MA400. September Introductory Course
(Financial Mathematics, Risk & Stochastics)
Exercises 3

1. Prove that three events A_1, A_2, A_3 are independent if

$$\begin{aligned}\mathbb{P}(A_1 \cap A_2 \cap A_3) &= \mathbb{P}(A_1)\mathbb{P}(A_2)\mathbb{P}(A_3),\\ \mathbb{P}(A_1 \cap A_2) &= \mathbb{P}(A_1)\mathbb{P}(A_2),\\ \mathbb{P}(A_1 \cap A_3) &= \mathbb{P}(A_1)\mathbb{P}(A_3),\\ \mathbb{P}(A_2 \cap A_3) &= \mathbb{P}(A_2)\mathbb{P}(A_3).\end{aligned}$$

2. Suppose that $\Omega = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $\mathcal{F}$ is the collection of all subsets of Ω, and the probability measure $\mathbb{P}$ assigns mass $\frac{1}{8}$ on each point of Ω.

(i) Are the following events independent?

$$A_1 = \{1, 2, 3, 4\}, \quad A_2 = \{5, 6, 7, 8\}.$$

(ii) Are the following events independent?

$$B_1 = \{1, 2, 3, 4\}, \quad B_2 = \{3, 4, 5, 6\}, \quad B_3 = \{2, 4, 6, 8\}.$$

(iii) Are the following events independent?

$$C_1 = \{1, 2, 3, 4\}, \quad C_2 = \{3, 4, 5, 6\}, \quad C_3 = \{3, 4, 7, 8\}.$$

(iv) Are the following events independent?

$$D_1 = \{1, 2, 3, 4\}, \quad D_2 = \{4, 5, 6, 7\}, \quad D_3 = \{4, 6, 7, 8\}.$$

3. Prove that if $A \cap B = \emptyset$, then A and B *cannot* be independent unless $\mathbb{P}(A) = 0$ or $\mathbb{P}(B) = 0$.

4. Suppose that the events $A, B, C \in \mathcal{F}$ form a partition of Ω and have probabilities $\mathbb{P}(A) = \mathbb{P}(B) = \mathbb{P}(C) = \frac{1}{3}$. Also, let X and Y be the random variables defined by

$$X(\omega) = \begin{cases} 1, & \text{if } \omega \in A, \\ 0, & \text{if } \omega \in B \cup C, \end{cases} \qquad Y(\omega) = \begin{cases} 1, & \text{if } \omega \in A, \\ 2, & \text{if } \omega \in B \\ 0, & \text{if } \omega \in C. \end{cases}$$

(i) Is it true that $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$?

(ii) Are X and Y independent?

5. Show that, if X and Y are independent random variables, then

$$\operatorname{var}(X + Y) = \operatorname{var}(X) + \operatorname{var}(Y).$$

6. Suppose that X has the geometric distribution with parameter $p \in (0, 1)$, i.e.,

$$\mathbb{P}(X = n) = (1 - p)p^{n-1}, \quad \text{for } n = 1, 2, \dots.$$

Calculate the expectation of the random variable $Y = \left(\frac{1}{2}\right)^X$.

7. The *moment generating function* M_X (or simply M if there is no ambiguity) of a random variable X is defined by

$$M_X(t) = \mathbb{E}\left[e^{tX}\right].$$

Assuming that $M_X(t)$ is finite for all $t \in [-\varepsilon, \varepsilon]$, for some $\varepsilon > 0$, show formally that

$$\mathbb{E}\big[X^n\big] = M^{(n)}(0), \quad \text{for } n = 1, 2, \dots,$$

where $M^{(n)}$ is the n-th derivative of M.

8. **(i)** Suppose that Z is a Bernoulli random variable with parameter p, i.e.,

$$\mathbb{P}(X = 0) = 1 - p \quad \text{and} \quad \mathbb{P}(X = 1) = p.$$

What is the moment generating function of Z?

(ii) Suppose that X is a binomial random variable with parameters n, p. Find the moment generating function of X.

Hint. There are two ways to derive this result:

(1) Start from the definition of moment generating functions and use the binomial expansion formula

$$(a + b)^n = \sum_{i=0}^{n} \frac{n!}{i!(n-i)!} a^i b^{n-i}.$$

(2) Use the fact that X has the same distribution as $Z_1 + Z_2 + \cdots + Z_n$, where $Z_1, Z_2, \dots, Z_n$ are independent Bernoulli random variables, each with parameter p.

(iii) Suppose that X is a binomial random variable with parameters n, p. Calculate its mean and its variance from its moment generating function.

(iv) Suppose that X and Y are independent binomial random variables with parameters n, p and m, p, respectively. What is the distribution of the random variable $X + Y$?

9. Suppose that a random variable X has the uniform distribution over the interval (a, b), i.e., X has the probability density function given by

$$f(x) = \begin{cases} 1/(b - a), & \text{if } a < x < b, \\ 0, & \text{otherwise.} \end{cases}$$

Find the moment generating function of X.

10. Suppose that X is a normal random variable with parameters m, σ. Show that the moment generating function of X is given by

$$M(t) = e^{mt + \frac{\sigma^2 t^2}{2}}.$$

Calculate: **(i)** the mean of X, **(ii)** the variance of X, and **(iii)** $\mathbb{E}[X^4]$.

Hint: You may use the result of Exercise 6 above.

11. Suppose that U is an *exponential* random variable with parameter $\mu > 0$, so that the probability density function of U is given by

$$f(u) = \mu e^{-\mu u}.$$

Calculate: **(i)** the moment generating function M_U, and **(ii)** the mean and the variance of U.

12. Prove that, if Z is a normal random variable with mean 0 and variance σ^2, then

$$\mathbb{E}\left[\exp\left(-\frac{\sigma^2}{2} - Z\right)\right] = 1.$$