Grid Ramsey Problem

Abstract

The Grid Graph $\Gamma_{m,n}$ is the graph product of $K_n$ and $K_m$, i.e. the graph with vertices $[m] \times [n]$ and $\{(i,j), (i',j')\}$ being an edge if either $i = j$ or $i' = j'$. For a positive integer $r$, Shelah’s number $G(r)$ is the smallest number $n$ such that every $r$-colouring of $E(\Gamma_{n,n})$ induces an alternating rectangle, i.e. a rectangle whose parallel edges receive the same colour.

These numbers first appeared in Shelah’s proof of the Hales-Jewett theorem in 1988, where he used the trivial bound $G(r) \leq r^{\binom{r+1}{2}} + 1$. The so far only improvement to the upper bound was made by Gyárfás in 1994, who showed $G(r) \leq r^{\binom{r+1}{2}} - r^{\binom{r-1}{2}+1} + 1$ for all $r \geq 3$. Conlon, Fox, Lee and Sudakov recently showed that $G(r)$ grows super-polynomially in $r$ destroying the hope for a significant improvement of the Hales-Jewett number via the Grid Ramsey Problem.

In this talk we try to explain why the problem of determining $G(r)$ is difficult and sketch a proof of the slightly better upper bound $G(r) \leq r^{\binom{r+1}{2}} - r^{\binom{3}{2}} + 1$ for all $r \geq 3$. 