MA400. September Introductory Course
(Financial Mathematics, Risk & Stochastics)
Exercises 1

1. Suppose that $\Omega = \mathbb{R}$. Which of the following families of sets are σ-algebras on Ω?

$$\begin{aligned}
\text{(i) } \mathcal{F} &= \big\{\Omega, \emptyset, (-\infty, a], (b, \infty) \mid \ a, b \in \mathbb{R}\big\}, \\
\text{(ii) } \mathcal{F} &= \big\{A \mid \ \text{either } A \text{ or } A^c \text{ is countable}\big\}, \\
\text{(iii) } \mathcal{F} &= \big\{\mathbb{R}, \emptyset, (-\infty, 5], (5, \infty), (-\infty, 3), [3, \infty), [3, 5], (-\infty, 3) \cup (5, \infty)\big\}.
\end{aligned}$$

2. Find the σ-algebra on Ω generated by $\mathcal{C}$ if

$$\begin{aligned}
\text{(i) } \Omega = \mathbb{R} \quad &\text{and} \quad \mathcal{C} = \big\{(-20, \sqrt{2}), (-15, \infty)\big\}, \\
\text{(ii) } \Omega = \mathbb{R} \quad &\text{and} \quad \mathcal{C} = \big\{(1, 2], \{2\}\big\}, \\
\text{(iii) } \Omega = \{1, 2, 3, 4\} \quad &\text{and} \quad \mathcal{C} = \big\{\emptyset, \{2, 3\}\big\}, \\
\text{(iv) } \Omega = \{1, 2, 3, 4\} \quad &\text{and} \quad \mathcal{C} = \big\{\{3\}, \{2, 3, 4\}\big\}.
\end{aligned}$$

3. Consider a measurable space $(\Omega, \mathcal{F})$ and any set $\Omega' \subseteq \Omega$. Prove that the family of sets

$$\mathcal{H} = \big\{A \cap \Omega' \mid \ A \in \mathcal{F}\big\}$$

is a σ-algebra on Ω'.

4. (*First Borel-Cantelli lemma*) Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and any sequence of events $A_1, A_2, \ldots, A_n, \ldots \in \mathcal{F}$ such that

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty. \tag{1}$$

Prove that

$$\mathbb{P}\left(\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m\right) = 0. \tag{2}$$

Note: The event $\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m$ is also called "A_n infinitely often (i.o.)" and is denoted by "$\limsup_{n\to\infty} A_n$".

Hint: Observe that the sequence of events $B_n := \bigcup_{m=n}^{\infty} A_m$ is decreasing, and use the "continuity" of a probability measure.

5. Let $(S, \mathcal{S}, \mu)$ be a measure space. Prove that μ is finitely-additive, i.e., show that

$$\mu\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu(A_i) \tag{3}$$

for any pairwise disjoint events $A_1, A_2, \dots, A_n \in \mathcal{S}$.

6. Consider two sets Ω, S and a function $f : \Omega \to S$. Given a set $A \subseteq S$, the inverse image $f^{-1}(A)$ of A under f is defined by

$$\begin{aligned} f^{-1}(A) &= \{\omega \in \Omega \mid \; f(\omega) \in A\} \\ &= \{\omega \in \Omega \mid \; \text{there exists } x \in A \text{ such that } f(\omega) = x\}\,. \end{aligned} \tag{4}$$

Given a collection $(A_i,\; i \in I)$ of subsets of S, where $I \neq \emptyset$ is an index set, we can see that

$$\begin{aligned} \omega \in f^{-1}\left(\bigcup_{i \in I} A_i\right) \quad &\Leftrightarrow \quad \text{there exists } x \in \bigcup_{i \in I} A_i \text{ such that } f(\omega) = x \\ &\Leftrightarrow \quad \text{for some } i \in I, \text{ there exists } x \in A_i \text{ such that } f(\omega) = x \\ &\Leftrightarrow \quad \text{there exists } i \in I \text{ such that } \omega \in f^{-1}(A_i) \\ &\Leftrightarrow \quad \omega \in \bigcup_{i \in I} f^{-1}(A_i). \end{aligned} \tag{5}$$

These equivalences prove that

$$f^{-1}\left(\bigcup_{i \in I} A_i\right) = \bigcup_{i \in I} f^{-1}(A_i). \tag{6}$$

Use a similar reasoning to show that

$$f^{-1}\left(\bigcap_{i \in I} A_i\right) = \bigcap_{i \in I} f^{-1}(A_i). \tag{7}$$

Also, show that

$$f^{-1}(S \setminus A) = \Omega \setminus f^{-1}(A) \quad \text{for all } A \subseteq S. \tag{8}$$