be the $s \times s$ matrix $D_2 = diag(\pi_2(\boldsymbol{\theta}))$ and let $\hat{D}_2 = diag(\pi_2(\hat{\boldsymbol{\theta}}_{PL}))$.
- The Pearson test statistic is given by

$$X_P^2 = N\hat{\mathbf{e}}_2'\hat{D}_2^{-1}\hat{\mathbf{e}}_2 = N(\mathbf{p}_2 - \boldsymbol{\pi}_2(\hat{\boldsymbol{\theta}}_{PL}))'\hat{D}_2^{-1}(\mathbf{p_2} - \boldsymbol{\pi_2}(\hat{\boldsymbol{\theta}}_{\mathbf{PL}})). \tag{15}$$

- The limiting distribution of $\sqrt{N}\hat{D}_2^{-0.5}\hat{\mathbf{e}}_2$ under the hypothesis that the model is correct is given by $N(0, {D_2}^{-0.5}\Omega_2 {D_2}^{-0.5})$.
- Hence X_P^2 has the limiting distribution of $\sum \delta_i W_i$, where the δ_i are eigenvalues of ${D_2}^{-0.5}\Omega_2 {D_2}^{-0.5}$ and the W_i are independent chi-square random variables, each with one degree of freedom.
- These eigenvalues can be estimated by the eigenvalues of $\hat{D}_2^{-0.5}\hat{\Omega}_2\hat{D}_2^{-0.5}$.
- A first and a second order Rao-Scott type test can be obtained.

Estimation of the covariance matrix under complex sampling

Estimation of the covariance matrix under complex sampling: stratified multistage sampling (1)

- $$\begin{aligned} \Sigma &= limvar\{\sqrt{N}(\mathbf{p} - \boldsymbol{\pi})\} \\ &= limvar\{\sqrt{N}(\frac{\sum_{h \in s} w_h \mathbf{y}^{(h)}}{\sum_{h \in s} w_h} - \boldsymbol{\pi})\} \end{aligned}$$

  where $limvar$ denotes the asymptotic covariance matrix.
- Using a usual linearization argument for a ratio:

$$\Sigma = limvar\{\sqrt{N}\frac{\sum_{h \in s} w_h(\mathbf{y}^{(h)} - \boldsymbol{\pi})}{E(\sum_{h \in s} w_h)}\}. \tag{16}$$

Estimation of the covariance matrix: stratified multistage sampling (2)

- Strata are labelled a and the primary sampling units are labelled $b = 1, \ldots, N_a$, where N_a is the number of primary sampling units selected in stratum a.
- Then we write

$$[\sum_{h \in s} w_h(\mathbf{y}^{(h)} - \boldsymbol{\pi})]/[E(\sum_{h \in s} w_h)] = \sum_a \sum_b \tilde{\mathbf{u}}_{ab}, \tag{17}$$

- where $\tilde{\mathbf{u}}_{ab} = \sum_{h \in s_{ab}} w_h(\mathbf{y}^{(h)} - \boldsymbol{\pi})/[E(\sum_{h \in s} w_h)]$ and s_{ab} is the set of sample units contained within primary sampling unit b within stratum a. So

$$\Sigma = limvar\{\sqrt{N} \sum_a \sum_b \tilde{\mathbf{u}}_{ab}\}. \tag{18}$$

Estimation of the covariance matrix: stratified multistage sampling (3)

- A standard estimator of $N^{-1}\Sigma$ is then given by

$$N^{-1}\hat{\Sigma} = \sum_a \frac{N_a}{N_a - 1} \sum_b (\mathbf{u}_{ab} - \bar{\mathbf{u}}_a)(\mathbf{u}_{ab} - \bar{\mathbf{u}}_a)' \tag{19}$$

- where $\mathbf{u}_{ab} = \sum_{h \in s_{ab}} w_h (\mathbf{y}^{(h)} - \mathbf{p}) / (\sum_{h \in s} w_h)$ and $\bar{\mathbf{u}}_a = N_a^{-1} \sum_b \mathbf{u}_{ab}$

Estimation of the covariance matrix under complex sampling (4)

- In order to compute the Wald and Pearson test statistic, we only require $\hat{\Sigma}_2 = T_2\hat{\Sigma}T_2'$.

$$N^{-1}\hat{\Sigma}_2 = \sum_a \frac{N_a}{N_a - 1}\sum_b (\mathbf{v}_{ab} - \bar{\mathbf{v}}_a)(\mathbf{v}_{ab} - \bar{\mathbf{v}}_a)' \tag{20}$$

where $\mathbf{v}_{ab} = \sum_{h \in s_{ab}} w_h(\mathbf{y}_2^{(h)} - \mathbf{p}_2)/(\sum_{h \in s} w_h)$, $\bar{\mathbf{v}}_a = N_a^{-1}\sum_b \mathbf{v}_{ab}$ and $\mathbf{y}_2^{(h)} = T_2\mathbf{y}^{(h)}$ is the $s \times 1$ vector containing indicator values $I(y_i^{(h)} = 1)$ and $I(y_i^{(h)} = y_j^{(h)} = 1)$ for different values of i and j.

Simulation study

Simulation A: data generated under SRS

- Four sample sizes ($n = 500, 1000, 2000, 3000$).
  1. $p = 5$ and $q = 1$ (1F 5V)
  2. $p = 8$ and $q = 1$ (1F 8V)
  3. $p = 15$ and $q = 1$ (1F 15V)
  4. $p = 10$ and $q = 2$, 5 indicators per factor (2F 10V)
  5. $p = 15$ and $q = 3$, 5 indicators per factor (3F 15V)
- Models 4 and 5 are confirmatory factor analysis models.
- The number of replications within each condition is 1000.
- Power analysis: a latent variable $z \sim N(0, 1)$ added to the data generating model.

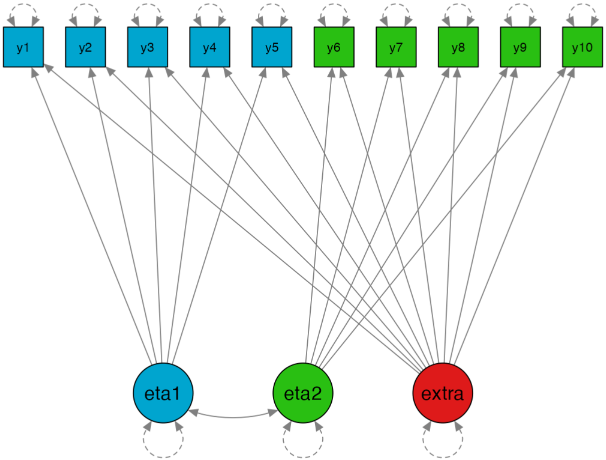
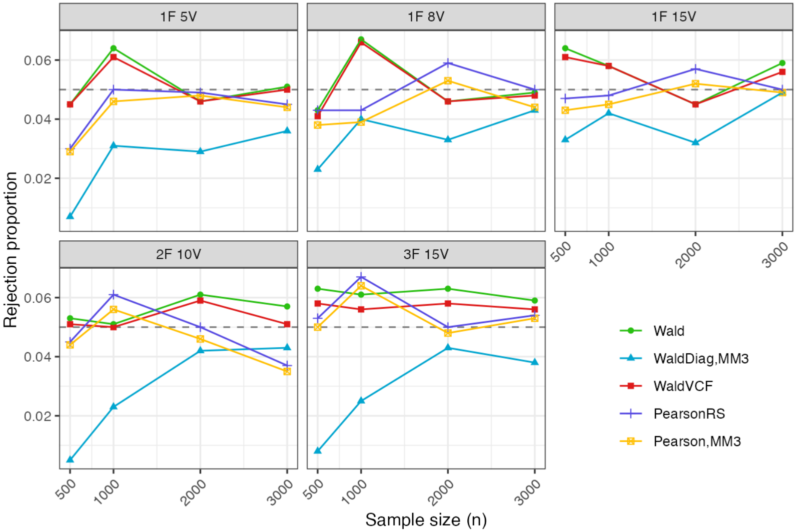


Figure: Model 4: Confirmatory factor analysis model

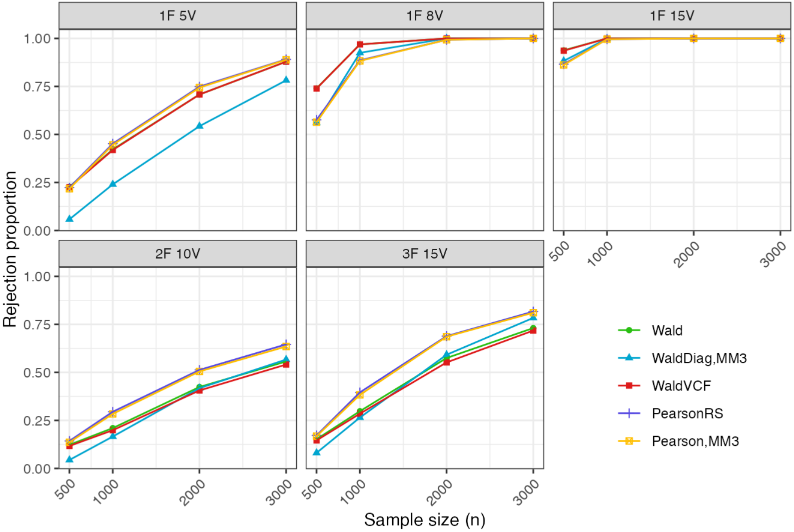
Simulation A: Test statistics computed

- The Wald test.
- The Wald v2 test (diagonal).
- The Wald v3 test (otrhogonal components)
- The Pearson test (PearsonRS).
- The first-and-second-moment adjusted (FSMadj) Pearson test statistic.

Type I errors ($\alpha = 0.05$)



Power ($\alpha = 0.05$)



Simulation A: Results

- The Wald v2 has the poorest performance. Both Pearson test statistics performed satisfactorily at all three significance levels $\alpha = 0.01, 0.05, 0.10$ and improved with the increase of the sample size.
- The power of all tests increases with the sample size but stayed at lower levels in the case of two and three-factor models.

Simulation B: data generated under complex sampling

- Four sample sizes ($n = 500, 1000, 2000, 3000$).
- We generate data for an entire population inspired by a sampling design used in large scale assessment surveys.
- The population consists of $2,000$ schools (Primary Sampling Units, PSU) of three types: "A" (400 units), "B" (1000 units), and "C" (600 units). The school type correlates with the average abilities of its students (stratification factor).
- Each school is assigned a random number of students from the normal distribution $N(500, 125^2)$ (the number then rounded down to a whole number).
- Students are then assigned randomly into classes of average sizes 15, 25 and 20 respectively for each school type A, B and C.
- The total population size is roughly 1 million students.

Simulation B: Sampling designs (1)

1. Stratified sampling: From each school type (strata), select 1000 students (PSU) using SRS. Let N_a be the total number of students in stratum $a \in \{1, 2, 3\}$. Probability of selection of a student in stratum a is $\Pr(\text{selection}) = \frac{1000}{N_a}$. The total sample size is $n = 3 \times 1000 = 3000$.

2. Two-stage cluster sampling: Select 140 schools (PSU; clusters) using probability proportional to size (PPS). For each school, select one class by SRS, and all students in that class. The probability of selection of a student in PSU $b = 1, \ldots, 2000$:

$$\Pr(\text{selection}) = \Pr(\text{weighted school selection}) \times \frac{1}{\#\text{ classes in school } b}.$$

   The total sample size will vary from sample to sample, but on average will be $n = 140 \times 21.5 = 3010$, where 21.5 is the average class size per school.

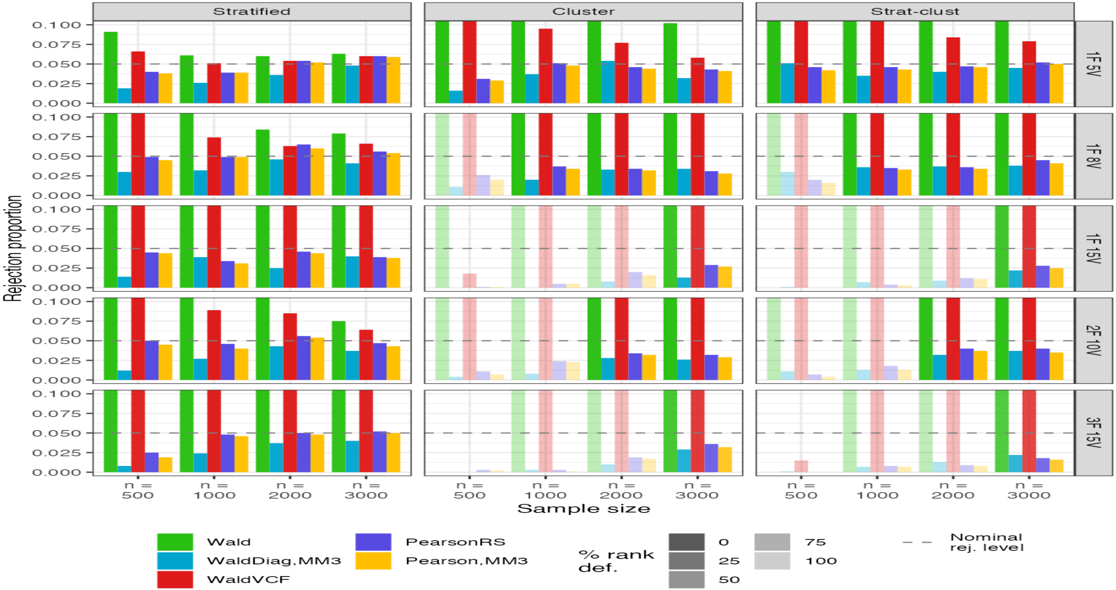
Simulation B: Sampling designs (2)

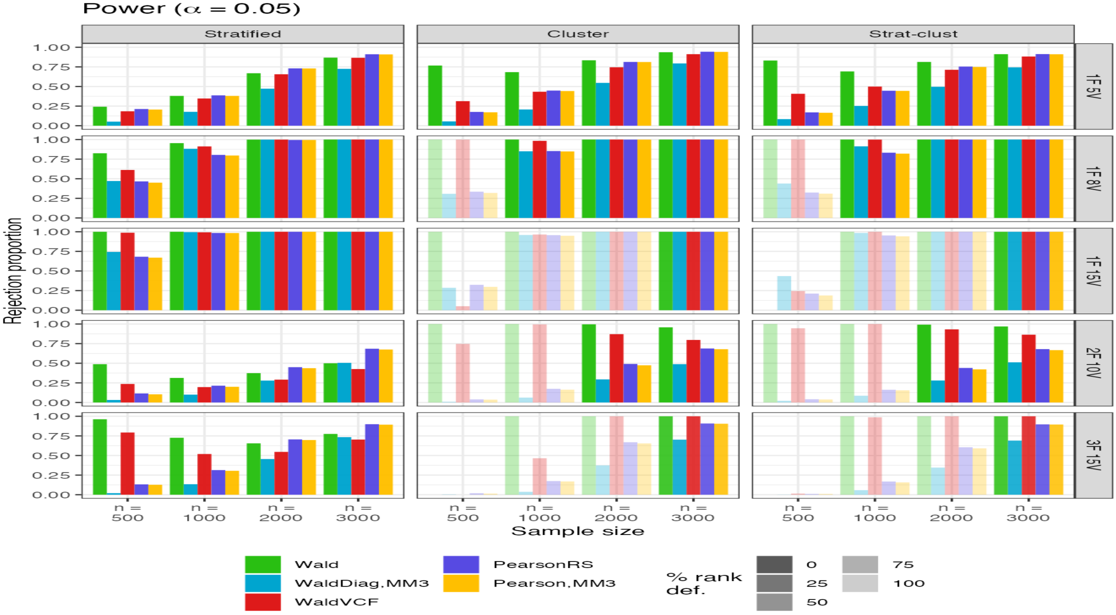
1. Two-stage stratified cluster sampling: For each school type (strata), select 50 schools using SRS. Then, within each school, select 1 class by SRS, and all students in that class are selected to the sample. The probability of selection of a student in PSU b from school type a is

$$\Pr(\text{selection}) = \frac{50}{\#\text{ schools of type } a} \times \frac{1}{\#\text{ classes in school } b}.$$

Here, the expected sample size is $n = 50 \times (15 + 25 + 20) = 3000$.

Type I errors ($\alpha = 0.05$)




Power ($\alpha = 0.05$)
Stratified
Cluster
Strat-clust
1F 5V
1F 8V
1F 15V
2F 10V
3F 15V
Rejection proportion
1.00
0.75
0.50
0.25
0.00
n = 500
n = 1000
n = 2000
n = 3000
Sample size
Wald
WaldDiag,MM3
WaldVCF
PearsonRS
Pearson,MM3
% rank def.
0
25
50
75
100

Simulation B: Results

- Type I error rates: Both Pearson tests performed satisfactorily under stratified sampling.
- In the cluster sampling and stratified cluster sampling and in samples sizes of 500 and 1000 we had a large proportion of rank deficiency issues with the estimated covariance matrix.
- The power of the test in the one-factor models and stratified sampling increased to 1 with the increase of the sample size.

Thank you for your attention!