MA400. September Introductory Course
(Financial Mathematics, Risk & Stochastics)
Exercises 7

1. Suppose that (X_t) is a continuous $(\mathcal{F}_t)$-local martingale such that $X_t \geq 0$ for all $t \geq 0$, $\mathbb{P}$-a.s.. Prove that (X_t) is an $(\mathcal{F}_t)$-supermartingale.

Hint. You may use the definition of a local martingale and Fatou's lemma.

2. Let Y be a random variable defined on $(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$Y \geq 0, \quad \mathbb{P}\text{-a.s.}, \quad \text{and} \quad \mathbb{E}^{\mathbb{P}}\big[Y\big] = 1. \tag{1}$$

Also, define the function $\mathbb{Q} : \mathcal{F} \to \mathbb{R}$ by

$$\mathbb{Q}(A) = \mathbb{E}^{\mathbb{P}}\left[Y \mathbf{1}_A\right] \quad \text{for all } A \in \mathcal{F}. \tag{2}$$

Prove that the function $\mathbb{Q}$ is a probability measure on $(\Omega, \mathcal{F})$.

3. Consider the probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ defined by (1)–(2) in the previous exercise. Prove that if Z is a random variable such that either $Z \geq 0$, $\mathbb{P}$-a.s., or $\mathbb{E}^{\mathbb{Q}}[|Z|] < \infty$, then

$$\mathbb{E}^{\mathbb{Q}}[Z] = \mathbb{E}^{\mathbb{P}}[YZ].$$

4. Consider the probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ defined by (1)–(2). Prove that

$$\mathbb{Q}\big(\{Y = 0\}\big) = 0.$$

5. Consider the probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ defined by (1)–(2), and suppose that $\mathbb{Q} \sim \mathbb{P}$. Show that if $d\mathbb{Q} = Y \, d\mathbb{P}$, then $Y > 0$, $\mathbb{P}$-a.s., and $d\mathbb{P} = Y^{-1} \, d\mathbb{Q}$.

6. Suppose that (L_t) is an $(\mathcal{F}_t)$-martingale with respect to the probability measure $\mathbb{P}$ such that $L_t > 0$, $\mathbb{P}$-a.s., and $\mathbb{E}^{\mathbb{P}}[L_t] = 1$ for all $t \geq 0$. Given a time $T > 0$, define the probability measure $\mathbb{Q}_T$ on the measurable space $(\Omega, \mathcal{F}_T)$ by

$$\mathbb{Q}_T(A) = \mathbb{E}^{\mathbb{P}}\left[L_T \mathbf{1}_A\right], \quad \text{for } A \in \mathcal{F}_T. \tag{3}$$

Given times $0 \leq s \leq t \leq T$, show that, if Z is an $\mathcal{F}_t$-measurable random variable, then

$$\mathbb{E}^{\mathbb{Q}_T}[Z \mid \mathcal{F}_s] = \frac{\mathbb{E}^{\mathbb{P}}[L_t Z \mid \mathcal{F}_s]}{L_s}. \tag{4}$$

7. Suppose that (L_t) is an $(\mathcal{F}_t)$-martingale as in Exercise 6 above. Also, suppose that (M_t) is an $(\mathcal{F}_t)$-martingale with respect to the probability measure $\mathbb{P}$. Prove that the process $(L_t^{-1} M_t,\ t \in [0, T])$, is an $(\mathcal{F}_t)$-martingale with respect to the probability measure $\mathbb{Q}_T$ defined by (3).

8. Given an n-dimensional $(\mathcal{F}_t)$-Brownian motion W and an n-dimensional, $(\mathcal{F}_t)$-progressively measurable process (X_t) satisfying

$$\int_0^t |X_s|^2 \, ds < \infty \quad \text{for all } t \geq 0, \quad \mathbb{P}\text{-a.s.},$$

show that the process (L_t) given by

$$L_t = \exp\left(-\frac{1}{2} \int_0^t |X_s|^2 \, ds + \int_0^t X_s \cdot dW_s \right) \tag{5}$$

satisfies the SDE

$$L_t = 1 + \int_0^t L_s X_s \cdot dW_s. \tag{6}$$

9. Suppose that (X_t) is an n-dimensional, $(\mathcal{F}_t)$-progressively measurable process that is bounded in the sense that

$$\left| X_t^i \right| \leq K \quad \text{for all } t \geq 0 \text{ and } i = 1, \ldots, n, \quad \mathbb{P}\text{-a.s.},$$

where $K > 0$ is a constant. Prove that the process (L_t) defined by (5) or (6) in the previous exercise is an $(\mathcal{F}_t)$-martingale.