# Geometric Graph Realizations and Positive Semidefinite Matrix Completion

Monique Laurent

2016 Colloquium in Combinatorics at LSE

Joint with M. E.-Nagy (Budapest) and A. Varvitsiotis (NTU)

# Goal

*Study geometric realizations of weighted graphs in low dimensional spheres*

- Positive semidefinite matrix completion

  and low rank solutions to semidefinite programs

- Geometric graph parameters for a graph $G$:
  - Gram dimension $\mathrm{gd}(G)$
  - Extreme Gram dimension $\mathrm{egd}(G)$

- Link to Colin de Verdière type graph parameters

# Notation

- A symmetric matrix $X$ is **positive semidefinite (psd: $X \succeq 0$)** iff $X$ is the **Gram matrix**: $X = (u_i^\mathsf{T} u_j)$ of some vectors $u_1, \ldots, u_n \in \mathbb{R}^k$. Smallest such $k$ is the rank of $X$.

- $\mathcal{E}_n =$ all $n \times n$ psd matrices with an **all-ones diagonal**, i.e., admitting a Gram representation by **unit** vectors.
  $\mathcal{E}_n$: elliptope

- For a graph $G = (V = [n], E)$
  $\mathcal{E}(G)$: projected elliptope = projection of $\mathcal{E}_n$ onto $\mathbb{R}^E$ =
  all $a = (u_i^\mathsf{T} u_j) \in \mathbb{R}^E$ for some **unit** vectors, i.e.,
  partial matrices $a \in \mathbb{R}^E$ completable to a matrix in $\mathcal{E}_n$.

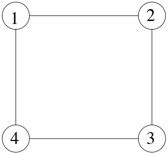
# Low rank solutions to semidefinite programs and matrix completion

# Positive semidefinite matrix completion

Given a partial matrix: $\begin{pmatrix} \mathbf{1} & 0 & * & -1 \\ 0 & \mathbf{1} & 1 & * \\ * & 1 & \mathbf{1} & 0 \\ -1 & * & 0 & \mathbf{1} \end{pmatrix}$

- Can it be completed to a psd matrix?
- What is the **smallest rank** of such a completion?

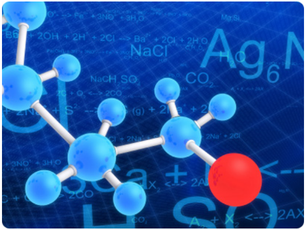
Give an answer depending on structural properties of the **graph of specified entries**:




Graph $C_4$

$\rightsquigarrow$ Gram dimension $\mathrm{gd}(G)$

# Matrix completion in distance geometry




Find the locations of objects in $\mathbb{R}^k$ from partial measurements of mutual distances: Find $u_1, \cdots, u_n \in \mathbb{R}^k$ such that

$$\|u_i - u_j\|^2 = d_{ij} \quad \forall ij \in E.$$

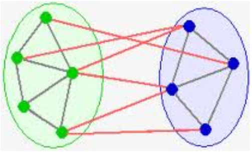
Equivalently: Find a solution of **rank at most k** to the SDP:

$$X \succeq 0, \; X_{ii} + X_{jj} - 2X_{ij} = d_{ij} \quad \forall ij \in E.$$

$\rightsquigarrow$ Smallest $k$ is the Euclidean dimension $\mathrm{ed}(G)$ [Belk-Connelly'07]

$\mathrm{ed}(G) \leq \mathrm{gd}(G) - 1$ [LV 2014]

# Approximating Max-Cut



SDP relaxation: optimize over the elliptope $\mathcal{E}_n$:

$$\max \sum_{ij \in E} w_{ij}(1 - X_{ij})/2 \;\; \text{s.t.} \;\; X \succeq 0, \;\; X_{ii} = 1 \; \forall i \in V$$

$\rightsquigarrow$ 0.878-approximation [Goemans-Williamson'95]

$\rightsquigarrow$ 0.881-approximation if $\operatorname{rank}(X) \leq 3$ [Avidor-Zwick'05]

$\rightsquigarrow$ Linear optimization over the projected elliptope $\mathcal{E}(G)$

# Ground state minimization

Given $n$ particles with paiwise interactions modelled by a graph $G = (V, E)$ and a function $A \in \mathbb{R}^E$.

Minimize the Hamiltonian function:

$$\min \ H(f) = \sum_{ij \in E} A_{ij} \, X_{ij} \ \text{ s.t. } X \in \mathcal{E}_n, \ \mathrm{rank}(X) \leq r.$$

- $r = 1$: Ising model.
- $r = 2$: $XY$ model.
- $r = 3$: Heisenberg model.

**Question:** For which graphs is the rank condition superfluous?

That is, for which graphs $G$ is it true that **every extreme point of $\mathcal{E}(G)$ has a completion to a matrix in $\mathcal{E}_n$ with rank at most** $r$?

$\rightsquigarrow$ Extreme Gram dimension $\mathrm{egd}(G)$

# (Extreme) Gram dimension of a graph

# (Extreme) Gram dimension of graph $G = (V, E)$

## Definition

1. The **Gram dimension** $\mathrm{gd}(G, a)$ of a partial matrix $a \in \mathcal{E}(G)$ is the *smallest rank* of a completion of $a$ in $\mathcal{E}_n$, i.e., *smallest* $k$

   s.t. $a_{ij} = u_i^\mathsf{T} u_j \ \forall ij \in E$ for some **unit** vectors $u_1, \ldots, u_n \in \mathbb{R}^k$.

2. The **Gram dimension** $\mathrm{gd}(G)$ of $G$ is

$$\mathrm{gd}(G) = \max_{a \in \mathcal{E}(G)} \mathrm{gd}(G, a).$$

3. The **extreme Gram dimension** $\mathrm{egd}(G)$ of $G$ is

$$\mathrm{egd}(G) = \max_{a \text{ extreme point of } \mathcal{E}(G)} \mathrm{gd}(G, a) \ \leq \ \mathrm{gd}(G).$$

# Gram dimension and orthogonality rank

$$\mathrm{gd}(G) \geq \mathrm{gd}(G, 0_E)$$

**(Orthogonality rank of $G$)**

*$\mathrm{gd}(G, 0_E)$ is the smallest integer $k$ for which there exist unit vectors $u_1, \cdots, u_n \in \mathbb{R}^k$ such that $u_i^\mathsf{T} u_j = 0 \;\; \forall ij \in E$.*

$\omega(G) \leq \mathrm{gd}(G, 0_E) \leq \chi(G)$

- For $G$ planar: $\mathrm{gd}(G, 0_E) \leq \chi(G) \leq 4$.
- For $G$ planar: Deciding if $\mathrm{gd}(G, 0_E) \leq 3$ is **NP-hard**.
  [Peeters 1997]
- For $G$ cycle: Deciding if $\mathrm{gd}(G, a) \leq 2$ is **NP-hard**.
  [ELV 2013]

# Example: the complete graph $K_n$

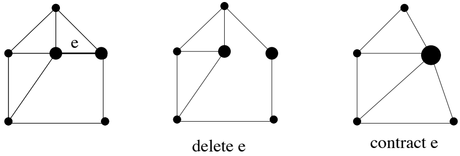
- $\mathrm{gd}(K_n) = n$.
- $\mathrm{egd}(K_n)$ = *max. rank of an extreme point* $X$ *of* $\mathcal{E}_n$

  *[Li-Tam'94]* $= \max\left\{ r : \binom{r+1}{2} \leq n \right\} = \left\lfloor \frac{\sqrt{8n+1}-1}{2} \right\rfloor$.

$\mathrm{egd}(K_2) = 1$, $\mathrm{egd}(K_n) = 2$ if $n \in [3,5]$, $\mathrm{egd}(K_n) = 3$ if $n \in [6,9]$

The elliptope $\mathcal{E}_3$

# Behaviour of $\mathrm{gd}(G)$, $\mathrm{egd}(G)$ under graph minors




## Lemma (LV 2014, ELV 2014)

*Both graph parameters* $\mathrm{gd}(G)$ *and* $\mathrm{egd}(G)$ *are* **minor monotone***:*

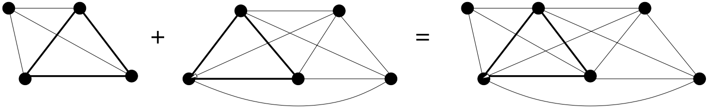
$$\mathrm{gd}(G\backslash e),\ \mathrm{gd}(G/e) \leq \mathrm{gd}(G), \quad \mathrm{egd}(G\backslash e),\ \mathrm{egd}(G/e) \leq \mathrm{egd}(G)$$

*Forbidden minors for small* $k$ *for graphs* $\mathrm{gd}(G) \leq k$? $\mathrm{egd}(G) \leq k$?

## Corollary

*If* $G$ *has* $n \leq \binom{r+1}{2}$ *nodes then* $\mathrm{egd}(G) \leq r$.

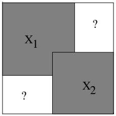
# Behaviour of $\mathrm{gd}(G)$ under clique sums




## Lemma (LV 2014)

*If $G$ is the* **clique $k$-sum** *of $G_1$ and $G_2$ then*

$$\mathrm{gd}(G) = \max\{\mathrm{gd}(G_1), \mathrm{gd}(G_2)\}.$$

The partial matrix: 

has a psd completion of rank $\max\{\mathrm{rank}(X_1), \mathrm{rank}(X_2)\}$.

## Corollary (LV 2014)

*For $G$ chordal, $\mathrm{gd}(G) = \omega(G)$ (maximum clique size).*

# Gram dimension and tree-width

$\mathrm{tw}(G)$ = smallest $k$ s.t. $G$ is contained in a clique sum of $K_{k+1}$'s.

**Lemma (LV 2014)**

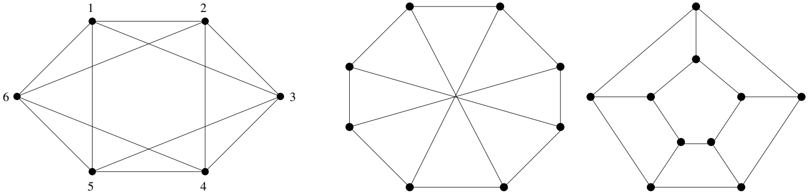
*For any graph $G$,* $\mathrm{gd}(G) \leq \mathrm{tw}(G) + 1$.

**Equality for $\mathrm{tw}(G) \leq 2$:**

$\mathrm{gd}(G) \leq 2 \iff$ no $K_3$ minor $\iff \mathrm{tw}(G) \leq 1$.
$\mathrm{gd}(G) \leq 3 \iff$ no $K_4$ minor $\iff \mathrm{tw}(G) \leq 2$.

[Arnborg, Prokurowski, Corneil 1990] show:

$\mathrm{tw}(G) \leq 3 \iff$ no $K_5, K_{2,2,2}, V_8, C_5 \square K_2$ minor.



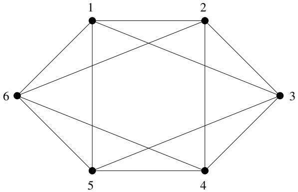
# Graphs with Gram dimension at most 4

[Arnborg, Prokurowski, Corneil 1990]:

$\mathrm{tw}(G) \leq 3 \iff$ no $K_5, K_{2,2,2}, V_8, C_5 \square K_2$ minor.

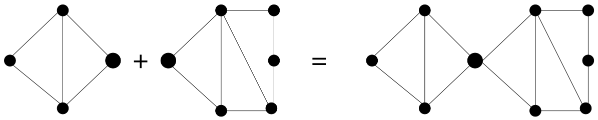
**Theorem (LV 2014)**

$\mathrm{gd}(G) \leq 4 \iff$ *no* $K_5, K_{2,2,2}$ *minor.*




- To see $\mathrm{gd}(K_{2,2,2}) \geq 5$: Construct $a \in \mathcal{E}(K_{2,2,2})$ having a **unique** completion $X \in \mathcal{E}_6$ with rank 5.
- **Hard part:** Show that $V_8$ and $C_5 \square K_2$ have Gram dimension 4.
- Links to [Belk-Connelly'07] for Euclidean distance embeddings.

# Behaviour of $\mathrm{egd}(G)$ under clique sums




**Lemma (ELV 2014)**

*If $G$ is the* **clique 0- or 1-sum** *of $G_1$ and $G_2$ then*

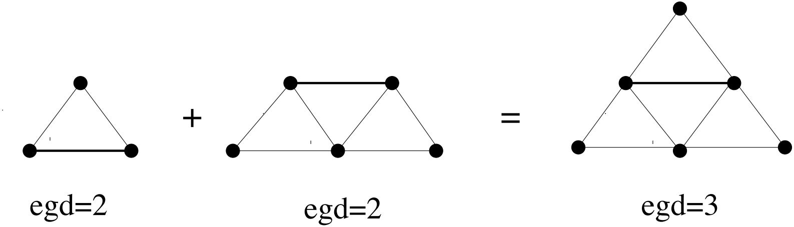
$$\mathrm{egd}(G) = \max\{\mathrm{egd}(G_1), \mathrm{egd}(G_2)\}.$$

$a$ extreme point of $\mathcal{E}(G) \iff a_i$ extreme point of $\mathcal{E}(G_i)$ for $i = 1, 2$

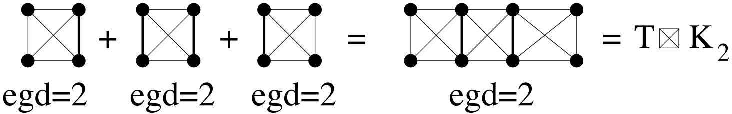
Not true for clique 2-sum...

# Behaviour of $\mathrm{egd}(G)$ under clique sums (continued)

General clique 2-sum does **not** preserve extreme Gram dimension:

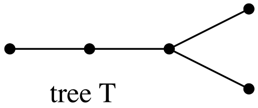


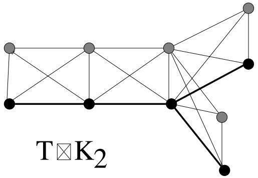

But **some** clique 2-sums do preserve the extreme Gram dimension:



# Extreme Gram dimension and strong largeur d'arborescence

The **strong product** $T \boxtimes K_r$ of a tree $T$ and a complete graph $K_r$:




**Theorem (ELV 2014)**

*For any tree $T$,* $\mathrm{egd}(T \boxtimes K_r) \leq r$. *Therefore:*

$$\mathrm{egd}(G) \leq \mathrm{la}_{\boxtimes}(G) := \min\ r \text{ s.t. } G \preceq T \boxtimes K_r \text{ for some tree } T.$$

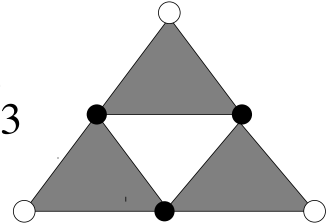
Example: For any cycle, $\mathrm{egd}(C_n) = 2$.

For $\mathrm{egd}(G) \leq 2$, this is *almost* an equality ....

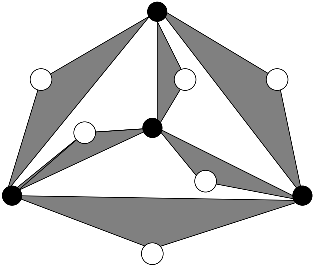
.... but first some classes of forbidden minors.

# Flower graphs $F_r$

F3



F4



$F_r$ is the complete graph $K_r$ on nodes $v_1, \cdots, v_r$ together with additional triangles on nodes $\{v_i, v_j, v_{ij}\}$.

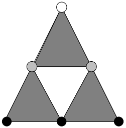
$F_r$ has $r + \binom{r}{2} = \binom{r+1}{2}$ nodes $\quad\quad \rightsquigarrow \mathrm{egd}(F_r) \leq r$.

Label the nodes $v_i$ by $e_i$, $v_{ij}$ by $(e_i + e_j)/\sqrt{2}$ to get $\mathrm{egd}(F_r) = r$.

**Minimality:**

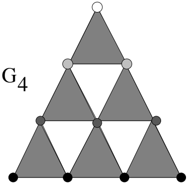
$$\mathrm{egd}(F_r/e) \leq r - 1, \quad \mathrm{egd}(F_r \backslash e) \leq \mathrm{la}_{\boxtimes}(F_r \backslash e) = r - 1.$$

# Supertriangles $G_r$



$G_3 = F_3$

$G_4$



$G_r$ has $r + (r-1) + \cdots + 1 = \binom{r+1}{2}$ nodes.

Label the nodes by vectors so that all shaded triangles are *minimally linear dependent* to get:

$$\mathrm{egd}(G_r) = r$$

Minimality: $\mathrm{egd}(G_r \backslash e) \leq r-1, \quad \mathrm{egd}(G_r/e) \leq \mathrm{la}_{\boxtimes}(G_r) = r-1.$

## Corollary

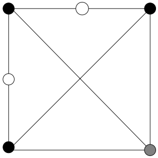
*The extreme Gram dimension is* **not bounded for planar graphs**.

# Graphs $H_r$

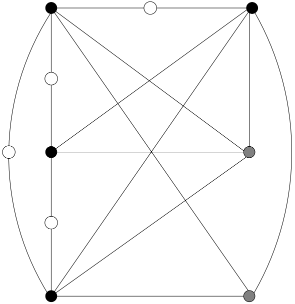
$H_3$ is a homeomorph of $K_4$.

$H_r$ has $\binom{r+1}{2}$ nodes.

$H_3$



$H_4$

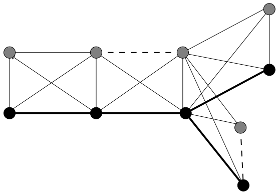


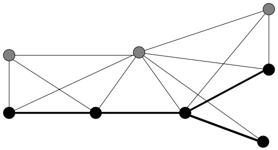
$$\mathrm{egd}(H_r) = r$$

Minimality for $H_3$: $\mathrm{egd}(H_3 \backslash e), \mathrm{egd}(H_3/e) \leq 2.$

Minimality for $H_r$?

# Characterizing $\mathrm{egd}(G) \leq 2$





**Theorem (ELV 2014)**

(1) $\mathrm{egd}(G) \leq 2 \iff$ (2) *no minor $F_3$ or $H_3$.*

Proof strategy:

- $\mathrm{egd}(K_5) = 2$, $\mathrm{egd}(K_{3,3}) = 2$.
- Wlog: $n \geq 6$, $G \neq K_{3,3}$, $G$ 2-connected. Show:

  (1) $\mathrm{egd}(G) \leq 2 \iff$ (2) no minor $F_3, H_3 \iff$ (3) $\mathrm{la}_{\boxtimes}(G) \leq 2$.

- Suffices to show: (2) $\implies$ (3).
- Note: $\mathrm{la}_{\boxtimes}(K_5) = \mathrm{la}_{\boxtimes}(K_{3,3}) = 3$.

# Link to Colin de Verdière type graph invariants

# On the "dual side": Colin de Verdière type parameters

Colin de Verdière (1990) introduced the spectral graph parameter $\mu(G)$ characterizing topological graph properties:

$$\mu(G) \leq 1 \iff G \text{ is a path}$$

$$\mu(G) \leq 2 \iff G \text{ is outerplanar}$$

$$\mu(G) \leq 3 \iff G \text{ is planar}$$

$$\mu(G) \leq 4 \iff G \text{ is linklessly embeddable in } \mathbb{R}^3$$

[Lovász-Schrijver 1998]

# Two related parameters $\nu(G), \nu_H(G)$

$$\mu(G) \le \nu(G) \le \nu_H(G)$$

**Definition**

- [Van der Holst 2003]

$\nu_H(G) = \max \ \ \mathrm{corank}(\Omega) \ \text{ s.t. } \Omega \succeq 0, \ \underbrace{\Omega_{ij} = 0 \ (ij \in \overline{E})}_{\mathcal{L}}, \ (SAP)$

- [CdV 1998] For $\nu(G)$: add: $\Omega_{ij} \neq 0$ if $ij \in E$.

**Definition (SAP: Strong Arnold property)**

$\Omega \in \mathcal{L}$ satisfies (SAP) if

$$\Omega Z = 0, \ Z_{ij} = 0 \ \forall ij \in V \cup E \implies Z = 0.$$

That is, *the tangent space at* $\Omega$ to the manifold of matrices whose rank is equal to $\mathrm{rank}(\Omega)$, intersects $\mathcal{L}$ transversally at $\Omega$.

# Link between $\mathrm{gd}(G)$ and $\nu_H(G)$

**Theorem (van der Holst 2003)**

1. *The parameter $\nu_H$ is minor monotone.*
2. $\nu_H(G) \leq 4 \iff$ *$G$ has no minor $K_5$ or $K_{2,2,2}$.*

*Same forbidden minors as for* $\mathrm{gd}(G) \leq 4$ *!*

**Theorem (LV 2014)**

$\nu_H(G) \leq \mathrm{gd}(G)$.

**Equality holds** if $G$ has no minor $K_5$ or $K_{2,2,2}$, but also for $K_5$, $K_{2,2,2}$, $G$ chordal,...

# Key tool: duality and geometry of SDP

Consider the primal and dual SDP's:

$$\max_X \ 0 \ \text{ s.t. } X_{ij} = a_{ij} \ (ij \in V \cup E), \ X \succeq 0 \qquad (P_a)$$

$$\min_\Omega \sum_{ij \in V \cup E} a_{ij}\Omega_{ij} \quad \text{i.e., } \Omega_{ij} = 0 \ (ij \in \overline{E}), \ \Omega \succeq 0. \qquad (D_a)$$

**Theorem (LV 2014)**

$\nu_H(G) = \max \mathrm{gd}(G, a)$, *taken over all partial matrices a for which* $(D_a)$ *has an optimal solution* $\Omega$ **satisfying (SAP)** *(i.e.,* $\Omega$ *is* **non-degenerate***) ...*

*... and then* $(P_a)$ *has a* **unique optimal solution**, *i.e., a has a* **unique psd completion.**

$\rightsquigarrow$ Link to universal rigidity.

*Hence:* $\mathrm{gd}(G) \geq \nu_H(G)$. *Does equality hold?*

## Link between $\mathrm{egd}(G)$ and $\nu(G)$

$$\nu(G) = \max_{\Omega \succeq 0 \ \& \ \mathrm{(SAP)}} \operatorname{corank}(\Omega) \ \text{ s.t. } \Omega_{ij} = 0 \ (ij \in \overline{E}), \Omega_{ij} \neq 0 \ (ij \in E).$$

---

[CdV 1998] bounds $\nu(G)$ by the **largeur d'arborescence** $\mathrm{la}(G)$:

$$\nu(G) \leq \min\{r : G \preceq T \square K_r \ \text{ for some tree } T\} = \mathrm{la}(G).$$

[ELV'14] bounds $\mathrm{egd}(G)$ by the **strong largeur d'arborescence:**

$$\mathrm{egd}(G) \leq \min\{r : G \preceq T \boxtimes K_r \ \text{ for some tree } T\} = \mathrm{la}_{\boxtimes}(G).$$

---

[Kotlov 2000]: $\nu(G) \leq 2 \iff$ no minor $F_3$ or $K_4 \iff \mathrm{la}(G) \leq 2$.

[ELV 2014]: $\mathrm{egd}(G) \leq 2 \iff$ no minor $F_3$ or $H_3$.

$\mathrm{la}_{\boxtimes}(G) \leq 2 \iff$ no minor $F_3$, $H_3$ or $W_5$.

---

**Question:** $\mathrm{egd}(G) \leq \nu(G)$? **Yes** if $\nu(G) \leq 2$.

[CdV 1998]: $\nu(G_r) = r$, $\nu(K_n) = n - 1$. Hence:

$$\mathrm{egd}(G_r) = \nu(G_r) = r, \quad \mathrm{egd}(K_n)(\sim \sqrt{2n}) \leq \nu(K_n) = n - 1.$$

# Concluding remarks

There are **strong links** between

- Low rank PSD matrix completions
- Topological graph parameters
- Geometry of SDP and rigidity theory

**Open questions:**

- $\mathrm{gd}(G) = \nu_H(G)$ ? $\quad \mathrm{egd}(G) \leq \nu(G)$ ?
- $\mathrm{gd}(G) = \mathrm{ed}(G) + 1$ ?
- $\mathrm{ed}(G)$ = smallest $k$ such that there exist $v_1, \ldots, v_n \in \mathbb{R}^k$ such that $d_{ij} = \|u_i - u_j\|^2$ ($ij \in E$) (if they exist in $\mathbb{R}^d$ for some $d$.).

  [LV 2014]: $\mathrm{gd}(G) = \mathrm{ed}(\nabla G) \geq \mathrm{ed}(G) + 1$.

# Based on the papers

- *A new graph parameter related to bounded rank positive semidefinite matrix completions.*
With A. Varvitsiotis. Mathematical Programming, 2014.

- *Forbidden minor characterizations for low-rank optimal solutions to semidefinite programs over the elliptope.*
With M. E.-Nagy and A. Varvitsiotis. Journal of Combinatorial Theory, 2014.

- *Semidefinite programming, universal rigidity and the Strong Arnold Property.*
With A. Varvitsiotis. Linear Algebra and its Applications, 2014.

- *Complexity of the positive semidefinite matrix completion problem with a rank constraint.*
With M. E.-Nagy and A. Varvitsiotis. Fields Institute Communications, 2013.