LSE Department of Statistics

High Frequency Trading in Kyle-Back Model

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Introduction

In this work it is considered a Kyle-Back model with a public dynamic signal and a private signal. The initial motivation of this work comes from the stylised fact that trading becomes more frequent when information is about to be released. Moreover, it is interesting to notice that we also produce a more realistic model in the sense that the insider is not the only source of information, but an agent that has a better signal than the others.

Agents

We shall consider an asset that is traded in the time interval $[0,1]$ whose true value, η, will be made public at time $t=1$ and the following agents:

Noise/liquidity traders: trade for liquidity reasons, and their demand is given by a standard $(\mathscr{F}_t)$-Brownian motion B independent of both B^M and B^I.

Market makers: observe the total demand

$$Y = \theta + B, \tag{1}$$

where θ is the demand of the informed trader, and a public signal given by

$$X_t^M = \int_0^t \sigma_M(s)dB_s^M + \int_0^t \sigma_M^2(s)\frac{\eta - X_s^M}{1-\Sigma_M(s)}ds \tag{2}$$

such that $\Sigma_M(t) = \int_0^t \sigma_M^2(s)ds$ and $\Sigma_M(1) = 1$. They set the process the price S considering the set of admissible pricing rules:

$$S(Y_{[0,t]}, X^M_{[0,t]}, t) = H(t, X_t, X_t^M) \quad \forall\, t \in [0,1), \tag{3}$$

where X is the unique strong solution to

$$dX_t = w(t)dY_t + \left(r_0(t) + r_1(t)X_t + r_2(t)X_t^M\right) dt \quad \forall\, t \in [0,1). \tag{4}$$

The informed investor: observes the price process, the public signal, and a private signal given by

$$X_t^I = X_0^I + \int_0^t \sigma_I(s)dB_s^I + \int_0^t \sigma_I^2(s)\frac{\eta - X_s^I}{1-\Sigma_I(s)}ds, \tag{5}$$

where $\Sigma_I(t) = c^2 + \int_0^t \sigma_I^2(s)ds$, $\Sigma_I(1) = 1$. Moreover, we require that $\eta \sim N(0,1)$ can be expressed as the sum of two independent random variable such that $\eta = X_0^I + \eta_1$ where $X_0^I \sim N(0,c^2)$. If $c=1$ the insider knows the true value of the asset at the start of the trading period.

Equilibrium

An admissible pricing rule is a linear function $H : [0,1] \times \mathbb{R}_+ \times \mathbb{R}_+$ of the form

$$H(t,x,x_1) = \beta_0(t) + \beta_1(t)x + \beta_2(t)x_1 \tag{6}$$

is a pricing rule if $\beta_1(t)$ is strictly positive for every $t \in [0,1)$.

An $\mathscr{F}^I$-adapted θ is said to be an admissible trading strategy for a given pricing rule H if it can be written as

$$d\theta_t = \alpha_t dt + \gamma_0(t)dB_t^I + \gamma_1(t)dB_t^M \tag{7}$$

and $\mathbb{E}\int_0^1 \left(H(s,X_s,X_s^M) - \eta\right)^2 ds < \infty$ $\mathbb{E}\int_0^1 \left(\int_{H^{-1}(s,z,X_s^M)}^{X_s} \frac{H_{x_1}(s,y,X_s^M)}{w(s)}dy\right)^2 \sigma_M^2(s)ds < \infty$, and $\mathbb{E}\int_0^1 \theta_t^2\sigma_Z^2(t)dt < \infty$.

The set of admissible trading strategies for the given H is denoted with $\mathcal{A}(H)$.

Definition

A couple (H^*, θ^*) is said to be an equilibrium if H^* is an admissible pricing rule, $\theta^* \in \mathcal{A}(H^*)$, and the following conditions are satisfied:

- Market efficiency condition: given θ^*, H^* is a rational pricing rule.
- Insider optimality condition: given H^*, θ^* solves the insider optimisation problem

$$\mathbb{E}[W_1^{\theta*}] = \sup_{\theta \in \mathcal{A}(H)} \mathbb{E}[W_1^\theta] \tag{8}$$

Theorem: Insider's Signal

Let

$$Z_t = \frac{1-\Sigma_Z(t)}{1-\Sigma_I(t)}X_t^I + \frac{1-\Sigma_Z(t)}{1-\Sigma_M(t)}X_t^M. \tag{9}$$

Then $\mathbb{E}\left[\eta|\mathscr{F}_t^I\right] = Z_t$. Furthermore, Z_t follows

$$Z_t = Z_0 + \int_0^t \sigma_Z(s)d\tilde{N}_s + \int_0^t \sigma_Z^2(s)\frac{\eta - Z_s}{1-\Sigma_Z(s)}ds, \text{ where} \tag{10}$$

$$\sigma_Z^2(t) = \left(\frac{1-\Sigma_Z(t)}{1-\Sigma_I(t)}\right)^2 \sigma_I^2(t) + \left(\frac{1-\Sigma_Z(t)}{1-\Sigma_M(t)}\right)^2 \sigma_M^2(t), \tag{11}$$

$$\sigma_Z(t)d\tilde{N}_t = \left(\frac{1-\Sigma_Z(t)}{1-\Sigma_I(t)}\right)\sigma_I(t)d\tilde{N}_t^{(1)} + \left(\frac{1-\Sigma_Z(t)}{1-\Sigma_M(t)}\right)\sigma_M(t)d\tilde{N}_t^{(2)}, \tag{12}$$

$\Sigma_Z(t) = c^2 + \int_0^t \sigma_Z^2(s)ds$, $\Sigma_Z(1) = 1$, and $\tilde{N}$ is a $(\mathscr{F}_t^I)$-Brownian motion and $\tilde{N}_t^{(1)}$ and $\tilde{N}_t^{(2)}$ are respectively the innovation processes of B^I and B^M considering the insider's filtration.

Theorem: Insider's Maximisation Problem

Assuming the insider receives a signal Z that converges to the true value of the asset $Z_1 = \eta$, the function H is a strictly increasing function with respect to the demand, and w is a nonnegative function. Then, for suitable values of r's, any admissible strategy for the insider such that $H(1, X_1, X_1^M) = \eta$, $\gamma_0 \equiv 0$, and $\gamma_1 \equiv 0$ is optimal.

Linear Pricing Rule

Since the only requirement about the functional form of α is that the price process is such that $H(1,X_1,X_1^M) = \eta$, we can without loss of generality assume that

$$\alpha_t = \alpha_0(t) + \alpha_1(t)X_t + \alpha_2(t)X_t^M + \alpha_3(t)\eta \tag{13}$$

Rationality Condition

If

$$\alpha_i(t) = -\beta_i(t)\alpha_3(t)\ \forall\ i = 0,1,2, \tag{14}$$

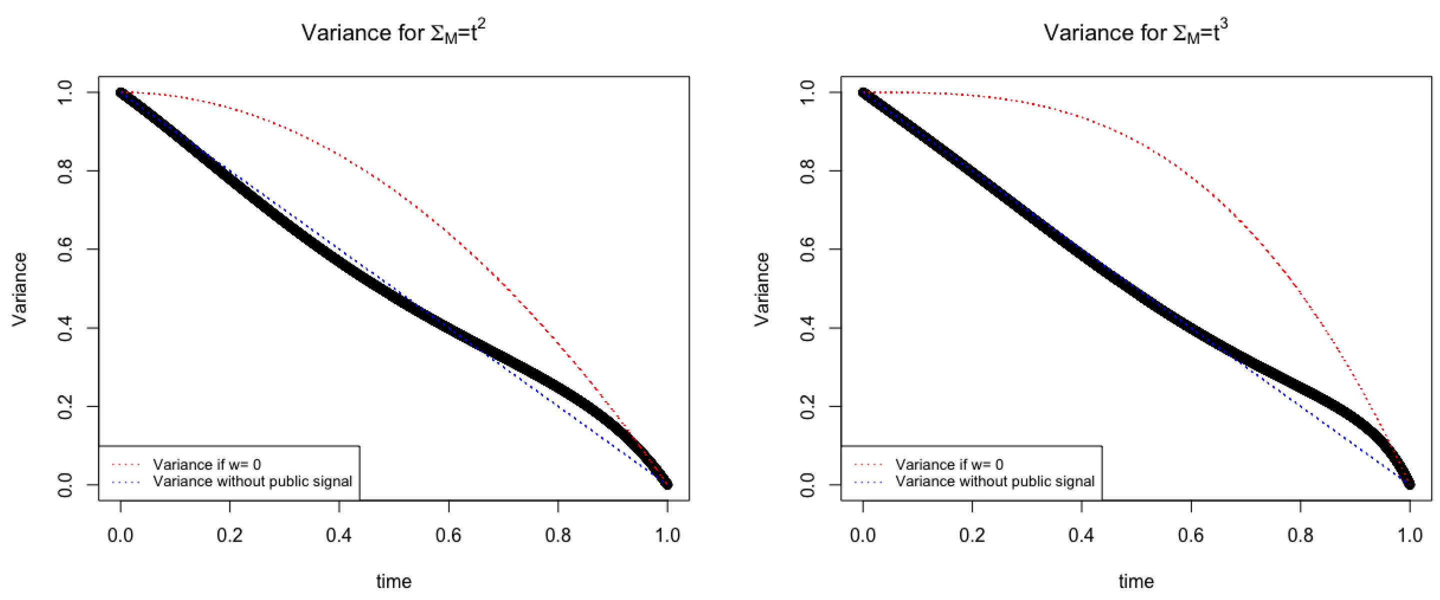
then the price process is a martingale given by the following SDE:

$$dS_t = \beta_1(t)w(t)dN_t^{(1)} + \beta_2(t)\sigma_M(t)dN_t^{(2)} \tag{15}$$

Where $N^{(1)}$ is the innovation process of B and $N^{(2)}$ is the innovation process of B^M considering the market maker's filtration.

Price Process in the Market Maker's Filtration

As $S_t = \mathbb{E}\left[\eta|\mathscr{F}_t^M\right] = \mathbb{E}\left[\mathbb{E}\left[\eta|\mathscr{F}_t^I\right]|\mathscr{F}_t^M\right] = \mathbb{E}\left[Z_t|\mathscr{F}_t^M\right]$ we can project both S and Z into the market maker's filtration to show that the martingale given by equation (15) is such that its variation goes from 1 to 0 in the trading period as we in the following example:



Furthermore, w and β_1 are an overparameterization so we set $\beta_1 \equiv 1$. We show that $\lim_{t\to 1}\beta_2(t) = 1$ and $X_t \to 0$ a.s.. Hence, β_0 is such that $\lim_{t\to 1}\beta_0(t) = 0$ to have

$$\lim_{t\to 1} S_t = 0 + 1\times 0 + 1 \times \eta = \eta \tag{16}$$

because by equation (2) is a bridge converging to η.

Therefore, we have all the equilibrium conditions being satisfied.

Conclusions

First, it is interesting to notice that the variance of the price process is always greater than the case when the market maker has no access to a public signal. This means that it goes faster to zero in this model when it is closer to the end of trading period showing a high-frequency trading during that period. It is also interesting to notice that even though the price is still a bridge in this model, it is not solely due to the insider. As $t \to 1$ the market maker gives more weight to the information coming from its own signal than the one from the insider coming from the demand.

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Suggestions and comments are welcome
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