Power laws in market microstructure

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- Models used by practitioners include square root and logarithm (e.g. Torre (1997), Potters and Bouchaud (2003), Almgren et al. (2005), Bershova and Rakhlin (2013), and Zarinelli et al. (2015)...).

More empirical evidence

- Several studies (Gopikrishnan et al. (2000), Lillo et al. (2005), Vaglica et al. (2008), Bershova and Rakhlin (2013)...) showed that metaorders have tail distribution following a power law (with exponents ranging from 1.56 to 1.74!).

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- To introduce non-uniform pricing one should consider a limit order market as in Glosten (1994) (*Is the electronic open limit order book inevitable?*, J. of F.).
- However, Biais, Hillon and Spatt (1995) and Sandas (2001) show that the empirical findings strongly contradict the predictions of many microstructure models on limit order markets including Glosten (1994).

Market structure

- Trading takes place at $t = 0$ and $t = 1$.
- Market consists of a riskless asset with $r = 0$ and a single risky asset. The fundamental value of the asset V will be revealed to the public at time 1.
  There are three types of agents on the market:
  - Competitive liquidity suppliers post limit orders and thereby construct the *limit order book* given by some function h.

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  - Noise traders with cumulative demand $Z \sim N(0, \sigma^2)$.
  - N Informed investors know V and are risk-neutral, i.e. they maximise their expected wealth at time 1.
  - A trading desk receiving orders from noise and informed traders. The desk does not trade in its own account and thus a market order of size y is priced at

    $$\int_Y^{Y+y} h(x)dx,$$

    where Y is the accumulated number of shares from earlier trades.

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where E^v is the expectation operator for the insider with the private information $V = v$.
- Since h is nondecreasing, the first order condition characterises the optimal X^* via $V = F(X^*)$, where

$$F(x) := \int_{-\infty}^{\infty} h(x+z)q(\sigma, z)dz \tag{1}$$

and $q(\sigma, \cdot)$ is the probability density function of a mean-zero Gaussian random variable with variance σ^2.

The case $N > 1$

- Assume all informed orders arrive after the noise.
- As every insider has symmetric information and is risk-neutral, in a symmetric equilibrium, the demand x^* for each insider must be the same and satisfy

$$v = E^v\left[\frac{h(Z+Nx^*)}{N} + \frac{N-1}{N^2x^*}\int_0^{Nx^*} h(Z+u)du\right].$$

- Denoting the total informed demand by X^*, the above can be rewritten as $V = F(X^*)$, where

$$F(x) := E^v\left[\frac{h(Z+x)}{N} + \frac{N-1}{Nx}\int_0^{x} h(Z+u)du\right], \qquad (2)$$

and $F(0)$ is interpreted by continuity to be

$$E^v\left[\frac{h(Z)}{N} + \frac{(N-1)h(Z)}{N}\right] = E^v[h(Z)].$$

The limit order book and equilibrium

Following Glosten, we assume limit prices are given by 'tail expectations:

$$h(y) = \left\{ \begin{array}{ll} E[V|Y \geq y], & \text{if } y > 0; \\ E[V|Y \leq y], & \text{if } y < 0. \end{array} \right\} \tag{3}$$

Definition 1

The pair (h^*, X^*) is said to be a Glosten equilibrium if h^* is non-decreasing and non-constant, $X^* \in \mathbb{R}$ and

i) h^* satisfies (3) with $Y = X^* + Z$;

ii) X^* is the profit maximising order size for the insider(s) given h^*. That is, $V = F(X^*)$, where F is given by (2).

Few objects of interest

- Suppose that (X^*, h^*) is an equilibrium and write h instead of h^* to ease exposition.
- Introduce the functions $\Phi^\pm$ and $\Pi^\pm$ via

$$\begin{aligned}\Phi^+(y) &:= E[V\mathbf{1}_{[V>y]}],\ \Pi^+(y) := P(V>y)\\ \Phi^-(y) &:= E[V\mathbf{1}_{[V\le y]}],\ \Pi^-(y) := P(V\le y) = 1-\Pi^+(y).\end{aligned}$$

- Define $\Psi^\pm(y) := \frac{\Phi^\pm(y)}{\Pi^\pm(y)}$ so that $\Psi^+(y) = E[V|V>y]$ and $\Psi^-(y) = E[V|V\le y]$.
- Since, for $y>0$, $h(y) = E[V|F^{-1}(V)+Z\ge y]$,

$$\begin{aligned}h(y) &= E[V|V\ge F(y-Z)] = \frac{E[V\mathbf{1}_{[V\ge F(y-Z)]}]}{P(V\ge F(y-Z))}\\ &= \frac{\int_{-\infty}^{\infty}\Phi^+(F(y-z))q(\sigma,z)dz}{\int_{-\infty}^{\infty}\Pi^+(F(y-z))q(\sigma,z)dz}\ (\neq E[\Psi^+(F(y-Z))]!)\end{aligned}$$

- An analogous representation holds for $y<0$.

An equation for F

- Define, for any continuous g, the mappings

$$\phi_g^{\pm}(x) := \frac{\int_{-\infty}^{\infty} \Phi^{\pm}(g(z))q(\sigma, x-z)dz}{\int_{-\infty}^{\infty} \Pi^{\pm}(g(z))q(\sigma, x-z)dz}.$$

  Let us also set

$$\phi_g(x) := \phi_g^{+}(x)\mathbf{1}_{x\geq 0} + \phi_g^{-}(x)\mathbf{1}_{x<0}. \tag{4}$$

- Now, combining all of the above yields an equation for F:

$$\begin{aligned} F(x) &= \frac{1}{N}\int_{-\infty}^{\infty} q(\sigma, x-z)\phi_F(z)dz \qquad (5) \\ &+ \frac{N-1}{Nx}\int_0^x dy \int_{-\infty}^{\infty} q(\sigma, y-z)\phi_F(z)dz. \end{aligned}$$

Characterisation of equilibrium

Given the above consideration the following now is obvious:

Theorem 2

Equilibrium exists if and only if there exists a function $F : \mathbb{R} \to \mathbb{R}$ that satisfies (5). Given such a solution F, (X^*, h^*) constitutes an equilibrium, where $X^* = F^{-1}(V)$ and h^* is defined via (4) and its counterpart for $y < 0$.

Therefore, finding an equilibrium boils down to finding a solution of (5).

Examples

- Suppose $P(V=1)=P(V=-1)=\frac{1}{2}$. Then, the unique symmetric solution of (5) is defined for $x>0$ by

$$F(x)=\frac{1}{N}\int_0^\infty q_0(\sigma,x,z)dz+\frac{N-1}{Nx}\int_0^x dy\int_0^\infty dz q_0(\sigma,y,z),$$

where $q_0(\sigma,y,z):=q(\sigma,y-z)-q(\sigma,y+z)$. Moreover, $X^*=\infty$ (resp. $X^*=-\infty$) if $V=1$ (resp. $V=-1$) and, thus, $h^*(y)=\mathbf{1}_{[y>0]}-\mathbf{1}_{[y<0]}$.
Nevertheless, insiders' profit remains finite:

$$\int_0^\infty E^1(1-h(Z+y))dy=2E^1\left(\int_0^\infty \mathbf{1}_{[Z<-y]}dy\right)=\sigma\sqrt{\frac{2}{\pi}}.$$

Examples

- Suppose $P(V=-1)=P(V=0)=P(V=1)=\frac{1}{3}$. Then, similar considerations yield

$$F(x)=\frac{1}{N}\int_0^\infty q_0(\sigma,x,z)\frac{1}{1+P(Z\geq z)}dz + \frac{N-1}{Nx}\int_0^x dy\int_0^\infty dz q_0(\sigma,y,z)\frac{1}{1+P(Z\geq z)}.$$

Again, $X*(1)=\infty$ and $X(-1)=-\infty$. But $X^*(0)=0$. Consequently, the order book will not be flat. In particular,, for $y>0$

$$h(y) = \frac{E[V\mathbf{1}_{[X^*(V)+Z\geq y]}]}{P(X^*(V)+Z\geq y)}=\frac{P(V=1)}{P(V=1)+P(V=0,Z\geq y)} = \frac{1}{1+P(Z\geq y)}.$$

Moreover, the bid-ask spread is given by $h(0+)-h(0-)=\frac{4}{3}$, independent of the noise variance.

Scaling property and uniqueness

- Due to the scaling property of q one should expect F exhibit similar scaling properties.
- Indeed, if $F(1; x)$ is a solution of (5) with $\sigma = 1$, then straightforward manipulations yield $F(1; \frac{x}{\sigma})$ solves (5).
- Thus, if (5) has a unique solution for one σ, it has a unique solution for all.
- This scaling property is also inherited by h: if (5) has a unique solution, $h(\sigma; x) = h(1; \frac{x}{\sigma})$ for all $x \neq 0$. As a consequence, $X^*(\sigma) = \sigma X^*(1)$.

Consequences of uniqueness

- The spread, i.e. $h(0+) - h(0-)$, is independent of σ!
- The spread associated with trade size $y > 0$, i.e. $h(y) - h(-y)$, and, therefore, the aggregate mid-spread S is decreasing with the amount of noise trading, consistent with the experimental findings of Bloomfield et al. (2009).

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- Thus, the order book gets flatten as σ increases and converges to a model with proportional transaction costs.

Towards existence

- Let's denote the interior of the support of V by (m, M), where $-\infty \leq m < M \leq \infty$, and recall on the support of V

$$\Psi^{\pm}(y) = \frac{\Phi^{\pm}(y)}{\Pi^{\pm}(y)} \tag{6}$$

Reminder

so that $\Psi^{+}(y) = E[V|V > y]$ and $\Psi^{-}(y) = E[V|V \leq y]$.

- For any continuous g let u^{+} (resp. u^{-}) be the unique solution of

$$u_t + \sigma^2 u_{xx} = 0, \qquad u(1, x) = \Pi^{+}(g(z)) \text{ (resp. } \Pi^{-}(g(z))). \tag{7}$$

The key lemma

Let g be as above. Then the following hold:

- There exits a solution B on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{Q})$ to the following SDE:

$$dB_t = \sigma dW_t + \sigma^2 \frac{u_x(t, B_t)}{u(t, B_t)} dt, \quad B_0 = x, \tag{8}$$

  where u is either u^+ or u^- and W is a Brownian motion with $W_0 = 0$.

- $\phi_g^+(x) = \mathbb{E}^{\mathbb{Q}^+}[\Psi^+(g(B_1))]$ and $\phi_g^-(x) = \mathbb{E}^{\mathbb{Q}^-}[\Psi^-(g(B_1))]$, where $(B, \mathbb{Q}^+)$ (resp. $(B, \mathbb{Q}^-)$) corresponds to the solution of (8) if $u = u^+$ (resp $u = u^-$) and $\mathbb{E}^{\mathbb{Q}}$ stands for the expectation under $\mathbb{Q}$. Equation for F

The key lemma (cont'd)

The following properties allow us to ensure in particular that the solution of (5) is increasing.

- $\phi_g^+(0) > \phi_g^-(0)$.
- Suppose further that g is non-decreasing. Then, $\phi_g^\pm$ are non-decreasing, too. Consequently, ϕ_g is non-decreasing. Moreover,

$$\phi_g^+(x) \leq \mathbb{E}^{\mathbb{Q}^+}\left[\Psi^+(g(\sigma W_1 + x))\right] \tag{9}$$

$$\phi_g^-(x) \geq \mathbb{E}^{\mathbb{Q}^-}\left[\Psi^-(g(\sigma W_1 + x))\right]. \tag{10}$$

Existence of equilibrium

Theorem 3

Suppose $-\infty < m < M < \infty$*. Then, there exists a Glosten equilibrium.*

The above theorem is proved by means of Schauder's fixed point theorem, hence no claim of uniqueness is given.

Asymptotics for F and h

- Although it is not possible to find explicitly F, it is possible to obtain its asymptotics.
- Recall that $g : (0, \infty) \to (0, \infty)$ is said to be *regularly varying of index* ρ at ∞ if

$$\lim_{\lambda \to \infty} \frac{g(\lambda x)}{g(\lambda)} = x^{\rho}, \quad \forall x > 0.$$

  Regular variation at $-\infty$ is defined analogously.
- It can be shown that F and h have the same regular variation index.

F is regularly varying

Suppose that $N > 1$, $-\infty < m < M < \infty$, and Π^+ has a continuous derivative. Note that $\Psi_x^+(M) := \frac{d\Psi^+(M-)}{dx} \leq 1$.

- Then, $M - F$ is regularly varying at ∞ with index

$$\rho^+ = \frac{\Psi_x^+(M) - 1}{1 - \frac{\Psi_x^+(M)}{N}}. \tag{11}$$

- Under above assumptions $F - m$ is regularly varying at $-\infty$ with index

$$\rho^- = \frac{\Psi_x^-(m) - 1}{1 - \frac{\Psi_x^-(m)}{N}}. \tag{12}$$

The case of slow variation

- The above shows that if $\Psi_x^+(M) < 1$, $-1 < \rho^+ < 0$ and $M - F(x) \sim x^{\rho^+}$ for large x.
- On the other hand, if $\Psi_x^+(M) = 1$, F, hence h, is slowly varying at ∞.
- To obtain a better understanding of how slow the variation of $M - F$ is, suppose that there exists an integer n and constant $k > 0$ such that

$$\frac{\Psi^+(x) - x}{(M - x)^n} \to \frac{1}{k} \quad \text{as } x \to M. \tag{13}$$

- Then, it can be shown that

$$M - F(x) \sim \left(\frac{N}{N-1}\frac{k}{n}\right)^{\frac{1}{n}} (\ln x)^{-\frac{1}{n}}.$$

Distribution of the volume

- Note for $x > 0$

$$P(X^* > x) = P(F^{-1}(V) > x) = P(V > F(x)) = \Pi^+(F(x)).$$

- However, assuming (13) also yields $\Pi^+(F)$ is regularly varying. Thus,

$$P(X^* > x) = x^{-\zeta^+} s(x),$$

where s is a slowly varying function and

$$\zeta^+ := \frac{\Psi_x^+(M)}{1 - \frac{\Psi_x^+(M)}{N}}$$

- Moreover, since $Y^* = X^* + Z$ and Z and V are independent, we have for $y > 0$

$$P(Y^* > y) = \int_{-\infty}^{\infty} dz P(X^* > z) q(\sigma, y - z),$$

which is regularly varying at infinity with the same index.

- Thus,

$$P(Y^* > y) = y^{-\zeta^+} s(y), \quad y > 0, \tag{14}$$

for some regularly varying s. In particular, if V has light tails, i.e. $\Psi_x^+(M) = 1$, $P(Y^* > y)$ is regularly varying of index $-\frac{N}{N-1}$. Fat tails

General signals

- Although the above theory is currently limited to bounded V, formal calculations in the general case show that F is regularly varying at ∞ of order ρ^+, where

$$\rho^+ = \frac{\Psi_x^+(\infty) - 1}{1 - \frac{\Psi_x^+(\infty)}{N}}. \tag{15}$$

- However, since ρ^+ must be non-negative, this places the restriction on N:

$$N > \Psi_x^+(\infty) \tag{16}$$

  unless $\Psi_x^+(\infty) = 1$.
- That is, equilibrium requires a sufficiently large amount of competition among insiders when the asset value has fat tails.

Table: Distributions with power-law impact

Distribution	Density	ρ^+
Beta prime	$x^{\lambda-1}(1+x)^{-(\lambda+\alpha)}$	$\left(\frac{N-1}{N}\alpha - 1\right)^{-1}$
Fréchet	$(x-\beta)^{-(1+\alpha)} \exp\left\{-\left(\frac{x-\beta}{s}\right)^{-\alpha}\right\}$	$\left(\frac{N-1}{N}\alpha - 1\right)^{-1}$
Lomax	$\left(1+\frac{x}{\lambda}\right)^{-(\alpha+1)}$	$\left(\frac{N-1}{N}\alpha - 1\right)^{-1}$
Pareto	$x^{-(\alpha+1)}$	$\left(\frac{N-1}{N}\alpha - 1\right)^{-1}$
Student	$\left(1+\frac{x^2}{\alpha}\right)^{-(\alpha+1)/2}$	$\left(\frac{N-1}{N}\alpha - 1\right)^{-1}$

In above probability densities are given up to a scaling factor and implicit constraints are enforced to ensure they are well defined with finite mean. Moreover, $N > \frac{\alpha}{\alpha-1}$ in all of the above.

Table: Distributions with logarithmic impact

Distribution	Density	Asymptotics
Exponential	$\exp(-\lambda x)$	$\frac{N}{\lambda(N-1)}\log x$
Gaussian	$\exp(-(x-\mu)^2/\Sigma)$	$\sqrt{\frac{2\Sigma N}{N-1}}\sqrt{\log x}$
Inverse Gaussian	$x^{-3/2}\exp\left(-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right)$	$\frac{2N\mu^2}{\lambda(N-1)}\log x$
Normal Inverse Gaussian	$\frac{K_1(\lambda\zeta(x))}{\pi\zeta(x)}\exp(\delta\gamma+\beta(x-\mu)$	$\frac{N}{(N-1)(\lambda+\beta-1)}\log x$
Weibull	$x^{d-1}\exp(-\lambda^p x^p)$	$\left(\frac{N}{\lambda^p(N-1)}\right)^{1/p}(\log x)^{1/p}$

In above probability densities are given up to a scaling factor and implicit constraints are enforced to ensure they are well defined with finite mean. Moreover, $\zeta(x) := \delta^2 + (x-\mu)^2$ for the Normal Inverse Gaussian distribution.

Equilibrium with Student signals

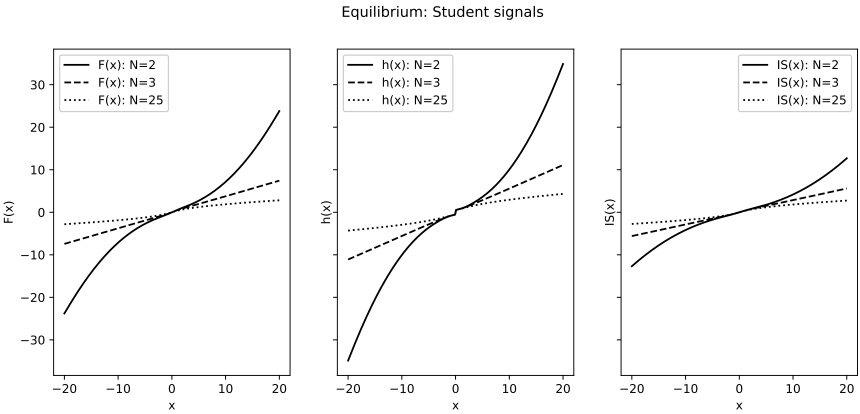

Figure: Equilibrium solutions for Student signals for the cases $N = 2$, $N = 3$ and $N = 25$.

Convergence to square root impact

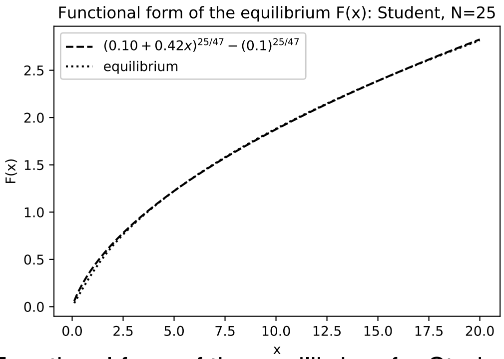

Figure: Functional form of the equilibrium for Student signals for $\alpha = 3$, $N = 25$.

Equilibrium with log-Normal signals

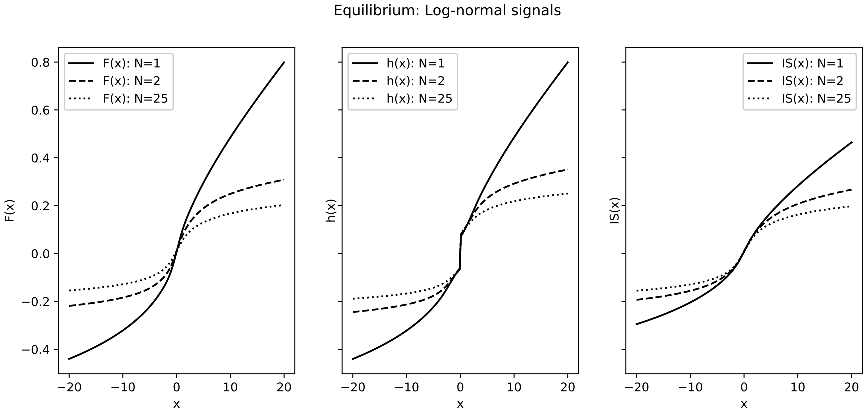

Figure: The discontinuity of $h(x)$ at the origin is the bid-ask spread.

Conclusion

- Solved in a one-period setting the equilibrium in a limit order market in the framework of Glosten (1994).
- Although the equilibrium is not explicitly solvable, we show that the impact of large trades show regular variation and find the exact exponent.
- The impact seems to be of power law for asset values exhibiting fat tails and logarithmic for the ones with light tails. This provides a testable answer to the debate among the practitioners on the nature of the price impact.
- The equilibrium volume for the assets is of regular variation. Thus, the model can be seen as a justification in an REE setting to several conclusions of the econophysics literature.

- Suppose that $\Pi^+(x)$ has a power-like behaviour in the sense that

$$\frac{\Pi^+_x(x)}{\Pi^+(x)} \sim -\alpha(M-x)^{-1}, \quad \alpha > 0.$$

- An application of L'Hospital rule shows that

$$\frac{\int_x^M \Pi^+(y)dy}{\Pi^+(x)(M-x)} \sim \frac{\Pi^+(x)}{\Pi^+(x) - (M-x)\Pi^+_x(x)}$$

- As

$$\frac{M-\Psi^+(x)}{M-x} = \frac{\int_x^M \Pi^+(y)dy}{(M-x)\Pi^+(x)} - 1,$$

we must have $\Psi^+_x(M) = \frac{\alpha}{\alpha+1} < 1$.