Edge Colouring of Multigraphs: Tashkinov Trees and Goldberg's Conjecture

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There are two trivial lower bounds for the **chromatic index** $\chi'(G)$ of a (multi)graph G, namely the **maximum degree** $\Delta(G)$ of G and the **density** $w(G)$ of G; the last graph parameter is defined by

$$w(G) = \max_{H \subseteq G, |V(H)| \geq 2} \left\lceil \frac{|E(H)|}{\lfloor \frac{1}{2}|V(H)| \rfloor} \right\rceil .$$

A famous conjecture made, independently, by Anderson, Goldberg, Gupta and Seymour in the 1970s says that every graph G satisfies

$$\chi'(G) \leq \max\{\Delta(G) + 1, w(G)\}.$$

In 1990 Nishizeki and Kashiwagi proved that $\chi'(G) \leq \max\{(11\Delta(G) + 8)/10, w(G)\}$ for every graph G. The proof was based on the so-called critical chain method. A shorter proof of this result was given by Tashkinov in 2000. The main tool in Tashkinov's proof are Tashkinov trees, a common generalization of both Vizing fans and Kierstaed paths. Based on Tashkinov's method Favrholdt, Stiebitz and Toft proved in 2006 that $\chi'(G) \leq \max\{(13\Delta(G) + 10)/12, w(G)\}$. In 2007 Scheide extended this result to $\chi'(G) \leq \max\{(15\Delta(G) + 12)/14, w(G)\}$. Furthermore, he proved that every graph G satisfy $\chi'(G) \leq \max\{\Delta(G) + \sqrt{\Delta(G)/2}, w(G)\}$ as well as $\chi'(G) \leq \chi'_f(G) + \sqrt{\chi'_f(G)/2}$, where $\chi'_f(G)$ denotes the fractional chromatic index of G. The last result extends a result of Kahn from 1996 as well as a result of Sanders and Steurer from 2005. The proofs of all these results are constructive and based on an extension of Tashkinov's method. In particular, the proof of the inequality $\chi'(G) \leq \max\{\Delta(G) + \sqrt{\Delta(G)/2}, w(G)\} =: \tau(G)$ yields an algorithm that computes, for every graph $G = (V, E)$, an edge colouring of G using at most $\tau(G)$ colours, where the algorithm has time complexity bounded from above by a polynomial in $|V|$ and $|E|$ (and also in Δ).