From perturbation theory to rigorous axioms: modern paradigm for studying foundations of QFT

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LSE, 27.10.2017

University of York

Outline of the talk

Physics $\cap$ Maths $\cap$ Philosophy

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- I had the pleasure to discuss with him about questions belonging to the intersection of Physics, Maths and Philosophy. In particular, about foundations of QFT.
- In this talk I will present my thoughts about the foundation of QFT from the (Physics ∩ Maths) perspective, but I hope to contribute also to the philosophical debate on this topic.

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- On the other hand, perturbative QFT (pQFT) produces numbers that agree with experiments with a remarkable precision, but its practitioners often are not concerned with mathematical rigor.
- So, is there a tension between axiomatic and perturbative QFT?

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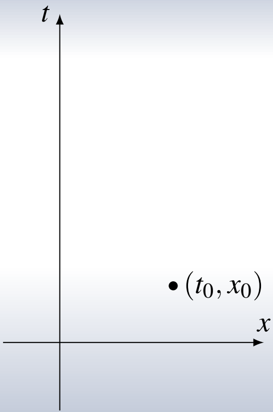
From this point of view, the Haag-Kastler framework is the conceptual foundation, whereas perturbation theory is a tool to produce models that fulfill (weakened version of) the Haag-Kastler axioms.

Space and time



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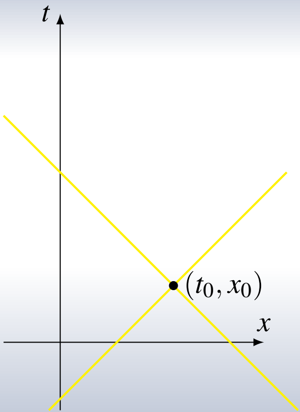

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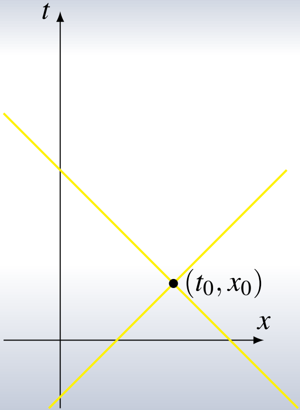
- On the spacetime diagram, we can draw at each point two lines (a cone) representing $|x - x_0| = |t - t_0|$, which limits the region of spacetime accessible from that point. This object is called the lightcone with apex (t_0, x_0).

Space and time



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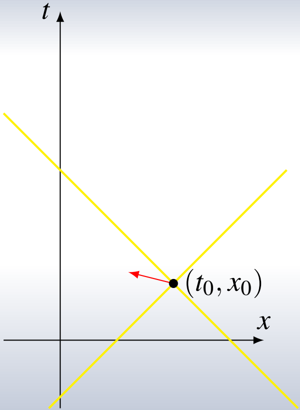

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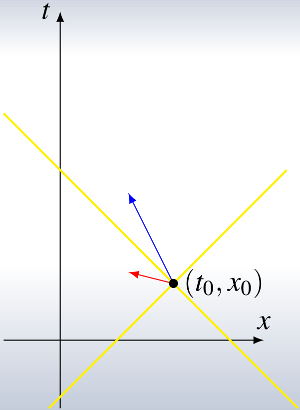

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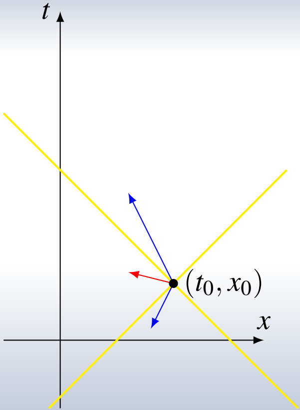

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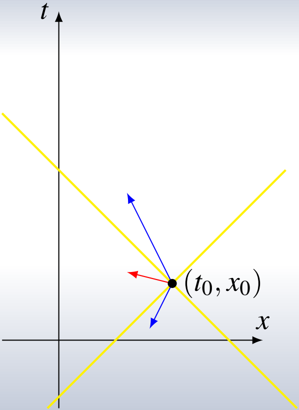

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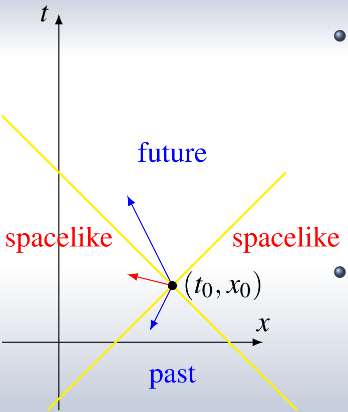

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- This way we divide the spacetime into regions that are in the future of (t_0, x_0), in its past, or are spacelike to (t_0, x_0).

Space and time

- To summarize: in special relativity at each point (t_0, x_0) the lighcone is described by the equation $|x - x_0| = |t - t_0|$, or equivalently $(t - t_0)^2 - (x - x_0)^2 = 0$.

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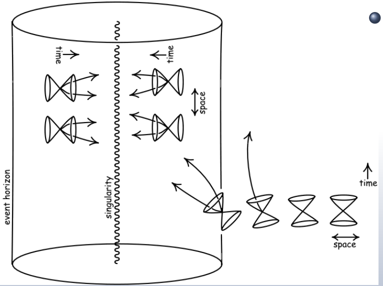

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- in general relativity we want to keep the idea of the lightcone, but the equation describing the lighcone changes from point to point. Lighcones at different points can be tilted and twisted, so observers at different points have different ideas what is future, past or spacelike.

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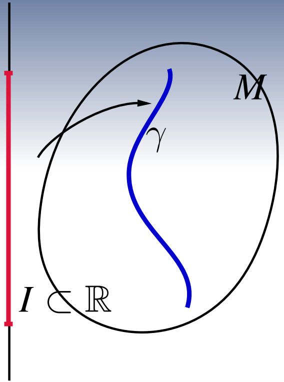

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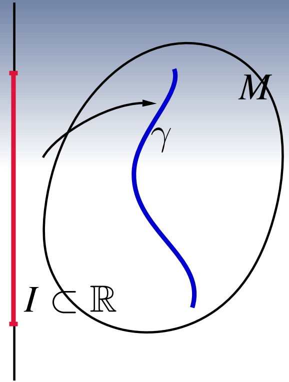

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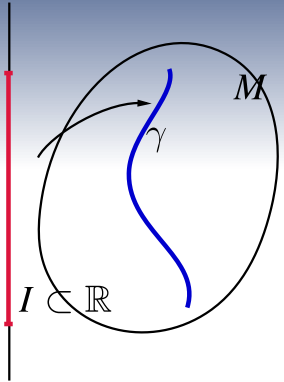

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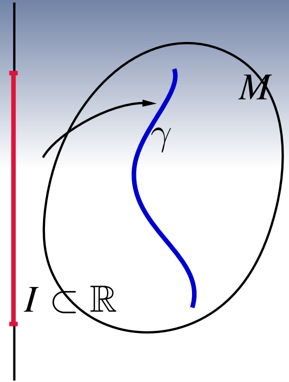

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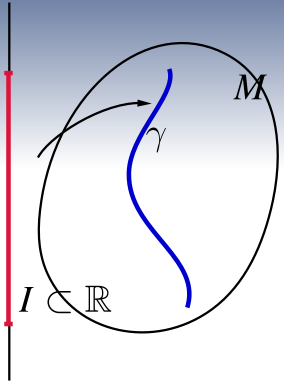

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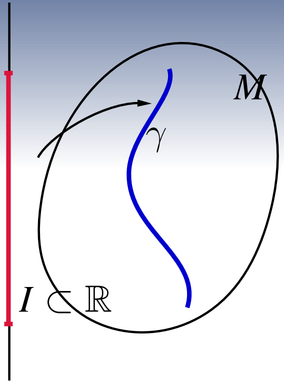

An important principle of general relativity states that observers can move only on timelike curves, so the causal structure given by the metric "tells particles where to go".

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- Idea: implement causality by considering algebras of observables that can be measured in bounded regions of spacetime.

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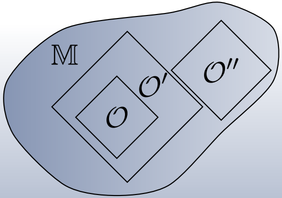

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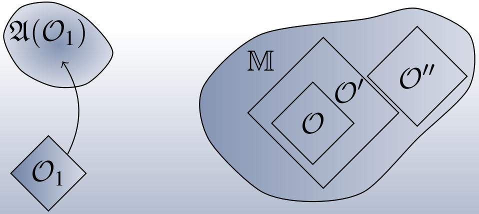

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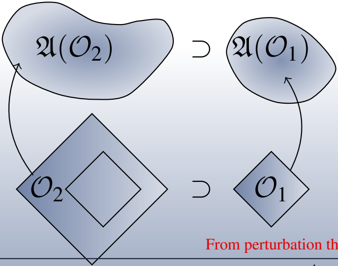

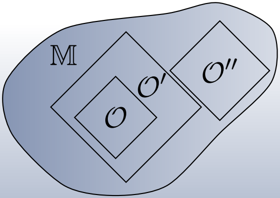

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- **Time slice axiom**: the algebra of a neighbourhood of a Cauchy surface of a given region (Cauchy surface = every inextendible causal curve intersects it exactly once). coincides with the algebra of the full region.
- **Spectrum condition**: for P, the generator of translations $e^{iaP} = U(a)$, $aP = a^\mu P_\mu$, the joint spectrum is contained in the forward lightcone: $\sigma(P) \subset \overline{V}_+$.

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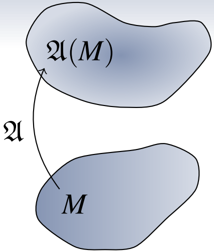

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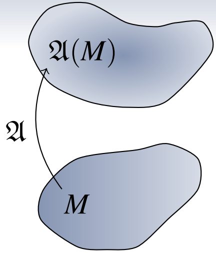

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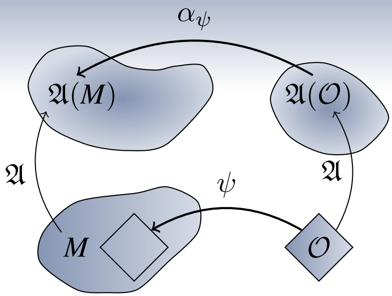

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- The covariance property reads:

$$\alpha_{\psi'} \circ \alpha_\psi = \alpha_{\psi' \circ \psi}\ , \quad \alpha_{\mathrm{id}_M} = \mathrm{id}_{\mathfrak{A}(M)}\ ,$$

Further axioms

One can also include two further axioms which are important in QFT: causality and time-slice axiom.

- **Causality**: If there exist admissible embeddings $\psi_j : M_j \to M, j = 1, 2$, such that the sets $\psi_1(M_1)$ and $\psi_2(M_2)$ are causally separated in M, then:
$$[\alpha_{\psi_1}(\mathfrak{A}(M_1)), \alpha_{\psi_2}(\mathfrak{A}(M_2))] = \{0\},$$
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One can also include two further axioms which are important in QFT: causality and time-slice axiom.

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$$[\alpha_{\psi_1}(\mathfrak{A}(M_1)), \alpha_{\psi_2}(\mathfrak{A}(M_2))] = \{0\},$$
where $[.,.]$ is the commutator of given C^* algebras.
- **Time-slice axiom:** If the morphism $\psi : M \to M'$ is such that $\psi(M)$ contains a Cauchy-surface in M', then α_ψ is an isomorphism.
- A stronger set of axioms that guarantees also operational independence between disjoint spacelike regions, was proposed by Gyeni and Rédei: *Categorial subsystem independence as morphism co-possibility*.

Perturbative algebraic quantum field theory

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- It combines Haag's idea of local quantum physics with methods of perturbation theory.
- The axioms of pAQFT are the same as the axioms of LCQFT, but we replace C^*-algebras with formal power series in $\hbar$ with coefficients in topological $*$-algebras.

Main contributions to pAQFT

- Free theory is obtained by the formal deformation quantization of Poisson (Peierls) bracket: $\star$-product ([Dütsch-Fredenhagen 00, Brunetti-Fredenhagen 00, Brunetti-Dütsch-Fredenhagen 09, ...]).

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- For a review see the book: *Perturbative algebraic quantum field theory. An introduction for mathematicians*, KR, Springer 2016.

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- Dynamics: we start with a Lagrangian L use a fully covariant modification of the Lagrangian formalism, adapted to the infinite dimensional situation.

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- For a given classical observable F one defines the interacting quantum observable F_{int} by using a formula that resembles a Dyson series and goes back to Bogoliubov.
- Constructing F_{int} requires renormalization and is done perturbatively. The method we use is the Epstein-Glaser renormalization. It is a *fully mathematically rigorous method* and it gives the same numerical results as "standard approaches" to renormalization.

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- Under some technical assumptions on M, P admits retarded and advanced Green's functions Δ^{R}, Δ^{A}. They satisfy: $P \circ \Delta^{\mathrm{R/A}} = \mathrm{id}_{\mathcal{D}(M)}$, $\Delta^{\mathrm{R/A}} \circ (P\big|_{\mathcal{D}(M)}) = \mathrm{id}_{\mathcal{D}(M)}$ and

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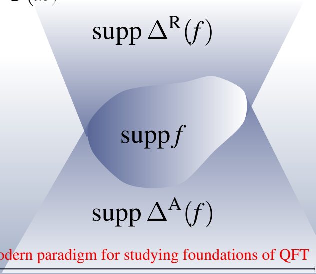

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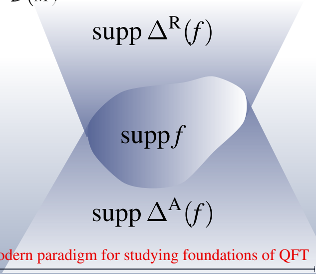

- Their difference is the causal propagator

$$\Delta \doteq \Delta^{\mathrm{R}} - \Delta^{\mathrm{A}}.$$

Poisson structure and the $\star$-product

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- The free QFT is defined as $\mathfrak{A}_0(M) \doteq (\mathcal{F}(M)[[\hbar]], \star, *)$, where $F^*(\varphi) \doteq \overline{F(\varphi)}$ and $\mathcal{F}(M)$ is an appropriate functional space (some WF set conditions on $F^{(n)}(\varphi)$s induced by W).

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$$\mathcal{T}F(\varphi) \doteq \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle F^{(2n)}(\varphi), (\tfrac{\hbar}{2}\Delta^{\mathrm{F}})^{\otimes n} \right\rangle ,$$

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- Define the time-ordered product $\cdot_{\mathcal{T}}$ on $\mathcal{F}_{\text{reg}}(M)[[\hbar]]$ by:

$$F \cdot_{\mathcal{T}} G \doteq \mathcal{T}(\mathcal{T}^{-1}F \cdot \mathcal{T}^{-1}G)$$

Interaction

- We now have an algebraic structure with two products $(\mathcal{F}_{\mathrm{reg}}(M)[[\hbar]], \star, \cdot_{\mathcal{T}})$, where $\star$ is non-commutative, $\cdot_{\mathcal{T}}$ is commutative and they are related by a causal relation:

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- Interaction is a functional $V \in \mathcal{F}_{\text{reg}}(M))$. Using the commutative product $\cdot_{\mathcal{T}}$ we define the S-matrix:
$$\mathcal{S}(\lambda V) \doteq e_{\mathcal{T}}^{i\lambda V/\hbar} = \mathcal{T}(e^{\mathcal{T}^{-1}(i\lambda V/\hbar)})\,,$$
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- Interacting fields are defined on $\mathcal{F}_{\mathrm{reg}}[[\hbar, \lambda]]$ by the formula of Bogoliubov:

$$F_{int} \doteq R_{\lambda V}(F) = -i\hbar \frac{d}{d\mu} \mathcal{S}(\lambda V)^{-1} \star \mathcal{S}(\lambda V + \mu F)\Big|_{\mu=0}$$

Interacting star product

- Using Møller maps $R_{V\lambda}$, one can define the following product on $\mathcal{F}_{\mathrm{reg}}[[\hbar,\lambda]]$:
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- Using $\star_{int}$, we construct the interacting quantum theory functor $\mathfrak{A}_{\lambda V}$.

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- To extend $\mathcal{T}_n$ to arbitrary local functionals we use e.g. the causal approach of Epstein and Glaser.

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- This opens up perspectives for better conceptual understanding of pQFT.
- Recent developments (e.g. Bahns & KR, CMP 2017) show that pAQFT can be used also to obtain non-perturbative results, i.e. to construct AQFT models.

Happy Birthday Miklós!