# MA400. September Introductory Course
# (Financial Mathematics, Risk & Stochastics)
# Exercises 2

**1.** Consider a real-valued random variable $X$ and a function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f^{-1}(C) = \big\{a \in \mathbb{R} \mid \ f(a) \in C\big\} \in \mathcal{B}(\mathbb{R}) \quad \text{for all } C \in \mathcal{B}(\mathbb{R}).$$

Show that $\sigma\big(f(X)\big) \subseteq \sigma(X)$, and conclude that $f(X)$ is a random variable.

**2.** Suppose that a random variable $X$ can take only four possible values, i.e., suppose that there exist distinct $x_1, x_2, x_3, x_4 \in \mathbb{R}$ such that

$$X(\omega) \in \{x_1, x_2, x_3, x_4\} \quad \text{for all } \omega \in \Omega.$$

Describe explicitly the $\sigma$-algebra $\sigma(X)$ generated by $X$.

**3.** Given a random variable $X$, prove that

**(i)** if $\mathcal{H} = \{\emptyset, \Omega\}$, then $X$ is $\mathcal{H}$-measurable if and only if $X$ is constant, and

**(ii)** if $\mathcal{H}$ is a $\sigma$-algebra such that $X$ is $\mathcal{H}$-measurable and $\mathbb{P}(A) = 0$ or $1$ for every $A \in \mathcal{H}$, then $\mathbb{P}(X = c) = 1$, for some constant $c$.

**4.** Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a measurable space $(S, \mathcal{S})$, and let $X$ be an $(S, \mathcal{S})$-valued random variable defined on $(\Omega, \mathcal{F})$, i.e., let $X$ be a function mapping $\Omega$ into $S$ such that

$$X^{-1}(A) = \big\{\omega \in \Omega \mid \ X(\omega) \in A\big\} \in \mathcal{F} \quad \text{for all } A \in \mathcal{S}.$$

Also, define the function $\overline{\mathbb{P}} : \mathcal{S} \to [0, 1]$ by

$$\overline{\mathbb{P}}(A) = \mathbb{P}\big(\{\omega \in \Omega \mid \ X(\omega) \in A\}\big) \quad \text{for } A \in \mathcal{S}.$$

Prove that $(S, \mathcal{S}, \overline{\mathbb{P}})$ is a probability space.

*Remark.* Suppose that $S = \mathbb{R}$ and $\mathcal{S} = \mathcal{B}(\mathbb{R})$, so that $X$ is a real-valued random variable. In this case, compare the relevant definitions to conclude that

$$\overline{\mathbb{P}}\big((-\infty, a]\big) = F(a) \quad \text{for all } a \in \mathbb{R},$$

where $F$ is the distribution function of $X$.

**5.** Consider a random variable $X$ with distribution function $F$. Prove the following results:

$$\text{(i) } \mathbb{P}(a < X \leq b) = F(b) - F(a),$$
$$\text{(ii) } \mathbb{P}(a \leq X \leq b) = F(b) - F(a-),$$
$$\text{(iii) } \mathbb{P}(X = a) = F(a) - F(a-).$$

In (ii) and (iii), $F(a-)$ is the left-hand limit of $F$ at $a$, i.e., $F(a-) = \lim_{c \uparrow a} F(c)$.

**6.** Consider tossing a coin that lands heads with probability $p \in (0,1)$ three times, and let $X$ be the number of heads observed. Determine the distribution function of $X$.

**7.** Which of the following functions are probability distribution functions?

$$\text{(i) } F(x) = \begin{cases} 0, & \text{if } x \le 0, \\ 1 - 0.3e^{-x}, & \text{if } x > 0, \end{cases}$$

$$\text{(ii) } F(x) = \begin{cases} 0, & \text{if } x < 0, \\ 0.5, & \text{if } 0 \le x < 2, \\ 0.3, & \text{if } 2 \le x < 4, \\ 1, & \text{if } 4 \le x, \end{cases}$$

$$\text{(iii) } F(x) = \begin{cases} 0, & \text{if } x < 0, \\ 0.3(1 - e^{-x}), & \text{if } x \ge 0. \end{cases}$$

**8.** **(i)** Give an example of a probability distribution function $F$ that has infinite discontinuities.

**(ii)** Prove that a probability distribution function $F$ has at most countably many discontinuities.

*Hint*: Recalling that $F$ is an increasing function with values in $[0,1]$, how many points $x$ such that $F(x) - F(x-) \in \left(\frac{1}{n+1}, \frac{1}{n}\right]$ can we have for each $n \ge 1$?