MA400. September Introductory Course
(Financial Mathematics, Risk & Stochastics)
P6. Stochastic processes

1. In what follows, we fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which all random variables considered are defined.

Stochastic Processes

2. *Definition.* A *stochastic process* is a family of random variables $(X_t,\ t \in \mathcal{T})$ indexed by a non-empty set $\mathcal{T}$.

   When the index set $\mathcal{T}$ is understood by the context, we usually write X or (X_t) instead of $(X_t,\ t \in \mathcal{T})$.

3. In this course, we consider only stochastic processes whose index set $\mathcal{T}$ is the set of natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$ or the set of positive real numbers $\mathbb{R}_+ = [0, \infty)$. In the first instance, we are talking about *discrete time processes*, in the second one, we are talking about *continuous time processes*.

4. Stochastic processes are mathematical models for quantities that evolve randomly over time. For example, we can use a stochastic process $(X_t,\ t \geq 0)$ to model the time evolution of the stock price of a given company. In this context, assuming that present time is 0, the random variable X_t is the stock price of the company at the future time t.

Filtrations and Stopping Times

5. *Definition.* A *filtration* on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a family $(\mathcal{F}_t,\ t \in \mathcal{T})$ of σ-algebras such that

$$\mathcal{F}_t \subseteq \mathcal{F} \text{ for all } t \in \mathcal{T}, \quad \text{and} \quad \mathcal{F}_s \subseteq \mathcal{F}_t \text{ for all } s, t \in \mathcal{T} \text{ such that } s \leq t. \tag{1}$$

   We usually write $(\mathcal{F}_t)$ or $\{\mathcal{F}_t\}$ instead of $(\mathcal{F}_t,\ t \in \mathcal{T})$.

   A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a filtration $(\mathcal{F}_t)$, often denoted by $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, is said to be a *filtered probability space*.

6. We have seen that σ-algebras are models for information. Accordingly, filtrations are models for *flows of information*. The inclusions in (**??**) reflect the idea that, as time progresses, more information becomes available, as well as the idea that "memory is perfect" in the sense that there is no information lost in the course of time.

7. *Definition.* The *natural filtration* $(\mathcal{F}_t^X)$ of a stochastic process (X_t) is defined by

$$\mathcal{F}_t^X = \sigma\big(X_s,\ s \in \mathcal{T},\ s \leq t\big), \quad t \in \mathcal{T}.$$

8. The natural filtration of a process (X_t) is the flow of information that the observation of the evolution in time of the process (X_t) yields, and only that.

9. *Definition*. We say that a process (X_t) is *adapted* to a filtration $(\mathcal{F}_t)$ if X_t is $\mathcal{F}_t$-measurable for all $t \in \mathcal{T}$, or equivalently, if $\mathcal{F}_t^X \subseteq \mathcal{F}_t$ for all $t \in \mathcal{T}$.

10. In the context of this definition, the information becoming available by the observation of the time evolution of an $(\mathcal{F}_t)$-adapted process (X_t) is (possibly strictly) included in the information flow modelled by $(\mathcal{F}_t)$.

11. Recalling that $\mathcal{T} = \mathbb{N}$ or $\mathcal{T} = \mathbb{R}_+$, a *random time* is any random variable with values in $\mathcal{T} \cup \{\infty\}$.

    We often use a "random time" τ to denote the time at which a given random event occurs. In this context, the set $\{\tau = \infty\}$ represents the event that the random event never occurs.

12. *Definition*. Given a filtration $(\mathcal{F}_t)$, we say that a random time τ is an $(\mathcal{F}_t)$-*stopping time* if

$$\tau^{-1}\big([0,t]\big) = \{\omega \in \Omega \mid \ \tau(\omega) \leq t\} \in \mathcal{F}_t \quad \text{for all } t \in \mathcal{T}. \tag{2}$$

13. We can think of an $(\mathcal{F}_t)$-stopping time as a random time with the property that, given any fixed time t, we know whether the random event that it represents has occurred or not in light of the information $\mathcal{F}_t$ that is available to us at time t.

    Note that the filtration $(\mathcal{F}_t)$ is essential for the definition of stopping times. Indeed, a random time can be a stopping time with respect to some filtration $(\mathcal{F}_t)$, but not with respect to some other filtration $(\mathcal{G}_t)$.

14. *Example*. Suppose that τ_1 and τ_2 are two $(\mathcal{F}_t)$-stopping times. Then the random time τ defined by $\tau = \min\big\{\tau_1, \tau_2\big\}$ is an $(\mathcal{F}_t)$-stopping time.

    Proof. The assumption that τ_1 and τ_2 are $(\mathcal{F}_t)$-stopping times implies that

$$\big\{\tau_1 \leq t\big\},\ \big\{\tau_2 \leq t\big\} \in \mathcal{F}_t \quad \text{for all } t \in \mathcal{T}.$$

    Therefore

$$\big\{\tau \leq t\big\} = \big\{\tau_1 \leq t\big\} \cup \big\{\tau_2 \leq t\big\} \in \mathcal{F}_t \quad \text{for all } t \in \mathcal{T},$$

    which proves the claim.

Martingales

15. *Definition.* An $(\mathcal{F}_t)$-adapted stochastic process (X_t) is an $(\mathcal{F}_t)$-*supermartingale* if

$$\begin{aligned} &\text{(i)} \quad \mathbb{E}\big[|X_t|\big] < \infty \quad \text{for all } t \in \mathcal{T}, \text{ and} \\ &\text{(ii)} \quad \mathbb{E}\big[X_t \mid \mathcal{F}_s\big] \le X_s, \quad \mathbb{P}\text{-a.s., for all } s, t \in \mathcal{T} \text{ such that } s < t. \end{aligned}$$

An $(\mathcal{F}_t)$-adapted stochastic process (X_t) is an $(\mathcal{F}_t)$-*submartingale* if

$$\begin{aligned} &\text{(i)} \quad \mathbb{E}\big[|X_t|\big] < \infty \quad \text{for all } t \in \mathcal{T}, \text{ and} \\ &\text{(ii)} \quad \mathbb{E}\big[X_t \mid \mathcal{F}_s\big] \ge X_s, \quad \mathbb{P}\text{-a.s., for all } s, t \in \mathcal{T} \text{ such that } s < t. \end{aligned}$$

An $(\mathcal{F}_t)$-adapted stochastic process (X_t) is an $(\mathcal{F}_t)$-*martingale* if

$$\begin{aligned} &\text{(i)} \quad \mathbb{E}\big[|X_t|\big] < \infty \quad \text{for all } t \in \mathcal{T}, \text{ and} \\ &\text{(ii)} \quad \mathbb{E}\big[X_t \mid \mathcal{F}_s\big] = X_s, \quad \mathbb{P}\text{-a.s., for all } s, t \in \mathcal{T} \text{ such that } s < t. \end{aligned}$$

16. Plainly, a process (X_t) is a submartingale if $(-X_t)$ is a supermartingale, and vice versa, while a process (X_t) is a martingale if it is both a sub and a supermartingale.

    A supermartingale "decreases on average". A submartingale "increases on average".

17. *Example.* A gambler bets repeatedly on a game of chance. If we denote by X_0 the gambler's initial capital and by X_n the gambler's total wealth after their n-th bet, then $X_n - X_{n-1}$ are the gambler's *net winnings* from their n-th bet $(n \ge 1)$.

    If (X_n) is a martingale, then the game series is *fair*.

    If (X_n) is a submartingale, then the game series is *favourable* to the gambler.

    If (X_n) is a supermartingale, then the game series is *unfavourable* to the gambler.

18. *Example.* Let $X_1, X_2, \ldots$ be a sequence of independent random variables in $\mathcal{L}^1$ such that $\mathbb{E}[X_n] = 0$ for all n. If we set

$$\begin{gathered} S_0 = 0, \quad S_n = X_1 + X_2 + \cdots + X_n, \quad \text{for } n \ge 1, \\ \mathcal{F}_0 = \{\emptyset, \Omega\} \quad \text{and} \quad \mathcal{F}_n = \sigma(X_1, X_2, \ldots, X_n), \quad \text{for } n \ge 1, \end{gathered}$$

then the process (S_n) is an $(\mathcal{F}_n)$-martingale.

Proof. Since $|X_1 + X_2 + \cdots + X_n| \le |X_1| + |X_2| + \cdots + |X_n|$, the assumption that $X_n \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ for all $n \ge 1$, implies that $\mathbb{E}[|S_n|] < \infty$ for all $n \ge 1$. Moreover, the assumption that $X_1, X_2, \ldots$ are independent implies that

$$\mathbb{E}\big[X_i \mid \mathcal{F}_m\big] = \begin{cases} X_i, & \text{if } i \le m, \\ \mathbb{E}[X_i], & \text{if } i > m. \end{cases}$$

It follows that, given any $m < n$,

$$\begin{aligned}
\mathbb{E}\big[S_n \mid \mathcal{F}_m\big] &= \sum_{i=1}^{n} \mathbb{E}\big[X_i \mid \mathcal{F}_m\big] \\
&= \sum_{i=1}^{m} X_i + \sum_{i=m+1}^{n} 0 \\
&= S_m.
\end{aligned}$$

19. *Example*. Let $(\mathcal{F}_t)$ be a filtration, and let any random variable $Y \in \mathcal{L}^1$. If we define

$$M_t = \mathbb{E}\big[Y \mid \mathcal{F}_t\big], \quad t \in \mathcal{T},$$

then M is a martingale.

Proof. By the definition of conditional expectation, $\mathbb{E}\big[|M_t|\big] < \infty$ for all $t \in \mathcal{T}$. Furthermore, given any times $s < t$, the tower property of conditional expectation implies

$$\begin{aligned}
\mathbb{E}\big[M_t \mid \mathcal{F}_s\big] &= \mathbb{E}\big[\mathbb{E}\big[Y \mid \mathcal{F}_t\big] \mid \mathcal{F}_s\big] \\
&= \mathbb{E}\big[Y \mid \mathcal{F}_s\big] \\
&= M_s.
\end{aligned}$$

Brownian motion

20. *Definition*. The *standard one-dimensional Brownian motion or Wiener process* (W_t) is the continuous time stochastic process described by the following properties:

    (i) $W_0 = 0$.

    (ii) *Continuity*: All of the sample paths $s \mapsto W_s(\omega)$ are continuous functions.

    (iii) *Independent increments*: The increments of (W_t) in non-overlapping time intervals are independent random variables. Specifically, given any times $t_1 < t_2 < \cdots < t_k$, the random variables $W_{t_2} - W_{t_1}, \ldots, W_{t_k} - W_{t_{k-1}}$ are independent.

    (iv) *Normality*: Given any times $s < t$, the random variable $W_t - W_s$ is normal with mean 0 and variance $t - s$, i.e., $W_t - W_s \sim N(0, t-s)$.

21. In this definition, it is important to observe that we must require *any finite number* of increments in non-overlapping time intervals to be independent. Indeed, there exists a process (X_t) which satisfies (i), (ii), (iv) of Definition **??**, and

    (iii') any two increments of (X_t) are independent, i.e., given any times $a < b < c$, the increments $X_b - X_a$ and $X_c - X_b$ are independent random variables,

    that is *not* a Brownian motion.

22. Given any times $s < t$,

$$\begin{aligned}
\mathbb{E}\left[W_s W_t\right] &= \mathbb{E}\left[W_s(W_s + W_t - W_s)\right] \\
&= \mathbb{E}\left[W_s^2\right] + \mathbb{E}\left[W_s(W_t - W_s)\right] \\
&= s + \mathbb{E}\left[W_s\right] \mathbb{E}\left[W_t - W_s\right] \\
&= s.
\end{aligned}$$

    Therefore, given any times s, t,

$$\mathbb{E}\left[W_s W_t\right] = \min(s, t).$$

23. *Time reversal*. The continuous time stochastic process $(B_t,\ t \in [0, T])$ defined by

$$B_t = W_T - W_{T-t}, \quad t \in [0, T],$$

    is a standard Brownian motion.

    Proof. We verify the requirements of the definition:

    - $B_0 = W_T - W_{T-0} = 0$.
    - Plainly, (B_t) has continuous sample paths because this is true for (W_t).

- Given $0 \leq t_1 < t_2 < \cdots < t_k \leq T$, observe that $T - t_k < \cdots < T - t_2 < T - T_1$, and $B_{t_i} - B_{t_{i-1}} = W_{T-t_{i-1}} - W_{T-t_i}$. Therefore, the increments $B_{t_2} - B_{t_1}$, $\ldots$, $B_{t_k} - B_{t_{k-1}}$ are independent random variables because this is true for the random variables $W_{T-t_1} - W_{T-t_2}$, $\ldots$, $W_{T-t_{k-1}} - W_{T-t_k}$ which are increments of the Brownian motion (W_t) in non-overlapping time intervals.
- Given any times $s < t$, since $B_t - B_s = -(W_{T-t} - W_{t-s})$, $B_t - B_s \sim N(0, t-s)$ because (W_t) is a Brownian motion, and so, $W_{T-t} - W_{t-s} \sim N(0, t-s)$.

24. *Example*. Given $s < t$, the properties of the normal distribution imply

$$\mathbb{E}[(W_{t+2} - W_t)^3] = 0 \quad \text{and} \quad \mathbb{E}[(W_t - W_s)^4] = 3(t-s)^2.$$

25. *Lemma*. The sample paths of the standard Brownian motion are nowhere differentiable functions, $\mathbb{P}$-a.s..

26. *Definition*. An n-dimensional standard Brownian motion (W_t) is a vector $(W_t^1, \ldots, W_t^n)$ of independent standard one-dimensional Brownian motions $(W_t^1), \ldots, (W_t^n)$.

27. We often want a stochastic process to be a Brownian motion with respect to the flow of information modelled by a filtration $(\mathcal{G}_t)$, which gives rise to the following definition.

    Definition. If $(\mathcal{G}_t)$ is a filtration, then a $(\mathcal{G}_t)$-adapted stochastic process (W_t) is called a $(\mathcal{G}_t)$-*Brownian motion* if

    (i) (W_t) is a Brownian motion, and

    (ii) for every time $t \geq 0$, the process $(W_{t+s} - W_t,\ s \geq 0)$ is independent of $\mathcal{G}_t$, i.e., the σ-algebras $\sigma\left(W_{t+s} - W_t,\ s \geq 0\right)$ and $\mathcal{G}_t$ are independent.

28. *Lemma*. Every $(\mathcal{G}_t)$-Brownian motion (W_t) is a $(\mathcal{G}_t)$-martingale.

    Proof. First, we note that the inequalities

$$\mathbb{E}\big[|W_t|\big] \leq 1 + \mathbb{E}\big[W_t^2\big] = 1 + t < \infty$$

imply that $W_t \in \mathcal{L}^1$ for all $t \geq 0$. Next, we observe that, given any times $s < t$,

$$\begin{aligned} \mathbb{E}[W_t \mid \mathcal{G}_s] &= \mathbb{E}[W_t - W_s \mid \mathcal{G}_s] + W_s \\ &= 0 + W_s \\ &= W_s, \end{aligned}$$

the second equality following because the random variable $W_t - W_s$ is independent of $\mathcal{G}_s$.