Weak convergence of stochastic integrals: an application to co-integrated time series

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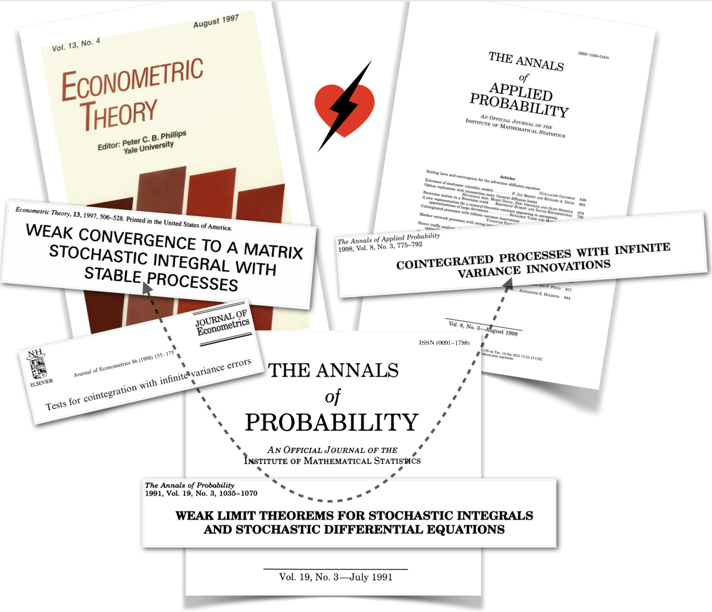
LSE Dept of Stats - Research Showcase

Econometrics meets probability



Co-integrated time series

- Co-integration with infinite variance innovations u_k and v_k

Time series (Park & Phillips, 1988)

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Least squares estimator

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- Let $\mathbb{T} = \mathrm{diag}(\frac{1}{n^{1/\alpha_1}}, \dots, \frac{1}{n^{1/\alpha_p}})$ and $\tilde{\mathbb{T}} = \mathrm{diag}(\frac{1}{n^{1/\tilde{\alpha}_1}}, \dots, \frac{1}{n^{1/\tilde{\alpha}_q}})$

Interested in asymptotics as $n \to \infty$

$$n\mathbb{T}(\hat{A} - A)\tilde{\mathbb{T}}^{-1} = \left(\mathbb{T}\mathbb{U}^\intercal \mathbb{X}\tilde{\mathbb{T}}\right)\left(\frac{1}{n}\tilde{\mathbb{T}}\mathbb{X}^\intercal \mathbb{X}\tilde{\mathbb{T}}\right)^{-1}$$

Asymptotic analysis

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- Rewriting $\mathbb{T}\mathbb{U}^\intercal\mathbb{X}\tilde{\mathbb{T}}$ leads to the **stochastic integrals**

$$\int_0^1 Z_{s-}^{n,i}\, d\tilde{Z}_s^{n,j} \quad \text{for } i = 1, \dots, p,\ j = 1, \dots, q.$$

Weak limit theorems for stochastic integrals

Famously, stochastic integration lacks continuity

$$\text{the map } (H_\cdot, X_\cdot) \mapsto \int_0^\cdot H_{s-} dX_s \text{ is } \textit{not} \text{ continuous}$$

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- This *works beautifully* in *Skorokhod's J1 topology*, **when**:
  (i) the X^n are **semimartingales** and have so-called **'uniformly controlled variations'** (also H^n adapted)
  (ii) the pairs (H^n, X^n) converge jointly in Skorokhod's J1 topology and do so **'sufficiently together'**

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  - ⇝ Leads to much more complicated **moving averages**

$$Z_t^{n,i} := \sum_{l=1}^{[nt]} \frac{1}{n^{1/\alpha_i}} u_l^i, \quad \tilde{Z}_t^{n,j} := \sum_{l=1}^{[nt]} \frac{1}{n^{1/\tilde{\alpha}_j}} v_l^j \quad (\alpha_i, \tilde{\alpha}_j > 1)$$

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  - ⇝ No longer convergence in J1! And UCV also problematic!

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- *Moving averages* **converge** in *Skorokhod's M1 topology*:

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  $\rightsquigarrow$ Can find $H^n_\cdot \Rightarrow H := 0$ in the **uniform norm** on $[0,1]$ so that the **integrals diverge** as $n \to \infty$!

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  ⇝ Assuming *independence of u and v*, then the **stochastic integrals** do indeed **converge** in the *M1 topology*!