MA400. September Introductory Course
(Financial Mathematics, Risk & Stochastics)
Exercises 6

1. Consider a standard one-dimensional Brownian motion (W_t). Use Itô's formula to calculate

$$W_t^2 = t + 2\int_0^t W_s \, dW_s,$$

and

$$W_t^{27} = 351\int_0^t W_s^{25} \, ds + 27\int_0^t W_s^{26} \, dW_s.$$

2. Consider a standard one-dimensional Brownian motion (W_t). Given $k \geq 2$ and $t \geq 0$, use Itô's formula to prove that

$$\mathbb{E}\left[W_t^k\right] = \frac{1}{2}k(k-1)\int_0^t \mathbb{E}\left[W_u^{k-2}\right] du.$$

Use this expression to calculate $\mathbb{E}\left[W_t^4\right]$ and $\mathbb{E}\left[W_t^6\right]$.

Hint: You may assume that all stochastic integrals with respect to a Brownian motion that you encounter in this exercise are martingales, so they have expectation 0.

3. Consider the following stochastic differential equation

$$Z_t = -\int_0^t Z_u \, du + \int_0^t e^{-u} \, dW_u.$$

Prove that its solution is given by

$$Z_t = e^{-t} W_t.$$

4. In Vasicek's interest rate model, the dynamics of the short rate process (r_t) are given by the stochastic differential equation

$$dr_t = k(\vartheta - r_t)\, dt + \sigma \, dW_t, \tag{1}$$

where k, ϑ and σ are strictly positive constants

(a) Show that the solution of (1) is given by

$$r_t = \vartheta + (r_0 - \vartheta)e^{-kt} + \sigma e^{-kt}\int_0^t e^{ks} \, dW_s.$$

Hint: Consider the Itô processes (X_t) and (Y_t) defined by

$$X_t = e^{kt} \quad \text{and} \quad Y_t = r_t,$$

and use the integration by parts formula.

(b) Calculate the mean $\mathbb{E}[r_t]$ and the variance $\mathrm{var}(r_t)$ of the random variable r_t.
Hint. To calculate the variance of r_t, you may use Itô's isometry. Also, you may assume that all stochastic integrals with respect to a Brownian motion that you encounter in this exercise are martingales, so they have expectation 0.

5. In the Cox-Ingersoll-Ross interest rate model, the dynamics of the short rate process (r_t) are given by the stochastic differential equation

$$dr_t = k(\vartheta - r_t)\,dt + \sigma\sqrt{r_t}\,dW_t,$$

where k, ϑ and σ are strictly positive constants. Prove that the stochastic process (r_t) satisfies

$$r_t = \vartheta + (r_0 - \vartheta)e^{-kt} + \sigma e^{-kt}\int_0^t e^{ks}\sqrt{r_s}\,dW_s,$$

and then calculate the mean $\mathbb{E}[r_t]$ and the variance $\mathrm{var}(r_t)$ of the random variable r_t.

Hint. You may assume that all stochastic integrals with respect to a Brownian motion that you encounter in this exercise are martingales, so they have expectation 0.

6. Consider a standard one-dimensional Brownian motion (W_t), and the Itô process given by

$$dX_t = te^{W_t}\,dt + \cos(t^2W_t)\,dW_t, \quad X_0 = \sqrt{\pi}.$$

Also, let (Z_t) be the Itô process defined by

(i) $Z_t = \sin(tX_t)$, or

(ii) $Z_t = X_t\exp(t^2X_t)$, or

(iii) $Z_t = X_t^3 + t\cos(X_t)$.

In each of these cases, use Itô's formula to provide expressions for the constant Z_0, and the processes (A_t) and (C_t) such that

$$Z_t = Z_0 + \int_0^t A_s\,ds + \int_0^t C_s\,dW_s.$$

7. Consider the exponential martingale (L_t) defined by the stochastic differential equation

$$dL_t = \vartheta L_t\,dW_t, \quad L_0 = 1,$$

where ϑ is a constant, and let (π_t) be the process defined by

$$\pi_t = \frac{L_t}{1+L_t}.$$

Prove that (π_t) satisfies the stochastic differential equation

$$d\pi_t = -\vartheta^2\pi_t^2\,(1-\pi_t)\,dt + \vartheta\pi_t\,(1-\pi_t)\,dW_t.$$