Packing bounded degree graphs

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joint work with
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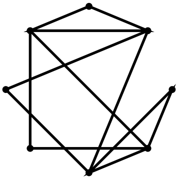
Decomposition of large/dense object into small/sparse objects.

Graph decompositions

G has a decomposition into $H_1, \ldots, H_s$ if there exist pairwise edge-disjoint copies of $H_1, \ldots, H_s$ in G which cover all edges of G.



General theme

Decomposition of large/dense object into small/sparse objects.

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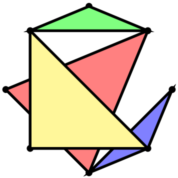

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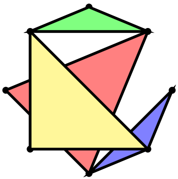


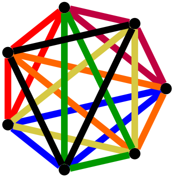
Graph packings

$H_1, \ldots, H_s$ pack into G if there exist pairwise edge-disjoint copies of $H_1, \ldots, H_s$ in G.

Background I: decompositions into small subgraphs

Theorem (Kirkman, 1847)

Complete graph K_n has a triangle decomposition $\Leftrightarrow$ $n \equiv 1$ *or* $3 \pmod 6$.



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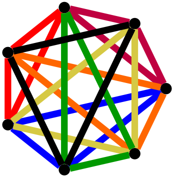

$n \equiv 1$ or $3 \pmod 6$

$\Leftrightarrow$

K_n satisfies necessary divisibility conditions for K_3-decomposition (i.e. 3 divides $e(K_n)$ and all degrees are even)

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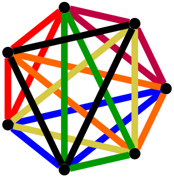


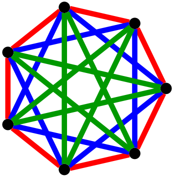
Generalizations include:

- **Wilson:** decompositions of cliques into general graphs
- **Keevash:** decompositions of quasi-random graphs and hypergraphs into cliques
- **Barber, Glock, Kühn, Lo, Montgomery & Osthus:** decompositions of graphs of large minimum degree into general graphs

Background II: Hamilton decompositions

Theorem (Walecki, 1892)

Complete graph K_n has a decomposition into Hamilton cycles $\Leftrightarrow$ n is odd.



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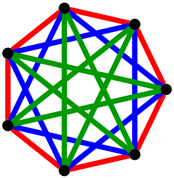


Theorem (Csaba, Kühn, Lo, Osthus & Treglown, 2016+)

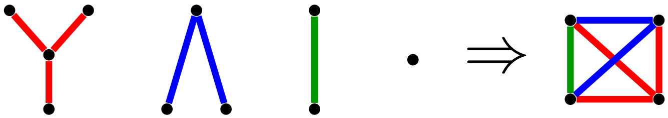
Every large n-vertex r-regular graph, where $r \geq \lfloor n/2 \rfloor$ is even, has a decomposition into Hamilton cycles.

proves Hamilton decomposition conjecture of Nash-Williams from 1970

Tree packing conjecture

Conjecture (Gyárfás & Lehel, 1976)

Given trees $T_1, \ldots, T_n$ such that T_i has i vertices, K_n has a decomposition into $T_1, \ldots, T_n$.



Note that $\sum_{i=1}^{n} e(T_i) = \binom{n}{2}$.

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Results on packing the smallest trees or the largest trees or very special families of trees

- **Gyárfás & Lehel:** $T_1, \dots, T_n$ pack into K_n if each T_i is either a path or star
- **Bollobás:** $T_1, \dots, T_{\frac{n}{\sqrt{2}}}$ pack into K_n
- **Balogh & Palmer:** $T_{n-n^{1/4}/10}, \dots, T_n$ pack into K_{n+1}
- **Zak:** $T_{n-4}, \dots, T_n$ pack into K_n
- ...

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approximate version for bounded degree trees:

Theorem (Böttcher, Hladkỳ, Piguet & Taraz, 2016)

If $1/n \ll \alpha, 1/\Delta$ and $T_1, \dots, T_t$ are trees such that

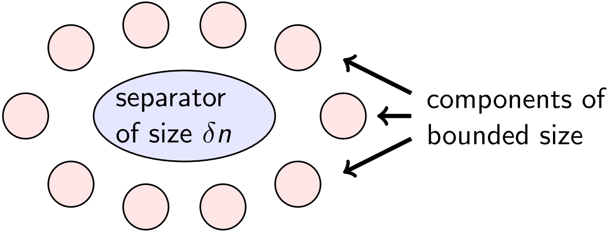
- $\Delta(T_i) \leq \Delta$ *and* $|T_i| \leq (1-\alpha)n$,
- $\sum_{i \in 1}^{t} e(T_i) \leq (1-\alpha)\binom{n}{2}$,

then $T_1, \dots, T_t$ pack into K_n.

Packings of well-separable graphs

Definition

H is (δ, K)-separable if by removing δn vertices one can split H into components of size at most K.
A family $\mathcal{H}$ of graphs is separable if for every $\delta > 0$ there is some K such that each $H \in \mathcal{H}$ is (δ, K)-separable.



Examples of separable graph families:
trees, planar graphs, minor closed families

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Theorem (Messuti, Rödl & Schacht, 2016+)

If $1/n \ll \alpha, 1/\Delta$ and if $\mathcal{H}$ is separable and $H_1, \ldots, H_t \in \mathcal{H}$ are such that

- $\Delta(H_i) \leq \Delta$ *and* $|H_i| \leq (1-\alpha)n$,
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Approximate decompositions into separable graphs

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next result allows for packing of spanning graphs:

Theorem (Ferber, Lee & Mousset, 2016+)

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Approx. decompositions into general bdd degree graphs

Theorem (Kim, Kühn, Osthus & Tyomkyn, 2016+)

If $1/n \ll \alpha, 1/\Delta$ and $H_1, \ldots, H_t$ are such that $\Delta(H_i) \leq \Delta$ and $|H_i| \leq n$ and $\sum_{i=1}^{t} e(H_i) \leq (1-\alpha)\binom{n}{2}$, then $H_1, \ldots, H_t$ pack into K_n.

in this general setup cannot ask for a decomposition into $H_1, \ldots, H_t$

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can replace host graph K_n by any quasi-random graph:

n-vertex graph G is (ϵ, d)-quasi-random if

- $d_G(v) = (1 \pm \epsilon)dn$ for every vertex v and
- $d_G(u, v) = (1 \pm \epsilon)d^2 n$ for every pair $u \neq v$ of vertices.

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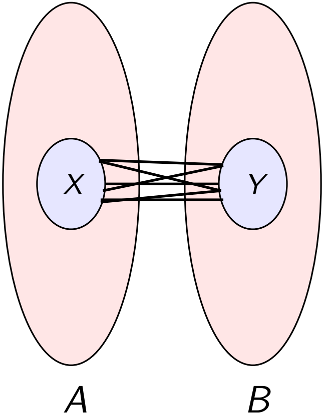
If $1/n \ll \epsilon \ll \alpha, d, 1/\Delta$, if G is (ϵ, d)-quasi-random and if $H_1, \dots, H_t$ are such that $\Delta(H_i) \le \Delta$ and $|H_i| \le n$ and $\sum_{i=1}^{t} e(H_i) \le (1-\alpha)e(G)$, then $H_1, \dots, H_t$ pack into G.

Actually, consider setting of ϵ-regularity:

$|A| = |B| = n$



(A, B) is ϵ-regular if

$$\frac{e(X, Y)}{|X||Y|} = (1 \pm \epsilon)\frac{e(A, B)}{|A||B|}$$

for not too small X, Y.

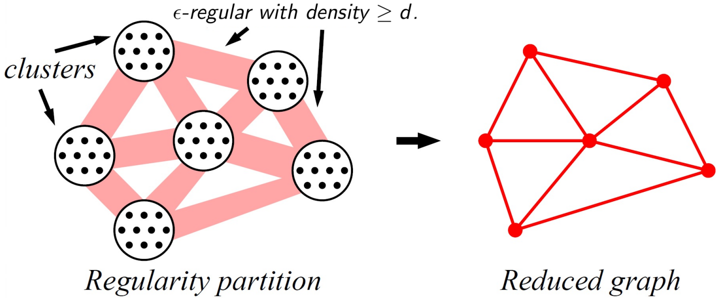
(A, B) is (ϵ, d)-super-regular if

- (A, B) ϵ-regular, density $d \pm \epsilon$,
- $d(a), d(b) = (d \pm \epsilon)n$.

Regularity lemma

Theorem (Szemerédi's regularity lemma, 1976)

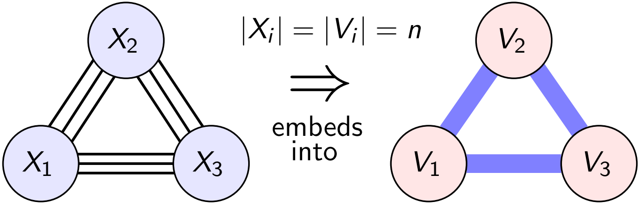
We can partition any large dense graph G into a bounded number of clusters so that almost all pairs are ϵ-regular.



Blow-up lemma

Theorem (Komlós, Sárközy & Szemerédi, 1997)

If $1/n \ll \epsilon \ll 1/\Delta, d$, *then the following embedding exists:*



$H : \Delta(H) \leq \Delta$ $\qquad$ G: (ϵ, d)-super-regular

Important tool to find spanning structures, e.g.

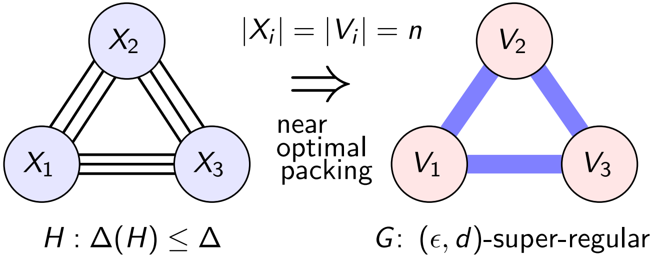
- powers of Hamilton cycles (Komlós, Sárközy & Szemerédi)
- H-factors (Komlós, Sárkőzy & Szemerédi, Kühn & Osthus)
- ...

Main result

can almost decompose G into copies of H:

Theorem (Kim, Kühn, Osthus & Tyomkyn, 2016+)

If $1/n \ll \epsilon \ll 1/\Delta, d$, then the following near-optimal packing exists:



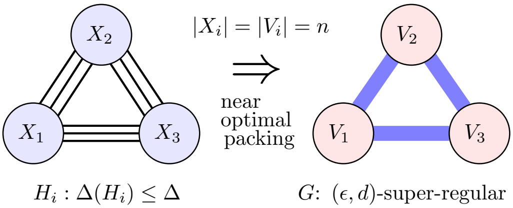
Main result

Theorem (Kim, Kühn, Osthus & Tyomkyn, 2016+)

Suppose $1/n \ll \epsilon \ll d, \alpha, 1/\Delta, 1/r$ *and that*

- *each of* $H_1, \dots, H_s$ *has vertex classes* $X_1, \dots, X_r$ *of size* n *and* $\Delta(H_i) \leq \Delta$,
- G *has vertex classes* $V_1, \dots, V_r$ *of size* n *such that all pairs* (V_i, V_j) *are* (ϵ, d)*-super-regular,*
- $\sum_{\ell=1}^{s} e(H_\ell) \leq (1-\alpha)e(G)$.

Then $H_1, \dots, H_s$ *pack into* G.



Main result

Actually prove a stronger version with 'bells and whistles' added, e.g.:

- allowed to specify 'target sets' for some of the vertices,
- allowed to have a bounded degree reduced graph with many clusters (i.e. much more than $1/\epsilon$),
- super-regular pairs in G allowed to have different densities,
- clusters allowed to have slightly different sizes.

Proof sketch: strategy

Strategy: pack $H_1, \ldots, H_s$ into G successively,
i.e. embed H_i into $G_i := G - H_1 \cdots - H_{i-1}$

Naive approach: Choose H_i 'uniformly at random' in G_i.

Aim to show:

(a) each edge of G_i equally likely to be chosen.

(b) G_{i+1} is ϵ_{i+1}-regular, where $\epsilon_{i+1} \sim \epsilon$.

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(b) G_{i+1} is ϵ_{i+1}-regular, where $\epsilon_{i+1} \sim \epsilon$.

Problems:

(a) is impossible (e.g. if H_i a triangle factor, G_i may have edges not in any triangle)

(b) seems infeasible, as ϵ_i increases too rapidly.

Proof sketch: using many rounds

To maintain ϵ-regularity of G: use bounded number of rounds

- choose embedding $\phi(H_i)$ of H_i independently for all i within the same round,
- update G only after each round

i.e. allow overlaps within a round

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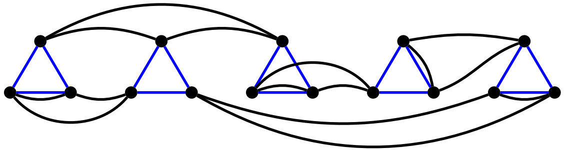
Example: each H_i is a triangle-factor

$\phi(H_i) \cup \phi(H_j)$



$\Rightarrow$ $\phi(H_i)$ and $\phi(H_j)$ are

- almost edge-disjoint if embedded in the same round,
- edge-disjoint if embedded in different rounds

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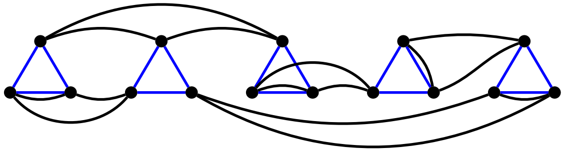

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- almost edge-disjoint if embedded in the same round,
- edge-disjoint if embedded in different rounds

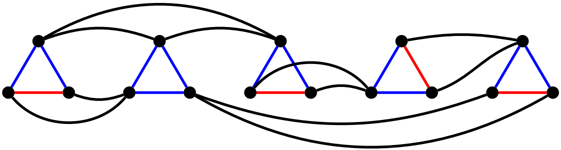
Aim: repair packing at the end to achieve edge-disjointness

Proof sketch: repairing the packing

Patching:

- set aside patching graph $P \subseteq G$ at the beginning of proof, ($P=$ thin edge-slice of G)
- use P to patch each H_i in turn

$\phi(H_i) \cup \phi(H_j)$



conflict edges

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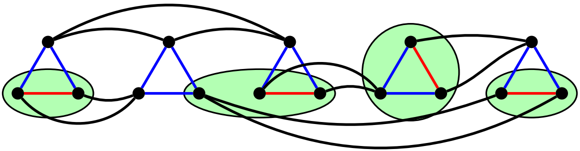
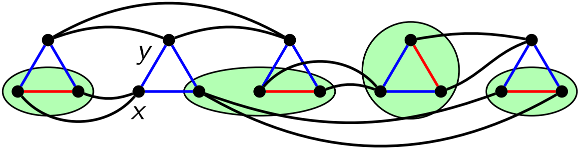


X= green set

Add small random vertex set to the vertices incident to conflict edges and re-embed $H_i[X]$ using P

Proof sketch: repairing the packing

$\phi(H_i) \cup \phi(H_j)$



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Problem: if x and y have common neighbours in X, they need to have many common neighbours in $P[X]$

Proof sketch: repairing the packing

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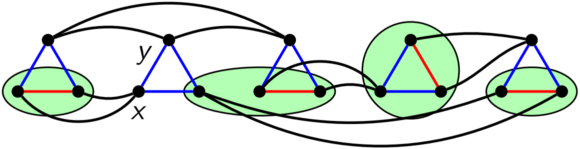


X= green set

Problem: if x and y have common neighbours in X, they need to have many common neighbours in $P[X]$

Solution: ensure that $\phi(H_i)$ behave well with respect to patching graph P already when choosing $\phi(H_i)$

Proof sketch: summary of strategy

Step 1: embed the H_i one by one using bounded number of rounds

- embed the H_i independently from each other within the same round, choosing a 'uniform' embedding of each H_i
- update G only after each round

i.e. allow overlaps within a round

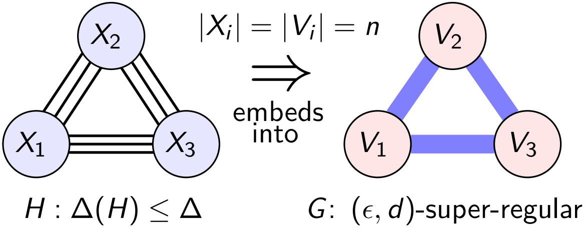
Step 2: deal with overlaps using patching graph

G will still be super-regular because

- choose 'uniform' embedding of each H_i
- perform only bounded number of updates of G

Proof sketch: uniform blow-up lemma

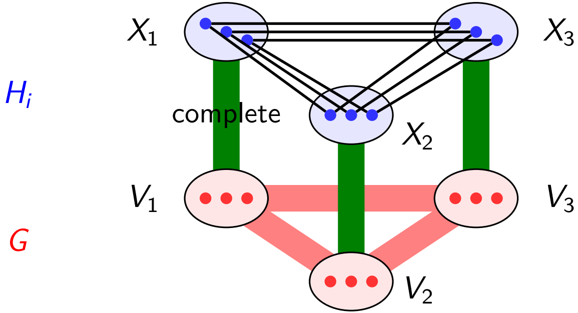
To embed a single H_i, prove uniform blow-up lemma which comes as close as possible to picking $e(H_i)$ of edges of G uniformly at random.



Proof of uniform blow up lemma:
develop approach introduced by Rödl and Ruciński
for simplicity only consider case when each of the bipartite subgraphs forming H_i is perfect matching

Proof sketch: uniform blow-up lemma

Aim: choose embedding of H_i which is as close as possible to picking $e(H_i)$ edges of G uniformly at random



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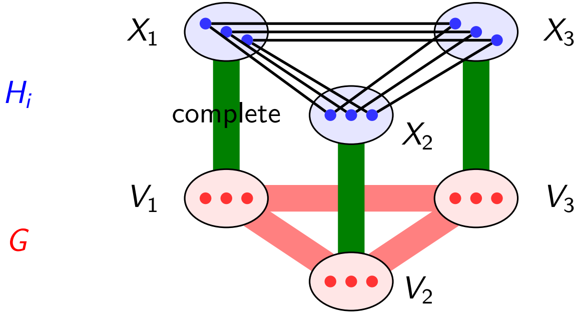
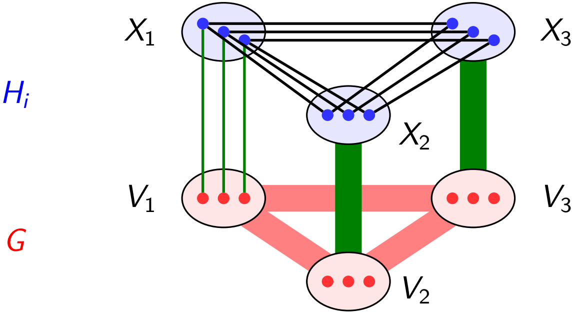


Consider candidacy bigraph between X_i and V_i containing an edge between x and v if it is ok to embed x to v.
Initially, it is complete bipartite.

Proof sketch: uniform blow-up lemma

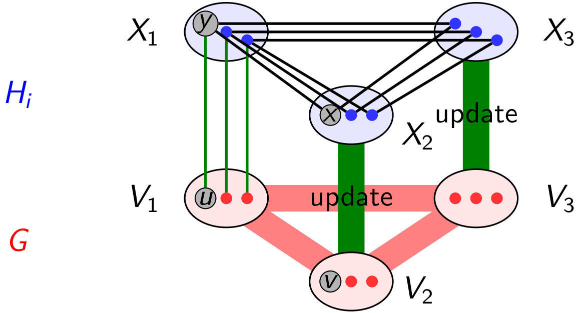
Aim: choose embedding of H_i which is as close as possible to picking $e(H_i)$ edges of G uniformly at random



Choose a random perfect matching between X_1 and V_1.

Proof sketch: uniform blow-up lemma

Aim: choose embedding of H_i which is as close as possible to picking $e(H_i)$ edges of G uniformly at random

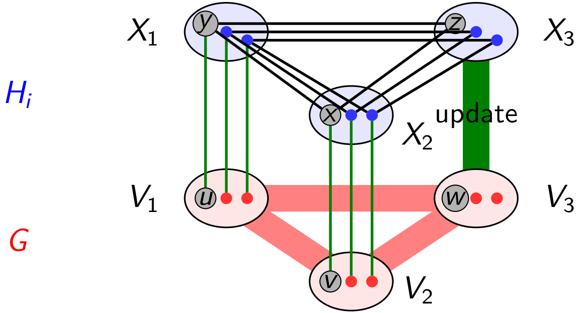


Update candidacy bigraphs: now x is adjacent to v only when v is a neighbour of u.

$\Longrightarrow$ updated candidacy bigraph is also super-regular.

Proof sketch: uniform blow-up lemma

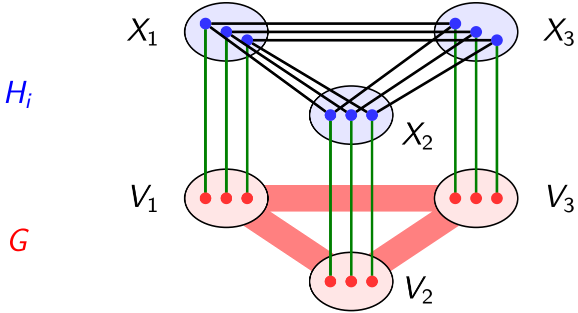
Aim: choose embedding of H_i which is as close as possible to picking $e(H_i)$ edges of G uniformly at random



Choose random perfect matching in (updated) candidacy bigraph between X_2 and V_2.

Update remaining candidacy bigraph: now z is adjacent to w only if w is common neighbour of v and u.

Aim: choose embedding of H_i which is as close as possible to picking $e(H_i)$ edges of G uniformly at random



With high probability, the updated candidacy bigraph is super-regular.
Choose another random perfect matching to complete the embedding.

Full statement of the uniform blow-up lemma

Lemma 1 (Uniform blow-up lemma). *Suppose*

$$0 < 1/n \ll c \ll \varepsilon \ll \gamma \ll \beta, d, d_0, 1/k, 1/\Delta_R, 1/(C+1) \text{ and } 1/n \ll 1/r.$$

Let $K := (k+1)^2\Delta_R$, *let* $w := K^2\Delta_R^2(\Delta_R+1)$ *and let* f *be the function defined in* (**??**). *Suppose that* R *is a graph on* $[r]$ *with* $\Delta(R) = \Delta_R$. *Let* $\vec{d}, \vec{\beta}, \vec{k}$ *be symmetric* $r \times r$ *matrices such that* $d = \min_{ij \in E(R)} d_{i,j}, \beta = \min_{ij\in E(R)} \beta_{i,j}, k = \max_{ij\in E(R)} k_{i,j}$ *and* $k_{i,j} \in \mathbb{N}$ *for all* $ij \in E(R)$. *Suppose that the following hold.*

- G *is an* $(\varepsilon, \vec{d})$*-super-regular graph with respect to* $(R, \mathcal{V})$, *where* $\mathcal{V} = (V_1, \dots, V_r)$, $\max_{i\in[r]} |V_i| = n$ *and* $n - C \le |V_i| \le n$ *for all* $i \in [r]$.
- P *is an* $(\varepsilon, \vec{\beta})$*-super-regular graph with respect to* $(R, \mathcal{V})$.
- H *is an* $(R, \vec{k}, C)$*-near-equiregular graph admitting the vertex partition* $(R, \mathcal{X})$ *with* $\mathcal{X} = (X_1, \dots, X_r)$ *where* $|X_i| = |V_i|$.
- A_0 *is a bipartite graph with bipartition* $(V(H), V(G))$ *such that* $N_{A_0}(X_i) = V_i$ *and* $A_0[X_i, V_i]$ *is* (ε, d_0)*-super-regular for each* $i \in [r]$.

Then there exists a randomised algorithm (the uniform embedding algorithm) which succeeds with probability at least $1 - (1-c)^n$ *in finding an embedding* ϕ *of* H *into* G *such that* $\phi(X_i) = V_i$ *for each* $i \in [r]$ *and* $\phi(x) \in N_{A_0}(x)$ *for each* $x \in V(H)$. *Conditional on being successful, this algorithm returns* $(\phi, \mathcal{Y}, \mathcal{U}, F, N)$ *with the following properties.*

(B1) *For all* $ij \in E(R)$, $v \in V_i$ *and* $S \subseteq V_j \cap N_G(v)$ *with* $|S| > f(\varepsilon)n$,

$$\mathbb{E}\left[|N_{\phi(H)}(v) \cap S|\right] = (1 \pm f(\varepsilon))\frac{k_{i,j}|S|}{d_{i,j}n}.$$

(B2) $\mathcal{U} = \{U_1, \dots, U_{Kr}\}$ *is a partition refining* $V_1, \dots, V_r$ *and* $\mathcal{Y} = \{Y_1, \dots, Y_{Kr}\}$ *is a partition refining* $X_1, \dots, X_r$ *with* $|Y_i| = |U_i|$, *with* $\max_{i\in[Kr]} |Y_i| = \lceil n/K \rceil$ *and* $|Y_i| - |Y_j| \le C$ *for all* $i \ne j$. F *is a disjoint union of bipartite graphs* $F_1, \dots, F_{Kr}$ *such that each* F_j *has bipartition* (Y_j, U_j). N *is a hypergraph with vertex set* $V(H)$ *and a hyperedge* N_x *for each* $x \in V(H)$. *Moreover, the following conditions hold.*

(B2.1) N *is an* $(H, R_K, \mathcal{Y})$*-candidacy hypergraph with* $\max\{|N_x| : N_x \in N\} \le K\Delta_R$.

(B2.2) *For all* $j \in [Kr]$ *and* $x \in Y_j$, $N_{F_j}(x) \subseteq U_j \cap N_{A_0}(x) \cap \bigcap_{y\in N_x} N_P(\phi(y))$, *thus* F *is a* $(H, P, R_K, A_0, \phi, \mathcal{Y}, \mathcal{U}, N)$*-candidacy bigraph.*

(B2.3) P *is* $(\varepsilon^{1/3}, \vec{\beta}')$*-super-regular with respect to* $(R_K, \mathcal{U})$, *where* $\vec{\beta}'$ *is the symmetric* $Kr \times Kr$ *matrix with entries* $\beta'_{\ell,\ell'} := \beta_{i,j}$ *whenever* $i = \lceil \ell/K \rceil$ *and* $j = \lceil \ell'/K \rceil$.

(B2.4) *For each* $j \in [Kr]$ *the graph* F_j *is* $(f(\varepsilon), d_0 p(R_K, \vec{\beta}', j))$*-super-regular.*

(B3) *For all* $u \ne v \in V(G)$, $\mathbb{P}[N_H(\phi^{-1}(u)) \cap N_H(\phi^{-1}(v)) \ne \emptyset] \le 1/\sqrt{n}$.

(B4) *Let* G'' *be a subgraph of* G *with* $V(G'') = V(G)$ *such that* $\Delta(G'') \le \gamma n$.

(B4.1) *For any vertex* $v \in V(G)$, $\mathbb{P}[v \in \phi_2(H, G, G'')] \le \gamma^{1/2}$.

(B4.2) $\mathbb{P}[|\{u : u \in U_j, u \in \phi_2(H, G, G'')\}| \le \gamma^{3/5} n$ *for all* $j \in [Kr]] \ge 1 - (1-2c)^n$.

(B5) *Suppose* $v_1, \dots, v_s \in V_i$ *where* $i \in [r]$ *with* $s \le K$. *For all* $j \in N_R(i)$ *and* $v \in V_j$, *let* B_v *be the random variable such that*

$$B_v := \begin{cases} 1 & \text{if there exists } \ell \in [s] \text{ with } v_\ell v \in \phi(E(H)), \\ 0 & \text{otherwise.} \end{cases}$$

Then for all $j \in N_R(i)$ *and all but at most* $2f(\varepsilon)n$ *vertices* $v \in N_G(v_1, \dots, v_s) \cap V_j$ *we have that*

$$\mathbb{P}[B_v = 1] = (1 \pm 2f(\varepsilon))\frac{k_{i,j}s}{d_{i,j}n}.$$

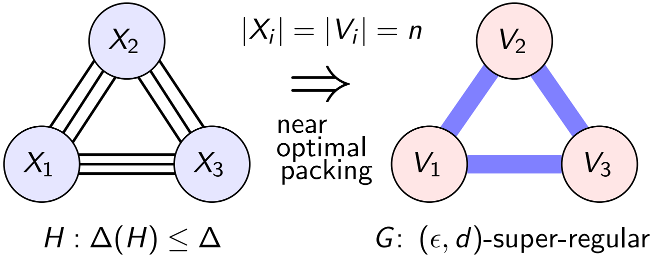
(B6) *For all* $i \in [r]$ *and all sets* $Q \subseteq X_i$ *and* $W \subseteq V_i$ *with* $|Q|, |W| > f(\varepsilon)n$,

$$\mathbb{P}\left[|\phi(Q) \cap W| = \frac{(1 \pm f(\varepsilon))|Q||W|}{n}\right] \ge 1 - (1-2c)^n.$$

can almost decompose G into copies of H:

Theorem (Kim, Kühn, Osthus & Tyomkyn, 2016+)

If $1/n \ll \epsilon \ll 1/\Delta, d$, then the following near-optimal packing exists:



Work in progress: further applications

Conjecture (Gyárfás & Lehel, 1978)

Given trees $T_1, \dots, T_n$ such that T_i has i vertices, K_n has a decomposition into $T_1, \dots, T_n$.

Tree packing conjecture holds for bounded degree trees:

Theorem (Joos, Kim, Kühn & Osthus, 2016+)

Suppose $1/n \ll 1/\Delta$. For each $i \in [n]$, let T_i be a tree with i vertices and $\Delta(T_i) \leq \Delta$. Then K_n decomposes into $T_1, \dots, T_n$.

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Proof uses

- 'bells and whistles' version of blow-up lemma for near optimal packings
- even-regular robust expanders have Hamilton decompositions (Kühn & Osthus)
- iterative absorption method

Work in progress: further applications

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Conjecture (Ringel 1963)

Let T be an $(n+1)$-vertex tree. Then K_{2n+1} decomposes into $2n+1$ copies of T.

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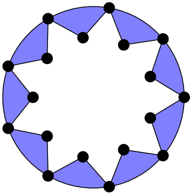
Suppose $1/n \ll 1/\Delta$. Let T be an $(n+1)$-vertex tree with $\Delta(T) \leq \Delta$. Then K_{2n+1} decomposes into $2n+1$ copies of T.

Theorem (Walecki, 1892)

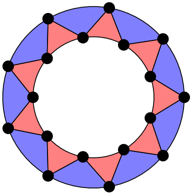
Complete graph K_n has a Hamilton decomposition $\Leftrightarrow$ n odd

Problem

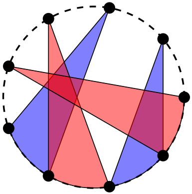
Prove a hypergraph version of Walecki's theorem.



loose Hamilton cycle

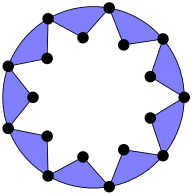


tight Hamilton cycle



Berge Hamilton cycle

Open problems: Hamilton decompositions of hypergraphs



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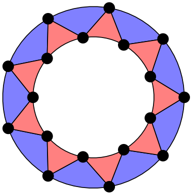
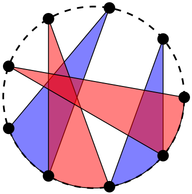


tight Hamilton cycle



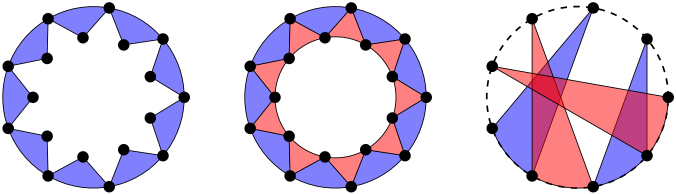
Berge Hamilton cycle

Conjecture (Kühn, Osthus, 2014)

If n is sufficiently large such that $k-1$ divides n and $n/(k-1)$ divides $\binom{n}{k}$ then $K_n^{(k)}$ has a decomposition into loose Hamilton cycles.

divisibility conditions are necessary:
k-uniform loose Hamilton cycles have $n/(k-1)$ edges and only exist if $k-1$ divides n

Open problems: Hamilton decompositions of hypergraphs

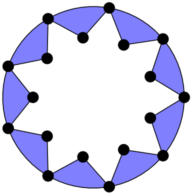


loose Hamilton cycle　　tight Hamilton cycle　　Berge Hamilton cycle

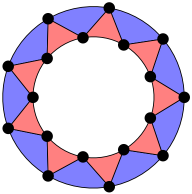
Conjecture (Bailey, Stevens, 2010)

If n is sufficiently large and n divides $\binom{n}{k}$ then $K_n^{(k)}$ has a decomposition into tight Hamilton cycles.

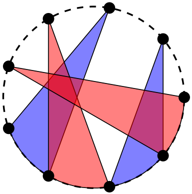
divisibility condition is necessary:
k-uniform tight Hamilton cycles have n edges



loose Hamilton cycle



tight Hamilton cycle



Berge Hamilton cycle

Theorem (Kühn, Osthus, 2014)

If $n \geq 30$ and n divides $\binom{n}{k}$ then $K_n^{(k)}$ has a decomposition into Hamilton Berge cycles.

- divisibility condition is necessary
- proves conjecture of Bermond, Germa, Heydemann, Sotteau from 1973
- case $k = 3$ already due to Verrall, building on results of Bermond