MA400. September Introductory Course
(Financial Mathematics, Risk and Stochastics)
Exercises 4

1. Prove the following statements:

(i) If (Z_k) is a sequence of positive random variables (i.e., $Z_k \geq 0$ for all k), then

$$\mathbb{E}\left[\sum_{k=1}^{\infty} Z_k\right] = \sum_{k=1}^{\infty} \mathbb{E}\big[Z_k\big] \leq \infty.$$

Hint: You may use the monotone convergence theorem.

(ii) If (Z_k) is a sequence of positive random variables such that $\sum_{k=1}^{\infty} \mathbb{E}\big[Z_k\big] < \infty$, then

$$\sum_{k=1}^{\infty} Z_k < \infty, \ \mathbb{P}\text{-a.s.}, \quad \text{which implies that} \quad \lim_{k\to\infty} Z_k = 0, \ \mathbb{P}\text{-a.s.}.$$

2. Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and an event $B \in \mathcal{F}$ with $\mathbb{P}(B) > 0$. Prove that the function $\mathbb{P}(\cdot \mid B) : \mathcal{F} \to \mathbb{R}$ defined by

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}, \quad \text{for } A \in \mathcal{F},$$

is a probability measure on $(\Omega, \mathcal{F})$.

3. A laboratory blood test is 95% effective in detecting a certain disease when it is present. However, the test also yields a 'false positive' result for 2% of healthy people tested. If 0.1% of the population actually have the disease, what is the probability that a person has the disease, given that his test result is positive?

4. An insurance company classifies drivers as class X, Y or Z. Experience indicates that the probability that a class X driver has at least one accident in any given year is 0.01, while the corresponding probabilities for classes Y and Z are 0.05 and 0.10, respectively. The company has also found that, of the drivers who apply for cover, 30% are class X, 60% class Y and 10% class Z.

i) A certain new client had an accident within one year. What is the probability that he is a class Z risk?

ii) Another client goes for n years without an accident. Assuming the incidence of accidents in different years to be independent, how large must n be before the company decides that she is more likely to belong to class X than to class Y?

5. Suppose that a random variable X has the geometric distribution with parameter p, so that

$$\mathbb{P}(X = j) = p(1-p)^{j-1}, \quad \text{for } j = 1, 2, \ldots.$$

Show that, given any $n, k = 1, 2, \ldots,$

$$\mathbb{P}(X = n + k \mid X > n) = \mathbb{P}(X = k).$$

6. If X and Y are independent Poisson random variables with parameters λ and μ, respectively, show that

(i) $X + Y$ is a Poisson random variable with parameter $\lambda + \mu$,

(ii) The conditional distribution of X given that $X + Y = n$ is binomial with parameters n, $\frac{\lambda}{\lambda+\mu}$.

7. A certain region is inhabited by two types of insect. Each insect caught will be of type 1 with probability p and type 2 with probability $1 - p$, independently of previous catches. Suppose that a random number N of catches are made, and the number of type 1 insects caught is X.

i) For $n = 0, 1, 2, \ldots$, find $\mathbb{E}[X \mid N = n]$.

ii) If $\mathbb{E}[N] = \mu$, find $\mathbb{E}[X]$.

8. Let X be a simple random variable. Given an event A, describe explicitly a version of the conditional probability $\mathbb{P}\big(A \mid \sigma(X)\big)$.

9. Suppose that a random variable X is equal to a constant c, $\mathbb{P}$-a.s.. Show that, given any σ-algebra $\mathcal{G}$, $\mathbb{E}[X \mid \mathcal{G}] = c$.

Hint. You may use the following property of the expectation operator that you are not required to prove here: if Z_1, Z_2 are random variables such that $Z_1 = Z_2$, $\mathbb{P}$-a.s., then $\mathbb{E}[Z_1] = \mathbb{E}[Z_2]$.

10. Suppose that X is a random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $X \geq 0$, $\mathbb{P}$-a.s., and $\mathbb{E}\big[X\big] < \infty$. Given a σ-algebra $\mathcal{H} \subseteq \mathcal{F}$, prove that

$$\mathbb{E}\big[X \mid \mathcal{H}\big] \geq 0, \quad \mathbb{P}\text{-a.s.}.$$

Hint. You may use the following property of the expectation operator that you are not required to prove here: if Z is a random variable such that $Z \geq 0$, $\mathbb{P}$-a.s., then $\mathbb{E}[Z] \geq 0$.

11. Suppose that X is a random variable in $\mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$. Prove that

$$\mathbb{E}\big[X \mid \{\Omega, \emptyset\}\big] = \mathbb{E}\big[X\big].$$

12. Consider random variables $X, X_1, X_2 \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ and two σ-algebras $\mathcal{G}, \mathcal{H} \subseteq \mathcal{F}$. Use the definition of conditional expectation to prove the following properties:

(i) If Y is a version of $\mathbb{E}[X \mid \mathcal{G}]$, then $\mathbb{E}[Y] = \mathbb{E}[X]$.

(ii) (*Linearity*) Given any constants $a_1, a_2 \in \mathbb{R}$,

$$\mathbb{E}[a_1 X_1 + a_2 X_2 \mid \mathcal{G}] = a_1 \mathbb{E}[X_1 \mid \mathcal{G}] + a_2 \mathbb{E}[X_2 \mid \mathcal{G}], \quad \mathbb{P}\text{-a.s.}.$$

Hint. To answer this question, you can use the linearity of expectation, which you are not required to prove here.

(iii) (*Tower property*) If $\mathcal{H} \subseteq \mathcal{G}$, then

$$\mathbb{E}\big[\mathbb{E}[X \mid \mathcal{G}] \mid \mathcal{H}\big] = \mathbb{E}[X \mid \mathcal{H}], \quad \mathbb{P}\text{-a.s.}.$$