Appendix to Deflation: Prevention and Cure
A Simple Model of Aggregate Demand

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Households

First, I consider the individual consumption behaviour of households that operate in perfect financial markets and can borrow freely against the security of their future disposable labour income. In the body of the paper these are called permanent income consumers. Next aggregate consumption is derived on the assumption that all consumers are permanent income consumers. The final step is to derive aggregate consumption behaviour when a constant fraction of households is constrained to consume their disposable labour income in each period.

A representative member of the generation born in period *t-s,* $s \geq 0$, maximizes at time t the expected utility of life-time sequences of consumption of domestic goods, imported goods and real money balances. The objective functional $v(t-s,t)$ is given in equations (1), (2) and (3). The period household budget identity is given in equations (8) or (9) and the household solvency constraint in (10). There is a constant (that is, both time-independent and age-independent) probability of survival till the next period, given by $\frac{1}{1+d}$, $d \geq 0$; d can be thought of as the death rate. The pure subjective rate of time preference is $r > 0$. Uncertainty about the time of death is the only uncertainty in the model, and its only effect on the objective functional is to raise the effective subjective discount factor from $1+r$ to $(1+r)(1+d)$. Period felicity of a household born at time $t-s,\ s \geq 0$ and surviving at time t is given by a constant intertemporal elasticity of substitution function of $z(t-s,t)$, where z is a CES function of a composite consumption good, $\tilde{c}(t-s,t)$,

and real money balances, $M(t-s,t)/\tilde{P}(t)$. The nominal stock of base money held by the household is $M(t-s,t)$ and $\tilde{P}$ is the money price of the composite consumption good. The composite consumption good is itself a CES function of consumption of domestic goods, $c_H(t-s,t)$ (with money price P) and consumption of imported goods $c_F(t-s,t)$ (with money price SP^*, where P^* is the foreign currency price of imports and S is the nominal spot exchange rate). The intertemporal substitution elasticity is denoted s.

For $r > 0; d \geq 0; s > 0, s \neq 1$

$$v(t-s,t) = \sum_{j=t}^{\infty} \left(\frac{1}{(1+r)(1+d)} \right)^{j-(t-1)} \frac{s}{s-1} z(t-s,j)^{\frac{s-1}{s}} \tag{1}$$

For $s = 1$

$$v(t,s) = \sum_{j=t}^{\infty} \left(\frac{1}{(1+r)(1+d)} \right)^{j-(t-1)} \ln z(t-s,j)$$

For $0 < a < 1;\ j > 0;\ j \neq 1$

$$z(t-s,j) = \left[a^{\frac{1}{j}} \tilde{c}(t-s,j)^{\frac{j-1}{j}} + (1-a)^{\frac{1}{j}} \left(\frac{M(t-s,j)}{\tilde{P}(j)} \right)^{\frac{j-1}{j}} \right]^{\frac{j}{j-1}}$$

(2)[1]

For $j = 1$

$$z(t-s,j) = \tilde{c}(t-s,j)^{a} \left(\frac{M(t-s,j)}{\tilde{P}(j)} \right)^{1-a}$$

[1] There would be no substantive difference to any of the results if the real money balances in the period felicity function were end-of-period rather than beginning-of-period money balances, that is, if

For $0 < h \le 1; q > 0; q \ne 1$

$$\tilde{c}(t-s, j) = \left(h^{\frac{1}{q}} c_H(t-s, j)^{\frac{q-1}{q}} + (1-h)^{\frac{1}{q}} c_F(t-s, j)^{\frac{q-1}{q}} \right)^{\frac{q}{q-1}} \tag{3}$$

For $q = 1$

$$\tilde{c}(t-s, j) = c_H(t-s, j)^h \, c_F(t-s, j)^{1-h}$$

$$\begin{aligned} \tilde{P} &= \left(h P^{1-q} + (1-h)\left(SP^*\right)^{1-q} \right)^{\frac{1}{1-q}} && \text{if } q \ne 1 \\ \tilde{P} &= P^h \left(SP^*\right)^{1-h} && \text{if } q = 1 \end{aligned} \tag{4}$$

Households have access to four stores of value, domestic base money, with nominal interest rate i_M , one-period debt denominated in domestic currency, $B(t-s,t)$, with nominal interest rate i, one-period debt denominated in foreign currency, $B^*(t-s,t)$, with nominal interest rate i^* and equity, claims to the domestic capital stock. The quantity of domestic capital held by the household is denoted $K(t-s,t)$ and the domestic currency price of a claim to one unit of domestic capital is P_K. A unit of capital purchased in period t entitles one to a dividend of $\Omega(t+1)$ in nominal terms in period *t+1*, plus the right to resell the undepreciated part of the capital at unit price $P_K(t+1)$. The proportional rate of depreciation of the capital stock is x . The rate of inflation of the composite good price $\tilde{p}$ is defined as $\tilde{p}(j) \equiv \frac{\tilde{P}(j)}{\tilde{P}(j-1)} - 1$, and the one-period domestic real interest rate in terms of the

equation (2) were replaced by

$$z(t-s, j) = \left[a^{\frac{1}{j}} \tilde{c}(t-s, j)^{\frac{j-1}{j}} + (1-a)^{\frac{1}{j}} \left(\frac{M(t-s, j+1)}{\tilde{P}(j)} \right)^{\frac{j-1}{j}} \right]^{\frac{j}{j-1}}.$$

composite consumption good $\tilde{r}$ is defined as $1+\tilde{r}(j) \equiv \frac{1+i(j)}{1+\tilde{p}(j)}$. We also define the real values (in terms of the composite consumption good) of the four asset stocks as $\tilde{m}(t-s, j) \equiv \frac{M(t-s, j)}{\tilde{P}(j-1)}$, $\tilde{b}(t-s, j) \equiv \frac{B(t-s, j)}{\tilde{P}(j-1)}$, $\tilde{b}^*(t-s, j) \equiv \frac{S(j-1)B^*(t-s, j)}{\tilde{P}(j-1)}$ and $\tilde{k}(t-s, j) \equiv \frac{P_K(j-1)K(t-s, j)}{\tilde{P}(j-1)}$. Real household financial wealth $\tilde{a}(t-s, j)$ is defined as

$$\tilde{a}(t-s, j) \equiv \left[1+r(j)\right](1+d)\left(\tilde{m}(t-s, j)+\tilde{b}(t-s, j)+\tilde{b}^*(t-s, j)+\tilde{k}(t-s, j)\right) \quad (5)$$

Nominal wage income is $W(t-s, j)$ and nominal taxes $T(t-s, j)$; real wage income in terms of the composite consumption good is $\tilde{w}(t-s, j) \equiv \frac{W(t-s, j)}{\tilde{P}(j)}$ and real taxes is $\tilde{t}(t-s, j) \equiv \frac{T(t-s, j)}{\tilde{P}(j)}$. We assume that the expected financial rates of return on all non-monetary assets are equalised, that is,

$$1+i(j+1) = [1+i^*(j+1)]\frac{E(j+1)}{E(j)} \quad (6)$$

$$1+i(j+1) = \frac{1}{P_K(j)}\left((1-x)P_K(j+1)+\Omega(j+1)\right) \quad (7)$$

We also assume that $(1+d)^{-1}$ is not only the individual period probability of survival, but also the fraction of each age cohort, and therefore of the population as a whole, that survives till the next period. With efficient annuities markets, the gross nominal rate of return earned on government debt by survivors is therefore $(1+i)(1+d)$, *and mutatis mutandis* for all financial assets.

The household budget identity is

$$
\begin{aligned}
&M(t-s,t+1)+B(t-s,t+1)+S(t)B^{*}(t-s,t+1)+P_K(t)K(t-s,t+1)\\
&\equiv(1+d)\begin{pmatrix}[1+i_M(t)]M(t-s,t)+[1+i(t)]B(t-s,t)\\ +[1+i^{*}(t)]S(t)\,B^{*}(t-s,t)+[\Omega(t)+\left(1-h\right)P_K(t)\,]K(t-s,t)\end{pmatrix}\\
&+W(t-s,t)-T(t-s,t)-P(t)c_H(t-s,t)-S(t)\,P^{*}(t)c_F(t-s,t)
\end{aligned}
\tag{8}
$$

or, using equations (5), (6) and (7),

$$
\begin{aligned}
&\tilde{a}(t-s,\ j+1)\\
&\equiv[1+\tilde{r}(j\ +1)](1+d)\begin{pmatrix}\tilde{a}(t-s,j)+\tilde{w}(t-s,j)-\tilde{t}(t-s,j)-\dfrac{P(j)}{\tilde{P}(j)}c_H(t-s,j)\\ -\dfrac{S(j)P^{*}(j)}{\tilde{P}(j)}c_F(t-s,j)-\left(i(j)-i_M(j)\right)(1+d)\dfrac{M(t-s,j)}{\tilde{P}(j)}\end{pmatrix}
\end{aligned}
\tag{9}
$$

$$
\lim_{j\to\infty}\tilde{\Xi}(j,t)\tilde{a}(t-s,j)=0 \tag{10}
$$

$$
\begin{aligned}
\tilde{\Xi}(j,t)&=\prod_{\ell=t}^{j}\frac{1}{[1+\tilde{r}(\ell)](1+d)} && \text{for } j>t\\
&=1 && \text{for } j=t
\end{aligned}
\tag{11}
$$

The solvency constraint and the budget identity imply that

$$
\begin{aligned}
&\tilde{a}(t-s,t)+\sum_{j=t}^{\infty}\tilde{\Xi}(j,t)\left(\tilde{w}(t-s,j)-\tilde{t}(t-s,j)\right)\\
&\equiv\sum_{j=t}^{\infty}\tilde{\Xi}(j,t)\begin{bmatrix}\dfrac{P(j)}{\tilde{P}(j)}c_H(t-s,j)+\dfrac{S(j)P^{*}(j)}{\tilde{P}(j)}c_F(t-s,j)\\ +\left(\dfrac{i(j)-i_M(j)}{1+\tilde{p}(j)}\right)(1+d)\tilde{m}(t-s,j)\end{bmatrix}
\end{aligned}
\tag{12}
$$

From the first-order conditions we obtain the following:

For $q\neq1$,

$$
c_H=h\left(\frac{P}{\tilde{P}}\right)^{-q}\tilde{c}=h\left[h+(1-h)\left(\frac{SP^{*}}{P}\right)^{1-q}\right]^{\frac{q}{1-q}}\tilde{c} \tag{13}
$$

$$
c_F=(1-h)\left(\frac{SP^{*}}{\tilde{P}}\right)^{-q}\tilde{c}=(1-h)\left[h\left(\frac{P}{SP^{*}}\right)^{1-q}+1-h\right]^{\frac{q}{1-q}}\tilde{c} \tag{14}
$$

For $q=1$,

$$c_H = h\left(\frac{P}{\tilde{P}}\right)^{-1}\tilde{c} = h\left(\frac{P}{SP^*}\right)^{h-1}\tilde{c} \tag{15}$$

$$c_F = (1-h)\left(\frac{SP^*}{\tilde{P}}\right)^{-1}\tilde{c} = (1-h)\left(\frac{SP^*}{P}\right)^{-h}\tilde{c} \tag{16}$$

$$\frac{\tilde{m}(t-s,j)}{1+\tilde{p}(j)} = \left(\frac{1-a}{a}\right)\left(\frac{1}{(1+d)\left[i(j)-i_M(j)\right]}\right)^{j}\tilde{c}(t-s,j) \tag{17}$$

$$\frac{\tilde{c}(t-s,\ j+1)}{\tilde{c}(t-s,\ j)} = \left(\frac{1+\tilde{r}(j+1)}{1+r}\right)^{s}\left(\frac{1+\left(\frac{1-a}{a}\right)\left(\frac{1}{(1+d)\left[i(j+1)-i_M(j+1)\right]}\right)}{1+\left(\frac{1-a}{a}\right)\left(\frac{1}{(1+d)\left[i(j)-i_M(j)\right]}\right)}\right)^{\frac{s-j}{j-1}} \tag{18}$$

Together with equations (5) and (9), this implies the following consumption function for households born at time $t-s$ that survive till time t:

$$\tilde{c}(t-s,t) = \tilde{m}(t)\left(\tilde{a}(t-s,t)+\tilde{h}(t-s,t)\right) \tag{19}$$

$$\tilde{h}(t-s,t) = \sum_{j=t}^{\infty}\tilde{\Xi}(j,t)[\tilde{w}(t-s,j)-\tilde{t}(t-s,j)] \tag{20}$$

$$\tilde{m}(t) = \left\{\sum_{j=t}^{\infty}\left[\begin{array}{l}\prod_{\ell=t}^{j}\left(\frac{1}{[1+\tilde{r}(\ell)](1+d)}\right)\left[1+\left(\frac{1-a}{a}\right)\left(\left(i(j)-i_M(j)\right)(1+d)\right)^{1-j}\right]\\ *\prod_{\ell=t}^{j}\left(\frac{1+\tilde{r}(\ell)}{1+r}\right)^{s}\left[\frac{1+\left(\frac{1-a}{a}\right)\left(\left(i(\ell)-i_M(\ell)\right)(1+d)\right)^{1-j}}{1+\left(\frac{1-a}{a}\right)\left(\left(i(\ell-1)-i_M(\ell-1)\right)(1+d)\right)^{1-j}}\right]^{\frac{s-j}{j-1}}\end{array}\right]\right\}^{-1} \tag{21}$$

If $s = j = 1$, the expression for the marginal propensity to consume out of comprehensive wealth simplifies to:

$$\tilde{m}(t) = a\left(\frac{(1+r)(1+d)-1}{(1+r)(1+d)}\right) \tag{22}$$

Aggregation

Normalise population and labour force size at time 0 to be $L(0)=1$. To every person alive in period t $b \geq 0$ children are born. The size of the surviving cohort at time t that was born at time $t-s,\ s\geq 0$ is $bL(t-s)\left(\frac{1}{1+d}\right)^{s} = b(1+b)^{t-s}\left(\frac{1}{1+d}\right)^{t}$.[2]

For any individual agent's stock or flow variable, $Y(t-s,t)$, we define the corresponding population aggregate $Y(t)$ as follows:

$$\begin{aligned} Y(t) &\equiv \sum_{s=1}^{\infty} b(1+b)^{t-s}\left(\frac{1}{1+d}\right)^{t} Y(t-s,t) && b>0 \\ &\equiv Y(0,t)\left(\frac{1}{1+d}\right)^{t} && b=0 \end{aligned} \tag{23}$$

We also assume that each surviving household, regardless of age, earns the same wage income and pays the same taxes, that is,

$$W(t-s,t)=W(t), \qquad s\geq 0 \tag{24}$$

and

$$T(t-s,t)=T(t), \qquad s\geq \overline{0} \tag{25}$$

It follows that each surviving household has the same human wealth:

$$\tilde{h}(t-s,t)=\tilde{h}(t) \qquad s\geq 0 \tag{26}$$

Finally, we assume that people are born with zero non-human wealth, that is,

$$\tilde{a}(t,t)=0 \tag{27}$$

It follows that aggregate consumption is given by

For $q\neq 1$,

$$c_H = h\left(\frac{P}{\tilde{P}}\right)^{-q}\tilde{c} = h\left[h+(1-h)\left(\frac{SP^*}{P}\right)^{1-q}\right]^{\frac{q}{1-q}}\tilde{c} \tag{28}$$

---

2 $L(t+1)=\left(\frac{1+b}{1+d}\right)L(t)$.

$$c_F = (1-h)\left(\frac{SP^*}{\tilde{P}}\right)^{-q}\tilde{c} = (1-h)\left[h\left(\frac{P}{SP^*}\right)^{1-q} + 1 - h\right]^{\frac{q}{1-q}}\tilde{c} \tag{29}$$

For $q = 1$,

$$c_H = h\left(\frac{P}{\tilde{P}}\right)^{-1}\tilde{c} = h\left(\frac{P}{SP^*}\right)^{h-1}\tilde{c} \tag{30}$$

$$c_F = (1-h)\left(\frac{SP^*}{\tilde{P}}\right)^{-1}\tilde{c} = (1-h)\left(\frac{SP^*}{P}\right)^{-h}\tilde{c} \tag{31}$$

$$\tilde{c}(t) = \tilde{m}(t)\left[\tilde{a}(t) + \tilde{h}(t)\right] \tag{32}$$

$$\tilde{a}(t+1) \equiv [1+\tilde{r}(t+1)]\left(\tilde{a}(t) + \tilde{w}(t) - \tilde{t}(t) - \tilde{c}(t) - \left(i(t) - i_M(t)\right)\frac{\tilde{m}(t)}{1+\tilde{p}(t)}\right) \tag{33}$$

$$\tilde{h}(t) = \sum_{j=t}^{\infty}\prod_{\ell=t}^{j}\left(\frac{1}{[1+\tilde{r}(\ell)][1+b]}\right)\left[\tilde{w}(j) - \tilde{t}(j)\right] \tag{34}$$

$$\frac{\tilde{m}(t)}{1+\tilde{p}(t)} = \left(\frac{1-a}{a}\right)\left(\frac{1}{i(t) - i_M(t)}\right)^{j}\tilde{c}(t) \tag{35}$$

$$\begin{aligned}\tilde{a}(t) &\equiv [1+\tilde{r}(t)]\left(\tilde{m}(t) + \tilde{b}(t) + \tilde{b}^*(t) + \tilde{k}(t)\right) \\ &= \frac{[1+i(t)]\left(M(t) + B(t)\right) + [1+i^*(t)]S(t)B^*(t) + \left[(1-x)P_K(t) + \Omega(t)\right]K(t)}{\tilde{P}(t)}\end{aligned} \tag{36}$$

, where M is the aggregate nominal base money stock, B the aggregate stock of non-monetary nominal government debt, B^* the aggregate stock of net foreign assets (denominated in foreign currency) held by the private sector and K the domestic capital stock. As we assume that the physical resources owned by households consist of their labour endowment and the domestic capital stock, which are both used exclusively in the production of domestic output, it is helpful, in order to identify the effect of changes in international relative prices, to rewrite equations (32) to (36) using domestic output rather than the composite consumption good as the numéraire.

For any stock variable, e.g. the capital stock, we define

$k(t) \equiv \dfrac{P_K(t-1)K(t)}{P(t-1)} = \dfrac{\tilde{P}(t-1)}{P(t-1)}\tilde{k}(t)$. For any flow variable, e.g. consumption of the composite commodity, we define $c(t) \equiv \dfrac{\tilde{P}(t)}{P(t)}\tilde{c}(t)$. The inflation rate of domestic output is p, that is, $1+p(t) \equiv \dfrac{P(t)}{P(t-1)}$. The real interest rate in terms of domestic output is r, that is, $1+r(t) \equiv [1+i(t)]\dfrac{P(t-1)}{P(t)} = \dfrac{1+i(t)}{1+p(t)} = \left[1+\tilde{r}(t)\right]\left[\dfrac{1+\tilde{p}(t)}{1+p(t)}\right]$.

This yields the following aggregate consumption behaviour:

For $q \neq 1$,

$$c_H = h\left(\frac{P}{\tilde{P}}\right)^{1-q} c = h\left[h + (1-h)\left(\frac{SP^*}{P}\right)^{1-q}\right]^{-1} c \tag{37}$$

$$c_F = (1-h)\left(\frac{SP^*}{\tilde{P}}\right)^{-q}\frac{P}{\tilde{P}}c = (1-h)\left[h\left(\frac{SP^*}{P}\right)^{q} + (1-h)\frac{SP^*}{P}\right]^{-1} c \tag{38}$$

For $q = 1$,

$$c_H = hc \tag{39}$$

$$c_F = (1-h)\left(\frac{SP^*}{P}\right)^{-1} c \tag{40}$$

$$c(t) = m(t)\left[a(t) + h(t)\right] \tag{41}$$

$$m(t) = \left\{\sum_{j=t}^{\infty}\left[\begin{array}{l}\prod_{\ell=t}^{j}\left(\dfrac{1}{[1+r(\ell)](1+d)}\right)\left[1+\left(\dfrac{1-a}{a}\right)\left(\left(i(j)-i_M(j)\right)(1+d)\right)^{1-j}\right] \\ *\prod_{\ell=t}^{j}\left(\dfrac{1+r(\ell)}{1+r}\right)^{s}\left[\dfrac{1+\left(\dfrac{1-a}{a}\right)\left(\left(i(\ell)-i_M(\ell)\right)(1+d)\right)^{1-j}}{1+\left(\dfrac{1-a}{a}\right)\left(\left(i(\ell-1)-i_M(\ell-1)\right)(1+d)\right)^{1-j}}\right]^{\frac{s-j}{j-1}}\end{array}\right]\right\}^{-1} \tag{42}$$

$$a(t+1) \equiv [1+r(t+1)]\left(a(t) + w(t) - t(t) - c(t) - \left(i(t) - i_M(t)\right)\frac{m(t)}{1+p(t)}\right) \tag{43}$$

$$h(t)=\sum_{j=t}^{\infty}\prod_{\ell=t}^{j}\left(\frac{1}{[1+r(\ell)][1+b]}\right)[w(j)-t(j)] \tag{44}$$

$$\frac{m(t)}{1+p(t)}=\left(\frac{1-a}{a}\right)\left(\frac{1}{i(t)-i_M(t)}\right)^j c(t) \tag{45}$$

$$\begin{aligned}a(t) &\equiv [1+r(t)]\big(m(t)+b(t)+b^*(t)+k(t)\big)\\ &=\frac{[1+i(t)]\big(M(t)+B(t)\big)+[1+i^*(t)]S(t)B^*(t)+\big[(1-x)P_K(t)+\Omega(t)\big]K(t)}{P(t)}\end{aligned} \tag{46}$$

Government

The government (that is, the consolidated central bank and general government sector) spends g_H on domestic output, g_F on foreign output, raises taxes T in nominal terms, issues base money with a nominal interest rate i_M, domestic currency debt with a nominal interest rate i and holds foreign exchange reserves D^*. We shall call $S(t)D^*(t)-B(t)-M(t)$ the financial net worth of the government.

$$\begin{aligned}&M(t+1)+B(t+1)-S(t)D^*(t+1)\\ &\equiv P(t)g_H(t)+S(t)P(t)g_F(t)-T(t)\\ &+[1+i_M(t)]M(t)+[1+i(t)]B(t)-S(t)[1+i^*(t)]D^*(t)\end{aligned} \tag{47}$$

Letting $\tilde{g}\equiv\frac{Pg_H+SP^*g_F}{\tilde{P}}$ and $\tilde{d}^*\equiv\frac{SD^*}{\tilde{P}}$, we can rewrite equation (47) as

$$\begin{aligned}&\tilde{b}(t+1)+\tilde{m}(t+1)-\tilde{d}^*(t+1)\\ &\equiv[1+\tilde{r}(t)][\tilde{b}(t)+\tilde{m}(t)-\tilde{d}^*(t)]+\tilde{g}(t)-\tilde{t}(t)+\frac{[i_M(t)-i(t)]}{1+\tilde{p}(t)}\tilde{m}(t)\end{aligned}.$$

Slightly more useful, for when below we ‘internalise’ the government’s intertemporal budget constraint by substituting it into the household intertemporal budget constraint is the following representation:

$$\tilde{b}(t+1)-\tilde{d}^*(t+1)\equiv[1+\tilde{r}(t)][\tilde{b}(t)-\tilde{d}^*(t)] \\ +\tilde{g}(t)-\tilde{t}(t)+\left(\frac{i_M(t)}{1+\tilde{p}(t)}\right)\tilde{m}(t)-\frac{\Delta M(t+1)}{\tilde{P}(t)} \tag{48}$$[3]

With the usual no-Ponzi finance solvency constraint:

$$\lim_{j\to\infty}\prod_{s=t}^{j}\left(\frac{1}{1+\tilde{r}(s)}\right)\left(\tilde{b}(j)-\tilde{d}^*(j)\right)=0 \tag{49}$$

this yields the government's intertemporal budget constraint:

$$\begin{aligned}[1+\tilde{r}(t)]\left(\tilde{b}(t)-\tilde{d}^*(t)\right)&=\frac{[1+i(t)]B(t)-\left[1+i^*(t)\right]S(t)D^*(t)}{P(t)} \\ &=\sum_{j=t}^{\infty}\prod_{s=t}^{j}\left(\frac{1}{1+\tilde{r}(s)}\right)\left(\tilde{t}(j)-\tilde{g}(j)-\frac{i_M(j)}{1+\tilde{p}(j)}\tilde{m}(j)+\frac{\Delta M(j+1)}{\tilde{P}(j)}\right)\end{aligned} \tag{50}$$

We assume that government spending on goods and services is determined, analogously to private consumption. With $0<\hbar<1;\ \hat{q}>0$,

$$\begin{aligned}g^H&=\hbar\left(\frac{P}{\tilde{P}}\right)^{-\hat{q}}\tilde{g}=\hbar\left[\hbar+(1-\hbar)\left(\frac{SP^*}{P}\right)^{1-\hat{q}}\right]^{\frac{\hat{q}}{1-\hat{q}}}\tilde{g} && \text{if } \hat{q}\neq 1 \\ &=\hbar\left(\frac{P}{\tilde{P}}\right)^{-1}\tilde{g}=\hbar\left(\frac{P}{SP^*}\right)^{\hbar-1}\tilde{g} && \text{if } \hat{q}=1\end{aligned} \tag{51}$$

$$\begin{aligned}g_F&=(1-\hbar)\left(\frac{SP^*}{\tilde{P}}\right)^{-\hat{q}}\tilde{g}=(1-\hbar)\left[\hbar\left(\frac{P}{SP^*}\right)^{1-\hat{q}}+1-\hbar\right]^{\frac{\hat{q}}{1-\hat{q}}}\tilde{g} && \text{if } \hat{q}\neq 1 \\ &=(1-\hbar)\left(\frac{SP^*}{\tilde{P}}\right)^{-1}\tilde{g}=(1-\hbar)\left(\frac{SP^*}{P}\right)^{-\hbar}\tilde{g} && \text{if } \hat{q}=1\end{aligned} \tag{52}$$

As the government's endowment stream (the tax base) consists of domestic output, we assume that government spending decisions can be represented by an exogenous sequence of aggregate public spending measured in domestic output,

[3] $\Delta M(t+1)\equiv M(t+1)-M(t)$

$\{g(j);\ j \geq t\}$ with $g = \frac{\tilde{P}}{P}\tilde{g}$. International relative prices then distribute this aggregate across domestic goods and imports according to equation (51), that is

$$g^H = \hat{h}\left(\frac{P}{\tilde{P}}\right)^{1-\hat{q}} g = \hat{h}\left[\hat{h} + (1-\hat{h})\left(\frac{SP^*}{P}\right)^{1-\hat{q}}\right]^{-1} g \qquad \text{if } \hat{q} \neq 1$$
$$= \hat{h} g \qquad \text{if } \hat{q} = 1 \tag{53}$$

$$g_F = (1-\hat{h})\left(\frac{SP^*}{\tilde{P}}\right)^{-\hat{q}} \frac{P}{\tilde{P}} g = (1-\hat{h})\left(\hat{h}\left(\frac{SP^*}{P}\right)^{\hat{q}} + (1-\hat{h})\frac{SP^*}{P}\right)^{-1} g \qquad \text{if } \hat{q} \neq 1$$
$$= (1-\hat{h})\left(\frac{SP^*}{P}\right)^{-1} g \qquad \text{if } \hat{q} = 1 \tag{54}$$

We can substitute the government's intertemporal budget constraint given in (50) into the private consumption function (41), using (44) and (46). This yields

$$c(t) = m(t)\left\{\begin{aligned} &\frac{\left(\begin{aligned}[1+i(t)]M(t) + [1+i^*(t)]S(t)[B^*(t)+D^*(t)]\\ +\left((1-x)P_K(t)+\Omega(t)\right)K(t)\end{aligned}\right)}{P(t)} \\ &+\sum_{j=t}^{\infty}\left[\prod_{\ell=t}^{j}\left(\frac{1}{[1+r(\ell)](1+b)}\right)w(j) - \prod_{\ell=t}^{j}\left(\frac{1}{1+r(\ell)}\right)g(j)\right] \\ &+\sum_{j=t}^{\infty}\left[\prod_{\ell=t}^{j}\left(\frac{1}{1+r(\ell)}\right)\frac{b}{1+b}\right]t(j) \\ &+\sum_{j=t}^{\infty}\left[\prod_{\ell=t}^{j}\left(\frac{1}{1+r(\ell)}\right)\left(\frac{\Delta M(j+1)}{P(j)} - \frac{i_M(j)m(j)}{1+p(j)}\right)\right] \end{aligned}\right\} \tag{55}$$

Equation (55) represents aggregate consumption behaviour when all consumers are permanent income consumers. If, instead, a fraction l of households always consume their current disposable income (and therefore always possesses zero financial net worth), aggregate consumption would be given by:

$$
c(t) = m(t)\left\{\begin{array}{l}
\dfrac{\left(\begin{array}{r}[1+i(t)]M(t)+[1+i^{*}(t)]S(t)[B^{*}(t)+D^{*}(t)] \\ +\left((1-x)P_K(t)+\Omega(t)\right)K(t)\end{array}\right)}{P(t)} \\
+\sum_{j=t}^{\infty}\left[(1-l)\prod_{\ell=t}^{j}\left(\dfrac{1}{[1+r(\ell)](1+b)}\right)w(j)-\prod_{\ell=t}^{j}\left(\dfrac{1}{1+r(\ell)}\right)g(j)\right] \\
+\sum_{j=t}^{\infty}\left[\prod_{\ell=t}^{j}\left(\dfrac{1}{1+r(\ell)}\right)\left(\dfrac{b+l}{1+b}\right)\right]t(j) \\
+\sum_{j=t}^{\infty}\left[\prod_{\ell=t}^{j}\left(\dfrac{1}{1+r(\ell)}\right)\left(\dfrac{\Delta M(j+1)}{P(j)}-\dfrac{i_M(j)m(j)}{1+p(j)}\right)\right]
\end{array}\right\}
+ l[w(t)-t(t)] \tag{56}
$$

If only permanent income households hold money balances, the demand for money becomes

$$
\begin{gathered}
\frac{m(t)}{1+p(t)} = \left(\frac{1-a}{a}\right)\left(\frac{1}{i(t+1)-i_M(t)}\right)^{j}\left[c(t)-l\left(w(t)-t(t)\right)\right] \\
i \geq i_M
\end{gathered} \tag{57}
$$

Private Investment

A competitive firm hires labour L and invests in a composite investment good $\tilde{i}$ each period to maximise the present discounted value of its future cash-flow, shown in equation (58), subject to the capital stock dynamics given in equation (59). The production function is constant returns to scale in capital, K, and labour, L, has positive but diminishing marginal products of capital and labour, is strictly concave and satisfies the Inada conditions. There are quadratic adjustment costs associated with investment, and the adjustment cost function is constant returns to scale. The money wage is w and $\tilde{P}_I$ is the price of the composite investment good. Θ is the level of total factor productivity and x is the proportional depreciation rate of the capital stock.

$$\sum_{j=t}^{\infty}\prod_{\ell=t}^{j}\left(\frac{1}{1+i(\ell)}\right)\begin{bmatrix} P(j)\left(\Theta(j)F[K(j),L(j)]-\frac{1}{2}v\frac{\tilde{\imath}(j)^2}{K(j)}\right)-w(j)L(j) \\ -\tilde{P}_I(j)\tilde{\imath}(j) \end{bmatrix} \tag{58}$$

$$v>0$$

$$K(t+1)=K(t)\frac{1}{1+x}+\tilde{\imath}(t) \tag{59}$$

$$1\geq x\geq 0$$

$$\begin{gathered}\tilde{\imath}=\left(\bar{h}^{\frac{1}{\bar{q}}}i_H^{\frac{\bar{q}-1}{\bar{q}}}+(1-\bar{h})^{\frac{1}{\bar{q}}}i_F^{\frac{\bar{q}-1}{\bar{q}}}\right)^{\frac{\bar{q}-1}{\bar{q}}} \\ 1>\bar{h}>0;\bar{q}>0;\bar{q}\neq 1 \\ \text{or, if } \bar{q}=1 \\ \tilde{\imath}=i_H^{\bar{h}}i_F^{1-\bar{h}}\end{gathered} \tag{60}$$

$$\begin{aligned}\tilde{P}_I &=\left[\bar{h}P^{1-\bar{q}}+(1-\bar{h})\left(SP^*\right)^{1-\bar{q}}\right]^{\frac{1}{1-\bar{q}}} && \text{if } \bar{q}\neq 1 \\ &=P^{\bar{h}}(SP^*)^{1-\bar{h}} && \text{if } \bar{q}=1\end{aligned} \tag{61}$$

The optimal investment rules are as follows:

$$\begin{aligned}\tilde{\imath}(t) &=\frac{1}{v}\left[\frac{P_K(t)-\tilde{P}_I(t)}{P(t)}\right]K(t) \\ &=\frac{1}{v}\left[\frac{P_K(t)}{P(t)}-\left[\bar{h}+(1-\bar{h})\left(\frac{S(t)P^*(t)}{P(t)}\right)^{1-\bar{q}}\right]^{\frac{1}{1-\bar{q}}}\right]K(t) && \text{if } \bar{q}\neq 1 \\ &=\frac{1}{v}\left[\frac{P_K(t)}{P(t)}-\left(\frac{S(t)P^*(t)}{P(t)}\right)^{1-\bar{h}}\right]K(t) && \text{if } \bar{q}=1\end{aligned} \tag{62}$$

$$\frac{P_K(t)}{P(t)}=\sum_{j=t}^{\infty}\prod_{\ell=t}^{j}\left(\frac{1}{[1+r(\ell)](1+x)}\right)\left[\Theta(j)F_K[K(j),L(j)]+\frac{1}{2}v\left(\frac{\tilde{\imath}(j)}{K(j)}\right)^2\right] \tag{63}$$

and either a competitive demand for labour function:

$$w(j)=P(j)\Theta(j)F_L[K(j),L(j)] \tag{64}$$

or output-determined employment, with the firm taking output as given (demand-determined):

$$y = \Theta F[K, L] \tag{65}$$

Let $i \equiv \dfrac{\tilde{P}_I}{P}\tilde{\imath}$ denote investment measured in units of domestic output, then

$$\begin{aligned}
i(t) &= \frac{1}{v}\left[\frac{P_K(t)}{\tilde{P}_I(t)} - 1\right]K(t) \\
&= \frac{1}{v}\left[\frac{P_K(t)}{P(t)}\left[\hbar + (1-\hbar)\left(\frac{S(t)P^*(t)}{P(t)}\right)^{1-\bar{q}}\right]^{\frac{1}{\bar{q}-1}} - 1\right]K(t) \qquad \text{if } \bar{q} \neq 1 \\
&= \frac{1}{v}\left[\frac{P_K(t)}{P(t)}\left(\frac{S(t)P^*(t)}{P(t)}\right)^{\hbar-1} - 1\right]K(t) \qquad \text{if } \bar{q} = 1
\end{aligned} \tag{66}$$

$$\frac{P_K(t)}{P(t)} = \sum_{j=t}^{\infty}\prod_{\ell=t}^{j}\left(\frac{1}{[1+r(\ell)](1+x)}\right)\left[\Theta(j)F_K[K(j), L(j)] + \frac{1}{2}v\left(\frac{\tilde{\imath}(j)}{K(j)}\right)^2\right] \tag{67}$$

If $\hat{q} \neq 1$

$$\begin{aligned}
i^H &= \hbar\left(\frac{P}{\tilde{P}_I}\right)^{1-\bar{q}} i = \hbar\left[\hbar + (1-\hbar)\left(\frac{SP^*}{P}\right)^{1-\bar{q}}\right]^{-1} i \\
i^F &= (1-\hbar)\left(\frac{SP^*}{\tilde{P}_I}\right)^{-\bar{q}} i = (1-\hbar)\left[\hbar\left(\frac{P}{SP^*}\right)^{\bar{q}} + (1-\hbar)\frac{P}{SP^*}\right]^{-1} i \\
&0 < \hbar < 1; \bar{q} > 0
\end{aligned} \tag{68}$$

Note that $w = \dfrac{wL}{P}$ and $\Omega = P\left(\Theta F[K, L] - \dfrac{1}{2}b\dfrac{\tilde{\imath}(j)^2}{K(j)}\right) - wL$.

Export demand

Without modelling the rest of the world in any detail, we want to specify export demand for domestic output, x, analogously to the home country import demand functions for private consumption, public spending and private investment. We assume that if all factors of production that produce foreign output (and none of the factors of production that produce domestic output) are owned by foreigners, and that aggregate demand in the rest of the world is for a composite commodity whose ideal consumer price index is $\tilde{P}^*$, where

$\tilde{P}^* = \left[h^* P^{*(1-q^*)} + (1-h^*)\left(\frac{P}{S}\right)^{1-q^*} \right]^{\frac{1}{1-q^*}}$ if $q^* \neq 1$ and $\tilde{P}^* = P^{*h^*} (\frac{P}{S})^{1-h^*}$ if $q^* = 1$. Finally, we assume that the small home country takes aggregate rest-of-the-world demand *measured in foreign output*, $f^* > 0$, as given. The export demand for home country output are:

$$\begin{aligned} x &= \left(1-h^*\right)\left[h^*\left(\frac{P}{SP^*}\right)^{q^*} + (1-h^*)\frac{P}{SP^*} \right]^{-1} f^* && \text{if } q^* \neq 1 \\ &= \left(1-h^*\right)\frac{SP^*}{P} f^* && \text{if } q^* = 1 \end{aligned} \tag{69}$$

Completing the model with some simple wage-price dynamics

The paper only considers the demand-side of the model. For completeness a sketch of a fairly New-Keynesian wage-price block is appended here. The results of the paper do not append on this particular view of the determination of output and employment. These would go through even if we assumed competitive labour and output markets and a flexible nominal wage and price of domestic output.[4]

The nominal wage in period *t* is predetermined. Output is demand-determined and employment is determined by output, y, and the production function:

$$e = y = \Theta F\left(K, L\right) \qquad (70)^5$$

Money wage determination follows a two-period staggered overlapping wage setting model in the spirit of Taylor [1979, 1980]. Instead of Taylor's relative *money* wage model, we use Buiter and Jewitt's overlapping *real* wage specification (Buiter and Jewitt [1981]). Half the labour force negotiates a two-period nominal contract wage each period. For instance, in period *t-1,* the nominal contract wage for periods *t*

[4] that is, equation (64) and equation (70), but with a flexible money wage, W and a flexible price of domestic output, *P*.
[5] that is, we use equation (65) rather than equation (64) to determine labour demand.

and *t+1* is negotiated. The nominal contract period negotiated in period *t-1* aims to achieve a given average real wage in periods *t* and *t+1*. That real wage depends on expected demand pressure in the labour market in the next two periods (proxied by the output gap, $y - \bar{y}$ over the life of the contract) and on the real wages achieved under the money wage contracts with which the contract negotiated in period *t-1* overlaps. Let $c(t)$ denote the logarithm of the nominal contract wage negotiated in period *t-1* for periods *t* and *t+1*. Unlike in the rest of the paper, we include the conditional expectation operator e_{t-1} explicitly, to underline the fact that the period *t* nominal contract wage is indeed predetermined.

$$\begin{aligned}
&c(t) - \frac{1}{2} e_{t-1}[\ln \tilde{P}(t) + \ln \tilde{P}(t+1)] \\
&= y\, e_{t-1}\left[y(t) - \bar{y}(t) + y(t+1) - \bar{y}(t+1)\right] + z\, e_{t-1}\left(c(t-1) - \frac{1}{2}[\ln \tilde{P}(t-1) + \ln \tilde{P}(t)] \right) \\
&+ (1-z)\, e_{t-1}\left(c(t+1) - \frac{1}{2}[\ln \tilde{P}(t+1) + \ln \tilde{P}(t+2)] \right) \\
&\qquad y > 0,\ 0 \le z \le 1
\end{aligned} \tag{71}$$

The money wage in period *t*, $w(t)$, is the average of the contract wages negotiated in the previous two periods.

$$\ln w(t) \equiv \frac{1}{2}\left(c(t) + c(t-1) \right) \tag{72}$$

Let $\bar{L}$ be the exogenous labour supply, then capacity output $\bar{y}$ is given by:

$$\bar{y} = \Theta F\left(K, \bar{L}\right) \tag{73}$$

The price of domestic output is determined by the mark-up rule in equation (74) specifying the price of domestic output as a constant proportional mark-up on unit labour cost.

$$P = \Psi\left(\frac{wL}{Y}\right) \qquad \Psi > 1 \tag{74}$$

References

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