MA400. September Introductory Course
(Financial Mathematics, Risk & Stochastics)
P7. Itô calculus

1. Throughout this chapter, we fix a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ carrying a standard $(\mathcal{F}_t)$-Brownian motion (W_t). We also denote by $(\mathcal{F}_t^W)$ the natural filtration of W.

   For technical reasons, we assume that every filtration $(\mathcal{G}_t)$ we consider is right-continuous, i.e., $\mathcal{G}_t = \bigcap_{\varepsilon>0} \mathcal{G}_{t+\varepsilon}$ for all $t \geq 0$, as well as augmented by the $\mathbb{P}$-negligible sets in $\mathcal{G}_\infty = \sigma\left(\bigcup_{t\geq 0} \mathcal{G}_t\right)$; we do not expand on such issues here.

Itô Integrals

2. The theory of Itô calculus presents one successful answer to how we can make sense to the integral

$$\int_0^t K_s \, dW_s.$$

   We assume that the integrand (K_t) is $(\mathcal{F}_t)$-*progressively measurable*. This measurability assumption is slightly stronger than assuming that (K_t) is $(\mathcal{F}_t)$-adapted.

   Note that every $(\mathcal{F}_t)$-adapted process with continuous (or, more generally, with right-continuous and left-limited) sample paths is $(\mathcal{F}_t)$-progressively measurable.

   We also assume that the sample paths of (K_t) are "reasonable" in the sense that

$$\int_0^t K_s^2 \, ds < \infty \quad \text{for all } t \geq 0, \ \mathbb{P}\text{-a.s.}.$$

3. *Definition.* (K_t) is a *simple* process if there exist times $0 = t_0 < t_1 < \cdots < t_n = T$ and $\mathcal{F}_{t_j}$-measurable random variables $\bar{K}_j$, $j = 0, 1, \ldots, n-1$, such that

$$K_t = \sum_{j=0}^{n-1} \bar{K}_j \mathbf{1}_{[t_j, t_{j+1})}(t).$$

4. *Definition.* The stochastic integral of a simple process (K_t) as in Definition 3 is defined by

$$\int_0^T K_s \, dW_s = \sum_{j=0}^{n-1} \bar{K}_j (W_{t_{j+1}} - W_{t_j}).$$

5. One construction of the Itô integral starts from stochastic integrals of simple processes as above, and then appeals to a density argument based on the Itô isometry. In particular, if (K_t) is an integrand satisfying the assumptions discussed informally in Paragraph 2 above, then its stochastic integral satisfies the *Itô isometry*:

$$\begin{aligned} \mathbb{E}\left[\left(\int_0^T K_s\,dW_s\right)^2\right] &= \mathbb{E}\left[\int_0^T K_s^2\,ds\right] \\ &= \int_0^T \mathbb{E}\left[K_s^2\right]\,ds. \end{aligned} \tag{1}$$

6. *Definition.* A stochastic process (X_t) with continuous sample paths such that X_0 is a constant, $\mathbb{P}$-a.s., is an $(\mathcal{F}_t)$-*local martingale* if there exists a sequence (τ_n) of $(\mathcal{F}_t)$-stopping times such that

   (i) $\lim_{n\to\infty} \tau_n = \infty, \quad \mathbb{P}$-a.s.,

   (ii) the process $(X_t^{\tau_n})$ defined by $X_t^{\tau_n} = X_{t\wedge\tau_n}$ is an $(\mathcal{F}_t)$-martingale.

   Here, $a \wedge b = \min(a, b)$.

7. It is important to remember that

   - every $(\mathcal{F}_t)$-martingale is an $(\mathcal{F}_t)$-local martingale;
   - there are $(\mathcal{F}_t)$-local martingales that are *NOT* $(\mathcal{F}_t)$-martingales;
   - $(\mathcal{F}_t)$-local martingales are "awkward" to work with, but they have important applications, e.g., in the modelling of financial bubbles.

8. Every Itô integral is an $(\mathcal{F}_t)$-local martingale.

   An Itô integral that satisfies

$$\mathbb{E}\left[\left(\int_0^T K_s\,dW_s\right)^2\right] < \infty \quad \text{for all } T \geq 0$$

   is a *square integrable* martingale.

   Note that *not every* martingale is a square integrable one.

Itô's formula

9. *Itô processes* follow from the definition of stochastic integrals. The expression

$$dX_t = a_t\,dt + b_t\,dW_t \tag{2}$$

is short for

$$X_t = X_0 + \int_0^t a_s\,ds + \int_0^t b_s\,dW_s, \quad t \geq 0.$$

Here, we assume that (a_t) and (b_t) are processes that satisfy assumptions ensuring that the two integrals in this expression are well-defined.

10. Itô's formula can be memorised by recalling Taylor's series expansion of a smooth function and using the expressions

$$(dW_t)^2 = dt, \quad dW_t\,dt = 0, \quad (dt)^2 = 0, \tag{3}$$

which imply that, if X is the Itô process given by (2), then

$$\begin{aligned}(dX_t)^2 &= a_t^2\,(dt)^2 + 2a_tb_t\,dW_t\,dt + b_t^2\,(dW_t)^2 \\ &= b_t^2\,dt.\end{aligned} \tag{4}$$

At this point, it should be stressed that the expressions (3) and (4) are not rigorous mathematics: one should use them as a mnemonic rule only!

11. Given a $C^{1,2}$ function $(t,x) \mapsto f(t,x)$ and the Itô process (X_t) given by (2), *Itô's lemma* states that the stochastic process (F_t) defined by $F_t = f(t,X_t)$ is also an Itô process. In particular, *Itô's formula* provides the expression

$$\begin{aligned}df(t,X_t) &= f_t(t,X_t)\,dt + f_x(t,X_t)\,dX_t + \frac{1}{2}f_{xx}(t,X_t)\,(dX_t)^2 \\ &= \left[f_t(t,X_t) + a_tf_x(t,X_t) + \frac{1}{2}b_t^2f_{xx}(t,X_t)\right]dt + b_tf_x(t,X_t)\,dW_t,\end{aligned} \tag{5}$$

where

$$f_t(t,x) = \frac{\partial f(t,x)}{\partial t}, \quad f_x(t,x) = \frac{\partial f(t,x)}{\partial x} \quad \text{and} \quad f_{xx}(t,x) = \frac{\partial^2 f(t,x)}{\partial x^2}.$$

The following is a useful special case:

$$\begin{aligned}df(t,W_t) &= f_t(t,W_t)\,dt + f_x(t,W_t)\,dW_t + \frac{1}{2}f_{xx}(t,W_t)\,(dW_t)^2 \\ &= \left[f_t(t,W_t) + \frac{1}{2}f_{xx}(t,W_t)\right]dt + f_x(t,W_t)\,dW_t.\end{aligned} \tag{6}$$

12. If f does not depend explicitly on time, i.e., if $x \mapsto f(x)$ is a C^2 function, then Itô's formula takes the form

$$
\begin{aligned}
df(X_t) &= f'(X_t)\,dX_t + \frac{1}{2}f''(X_t)\,(dX_t)^2 \\
&= \left[a_t f'(X_t) + \frac{1}{2}b_t^2 f''(X_t)\right] dt + b_t f'(X_t)\,dW_t,
\end{aligned} \tag{7}
$$

where f' and f'' are the first and the second derivative of f, respectively. Also,

$$
\begin{aligned}
df(W_t) &= f'(W_t)\,dW_t + \frac{1}{2}f''(W_t)\,(dW_t)^2 \\
&= \frac{1}{2}f''(W_t)\,dt + f'(W_t)\,dW_t.
\end{aligned} \tag{8}
$$

13. *Example*. The solution of the stochastic equation

$$
dS_t = \mu_t S_t\,dt + \sigma_t S_t\,dW_t \tag{9}
$$

is given by

$$
S_t = S_0 \exp\left(\int_0^t \left(\mu_u - \frac{1}{2}\sigma_u^2\right) du + \int_0^t \sigma_u\,dW_u\right). \tag{10}
$$

We can verify this claim in two ways:

Way 1. Noting that

$$
\frac{d\ln s}{ds} = \frac{1}{s} \quad \text{and} \quad \frac{d^2 \ln s}{ds^2} = -\frac{1}{s^2},
$$

we can use (7) to calculate

$$
\begin{aligned}
d\ln S_t &= \frac{1}{S_t}\,dS_t + \frac{1}{2}\left(-\frac{1}{S_t^2}\right)(dS_t)^2 \\
&= \frac{1}{S_t}\left[\mu_t S_t\,dt + \sigma_t S_t\,dW_t\right] - \frac{1}{2S_t^2}\left(\sigma_t S_t\right)^2\,dt \\
&= \left(\mu_t - \frac{1}{2}\sigma_t^2\right) dt + \sigma_t\,dW_t,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\ln S_t - \ln S_0 &= \int_0^t d\ln S_u \\
&= \int_0^t \left(\mu_u - \frac{1}{2}\sigma_u^2\right) du + \int_0^t \sigma_u\,dW_u.
\end{aligned}
$$

It follows that

$$\begin{aligned}
S_t &= e^{\ln S_t} \\
&= \exp\left(\ln S_0 + \int_0^t \left(\mu_u - \frac{1}{2}\sigma_u^2\right) du + \int_0^t \sigma_u \, dW_u\right),
\end{aligned}$$

which establishes that the solution of (9) is given by (10).

Way 2. We consider the Itô process

$$dX_t = \left(\mu_t - \frac{1}{2}\sigma_t\right) dt + \sigma_t \, dW_t,$$

and we define $f(x) = S_0 e^x$, so that

$$f'(x) = f''(x) = f(x).$$

Using Itô's formula (7), we can see that the process (S_t) defined by (10) satisfies

$$\begin{aligned}
dS_t &= df(X_t) \\
&= f'(X_t) \, dX_t + \frac{1}{2} f''(X_t) \, (dX_t)^2 \\
&= \left(\mu_t - \frac{1}{2}\sigma_t^2\right) f(X_t) \, dt + \sigma_t f(X_t) \, dW_t + \frac{1}{2}\sigma_t^2 f(X_t) \, dt \\
&= \mu_t S_t \, dt + \sigma_t S_t \, dW_t, \qquad (11)
\end{aligned}$$

which proves that (S_t) satisfies (9).

14. The results above can be generalised in a straightforward way to account for multi-dimensional Itô processes.

To fix ideas, let (W_t) be an n-dimensional Brownian motion, and consider the Itô processes (X_t^1), ..., (X_t^m) given by

$$dX_t^i = a_t^i \, dt + \sum_{j=1}^n b_t^{ij} \, dW_t^j, \quad \text{for } i = 1, \ldots, m,$$

where (a_t^i) and $\left(b_t^{ij}\right)$, for $i = 1, \ldots, m$ and $j = 1, \ldots, n$, are appropriate stochastic processes.

To simplify the notation, we can introduce a "vector formalism" and write

$$dX_t^i = a_t^i \, dt + b_t^i \cdot dW_t, \quad \text{for } i = 1, \ldots, m,$$

where (b_t^i) is the vector process given by $b_t^i = (b_t^1, \ldots, b_t^m)'$.

If f is a $C^{1,2,\ldots,2}$ function, then Itô's formula provides the expression

$$\begin{aligned}
df(t, X_t^1, \ldots, X_t^m) &= f_t(t, X_t^1, \ldots, X_t^m)\,dt + \sum_{i=1}^m f_{x_i}(t, X_t^1, \ldots, X_t^m)\,dX_t^i \\
&\quad + \frac{1}{2}\sum_{i,k=1}^m f_{x_i x_k}(t, X_t^1, \ldots, X_t^m)\left(dX_t^i\right)\left(dX_t^k\right) \\
&= \Bigg[f_t(t, X_t^1, \ldots, X_t^m) + \sum_{i=1}^m a_t^i f_{x_i}(t, X_t^1, \ldots, X_t^m) \\
&\quad + \frac{1}{2}\sum_{i,k=1}^m \left(\sum_{\ell=1}^n b_t^{i\ell} b_t^{k\ell}\right) f_{x_i x_k}(t, X_t^1, \ldots, X_t^m)\Bigg]\,dt \\
&\quad + \sum_{i=1}^m f_{x_i}(t, X_t^1, \ldots, X_t^m) b_t^i \cdot dW_t. \qquad (12)
\end{aligned}$$

It is worth noting that the second expression here follows immediately from the first one if we consider the *formal* expressions

$$(dt)^2 = 0, \quad dW_t^i\,dt = 0 \quad \text{and} \quad dW_t^i\,dW_t^j = \begin{cases} dt, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases} \qquad (13)$$

15. Another useful result of stochastic analysis is the *integration by parts formula*. Given the pair of Itô processes

$$\begin{aligned}
dX_t &= a_t\,dt + b_t \cdot dW_t, \quad t \geq 0, \\
dY_t &= c_t\,dt + d_t \cdot dW_t, \quad t \geq 0,
\end{aligned}$$

the product process $(X_t Y_t)$ is again an Itô process, and

$$\begin{aligned}
d\,(X_t Y_t) &= X_t\,dY_t + Y_t\,dX_t + (dX_t)(dY_t) \\
&= [Y_t a_t + X_t c_t + b_t' d_t]\,dt + [Y_t b_t + X_t d_t] \cdot dW_t, \qquad (14)
\end{aligned}$$

where we have used the formal expressions (13).

Martingale representation theorem

16. Stochastic integrals are local martingales. Is it true that every local martingale can be written as a stochastic integral? The answer is yes if information coincides with the natural filtration $(\mathcal{F}_t^W)$ of the Brownian motion (W_t).

17. Suppose that (W_t) is an n-dimensional Brownian motion. The *martingale representation theorem* states that, given any $(\mathcal{F}_t^W)$-local martingale (M_t), there exist a constant M_0 and an appropriate process (K_t) such that

$$M_t = M_0 + \int_0^t K_s \cdot dW_s.$$

Changes of probability measures

18. We can have many probability measures other than $\mathbb{P}$ defined on the measurable space $(\Omega, \mathcal{F})$. Indeed, let Y be any random variable defined on $(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$Y \geq 0, \quad \mathbb{P}\text{-a.s.}, \quad \text{and} \quad \mathbb{E}^{\mathbb{P}}\big[Y\big] = 1. \tag{15}$$

Here, we write $\mathbb{E}^{\mathbb{P}}$ instead of just $\mathbb{E}$ to indicate that we compute expectations with respect to the probability measure $\mathbb{P}$. We can then define the probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ by

$$\mathbb{Q}(A) = \mathbb{E}^{\mathbb{P}}\left[Y \mathbf{1}_A\right] \quad \text{for all } A \in \mathcal{F}. \tag{16}$$

19. The probability measure $\mathbb{Q}$ defined by (16) has the property that

$$\text{given any } A \in \mathcal{F}, \quad \mathbb{P}(A) = 0 \quad \Rightarrow \quad \mathbb{Q}(A) = 0. \tag{17}$$

The construction in Paragraph 18 has a converse:

Let $\mathbb{Q}$ be any probability measure on $(\Omega, \mathcal{F})$ such that (17) is true. Then, there exists a random variable Y satisfying (15) such that (16) is true. Moreover, this random variable is unique $\mathbb{P}$-a.s..

In this case, we say that $\mathbb{Q}$ is *absolutely continuous* with respect to $\mathbb{P}$, and we write $\mathbb{Q} \ll \mathbb{P}$. The random variable Y is the associated *Radon-Nikodym derivative*, and we write

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = Y \quad \text{or} \quad d\mathbb{Q} = Y \, d\mathbb{P}. \tag{18}$$

If $\mathbb{Q} \ll \mathbb{P}$ and $\mathbb{P} \ll \mathbb{Q}$, then we say that $\mathbb{Q}$ and $\mathbb{P}$ are *equivalent*, and we write $\mathbb{Q} \sim \mathbb{P}$.

20. Suppose that (L_t) is an $(\mathcal{F}_t)$-martingale with respect to the probability measure $\mathbb{P}$ such that $L_t > 0$, $\mathbb{P}$-a.s., and $\mathbb{E}^{\mathbb{P}}[L_t] = 1$ for all $t \geq 0$. Given a time $T > 0$, we can define a probability measure $\mathbb{Q}_T$ on the measurable space $(\Omega, \mathcal{F}_T)$ by

$$\mathbb{Q}_T(A) = \mathbb{E}^{\mathbb{P}}\left[L_T \mathbf{1}_A\right] \quad \text{for all } A \in \mathcal{F}_T. \tag{19}$$

The family of probability measures $\{\mathbb{Q}_T, \ T \geq 0\}$ thus arising is consistent in the following sense. Given any times $t < T$, we can use the properties of conditional expectation to calculate

$$\begin{aligned}
\text{for all } A \in \mathcal{F}_t, \quad \mathbb{Q}_T(A) &= \mathbb{E}^{\mathbb{P}}\left[L_T \mathbf{1}_A\right] \\
&= \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}\left[L_T \mathbf{1}_A \mid \mathcal{F}_t\right]\right] \\
&= \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}\left[L_T \mid \mathcal{F}_t\right] \mathbf{1}_A\right] \\
&= \mathbb{E}^{\mathbb{P}}\left[L_t \mathbf{1}_A\right] \\
&= \mathbb{Q}_t(A).
\end{aligned}$$

In other words, if we consider the *restriction* of the probability measures $\mathbb{P}$ and $\mathbb{Q}_T$ on the σ-algebra $\mathcal{F}_t$, then $d\mathbb{Q}_T = L_t\, d\mathbb{P}$ for all $t \in [0, T]$.

This observation shows that martingales are intimately related to changes of probability measures.

21. In the context of the previous paragraph, if Z is an $\mathcal{F}_T$-measurable random variable satisfying appropriate integrability conditions, then

$$\mathbb{E}^{\mathbb{Q}_T}[Z \mid \mathcal{F}_s] = \frac{\mathbb{E}^{\mathbb{P}}[L_T Z \mid \mathcal{F}_s]}{L_s} \quad \text{for all } s \in [0, T]. \tag{20}$$

This is a very useful result!

To see (20), we consider the definition of conditional expectation, we observe that both sides of this identity are $\mathcal{F}_s$-measurable random variables in $\mathcal{L}^1$, and we note that, given any event $A \in \mathcal{F}_s$,

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}_T}\left[\frac{\mathbb{E}^{\mathbb{P}}[L_T Z \mid \mathcal{F}_s]}{L_s}\mathbf{1}_A\right] &= \mathbb{E}^{\mathbb{P}}\left[L_T\frac{\mathbb{E}^{\mathbb{P}}[L_T Z \mid \mathcal{F}_s]}{L_s}\mathbf{1}_A\right] \\
&= \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}\left[L_T\frac{\mathbb{E}^{\mathbb{P}}[L_T Z \mid \mathcal{F}_s]}{L_s}\mathbf{1}_A \mid \mathcal{F}_s\right]\right] \\
&= \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}\left[L_T \mid \mathcal{F}_s\right]\frac{\mathbb{E}^{\mathbb{P}}[L_T Z \mid \mathcal{F}_s]}{L_s}\mathbf{1}_A\right] \\
&= \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}[L_T Z \mid \mathcal{F}_s]\mathbf{1}_A\right] \\
&= \mathbb{E}^{\mathbb{P}}\left[L_T Z \mathbf{1}_A\right] \\
&= \mathbb{E}^{\mathbb{Q}_T}\left[Z \mathbf{1}_A\right] \\
&= \mathbb{E}^{\mathbb{Q}_T}\left[\mathbb{E}^{\mathbb{Q}_T}[Z \mid \mathcal{F}_s]\mathbf{1}_A\right].
\end{aligned}$$

Girsanov's theorem

22. Suppose that the $(\mathcal{F}_t)$-Brownian motion is n-dimensional, and let (X_t) be an n-dimensional, $(\mathcal{F}_t)$-progressively measurable process satisfying

$$\int_0^t |X_s|^2 \, ds < \infty \quad \text{for all } t \geq 0, \quad \mathbb{P}\text{-a.s.}.$$

Under this assumption, the process (L_t) given by

$$L_t = \exp\left(-\frac{1}{2} \int_0^t |X_s|^2 \, ds + \int_0^t X_s \cdot dW_s \right)$$

is well-defined for all t. Using Itô's formula, we can verify that

$$L_t = 1 + \int_0^t L_s X_s \cdot dW_s \quad \text{for all } t \geq 0, \tag{21}$$

so (L_t) is an $(\mathcal{F}_t)$-local martingale.

23. Under appropriate conditions, the process (L_t) defined by (21) is a martingale, in which case $\mathbb{E}\big[L_T\big] = 1$ for all $T \geq 0$. One sufficient condition for (L_t) to be a martingale is *Novikov's condition*:

$$\mathbb{E}\left[\exp\left(\frac{1}{2} \int_0^t |X_s|^2 \, ds \right) \right] < \infty \quad \text{for all } t \geq 0.$$

24. If (L_t) is a martingale, then, given any fixed time $T > 0$, we can define a probability measure $\mathbb{Q}_T$ on $(\Omega, \mathcal{F}_T)$ by

$$\mathbb{Q}_T(A) = \mathbb{E}\left[L_T \mathbf{1}_A \right] \quad \text{for all } A \in \mathcal{F}_T. \tag{22}$$

Girsanov's theorem states that, given any fixed time $T > 0$, the process $(\tilde{W}_t)$ defined by

$$\tilde{W}_t = W_t - \int_0^t X_s \, ds, \quad t \in [0, T]$$

is an n-dimensional $(\mathcal{F}_t)$-Brownian motion with respect to $\mathbb{Q}_T$.

25. Given a constant ϑ, the process

$$L_t = \exp\left(-\frac{1}{2} \vartheta^2 t - \vartheta W_t \right),$$

where (W_t) is a one-dimensional Brownian motion, is an $(\mathcal{F}_t)$-martingale. It follows that, if $\mathbb{Q}_T$ is the probability measure defined on the measurable space $(\Omega, \mathcal{F}_T)$ by (22), then the process

$$W_t^{\vartheta} = \vartheta t + W_t, \quad \text{for } t \in [0, T],$$

is a one-dimensional $(\mathcal{F}_t)$-Brownian motion with respect to $\mathbb{Q}_T$.

In this context,

$$\mathbb{E}^{\mathbb{P}}\left[W_t\right] = 0, \quad \mathbb{E}^{\mathbb{Q}_T}\left[W_t\right] = -\vartheta t,$$
$$\mathbb{E}^{\mathbb{P}}\left[W_t^{\vartheta}\right] = \vartheta t \quad \text{and} \quad \mathbb{E}^{\mathbb{Q}_T}\left[W_t^{\vartheta}\right] = 0.$$

Also, if (K_t) is a process such that all associated stochastic integrals are well-defined, and all integrals with respect to the associated Brownian motions are martingales, then

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left[\int_0^t K_u\, dW_u\right] &= 0, \\
\mathbb{E}^{\mathbb{Q}_T}\left[\int_0^t K_u\, dW_u\right] &= \mathbb{E}^{\mathbb{Q}_T}\left[\int_0^t K_u\, dW_u^{\vartheta} - \int_0^t K_u \vartheta\, du\right] \\
&= -\vartheta \mathbb{E}^{\mathbb{Q}_T}\left[\int_0^t K_u\, du\right], \\
\mathbb{E}^{\mathbb{P}}\left[\int_0^t K_u\, dW_u^{\vartheta}\right] &= \mathbb{E}^{\mathbb{P}}\left[\int_0^t K_u\, dW_u + \int_0^t K_u \vartheta\, du\right] \\
&= \vartheta \mathbb{E}^{\mathbb{P}}\left[\int_0^t K_u\, du\right]
\end{aligned}$$

and

$$\mathbb{E}^{\mathbb{Q}_T}\left[\int_0^t K_u\, dW_u^{\vartheta}\right] = 0.$$