Forbidden induced bipartite graphs

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Introduction

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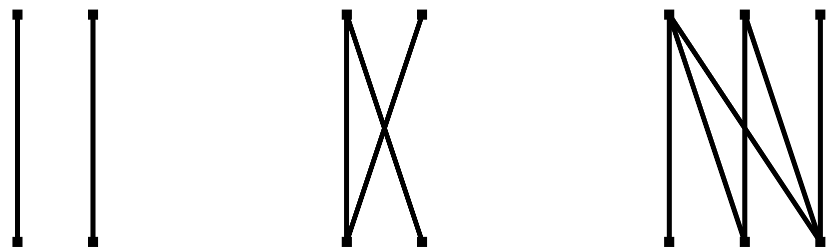

$2K_2$ $\qquad$ P_4

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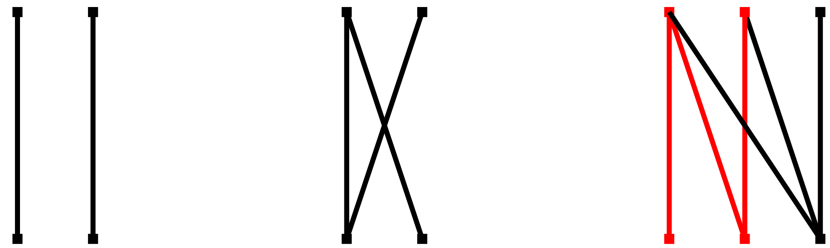


$2K_2$ $\qquad\qquad$ P_4

Let *Forb*$_n(H)$ be the number of bipartite graphs on n vertices which do not contain H as an induced subgraph.

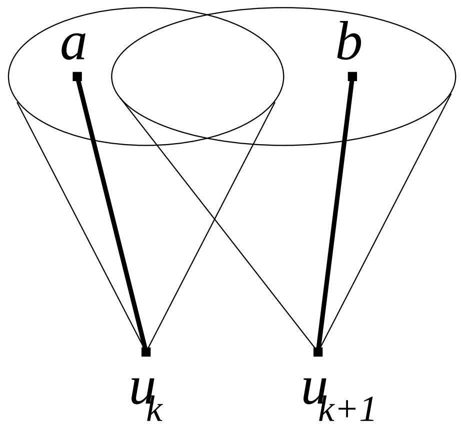
We want to find bounds on *Forb*$_n(H)$ for every bipartite graph H. We are especially interested in those H for which there are bounds of the form n^{cn}.

The neighbourhood $\Gamma(v)$ of a vertex in a graph is the set of vertices adjacent to v. The degree of v is $d(v) = |\Gamma(v)|$.

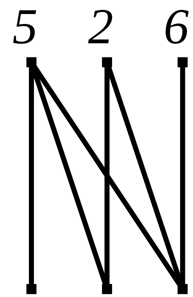
Theorem 1. *Forb*$_n(2K_2) = n^{n+o(n)}$.

Proof. Let G be a $2K_2$-free bipartite graph with parts U, V. Let $u_1, u_2, \ldots$ be the vertices in U in order of increasing degree.

Suppose $\exists a \in \Gamma(u_k) - \Gamma(u_{k+1})$. Then since u_{k+1} has at least as many neighbours as u_k, there exists $b \in \Gamma(u_{k+1}) - \Gamma(u_k)$. But then $\{a, b, u_k, u_{k+1}\}$ induces a copy of $2K_2$.



So $\Gamma(u_k) \subset \Gamma(u_{k+1})$ for each k. We can record the bipartition of G and the following list of n vertices: $\Gamma(u_1), u_1, \Gamma(u_2) - \Gamma(u_1), u_2, \ldots$.



For example, 1 3 4 would give the list $5, 1, 2, 3, 6, 4$.

We can recover G from this recording, so $\mathit{Forb}_n(2K_2) \leq 2^n n^n = n^{n+o(n)}$. It is easy to see that this upper bound is correct. $\square$

General H

Although there are more than $2^{\frac{n^2}{4}}$ bipartite graphs, it is not hard to show, using the Szemerédi Regularity Lemma, that for any H, $\mathit{Forb}_n(H) = 2^{o(n^2)}$.

We want to find lower bounds and better upper bounds.

Cycles

It is easy to show that there are more than n^{cn} bipartite graphs with girth greater than g, for any fixed c and g.

If H contains a cycle of length g, then certainly a graph with girth greater than g does not contain an induced copy of H.

So if H contains a cycle, then *Forb*$_n(H)$ grows rapidly. In fact, *Forb*$_n(H) = 2^{\Omega(n^{\frac{6}{5}})}$.

The bipartite complement $\overline{H}$ of a graph H with bipartition $X \sqcup Y$ is the bipartite graph with the same bipartition and edges between X and Y exactly where edges are not in H.

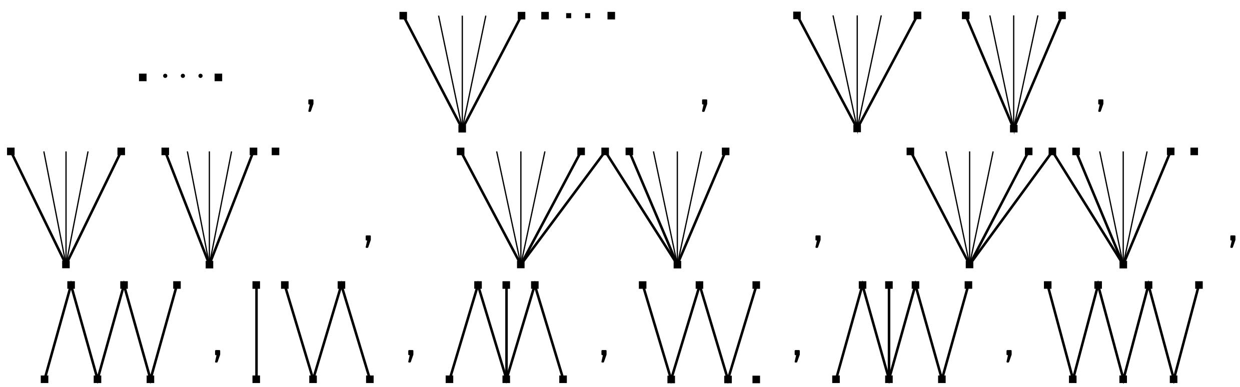
If $\overline{H}$ contains a cycle, then similarly *Forb*$_n(H)$ grows rapidly.

So unless H is a forest whose bipartite complement is a forest, *Forb*$_n(H) = 2^{\Omega(n^{\frac{6}{5}})}$.

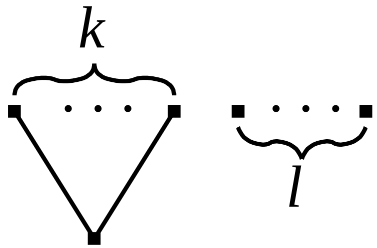
A forest on n vertices has at most $n-1$ edges. The sum of the number of edges in H and in $\overline{H}$ is $|X|.|Y| = |X|(n-|X|)$.

So if $2n-2 < |X|(n-|X|)$, either H or $\overline{H}$ contains a cycle. This inequality must hold unless either $n \leq 7$ or $|X| \leq 2$ (w.l.o.g. $|X| \leq |Y|$).

This deals with all but the following graphs:



If H is a k-star plus l isolated vertices:



then $Forb_n(H) = n^{\frac{n}{2}\max(k-1,l-1)+o(n)}$.

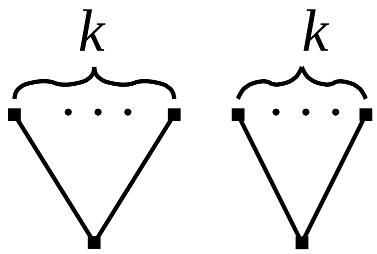
Proof. Let G be an H-free bipartite graph with bipartition $U \sqcup V$. If $u \in U$, then either u has degree at most $k-1$ in G or at most $l-1$ in $\overline{G}$.

W.l.o.g. $|U| \leq \frac{n}{2}$. It follows that

$Forb_n(H) \leq 2^n \left(\binom{|V|}{0} + \ldots + \binom{|V|}{k-1} + \binom{|V|}{l-1} + \ldots + \binom{|V|}{0}\right)^{\frac{n}{2}}$

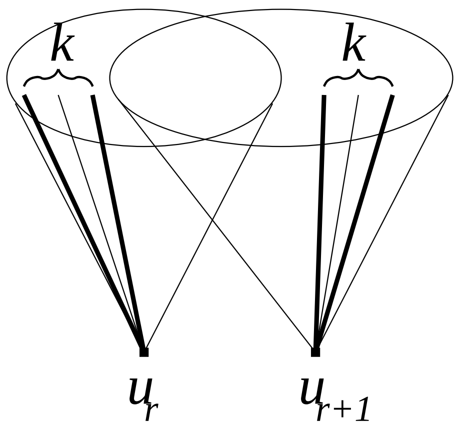
which gives the required upper bound. The lower bound is trivial. $\square$

If H is a disjoint union of two k-stars:



then consider a bipartite graph G with bipartition $U \sqcup V$ which is H-free.

Let $u_1, u_2, \ldots$ be the vertices in U in order of increasing degree. Then $|\Gamma(u_r) - \Gamma(u_{r+1})| \leq k-1$, or u_r, u_{r+1} and their appropriate neighbours would induce a copy of H:



We can think of this as follows: $\Gamma(u_{r+1})$ is $\Gamma(u_r)$ with at most $k-1$ vertices changed, plus $d(u_{r+1}) - d(u_r)$ more vertices.

So we can record the graph G in a similar way to the recording of a $2K_2$-free bipartite graph; we simply have also to record the vertices changed at each r.

There are $\binom{d(u_r)}{k-1}\binom{|V|-d(u_r)}{k-1} \le n^{2k-2+o(1)}$ ways to change $k-1$ vertices. We can assume w.l.o.g. that $|U| \le \frac{n}{2}$, and we get
$\mathit{Forb}_n(H) \le 2^n n^n n^{(2k-2+o(1))\frac{n}{2}} = n^{kn+o(n)}$.

By examining more pairs of vertices in U we can show that $\mathit{Forb}_n(H) \leq n^{\frac{k+1}{2}n+o(n)}$.

This is the correct bound: we can construct an interesting family of graphs which provides a matching lower bound.

In fact, for every bipartite graph H we can prove one of the following:

$$\textit{Forb}_n(H) = 2^{\Omega(n^{\frac{6}{5}})},$$

or we can find both lower and upper bounds of the form $n^{cn+o(n)}$ (we either know the correct value of c or at worst have a gap of 1 between the bounds),

or $H = P_7$, the path on seven vertices.
At present we can only say that
$n^{n+o(n)} \leq \textit{Forb}_n(P_7) \leq 2^{o(n^2)}$.

$\sim\sim\sim\sim$ End $\sim\sim\sim\sim$