MA400. September Introductory Course (Financial Mathematics, Risk & Stochastics) Exercises 5

1. Let $X_0 = 1$, and let $X_1, X_2, \ldots$ be a sequence of independent positive random variables with

$$\mathbb{E}\big[X_k\big] = 1 \quad \text{for all } k.$$

Define

$$M_n = X_0 X_1 \cdots X_n, \quad \text{for } n \geq 0,$$

and let $\mathcal{F}_n$ be the σ-algebra generated by the the random variables $X_0, X_1, \ldots, X_n$. Prove that the process (M_n) is an $(\mathcal{F}_n)$-martingale.

2. Let $X_0 = 0$, and let $X_1, X_2, \ldots$ be a sequence of random variables such that $\mathbb{E}[|X_k|] < \infty$ for all $k \geq 1$. Also, let $\mathcal{F}_n$ be the σ-algebra generated by the the random variables $X_0, X_1, \ldots, X_n$, and define

$$M_0 = X_0 \quad \text{and} \quad M_n = \sum_{i=1}^{n} \big(X_i - \mathbb{E}\big[X_i \mid \mathcal{F}_{i-1}\big]\big), \quad \text{for } n \geq 1.$$

Prove that the process (M_n) is an $(\mathcal{F}_n)$-martingale.

3. Consider a filtration $(\mathcal{F}_n)$ and an $(\mathcal{F}_n)$-adapted stochastic process (X_n) such that $X_0 = 0$ and $\mathbb{E}\big[|X_n|\big] < \infty$ for all $n \geq 0$. Also, let (c_n) be a sequence of constants. Define $M_0 = 0$ and

$$M_n = c_n X_n - \sum_{j=1}^{n} c_j \mathbb{E}\big[X_j - X_{j-1} \mid \mathcal{F}_{j-1}\big] - \sum_{j=1}^{n} \big(c_j - c_{j-1}\big) X_{j-1}, \quad \text{for } n \geq 1.$$

Prove that (M_n) is an $(\mathcal{F}_n)$-martingale.

4. Let (W_t) be a standard one-dimensional Brownian motion. Given times $r < s < t < u$, calculate the expectations

(i) $\mathbb{E}\left[(W_t - W_s)(W_s - W_r)\right]$,

(ii) $\mathbb{E}\left[(W_u - W_t)^2 (W_s - W_r)^2\right]$,

(iii) $\mathbb{E}\left[(W_u - W_s)(W_t - W_r)\right]$,

(iv) $\mathbb{E}\left[(W_t - W_r)(W_s - W_r)^2\right]$, and

(v) $\mathbb{E}\left[W_r W_s W_t\right]$.

5. *Scaling of the standard Brownian motion.* Let (W_t) be a standard Brownian motion. Given a constant $c > 0$, show that the stochastic process (X_t) defined by

$$X_t = \frac{1}{\sqrt{c}} W_{ct}, \quad \text{for } t \geq 0,$$

is a standard Brownian motion.

6. Suppose that the process (W_t) is a standard one-dimensional $(\mathcal{F}_t)$-Brownian motion.

(I) Prove that the process (X_t) defined by

$$X_t = W_t^2 - t, \quad \text{for } t \geq 0,$$

is an $(\mathcal{F}_t)$-martingale.
Hint. Observe that the $(\mathcal{F}_t)$-martingale property of (X_t) is equivalent to

$$\mathbb{E}[W_t^2 \mid \mathcal{F}_s] - W_s^2 = t - s \quad \text{for all } s < t. \tag{1}$$

Then consider $\mathbb{E}[(W_t - W_s)^2 \mid \mathcal{F}_s]$ and prove (1).

(II) Prove that the process (Y_t) defined by

$$Y_t = \exp\left(-\frac{1}{2}\theta^2 t - \theta W_t\right), \quad \text{for } t \geq 0,$$

is an $(\mathcal{F}_t)$-martingale.
Hint. Given times $s < t$, you can use Exercise 3.11 to calculate $\mathbb{E}[Y_t Y_s^{-1} \mid \mathcal{F}_s]$.