Diameter of Polytopes: Algorithmic and Combinatorial Aspects

Laura Sanità

Department of Mathematics and Computer Science
TU Eindhoven (Netherlands)

Department of Combinatorics and Optimization
University of Waterloo (Canada)

IPCO Summer School, 2020

Linear Programming

Linear Programming

- Linear Programming is concerned with the problem of
  - minimize/maximize a linear function on d continuous variables
  - subject to a finite set of linear constraints
- Example:

$$
\begin{array}{llll}
\max & 5x_1 & -3x_2 & \\
 & 2x_1 & +3x_2 & \leq 2 \\
 & -x_1 & +4x_2 & \leq 3 \\
 & & -3x_2 & \leq 0
\end{array}
$$

Linear Programming

- Linear Programming is concerned with the problem of
  - minimize/maximize a linear function on d continuous variables
  - subject to a finite set of linear constraints
- Example:

$$\begin{array}{llll} \max & 5x_1 & -3x_2 & \\ & 2x_1 & +3x_2 & \leq 2 \\ & -x_1 & +4x_2 & \leq 3 \\ & & -3x_2 & \leq 0 \end{array}$$

$$\begin{array}{lll} \max & c^T x & \\ & Ax & \leq b \end{array}$$

  - $x \in \mathbb{R}^d$ is the vector of variables
  - $c \in \mathbb{R}^d, b \in \mathbb{R}^n, A \in \mathbb{R}^{n \times d}$ are given
- The above problem instances are called Linear Programs (LP).

Is Linear Programming useful?

Is Linear Programming useful?

- LPs can be used to model several optimization problems:
  - shortest path in a graph
  - network flows
  - assignment
  - ...

Is Linear Programming useful?

- LPs can be used to model several optimization problems:
  - shortest path in a graph
  - network flows
  - assignment
  - ...
- LPs are a fundamental tool for solving harder problems.

Is Linear Programming useful?

- LPs can be used to model several optimization problems:
  - shortest path in a graph
  - network flows
  - assignment
  - ...
- LPs are a fundamental tool for solving harder problems. For example:
  - Optimization problems with integer variables (via Branch&Bound, Cutting planes,...)
  - Approximation algorithms for NP-hard problems.
  - Commercial solvers (CPLEX, GUROBI, XPRESS, ...), Operations Research Industry, Data Science.

Algorithms for solving LPs?

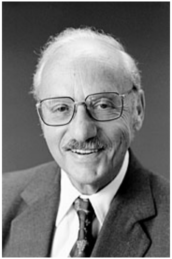
Algorithms for solving LPs?

- The development of algorithms for solving LPs started in the 40's. Some pioneers: Kantorovich&Koopmans, Dantzig, Von Neumann, Ford&Fulkerson...

Algorithms for solving LPs?

• The development of algorithms for solving LPs started in the 40's. Some pioneers: Kantorovich&Koopmans, Dantzig, Von Neumann, Ford&Fulkerson...

- **George Dantzig**: published the Simplex Algorithm for solving LPs in 1947

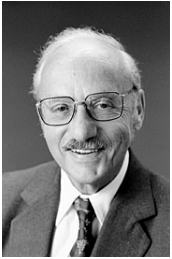


• Nowadays, the simplex algorithm is extremely popular and used in practice,

Algorithms for solving LPs?

• The development of algorithms for solving LPs started in the 40's. Some pioneers: Kantorovich&Koopmans, Dantzig, Von Neumann, Ford&Fulkerson...

- **George Dantzig**: published the Simplex Algorithm for solving LPs in 1947



• Nowadays, the simplex algorithm is extremely popular and used in practice, named as one of the "top 10 algorithms" of the 20th century.

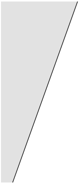
The Simplex Algorithm

The Simplex Algorithm

- The set of possible solutions of an LP has a very nice structure:

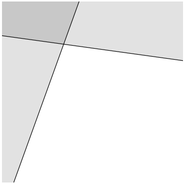
The Simplex Algorithm

- The set of possible solutions of an LP has a very nice structure:



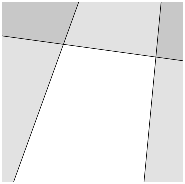
The Simplex Algorithm

- The set of possible solutions of an LP has a very nice structure:



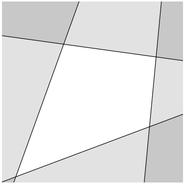
The Simplex Algorithm

- The set of possible solutions of an LP has a very nice structure:



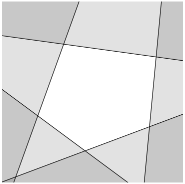
The Simplex Algorithm

- The set of possible solutions of an LP has a very nice structure:



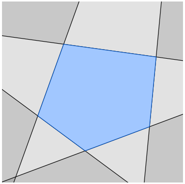
The Simplex Algorithm

- The set of possible solutions of an LP has a very nice structure:



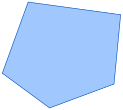
The Simplex Algorithm

- The set of possible solutions of an LP has a very nice structure:



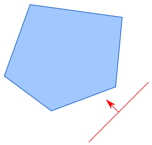
The Simplex Algorithm

• The set of possible solutions of an LP has a very nice structure: it is a convex set called a polyhedron (or a polytope, if bounded)



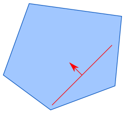
The Simplex Algorithm

• The set of possible solutions of an LP has a very nice structure: it is a convex set called a polyhedron (or a polytope, if bounded)



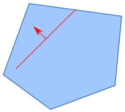
The Simplex Algorithm

• The set of possible solutions of an LP has a very nice structure: it is a convex set called a polyhedron (or a polytope, if bounded)



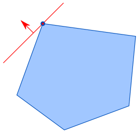
The Simplex Algorithm

• The set of possible solutions of an LP has a very nice structure: it is a convex set called a polyhedron (or a polytope, if bounded)



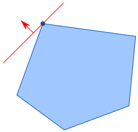
The Simplex Algorithm

• The set of possible solutions of an LP has a very nice structure: it is a convex set called a polyhedron (or a polytope, if bounded)



The Simplex Algorithm

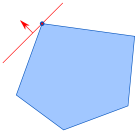
• The set of possible solutions of an LP has a very nice structure: it is a convex set called a polyhedron (or a polytope, if bounded)



• It is not difficult to realize that an optimal solution of such an LP can be found at one of the extreme points of the feasible region.

The Simplex Algorithm

• The set of possible solutions of an LP has a very nice structure: it is a convex set called a polyhedron (or a polytope, if bounded)

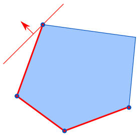


• It is not difficult to realize that an optimal solution of such an LP can be found at one of the extreme points of the feasible region.

• **Simplex Algorithm's idea:** move from an extreme point to an improving adjacent one, until the optimum is found!

The Simplex Algorithm

• The set of possible solutions of an LP has a very nice structure: it is a convex set called a polyhedron (or a polytope, if bounded)

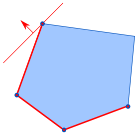


• It is not difficult to realize that an optimal solution of such an LP can be found at one of the extreme points of the feasible region.

• **Simplex Algorithm's idea:** move from an extreme point to an improving adjacent one, until the optimum is found!

The Simplex Algorithm

• The set of possible solutions of an LP has a very nice structure: it is a convex set called a polyhedron (or a polytope, if bounded)



• It is not difficult to realize that an optimal solution of such an LP can be found at one of the extreme points of the feasible region.

• **Simplex Algorithm's idea:** move from an extreme point to an improving adjacent one, until the optimum is found!

• The operation of moving from one extreme point to the next is called pivoting

Pivoting

Pivoting

- Clearly, the path followed by the algorithm depends on the pivoting rule:
  How do we choose the next (improving) extreme point?

Pivoting

- Clearly, the path followed by the algorithm depends on the pivoting rule:
  How do we choose the next (improving) extreme point?

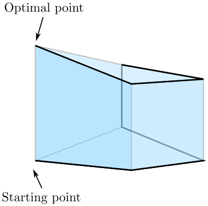
- Dantzig's pivoting rule: move along the edge that "seems" more promising in term of cost-function improvement

Pivoting

- Clearly, the path followed by the algorithm depends on the pivoting rule:
  How do we choose the next (improving) extreme point?

- Dantzig's pivoting rule: move along the edge that "seems" more promising in term of cost-function improvement

  - [Klee & Minty'72] showed that pivoting according to that rule requires an exponential in d number of steps.

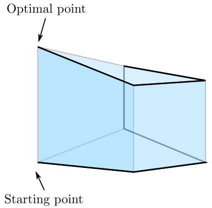


Pivoting

- Clearly, the path followed by the algorithm depends on the pivoting rule:
  How do we choose the next (improving) extreme point?

- Dantzig's pivoting rule: move along the edge that "seems" more promising in term of cost-function improvement

  - [Klee & Minty'72] showed that pivoting according to that rule requires an exponential in d number of steps.



- Other pivoting rules?

Pivoting

- Many pivoting rules have been proposed in the literature in the past decades
  - Dantzig's rule
  - Greatest improvement
  - Bland's rule
  - Steepest-edge
  - Random pivot rules
  - Cunningham's pivot rule
  - Zadeh's pivot rule
  - ...

Pivoting

- Many pivoting rules have been proposed in the literature in the past decades
  - Dantzig's rule
  - Greatest improvement
  - Bland's rule
  - Steepest-edge
  - Random pivot rules
  - Cunningham's pivot rule
  - Zadeh's pivot rule
  - ...

...exhibiting a worst-case (sub)exponential behaviour for the Simplex algorithm [Klee&Minty'72, Jeroslow'73, Avis&Chvàtal'78, Goldfarb&Sit'79, Friedmann&Hansen&Zwick'11, Friedmann'11, Avis&Friedmann'17, Disser&Hopp'19]

Pivoting

- Many pivoting rules have been proposed in the literature in the past decades
  - Dantzig's rule
  - Greatest improvement
  - Bland's rule
  - Steepest-edge
  - Random pivot rules
  - Cunningham's pivot rule
  - Zadeh's pivot rule
  - . . .

...exhibiting a worst-case (sub)exponential behaviour for the Simplex algorithm [Klee&Minty'72, Jeroslow'73, Avis&Chvàtal'78, Goldfarb&Sit'79, Friedmann&Hansen&Zwick'11, Friedmann'11, Avis&Friedmann'17, Disser&Hopp'19]

- The Simplex algorithm (with e.g. Dantzig's rule) can 'implicitly' solve hard problems [Adler,Papadimitriou&Rubinstein'14, Skutella&Disser'15, Fearnley&Savani'15]

Is there a polynomial pivoting rule?

Is there a polynomial pivoting rule?

- After more than 70 years of use/studies, we still do **not** know the answer!

Is there a polynomial pivoting rule?

- After more than 70 years of use/studies, we still do **not** know the answer!
- LPs can be solved in polynomial-time using other algorithms (Ellipsoid algorithm [Khachiyan'79], Interior-point methods [Karmarkar'84])

Is there a polynomial pivoting rule?

- After more than 70 years of use/studies, we still do **not** know the answer!

- LPs can be solved in polynomial-time using other algorithms (Ellipsoid algorithm [Khachiyan'79], Interior-point methods [Karmarkar'84])
  - However, such algorithms run in weakly polynomial-time ($poly(d, n, \log L)$ where $L :=$ largest coefficient)

Is there a polynomial pivoting rule?

- After more than 70 years of use/studies, we still do **not** know the answer!

- LPs can be solved in polynomial-time using other algorithms (Ellipsoid algorithm [Khachiyan'79], Interior-point methods [Karmarkar'84])
  - However, such algorithms run in weakly polynomial-time ($poly(d, n, \log L)$ where $L :=$ largest coefficient)

- The existence of a polynomial pivoting rule could imply a strongly polynomial-time algorithm for solving LPs
(# elementary operations polynomial in the number of different input values)

Is there a polynomial pivoting rule?

- After more than 70 years of use/studies, we still do **not** know the answer!

- LPs can be solved in polynomial-time using other algorithms (Ellipsoid algorithm [Khachiyan'79], Interior-point methods [Karmarkar'84])
  - However, such algorithms run in weakly polynomial-time ($poly(d, n, \log L)$ where $L :=$ largest coefficient)

- The existence of a polynomial pivoting rule could imply a strongly polynomial-time algorithm for solving LPs (# elementary operations polynomial in the number of different input values)
  - Mentioned as one of the mathematical problems for next century by Fields Medalist S. Smale in 2000

Is there a polynomial pivoting rule?

• After more than 70 years of use/studies, we still do **not** know the answer!

• LPs can be solved in polynomial-time using other algorithms (Ellipsoid algorithm [Khachiyan'79], Interior-point methods [Karmarkar'84])

- However, such algorithms run in weakly polynomial-time ($poly(d, n, \log L)$ where $L :=$ largest coefficient)

• The existence of a polynomial pivoting rule could imply a strongly polynomial-time algorithm for solving LPs
(# elementary operations polynomial in the number of different input values)

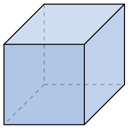
- Mentioned as one of the mathematical problems for next century by Fields Medalist S. Smale in 2000

Related Question: *What is the maximum length of a 'shortest path' between two extreme points of a polytope?*

Diameter of polytopes

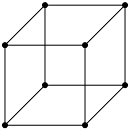
Diameter of polytopes

- We can naturally associate an undirected graph to a given polytope $P \subseteq \mathbb{R}^d$:
  - the vertices correspond to the extreme points of P
  - the edges are given by the 1-dimensional faces of P



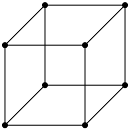
Diameter of polytopes

- We can naturally associate an undirected graph to a given polytope $P \subseteq \mathbb{R}^d$:
  - the vertices correspond to the extreme points of P
  - the edges are given by the 1-dimensional faces of P



Diameter of polytopes

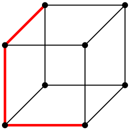
- We can naturally associate an undirected graph to a given polytope $P \subseteq \mathbb{R}^d$:
  - the vertices correspond to the extreme points of P
  - the edges are given by the 1-dimensional faces of P



- The diameter of P is the maximum value of a shortest path between a pair of vertices on this graph

Diameter of polytopes

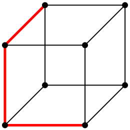
- We can naturally associate an undirected graph to a given polytope $P \subseteq \mathbb{R}^d$:
  - the vertices correspond to the extreme points of P
  - the edges are given by the 1-dimensional faces of P



- The diameter of P is the maximum value of a shortest path between a pair of vertices on this graph

Diameter of polytopes

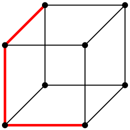
- We can naturally associate an undirected graph to a given polytope $P \subseteq \mathbb{R}^d$:
  - the vertices correspond to the extreme points of P
  - the edges are given by the 1-dimensional faces of P



- The diameter of P is the maximum value of a shortest path between a pair of vertices on this graph (1-skeleton of P).

Diameter of polytopes

- We can naturally associate an undirected graph to a given polytope $P \subseteq \mathbb{R}^d$:
  - the vertices correspond to the extreme points of P
  - the edges are given by the 1-dimensional faces of P



- The diameter of P is the maximum value of a shortest path between a pair of vertices on this graph (1-skeleton of P).

Remark: *In order for a polynomial pivoting rule to exist, a necessary condition is a polynomial bound on the value of the diameter!*

Diameter of polytopes

• A famous conjecture, proposed by [Hirsch'57], was the Hirsch conjecture, stating that the diameter of a d-dimensional polytope with n facets is $\leq n - d$.

Diameter of polytopes

• A famous conjecture, proposed by [Hirsch'57], was the Hirsch conjecture, stating that the diameter of a d-dimensional polytope with n facets is $\leq n - d$.

- Disproved first for unbounded polyhedra [Klee&Walkup'67]
- ..and later for bounded ones [Santos'12] (awarded Fulkerson Prize in 2015)

Diameter of polytopes

• A famous conjecture, proposed by [Hirsch'57], was the Hirsch conjecture, stating that the diameter of a d-dimensional polytope with n facets is $\leq n - d$.

- Disproved first for unbounded polyhedra [Klee&Walkup'67]
- ..and later for bounded ones [Santos'12] (awarded Fulkerson Prize in 2015)
- holds e.g. for for 0/1-polytopes [Naddef'89]

Diameter of polytopes

- A famous conjecture, proposed by [Hirsch'57], was the Hirsch conjecture, stating that the diameter of a d-dimensional polytope with n facets is $\leq n - d$.
  - Disproved first for unbounded polyhedra [Klee&Walkup'67]
  - ..and later for bounded ones [Santos'12] (awarded Fulkerson Prize in 2015)
  - holds e.g. for for 0/1-polytopes [Naddef'89]

- Importantly, its polynomial-version is still open:

Is the diameter of a d-dimensional polytope with n facets bounded by $f(d, n)$, for some polynomial function $f(d, n)$?

Diameter of polytopes

• A famous conjecture, proposed by [Hirsch'57], was the Hirsch conjecture, stating that the diameter of a d-dimensional polytope with n facets is $\leq n - d$.

- Disproved first for unbounded polyhedra [Klee&Walkup'67]
- ..and later for bounded ones [Santos'12] (awarded Fulkerson Prize in 2015)
- holds e.g. for for 0/1-polytopes [Naddef'89]

• Importantly, its polynomial-version is still open:

Is the diameter of a d-dimensional polytope with n facets bounded by $f(d, n)$, for some polynomial function $f(d, n)$?

• Best bound: $\sim (n-d)^{\log O(d/\log d)}$ [Sukegawa'18] (strengthening [Kalai&Kleitman'92, Todd'14, Sukegawa&Kitahara'15])

Diameter of polytopes

- A famous conjecture, proposed by [Hirsch'57], was the Hirsch conjecture, stating that the diameter of a d-dimensional polytope with n facets is $\leq n - d$.
  - Disproved first for unbounded polyhedra [Klee&Walkup'67]
  - ..and later for bounded ones [Santos'12] (awarded Fulkerson Prize in 2015)
  - holds e.g. for for 0/1-polytopes [Naddef'89]

- Importantly, its polynomial-version is still open:

Is the diameter of a d-dimensional polytope with n facets bounded by $f(d, n)$, for some polynomial function $f(d, n)$?

- Best bound: $\sim (n-d)^{\log O(d/\log d)}$ [Sukegawa'18] (strengthening [Kalai&Kleitman'92, Todd'14, Sukegawa&Kitahara'15])

- The diameter of a polytope has been studied from many different perspectives...

Diameter of polytopes

Diameter of polytopes

• Many researchers studied the diameter of polytopes describing feasible solutions of combinatorial optimization problems (and their relaxations).

Diameter of polytopes

• Many researchers studied the diameter of polytopes describing feasible solutions of combinatorial optimization problems (and their relaxations).

→ Just to mention a few: Matching, TSP, Flow and Transportation, Edge-cover, Stable marriage, Stable set, Partition, *and many more*...

Diameter of polytopes

• Many researchers studied the diameter of polytopes describing feasible solutions of combinatorial optimization problems (and their relaxations).

$\rightarrow$ Just to mention a few: Matching, TSP, Flow and Transportation, Edge-cover, Stable marriage, Stable set, Partition, *and many more*. . .

• The diameter of a polytope has been investigated also from a computational complexity point of view.

Diameter of polytopes

• Many researchers studied the diameter of polytopes describing feasible solutions of combinatorial optimization problems (and their relaxations).

$\rightarrow$ Just to mention a few: Matching, TSP, Flow and Transportation, Edge-cover, Stable marriage, Stable set, Partition, *and many more*...

• The diameter of a polytope has been investigated also from a computational complexity point of view.

- [Frieze&Teng'94]: Computing the diameter of a polytope is weakly NP-hard.
- [S.'18]: Computing the diameter of a polytope is strongly NP-hard. Computing a pair of vertices at maximum distance is APX-hard.

Diameter of polytopes

• Many researchers studied the diameter of polytopes describing feasible solutions of combinatorial optimization problems (and their relaxations).

→ Just to mention a few: Matching, TSP, Flow and Transportation, Edge-cover, Stable marriage, Stable set, Partition, *and many more*. . .

• The diameter of a polytope has been investigated also from a computational complexity point of view.

- [Frieze&Teng'94]: Computing the diameter of a polytope is weakly NP-hard.
- [S.'18]: Computing the diameter of a polytope is strongly NP-hard. Computing a pair of vertices at maximum distance is APX-hard.

  → The latter result holds for half-integral polytopes with a very easy description (fractional matching polytope).

In this lecture

In this lecture

• Characterization of the diameter of two polytopes (well-known in the combinatorial optimization community):

- the matching polytope
- the fractional matching polytope

In this lecture

- Characterization of the diameter of two polytopes (well-known in the combinatorial optimization community):
  - the matching polytope
  - the fractional matching polytope

- Discuss general algorithmic and hardness implications

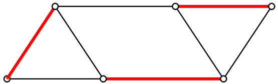
In this lecture

- Characterization of the diameter of two polytopes (well-known in the combinatorial optimization community):
  - the matching polytope
  - the fractional matching polytope
- Discuss general algorithmic and hardness implications
- Highlight open questions

The matching polytope

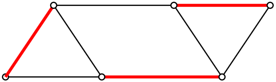
The matching polytope

- For a graph $G = (V, E)$, a matching is a subset of edges that have no node in common.

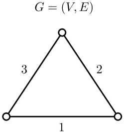


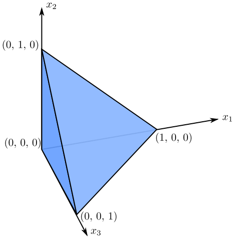
The matching polytope

• For a graph $G = (V, E)$, a matching is a subset of edges that have no node in common.



• The matching polytope ($\mathcal{P}_M$) is given by the convex hull of characteristic vectors of matchings of G.





The matching polytope

- [Edmonds'65] gave an LP-description of $\mathcal{P}_M$:

$$\mathcal{P}_M := \{x \in \mathbb{R}^E : \quad \begin{array}{rl} \sum_{e \in \delta(v)} x_e \leq 1 & \forall v \in V, \\ \sum_{e \in E[S]} x_e \leq \frac{|S|-1}{2} & \forall S \subseteq V : |S| \text{ odd} \\ x \geq 0\} & \end{array}$$

The matching polytope

- [Edmonds'65] gave an LP-description of $\mathcal{P}_M$:

$$\mathcal{P}_M := \{x \in \mathbb{R}^E : \begin{array}{rll} \sum_{e \in \delta(v)} x_e \leq 1 & & \forall v \in V, \\ \sum_{e \in E[S]} x_e \leq \frac{|S|-1}{2} & & \forall S \subseteq V : |S| \text{ odd} \\ x \geq 0\} & & \end{array}$$

How do we characterize adjacency of extreme points?

The matching polytope

- [Edmonds'65] gave an LP-description of $\mathcal{P}_M$:

$$\mathcal{P}_M := \{x \in \mathbb{R}^E : \begin{array}{rl} \sum_{e\in\delta(v)} x_e \leq 1 & \forall v \in V, \\ \sum_{e\in E[S]} x_e \leq \frac{|S|-1}{2} & \forall S \subseteq V : |S| \text{ odd} \\ x \geq 0\} & \end{array}$$

How do we characterize adjacency of extreme points?

Note: For a polyhedron $\mathcal{P} := \{x \in \mathbb{R}^d : Ax \leq b\}$ the following are equivalent:

- $z, y \in \mathcal{P}$ are adjacent extreme points on $\mathcal{P}$;
- There exists an cost vector $c \in \mathbb{R}^d$ such that z, y are the only optimal extreme points of $\max\{c^T x \ : \ x \in \mathcal{P}\}$;

The matching polytope

- [Edmonds'65] gave an LP-description of $\mathcal{P}_M$:

$$\mathcal{P}_M := \{x \in \mathbb{R}^E : \begin{array}{rl} \sum_{e \in \delta(v)} x_e \leq 1 & \forall v \in V, \\ \sum_{e \in E[S]} x_e \leq \frac{|S|-1}{2} & \forall S \subseteq V : |S| \text{ odd} \\ x \geq 0\} & \end{array}$$

How do we characterize adjacency of extreme points?

Note: For a polyhedron $\mathcal{P} := \{x \in \mathbb{R}^d : Ax \leq b\}$ the following are equivalent:

- $z, y \in \mathcal{P}$ are adjacent extreme points on $\mathcal{P}$;
- There exists an cost vector $c \in \mathbb{R}^d$ such that z, y are the only optimal extreme points of $\max\{c^T x \ : \ x \in \mathcal{P}\}$;
- The matrix corresp. to the constraints tight for both y, z has rank $d-1$.

The matching polytope

- [Edmonds'65] gave an LP-description of $\mathcal{P}_M$:

$$\mathcal{P}_M := \{x \in \mathbb{R}^E : \begin{array}{rl} \sum_{e \in \delta(v)} x_e \leq 1 & \forall v \in V, \\ \sum_{e \in E[S]} x_e \leq \frac{|S|-1}{2} & \forall S \subseteq V : |S| \text{ odd} \\ x \geq 0\} & \end{array}$$

How do we characterize adjacency of extreme points?

Note: For a polyhedron $\mathcal{P} := \{x \in \mathbb{R}^d : Ax \leq b\}$ the following are equivalent:

- $z, y \in \mathcal{P}$ are adjacent extreme points on $\mathcal{P}$;
- There exists an cost vector $c \in \mathbb{R}^d$ such that z, y are the only optimal extreme points of $\max\{c^T x \ : \ x \in \mathcal{P}\}$;
- The matrix corresp. to the constraints tight for both y, z has rank $d-1$.

- Matching is a graph problem. Any graphical characterization of adjacency?

The matching polytope

Theorem [Balinski&Russakoff'74,Chvàtal'75]

Two vertices of $\mathcal{P}_M$ are adjacent **iff** *the symmetric difference of the corresponding matchings induces* **one** *component.*

The matching polytope

Theorem [Balinski&Russakoff'74,Chvàtal'75]

Two vertices of $\mathcal{P}_M$ are adjacent **iff** *the symmetric difference of the corresponding matchings induces* **one** *component.*

Proof: The symmetric difference of two matchings is a union of alternating cycles and alternating paths.

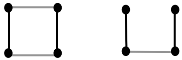
(a) (b)

The matching polytope

Theorem [Balinski&Russakoff'74,Chvàtal'75]

Two vertices of $\mathcal{P}_M$ are adjacent **iff** *the symmetric difference of the corresponding matchings induces* **one** *component.*

Proof: The symmetric difference of two matchings is a union of alternating cycles and alternating paths.



(a) (b)

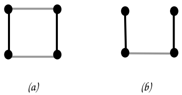
- *Sufficiency*: There is an objective function for which these matchings are the only optimal extreme point solutions.

The matching polytope

Theorem [Balinski&Russakoff'74,Chvàtal'75]

Two vertices of $\mathcal{P}_M$ are adjacent **iff** *the symmetric difference of the corresponding matchings induces* **one** *component.*

Proof: The symmetric difference of two matchings is a union of alternating cycles and alternating paths.



- *Sufficiency*: There is an objective function for which these matchings are the only optimal extreme point solutions.
- *Necessity*: If not, such an objective function can't exist! □

The matching polytope

Corollary

The diameter of $\mathcal{P}_M$ is equal to the size of a maximum matching of G.

The matching polytope

Corollary

The diameter of $\mathcal{P}_M$ is equal to the size of a maximum matching of G.

Proof:

- Let M be a maximum matching. The distance between any two matchings is at most $|M|$.

The matching polytope

Corollary

The diameter of $\mathcal{P}_M$ is equal to the size of a maximum matching of G.

Proof:

- Let M be a maximum matching. The distance between any two matchings is at most $|M|$.
- The distance between the empty matching (extreme point $\mathbf{0}$) and the matching M is $|M|$.

$\square$

The matching polytope

Corollary

The diameter of $\mathcal{P}_M$ is equal to the size of a maximum matching of G.

Proof:

- Let M be a maximum matching. The distance between any two matchings is at most $|M|$.
- The distance between the empty matching (extreme point $\mathbf{0}$) and the matching M is $|M|$.

$\square$

• **Obs 1:** From [Edmonds'65] it follows that the diameter of the matching polytope can be computed in polynomial time.

The matching polytope

Corollary

The diameter of $\mathcal{P}_M$ is equal to the size of a maximum matching of G.

Proof:

- Let M be a maximum matching. The distance between any two matchings is at most $|M|$.
- The distance between the empty matching (extreme point $\mathbf{0}$) and the matching M is $|M|$.

$\square$

• **Obs 1:** From [Edmonds'65] it follows that the diameter of the matching polytope can be computed in polynomial time.

• **Obs 2:** We can restate as:

$$diameter(\mathcal{P}_M) = \max_{x \in vertices(\mathcal{P}_M)} \{\mathbf{1}^T x\}$$

The fractional matching polytope

The fractional matching polytope

- The fractional matching polytope is given by a standard *LP-relaxation*:

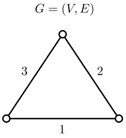
$$\mathcal{P}_{FM} := \{x \in \mathbb{R}^E : \sum_{e \in \delta(v)} x_e \leq 1 \ \forall v \in V, \ \ x \geq 0\}$$

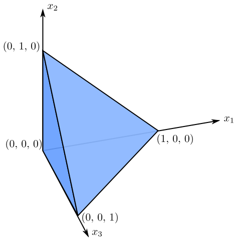
The fractional matching polytope

- The fractional matching polytope is given by a standard *LP-relaxation*:

$$\mathcal{P}_{FM} := \{x \in \mathbb{R}^E : \sum_{e \in \delta(v)} x_e \leq 1 \ \ \forall v \in V, \ \ x \geq 0\}$$

- This polytope can have fractional extreme points



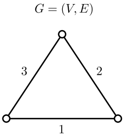


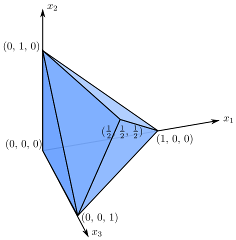
The fractional matching polytope

- The fractional matching polytope is given by a standard *LP-relaxation*:

$$\mathcal{P}_{FM} := \{x \in \mathbb{R}^E : \sum_{e \in \delta(v)} x_e \leq 1 \;\; \forall v \in V, \;\; x \geq 0\}$$

- This polytope can have fractional extreme points



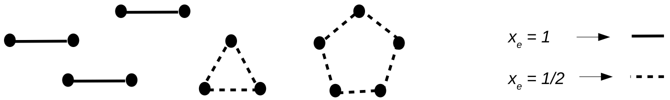


The fractional matching polytope

- [Balinski'65]: $\mathcal{P}_{FM}$ is a half-integral polytope.

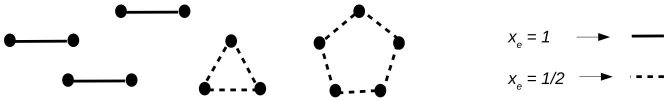
The fractional matching polytope

- [Balinski'65]: $\mathcal{P}_{FM}$ is a half-integral polytope. For a vertex x of $\mathcal{P}_{FM}$
  - the edges $\{e \in E : x_e = 1\} \rightarrow$ induce a matching $(\mathcal{M}_x)$
  - the edges $\{e \in E : x_e = \frac{1}{2}\} \rightarrow$ induce a collection of odd cycles $(\mathcal{C}_x)$



The fractional matching polytope

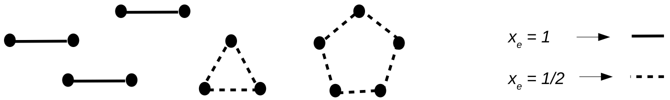
- [Balinski'65]: $\mathcal{P}_{FM}$ is a half-integral polytope. For a vertex x of $\mathcal{P}_{FM}$
  - the edges $\{e \in E : x_e = 1\} \rightarrow$ induce a matching ($\mathcal{M}_x$)
  - the edges $\{e \in E : x_e = \frac{1}{2}\} \rightarrow$ induce a collection of odd cycles ($\mathcal{C}_x$)



- Adjacency relations have also been studied (see e.g. Behrend'13)

The fractional matching polytope

- [Balinski'65]: $\mathcal{P}_{FM}$ is a half-integral polytope. For a vertex x of $\mathcal{P}_{FM}$
  - the edges $\{e \in E : x_e = 1\} \rightarrow$ induce a matching ($\mathcal{M}_x$)
  - the edges $\{e \in E : x_e = \frac{1}{2}\} \rightarrow$ induce a collection of odd cycles ($\mathcal{C}_x$)



- Adjacency relations have also been studied (see e.g. Behrend'13)

  $\rightarrow$ Let's derive some graphical properties of adjacent extreme points!

Adjacency

Adjacency

- Consider again the LP-description.

$\mathcal{P}_{FM}$:

$$\sum_{e \in \delta(v)} x_e \leq 1 \qquad \forall v \in V$$

$$x_e \geq 0 \qquad \forall e \in E$$

Adjacency

- Consider again the LP-description.

$\mathcal{P}_{FM}$:

$$\sum_{e \in \delta(v)} x_e \leq 1 \qquad \forall v \in V$$

$$x_e \geq 0 \qquad \forall e \in E$$

$\bar{\mathcal{P}}_{FM}$:

$$\sum_{e \in \delta(v)} x_e + x_v = 1 \qquad \forall v \in V$$

$$x_f \geq 0 \qquad \forall f \in E \cup V$$

- Add slack variables $\rightarrow$

Adjacency

- Consider again the LP-description.

$\mathcal{P}_{FM}$:

$$\sum_{e\in\delta(v)} x_e \leq 1 \qquad \forall v \in V$$

$$x_e \geq 0 \qquad \forall e \in E$$

$\bar{\mathcal{P}}_{FM}$:

$$\sum_{e\in\delta(v)} x_e + x_v = 1 \qquad \forall v \in V$$

$$x_f \geq 0 \qquad \forall f \in E \cup V$$

- Add slack variables $\rightarrow$ corresponds to adding one loop edge on each node!

Adjacency

- Consider again the LP-description.

$\mathcal{P}_{FM}$:

$$\sum_{e \in \delta(v)} x_e \leq 1 \qquad \forall v \in V$$

$$x_e \geq 0 \qquad \forall e \in E$$

$\bar{\mathcal{P}}_{FM}$:

$$\sum_{e \in \delta(v)} x_e + x_v = 1 \qquad \forall v \in V$$

$$x_f \geq 0 \qquad \forall f \in E \cup V$$

- Add slack variables $\rightarrow$ corresponds to adding one loop edge on each node!
- Let $\bar{y}$,$\bar{z}$ be adjacent vertices of $\bar{\mathcal{P}}_{FM}$, and $\bar{\mathcal{C}}_{\bar{y}}$, $\bar{\mathcal{C}}_{\bar{z}}$ the set of their odd cycles.

Adjacency

- Consider again the LP-description.

$\mathcal{P}_{FM}$:

$$\sum_{e\in\delta(v)} x_e \leq 1 \qquad \forall v \in V$$

$$x_e \geq 0 \qquad \forall e \in E$$

$\bar{\mathcal{P}}_{FM}$:

$$\sum_{e\in\delta(v)} x_e + x_v = 1 \qquad \forall v \in V$$

$$x_f \geq 0 \qquad \forall f \in E \cup V$$

- Add slack variables $\rightarrow$ corresponds to adding one loop edge on each node!
- Let $\bar{y}$,$\bar{z}$ be adjacent vertices of $\bar{\mathcal{P}}_{FM}$, and $\bar{\mathcal{C}}_{\bar{y}}$, $\bar{\mathcal{C}}_{\bar{z}}$ the set of their odd cycles.
- **Claim:** $|\bar{\mathcal{C}}_{\bar{z}} \Delta \bar{\mathcal{C}}_{\bar{y}}| \leq 2$.

Adjacency

- Consider again the LP-description.

$\mathcal{P}_{FM}$:

$$\sum_{e\in\delta(v)} x_e \leq 1 \qquad \forall v \in V$$

$$x_e \geq 0 \qquad \forall e \in E$$

$\bar{\mathcal{P}}_{FM}$:

$$\sum_{e\in\delta(v)} x_e + x_v = 1 \qquad \forall v \in V$$

$$x_f \geq 0 \qquad \forall f \in E \cup V$$

- Add slack variables $\rightarrow$ corresponds to adding one loop edge on each node!
- Let $\bar{y}$,$\bar{z}$ be adjacent vertices of $\bar{\mathcal{P}}_{FM}$, and $\bar{\mathcal{C}}_{\bar{y}}$, $\bar{\mathcal{C}}_{\bar{z}}$ the set of their odd cycles.
- **Claim:** $|\bar{\mathcal{C}}_{\bar{z}} \Delta \bar{\mathcal{C}}_{\bar{y}}| \leq 2$. Why?

Adjacency

- Consider again the LP-description.

$\mathcal{P}_{FM}$:

$$\sum_{e \in \delta(v)} x_e \leq 1 \qquad \forall v \in V$$

$$x_e \geq 0 \qquad \forall e \in E$$

$\bar{\mathcal{P}}_{FM}$:

$$\sum_{e \in \delta(v)} x_e + x_v = 1 \qquad \forall v \in V$$

$$x_f \geq 0 \qquad \forall f \in E \cup V$$

- Add slack variables $\rightarrow$ corresponds to adding one loop edge on each node!
- Let $\bar{y}$,$\bar{z}$ be adjacent vertices of $\bar{\mathcal{P}}_{FM}$, and $\bar{\mathcal{C}}_{\bar{y}}$, $\bar{\mathcal{C}}_{\bar{z}}$ the set of their odd cycles.
- **Claim:** $|\bar{\mathcal{C}}_{\bar{z}} \Delta \bar{\mathcal{C}}_{\bar{y}}| \leq 2$. Why?
  - support($\bar{y}$) $\cup$ support($\bar{z}$) contains one component $\bar{K}$ with some f: $\bar{y}_f \neq \bar{z}_f$.

Adjacency

- Consider again the LP-description.

$\mathcal{P}_{FM}$:

$$\sum_{e\in\delta(v)} x_e \leq 1 \qquad \forall v \in V$$

$$x_e \geq 0 \qquad \forall e \in E$$

$\bar{\mathcal{P}}_{FM}$:

$$\sum_{e\in\delta(v)} x_e + x_v = 1 \qquad \forall v \in V$$

$$x_f \geq 0 \qquad \forall f \in E \cup V$$

- Add slack variables $\rightarrow$ corresponds to adding one loop edge on each node!
- Let $\bar{y}$,$\bar{z}$ be adjacent vertices of $\bar{\mathcal{P}}_{FM}$, and $\bar{\mathcal{C}}_{\bar{y}}$, $\bar{\mathcal{C}}_{\bar{z}}$ the set of their odd cycles.
- **Claim:** $|\bar{\mathcal{C}}_{\bar{z}} \Delta \bar{\mathcal{C}}_{\bar{y}}| \leq 2$. Why?
  - support($\bar{y}$) $\cup$ support($\bar{z}$) contains one component $\bar{K}$ with some f: $\bar{y}_f \neq \bar{z}_f$.
  - $\bar{y}$,$\bar{z}$ adjacent $\Leftrightarrow$ there are $|E \cup V| - 1$ linearly independent constraints of $\bar{\mathcal{P}}_{FM}$ that are tight for both $\bar{y}$ and $\bar{z}$

Adjacency

- Consider again the LP-description.

$\mathcal{P}_{FM}$:

$$\sum_{e\in\delta(v)} x_e \leq 1 \qquad \forall v \in V$$
$$x_e \geq 0 \qquad \forall e \in E$$

$\bar{\mathcal{P}}_{FM}$:

$$\sum_{e\in\delta(v)} x_e + x_v = 1 \qquad \forall v \in V$$
$$x_f \geq 0 \qquad \forall f \in E \cup V$$

- Add slack variables $\rightarrow$ corresponds to adding one loop edge on each node!
- Let $\bar{y}$,$\bar{z}$ be adjacent vertices of $\bar{\mathcal{P}}_{FM}$, and $\bar{\mathcal{C}}_{\bar{y}}$, $\bar{\mathcal{C}}_{\bar{z}}$ the set of their odd cycles.
- **Claim:** $|\bar{\mathcal{C}}_{\bar{z}} \Delta \bar{\mathcal{C}}_{\bar{y}}| \leq 2$. Why?
  - support($\bar{y}$) $\cup$ support($\bar{z}$) contains one component $\bar{K}$ with some f: $\bar{y}_f \neq \bar{z}_f$.
  - $\bar{y}$,$\bar{z}$ adjacent $\Leftrightarrow$ there are $|E \cup V| - 1$ linearly independent constraints of $\bar{\mathcal{P}}_{FM}$ that are tight for both $\bar{y}$ and $\bar{z}$ $\Rightarrow$ $\bar{K}$ has at most $\leq |\bar{K}|+1$ edges.

Adjacency

- Consider again the LP-description.

$\mathcal{P}_{FM}$:

$$\sum_{e\in\delta(v)} x_e \leq 1 \qquad \forall v \in V$$

$$x_e \geq 0 \qquad \forall e \in E$$

$\bar{\mathcal{P}}_{FM}$:

$$\sum_{e\in\delta(v)} x_e + x_v = 1 \qquad \forall v \in V$$

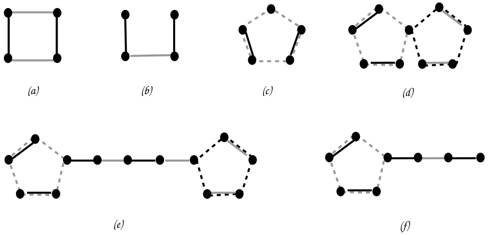
$$x_f \geq 0 \qquad \forall f \in E \cup V$$

- Add slack variables $\rightarrow$ corresponds to adding one loop edge on each node!
- Let $\bar{y}$,$\bar{z}$ be adjacent vertices of $\bar{\mathcal{P}}_{FM}$, and $\bar{\mathcal{C}}_{\bar{y}}$, $\bar{\mathcal{C}}_{\bar{z}}$ the set of their odd cycles.
- **Claim:** $|\bar{\mathcal{C}}_{\bar{z}} \Delta \bar{\mathcal{C}}_{\bar{y}}| \leq 2$. Why?
  - support($\bar{y}$) $\cup$ support($\bar{z}$) contains one component $\bar{K}$ with some f: $\bar{y}_f \neq \bar{z}_f$.
  - $\bar{y}$,$\bar{z}$ adjacent $\Leftrightarrow$ there are $|E \cup V| - 1$ linearly independent constraints of $\bar{\mathcal{P}}_{FM}$ that are tight for both $\bar{y}$ and $\bar{z}$ $\Rightarrow$ $\bar{K}$ has at most $\leq |\bar{K}|+1$ edges.
  - **Obs:** An n-connected graph with $n+1$ edges has ≤ 2 odd cycles!

$\square$

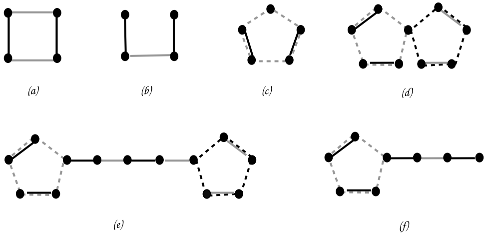
Adjacency

- We explicitly highlight the following adjacencies:



Adjacency

- We explicitly highlight the following adjacencies:



Exercise: *Prove that these fractional matchings are adjacent extreme points!*

Diameter

Diameter

Theorem [S.'18]

$diameter(\mathcal{P}_{FM}) = \max_{x \in vertices(\mathcal{P}_{FM})}\{\mathbf{1}^T x + \frac{|\mathcal{C}_x|}{2}\}$

Diameter

Theorem [S.'18]

$diameter(\mathcal{P}_{FM}) = \max_{x \in vertices(\mathcal{P}_{FM})}\{\mathbf{1}^T x + \frac{|\mathcal{C}_x|}{2}\}$

- **Obs:** For a bipartite graph $\mathcal{C}_x = \emptyset$

Diameter

Theorem [S.'18]

$diameter(\mathcal{P}_{FM}) = \max_{x \in vertices(\mathcal{P}_{FM})}\{\mathbf{1}^T x + \frac{|\mathcal{C}_x|}{2}\}$

- **Obs:** For a bipartite graph $\mathcal{C}_x = \emptyset \quad \rightarrow diameter(\mathcal{P}_{FM}) = diameter(\mathcal{P}_M)$.

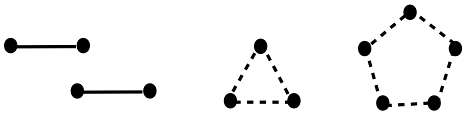
Diameter

Theorem [S.'18]

$diameter(\mathcal{P}_{FM}) = \max_{x \in vertices(\mathcal{P}_{FM})}\{\mathbf{1}^T x + \frac{|\mathcal{C}_x|}{2}\}$

- **Obs:** For a bipartite graph $\mathcal{C}_x = \emptyset \quad \rightarrow diameter(\mathcal{P}_{FM}) = diameter(\mathcal{P}_M)$.

Lower bound: Let w be any vertex.



Diameter

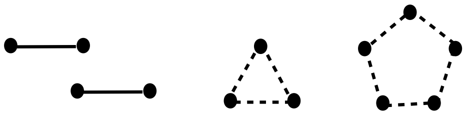
Theorem [S.'18]

$diameter(\mathcal{P}_{FM}) = \max_{x \in vertices(\mathcal{P}_{FM})}\{\mathbf{1}^T x + \frac{|\mathcal{C}_x|}{2}\}$

- **Obs:** For a bipartite graph $\mathcal{C}_x = \emptyset \quad \rightarrow diameter(\mathcal{P}_{FM}) = diameter(\mathcal{P}_M)$.

Lower bound: Let w be any vertex.

- Show: the distance between w and the $\mathbf{0}$-vertex is $\geq \mathbf{1}^T w + \frac{|\mathcal{C}_w|}{2}$.



Diameter

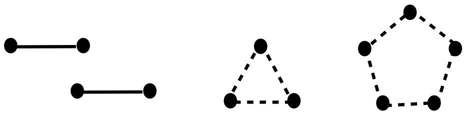
Theorem [S.'18]

$diameter(\mathcal{P}_{FM}) = \max_{x \in vertices(\mathcal{P}_{FM})}\{\mathbf{1}^T x + \frac{|\mathcal{C}_x|}{2}\}$

- **Obs:** For a bipartite graph $\mathcal{C}_x = \emptyset \quad \rightarrow diameter(\mathcal{P}_{FM}) = diameter(\mathcal{P}_M)$.

Lower bound: Let w be any vertex.

- Show: the distance between w and the $\mathbf{0}$-vertex is $\geq \mathbf{1}^T w + \frac{|\mathcal{C}_w|}{2}$.
  - Add a loop edge on each node v in support(w).



Diameter

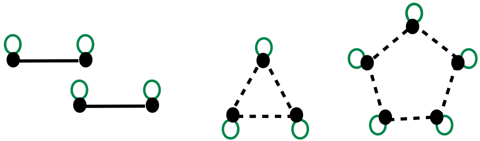
Theorem [S.'18]

$diameter(\mathcal{P}_{FM}) = \max_{x \in vertices(\mathcal{P}_{FM})}\{\mathbf{1}^T x + \frac{|\mathcal{C}_x|}{2}\}$

- **Obs:** For a bipartite graph $\mathcal{C}_x = \emptyset \quad \rightarrow diameter(\mathcal{P}_{FM}) = diameter(\mathcal{P}_M)$.

Lower bound: Let w be any vertex.

- Show: the distance between w and the $\mathbf{0}$-vertex is $\geq \mathbf{1}^T w + \frac{|\mathcal{C}_w|}{2}$.
  - Add a loop edge on each node v in support(w).



Diameter

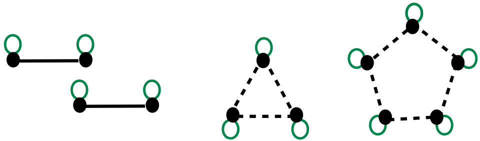
Theorem [S.'18]

$diameter(\mathcal{P}_{FM}) = \max_{x \in vertices(\mathcal{P}_{FM})}\{\mathbf{1}^T x + \frac{|\mathcal{C}_x|}{2}\}$

- **Obs:** For a bipartite graph $\mathcal{C}_x = \emptyset \quad \rightarrow diameter(\mathcal{P}_{FM}) = diameter(\mathcal{P}_M)$.

Lower bound: Let w be any vertex.

- Show: the distance between w and the $\mathbf{0}$-vertex is $\geq \mathbf{1}^T w + \frac{|\mathcal{C}_w|}{2}$.
  - Add a loop edge on each node v in support(w).
  - **Note**: $\Big($# of odd cycles $\Big) = 2\Big(\mathbf{1}^T w + \frac{|\mathcal{C}_w|}{2}\Big)$



Diameter

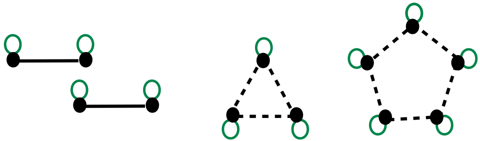
Theorem [S.'18]

$diameter(\mathcal{P}_{FM}) = \max_{x \in vertices(\mathcal{P}_{FM})} \{\mathbf{1}^T x + \frac{|\mathcal{C}_x|}{2}\}$

- **Obs:** For a bipartite graph $\mathcal{C}_x = \emptyset \quad \rightarrow diameter(\mathcal{P}_{FM}) = diameter(\mathcal{P}_M)$.

Lower bound: Let w be any vertex.

- Show: the distance between w and the $\mathbf{0}$-vertex is $\geq \mathbf{1}^T w + \frac{|\mathcal{C}_w|}{2}$.
  - Add a loop edge on each node v in support(w).
  - **Note**: $\Big(\#$ of odd cycles $\Big) = 2\Big(\mathbf{1}^T w + \frac{|\mathcal{C}_w|}{2}\Big)$
  - At each move, the above quantity can decrease by at most 2

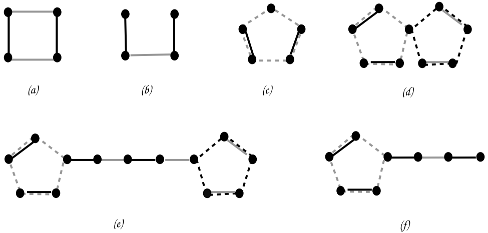


Upper bound

- Given two distinct vertices z and y of $\mathcal{P}_{FM}$, the selection of the moves to take is not straightforward.

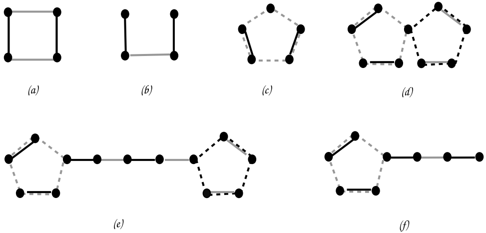
Upper bound

• Given two distinct vertices z and y of $\mathcal{P}_{FM}$, the selection of the moves to take is not straightforward. Recall the adjacencies we mentioned:



Upper bound

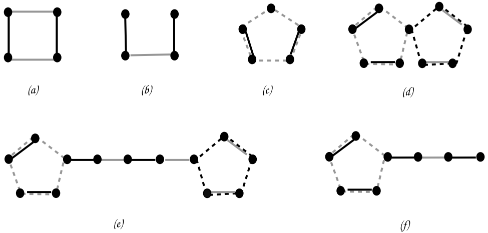
- Given two distinct vertices z and y of $\mathcal{P}_{FM}$, the selection of the moves to take is not straightforward. Recall the adjacencies we mentioned:



- An easy "attempt" to go from z to y would be to define:

Upper bound

• Given two distinct vertices z and y of $\mathcal{P}_{FM}$, the selection of the moves to take is not straightforward. Recall the adjacencies we mentioned:

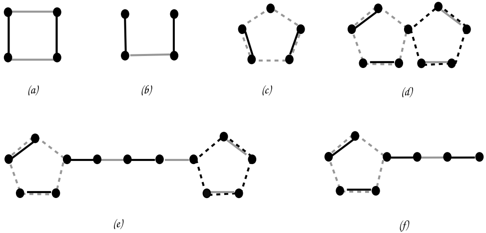


• An easy "attempt" to go from z to y would be to define:

- (i) a path from z to a 0/1-vertex $\bar{z}$ by removing one $C \in \mathcal{C}_z$ at each step

Upper bound

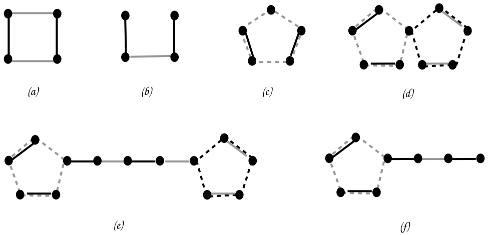
- Given two distinct vertices z and y of $\mathcal{P}_{FM}$, the selection of the moves to take is not straightforward. Recall the adjacencies we mentioned:



- An easy "attempt" to go from z to y would be to define:
  - (i) a path from z to a 0/1-vertex $\bar{z}$ by removing one $C \in \mathcal{C}_z$ at each step
  - (ii) a path from y to a 0/1-vertex $\bar{y}$ by removing one $C \in \mathcal{C}_y$ at each step

Upper bound

• Given two distinct vertices z and y of $\mathcal{P}_{FM}$, the selection of the moves to take is not straightforward. Recall the adjacencies we mentioned:

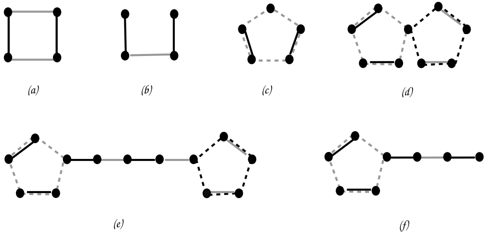


• An easy "attempt" to go from z to y would be to define:

- (i) a path from z to a 0/1-vertex $\bar{z}$ by removing one $C \in \mathcal{C}_z$ at each step
- (ii) a path from y to a 0/1-vertex $\bar{y}$ by removing one $C \in \mathcal{C}_y$ at each step
- (iii) a path from $\bar{z}$ to $\bar{y}$ (e.g. using the 1-skeleton of $\mathcal{P}_M$)

Upper bound

• Given two distinct vertices z and y of $\mathcal{P}_{FM}$, the selection of the moves to take is not straightforward. Recall the adjacencies we mentioned:



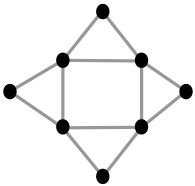
• An easy "attempt" to go from z to y would be to define:

- (i) a path from z to a 0/1-vertex $\bar{z}$ by removing one $C \in \mathcal{C}_z$ at each step
- (ii) a path from y to a 0/1-vertex $\bar{y}$ by removing one $C \in \mathcal{C}_y$ at each step
- (iii) a path from $\bar{z}$ to $\bar{y}$ (e.g. using the 1-skeleton of $\mathcal{P}_M$)

...but unfortunately this may lead to paths longer than the claimed bound!

Upper bound

- Bad example:

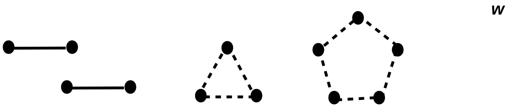


Upper bound

- Given two distinct vertices z and y of $\mathcal{P}_{FM}$, we
  - Define a path of the form: $z \to w \to y$ for some maximal vertex w of $\mathcal{P}_{FM}$ satisfying: $\text{support}(w) \subseteq \text{support}(z) \cup \text{support}(y)$

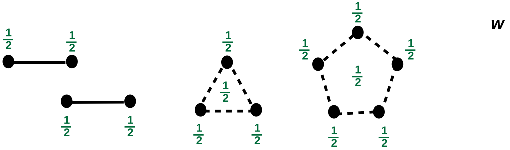
Upper bound

- Given two distinct vertices z and y of $\mathcal{P}_{FM}$, we
  - Define a path of the form: $z \to w \to y$ for some maximal vertex w of $\mathcal{P}_{FM}$ satisfying: support$(w) \subseteq$ support$(z) \cup$ support(y)
  - Rely on a token argument: assign a token of value $\frac{1}{2}$ to each node v and each cycle C in support(w)



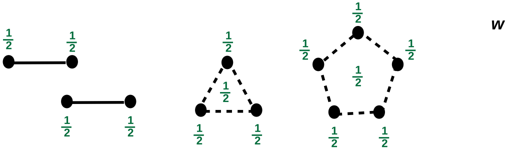
Upper bound

- Given two distinct vertices z and y of $\mathcal{P}_{FM}$, we
  - Define a path of the form: $z \to w \to y$ for some maximal vertex w of $\mathcal{P}_{FM}$ satisfying: $\text{support}(w) \subseteq \text{support}(z) \cup \text{support}(y)$
  - Rely on a token argument: assign a token of value $\frac{1}{2}$ to each node v and each cycle C in $\text{support}(w)$



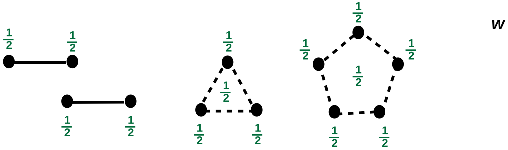
Upper bound

- Given two distinct vertices z and y of $\mathcal{P}_{FM}$, we
  - Define a path of the form: $z \rightarrow w \rightarrow y$ for some maximal vertex w of $\mathcal{P}_{FM}$ satisfying: support$(w) \subseteq$ support$(z) \cup$ support(y)
  - Rely on a token argument: assign a token of value $\frac{1}{2}$ to each node v and each cycle C in support(w) (**Note**: total token value $= \mathbf{1}^T w + \frac{|\mathcal{C}_w|}{2}$)



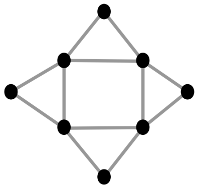
Upper bound

- Given two distinct vertices z and y of $\mathcal{P}_{FM}$, we
  - Define a path of the form: $z \to w \to y$ for some maximal vertex w of $\mathcal{P}_{FM}$ satisfying: $\text{support}(w) \subseteq \text{support}(z) \cup \text{support}(y)$
  - Rely on a token argument: assign a token of value $\frac{1}{2}$ to each node v and each cycle C in support(w) (**Note**: total token value $= \mathbf{1}^T w + \frac{|\mathcal{C}_w|}{2}$)
  - Show: each move on the path can be payed using two tokens of nodes/cycles



Upper bound

- Example:



Algorithmic and hardness implications

Hardness

Theorem [S.'18]

Computing the diameter of a polytope is a strongly NP-hard problem.

Hardness

Theorem [S.'18]

Computing the diameter of a polytope is a strongly NP-hard problem.

- Reduction from the (strongly) NP-hard problem Partition Into Triangles.
  - **Given:** A graph $G = (V, E)$
  - **Decide:** V can be partitioned into $\{V_1, \ldots, V_q\}$: $\forall i,\ V_i$ induces a triangle

Hardness

Theorem [S.'18]

Computing the diameter of a polytope is a strongly NP-hard problem.

- Reduction from the (strongly) NP-hard problem Partition Into Triangles.
  - **Given:** A graph $G = (V, E)$
  - **Decide:** V can be partitioned into $\{V_1, \dots, V_q\}$: $\forall i,\ V_i$ induces a triangle
- Given G, consider the fractional matching polytope $\mathcal{P}_{FM}$ associated to G.
- Let x be a vertex of $\mathcal{P}_{FM}$. Then: (i) $\mathbf{1}^T x \leq \frac{|V|}{2}$ (ii) $\frac{|\mathcal{C}_x|}{2} \leq \frac{|V|}{6}$

Hardness

Theorem [S.'18]

Computing the diameter of a polytope is a strongly NP-hard problem.

- Reduction from the (strongly) NP-hard problem Partition Into Triangles.
  - **Given:** A graph $G = (V, E)$
  - **Decide:** V can be partitioned into $\{V_1, \ldots, V_q\}$: $\forall i,\ V_i$ induces a triangle
- Given G, consider the fractional matching polytope $\mathcal{P}_{FM}$ associated to G.
- Let x be a vertex of $\mathcal{P}_{FM}$. Then: (i) $\mathbf{1}^T x \leq \frac{|V|}{2}$ (ii) $\frac{|\mathcal{C}_x|}{2} \leq \frac{|V|}{6}$

Proposition: $diam(\mathcal{P}_{FM}) = \frac{2}{3}|V| \Leftrightarrow G$ is a yes-instance to PIT.

$\square$

Hardness

Theorem [S.'18]

Computing the diameter of a polytope is a strongly NP-hard problem.

- Reduction from the (strongly) NP-hard problem Partition Into Triangles.
  - **Given:** A graph $G = (V, E)$
  - **Decide:** V can be partitioned into $\{V_1, \dots, V_q\}$: $\forall i,\ V_i$ induces a triangle

- Given G, consider the fractional matching polytope $\mathcal{P}_{FM}$ associated to G.
- Let x be a vertex of $\mathcal{P}_{FM}$. Then: (i) $\mathbf{1}^T x \leq \frac{|V|}{2}$ (ii) $\frac{|\mathcal{C}_x|}{2} \leq \frac{|V|}{6}$

Proposition: $diam(\mathcal{P}_{FM}) = \frac{2}{3}|V| \Leftrightarrow G$ is a yes-instance to PIT.

$\square$

- With some extra effort, we can strengthen the result to show APX-hardness.

Hardness

Hardness

• Do the previous results have some hardness implication on the performance Simplex algorithm? Not in the current form...

Hardness

• Do the previous results have some hardness implication on the performance Simplex algorithm? Not in the current form... but some implications can be derived easily with a little extra work!

Hardness

- Do the previous results have some hardness implication on the performance Simplex algorithm? Not in the current form... but some implications can be derived easily with a little extra work!
- In particular, one can observe the following (see [De Loera, Kafer, S.'19]):

Given a vertex of a bipartite matching polytope and an objective function, deciding if there exists a neighboring optimal vertex is NP-hard.

Hardness

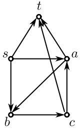
• Do the previous results have some hardness implication on the performance Simplex algorithm? Not in the current form… but some implications can be derived easily with a little extra work!

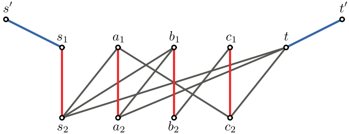
• In particular, one can observe the following (see [De Loera, Kafer, S.'19]):

Given a vertex of a bipartite matching polytope and an objective function, deciding if there exists a neighboring optimal vertex is NP-hard.

• **Proof.** Reduction: Given a directed graph H we:

- construct a bipartite graph G, extreme point x of $\mathcal{P}_{FM}(G)$, obj function c.
- show that $\exists$ a neighboring optimal extreme point of x iff H is Hamiltonian.





Hardness

• Do the previous results have some hardness implication on the performance Simplex algorithm? Not in the current form... but some implications can be derived easily with a little extra work!

• In particular, one can observe the following (see [De Loera, Kafer, S.'19]):

Given a vertex of a bipartite matching polytope and an objective function, deciding if there exists a neighboring optimal vertex is NP-hard.

• **Note:** Similar observation in [Barahona&Tardos'89] for circulation polytope.

Hardness

• Do the previous results have some hardness implication on the performance Simplex algorithm? Not in the current form... but some implications can be derived easily with a little extra work!

• In particular, one can observe the following (see [De Loera, Kafer, S.'19]):

Given a vertex of a bipartite matching polytope and an objective function, deciding if there exists a neighboring optimal vertex is NP-hard.

• **Note:** Similar observation in [Barahona&Tardos'89] for circulation polytope.

Corollary

Finding the shortest monotone path to an optimal solution is NP-hard, and hard-to-approximate within a factor better than 2.

Hardness

- Do the previous results have some hardness implication on the performance Simplex algorithm? Not in the current form... but some implications can be derived easily with a little extra work!
- In particular, one can observe the following (see [De Loera, Kafer, S.'19]):

Given a vertex of a bipartite matching polytope and an objective function, deciding if there exists a neighboring optimal vertex is NP-hard.

- **Note:** Similar observation in [Barahona&Tardos'89] for circulation polytope.

Corollary

Finding the shortest monotone path to an optimal solution is NP-hard, and hard-to-approximate within a factor better than 2.

- **Consequences** (unless P=NP):
  - For any efficient pivoting rule, an edge-augmentation algorithm (like Simplex) can't reach the optimum with a min number of augmentations.

Final remarks

- **Main questions:**
  - Is the polynomial-Hirsch conjecture true?
  - Is there a polynomial pivoting rule for the Simplex algorithm?

Final remarks

- **Main questions:**
  - Is the polynomial-Hirsch conjecture true?
  - Is there a polynomial pivoting rule for the Simplex algorithm?
- Diameter of the perfect matching polytope?

Final remarks

- **Main questions:**
  - Is the polynomial-Hirsch conjecture true?
  - Is there a polynomial pivoting rule for the Simplex algorithm?
- Diameter of the perfect matching polytope?
- All the hardness results discussed are for non simple polytopes.

$\rightarrow$ A d-dimensional polytope is simple if every vertex is in exactly d facets

Final remarks

- **Main questions:**
  - Is the polynomial-Hirsch conjecture true?
  - Is there a polynomial pivoting rule for the Simplex algorithm?
- Diameter of the perfect matching polytope?
- All the hardness results discussed are for non simple polytopes.

$\rightarrow$ A d-dimensional polytope is simple if every vertex is in exactly d facets

  - Can one extend them to simple polytopes?
  - Note: the complexity of computing the diameter of a simple polytope is mentioned as an open question in the survey of [Kaibel&Pfetsch'03]

Final remarks

- **Main questions:**
  - Is the polynomial-Hirsch conjecture true?
  - Is there a polynomial pivoting rule for the Simplex algorithm?

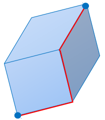
- Diameter of the perfect matching polytope?

- All the hardness results discussed are for non simple polytopes.

$\rightarrow$ A d-dimensional polytope is simple if every vertex is in exactly d facets

  - Can one extend them to simple polytopes?
  - Note: the complexity of computing the diameter of a simple polytope is mentioned as an open question in the survey of [Kaibel&Pfetsch'03]

Thank you!



Diameter of Polytopes:
Algorithmic and Combinatorial Aspects

Laura Sanità

Department of Mathematics and Computer Science
TU Eindhoven (Netherlands)

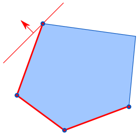
Department of Combinatorics and Optimization
University of Waterloo (Canada)

IPCO Summer School, 2020

From last lecture...

• The Simplex algorithm is an extremely popular method to solve Linear Programs (LP) (named as one of the “top 10 algorithms” of the 20th century).

• It exploits the fact that an optimal solution of an LP defined on a polytope can be found at one of its extreme points



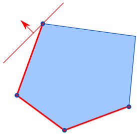
• **Simplex Algorithm's idea:** pivot from an extreme point to an improving adjacent one, until the optimum is found!

• Related concept: **Diameter** of a polytope $\rightarrow$ Maximum length of a ‘shortest path’ between two extreme points of a polytope.

From last lecture...

• The Simplex algorithm is an extremely popular method to solve Linear Programs (LP) (named as one of the "top 10 algorithms" of the 20th century).

• It exploits the fact that an optimal solution of an LP defined on a polytope can be found at one of its extreme points



• **Simplex Algorithm's idea:** pivot from an extreme point to an improving adjacent one, until the optimum is found!

• Related concept: **Diameter** of a polytope $\rightarrow$ Maximum length of a 'shortest path' between two extreme points of a polytope.

...Can we get new insights by enlarging the set of directions?

Circuits

- One interesting way to enlarge the set of directions is to look at circuits.

Circuits

• One interesting way to enlarge the set of directions is to look at circuits.

• For a given polytope, the circuits are given by the set of all potential edge-directions that can arise by translating some of its facets.

Circuits

• One interesting way to enlarge the set of directions is to look at circuits.

• For a given polytope, the circuits are given by the set of all potential edge-directions that can arise by translating some of its facets.

Circuits

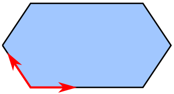
• One interesting way to enlarge the set of directions is to look at circuits.

• For a given polytope, the circuits are given by the set of all potential edge-directions that can arise by translating some of its facets.

Circuits

• One interesting way to enlarge the set of directions is to look at circuits.

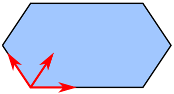
• For a given polytope, the circuits are given by the set of all potential edge-directions that can arise by translating some of its facets.



Circuits

• One interesting way to enlarge the set of directions is to look at circuits.

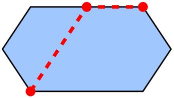
• For a given polytope, the circuits are given by the set of all potential edge-directions that can arise by translating some of its facets.



Circuits

- One interesting way to enlarge the set of directions is to look at circuits.

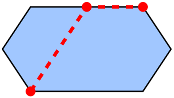
- For a given polytope, the circuits are given by the set of all potential edge-directions that can arise by translating some of its facets.



Circuits

- One interesting way to enlarge the set of directions is to look at circuits.

- For a given polytope, the circuits are given by the set of all potential edge-directions that can arise by translating some of its facets.

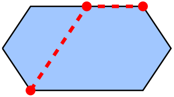


- Circuits have a long history [Rockafellar'69,Graver'75,Bland'76].

Circuits

• One interesting way to enlarge the set of directions is to look at circuits.

• For a given polytope, the circuits are given by the set of all potential edge-directions that can arise by translating some of its facets.



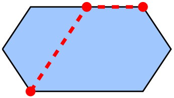
• Circuits have a long history [Rockafellar'69,Graver'75,Bland'76].

• Circuits and circuit-augmentation algorithms have appeared in several papers on linear/integer optimization (see e.g. [Hemmecke, Onn, Weismantel'11] [Hemmecke, Onn, Romanchuk'13] [De Loera, Hemmecke, Lee'15] [Borgwardt, Viss'19])

Circuits

• One interesting way to enlarge the set of directions is to look at circuits.

• For a given polytope, the circuits are given by the set of all potential edge-directions that can arise by translating some of its facets.



• Circuits have a long history [Rockafellar'69,Graver'75,Bland'76].

• Circuits and circuit-augmentation algorithms have appeared in several papers on linear/integer optimization (see e.g. [Hemmecke, Onn, Weismantel'11] [Hemmecke, Onn, Romanchuk'13] [De Loera, Hemmecke, Lee'15] [Borgwardt, Viss'19])

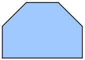
Circuits

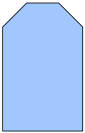
• [Borgwardt,Finhold,Hemmecke'14] formalized the notion of **circuit-diameter**: *max-value of a shortest path between two extreme points, assuming that at any given point we can move maximally along any circuit.*

Circuits

• [Borgwardt,Finhold,Hemmecke'14] formalized the notion of **circuit-diameter**: *max-value of a shortest path between two extreme points, assuming that at any given point we can move maximally along any circuit.*

• **Note:** Polytopes with the same combinatorial structure might have different circuit-diameter values.





Circuits

- [Borgwardt,Finhold,Hemmecke'14] formalized the notion of **circuit-diameter**: *max-value of a shortest path between two extreme points, assuming that at any given point we can move maximally along any circuit.*

- **Note:** Polytopes with the same combinatorial structure might have different circuit-diameter values.

Circuits

• [Borgwardt,Finhold,Hemmecke'14] formalized the notion of **circuit-diameter**: *max-value of a shortest path between two extreme points, assuming that at any given point we can move maximally along any circuit.*

• **Note:** Polytopes with the same combinatorial structure might have different circuit-diameter values.

- [Borgwardt,Finhold,Hemmecke'14] conjectured that the circuit-diameter satisfies the Hirsch bound.
- [Stephen&Yusun'15] showed that the Klee-Walkup polyhedron satisfies it.

Interesting aspects

Interesting aspects

- Algorithmic aspects:
  - Can we exploit circuit-augmentation algorithms to make conclusions about the perfomance of the Simplex algorithm?

Interesting aspects

- Algorithmic aspects:
  - Can we exploit circuit-augmentation algorithms to make conclusions about the perfomance of the Simplex algorithm?

- Diameter-related aspects:
  - Can we gain insights from the generalized notion of circuit-diameter on long-standing conjectures in the literature about diameters?

Interesting aspects – In this lecture

- Algorithmic aspects:
  - Can we exploit circuit-augmentation algorithms to make conclusions about the perfomance of the Simplex algorithm?

  $\rightarrow$ Emphasis: LPs defined on 0/1 polytopes

- Diameter-related aspects:
  - Can we gain insights from the generalized notion of circuit-diameter on long-standing conjectures in the literature about diameters?

  $\rightarrow$ Emphasis: TSP polytope

Circuits

Circuits

- Formally, for a polyhedron P of the form $P = \{x \in \mathbb{R}^n : Ax = b,\ Bx \leq d\}$, a non-zero vector $g \in \mathbb{R}^n$ is a **circuit** if
  - $g \in$ Kernel(A)
  - Bg is support-minimal in the set $\{By : y \in \text{ Kernel}(A),\ y \neq 0\}$

Circuits

• Formally, for a polyhedron P of the form $P = \{x \in \mathbb{R}^n : Ax = b,\ Bx \leq d\}$, a non-zero vector $g \in \mathbb{R}^n$ is a **circuit** if

- $g \in$ Kernel(A)
- Bg is support-minimal in the set $\{By : y \in$ Kernel$(A),\ y \neq 0\}$

$\rightarrow$ *Circuits correspond to all edge-directions obtainable by possibly translating facets.*

Circuits

• Formally, for a polyhedron P of the form $P = \{x \in \mathbb{R}^n : Ax = b,\ Bx \leq d\}$, a non-zero vector $g \in \mathbb{R}^n$ is a **circuit** if

- $g \in \text{Kernel}(A)$
- Bg is support-minimal in the set $\{By : y \in \text{Kernel}(A),\ y \neq 0\}$

$\rightarrow$ *Circuits correspond to all edge-directions obtainable by possibly translating facets.*

• **Note**: If g is a circuit, then αg is a circuit (for any non zero $\alpha \in \mathbb{R}$).

Circuits

- Formally, for a polyhedron P of the form $P = \{x \in \mathbb{R}^n : Ax = b,\ Bx \leq d\}$, a non-zero vector $g \in \mathbb{R}^n$ is a **circuit** if
  - $g \in$ Kernel(A)
  - Bg is support-minimal in the set $\{By : y \in \text{ Kernel}(A),\ y \neq 0\}$

$\rightarrow$ *Circuits correspond to all edge-directions obtainable by possibly translating facets.*

- **Note**: If g is a circuit, then αg is a circuit (for any non zero $\alpha \in \mathbb{R}$).

- The set of circuits can be made *finite* by normalizing in some way, e.g.
  - (optional:) g has co-prime integer components

Circuits

Circuits

- Consider the fractional matching polytope:

$$\sum_{e\in\delta(v)} x_e \leq 1 \qquad \forall v \in V$$
$$x_e \geq 0 \qquad \forall e \in E$$

Circuits

- Consider the fractional matching polytope:

$$\sum_{e \in \delta(v)} x_e \leq 1 \qquad \forall v \in V$$

$$x_e \geq 0 \qquad \forall e \in E$$

- Let g be a circuit.

What can we say about its support?

Circuits

- Consider the fractional matching polytope:

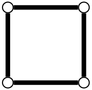
$$\sum_{e \in \delta(v)} x_e \leq 1 \qquad \forall v \in V$$

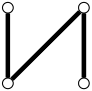
$$x_e \geq 0 \qquad \forall e \in E$$

- Let g be a circuit.

What can we say about its support?

- ▶ $\sup(g)$ induces a connected graph.





Circuits

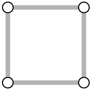
- Consider the fractional matching polytope:
- Let g be a circuit.

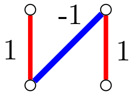
What can we say about its support?

$$\sum_{e \in \delta(v)} x_e \leq 1 \qquad \forall v \in V$$

$$x_e \geq 0 \qquad \forall e \in E$$

- ▶ $\sup(g)$ induces a connected graph.





Circuits

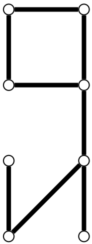
- Consider the fractional matching polytope:
- Let g be a circuit.

What can we say about its support?

$$\sum_{e \in \delta(v)} x_e \leq 1 \qquad \forall v \in V$$

$$x_e \geq 0 \qquad \forall e \in E$$

- sup(g) induces a connected graph.
- If sup(g) contains an even cycle, sup(g) is an even cycle.



Circuits

- Consider the fractional matching polytope:

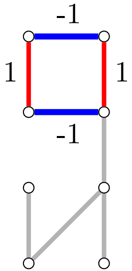
$$\sum_{e \in \delta(v)} x_e \leq 1 \qquad \forall v \in V$$

$$x_e \geq 0 \qquad \forall e \in E$$

- Let g be a circuit.

What can we say about its support?

  - $\sup(g)$ induces a connected graph.
  - If $\sup(g)$ contains an even cycle, $\sup(g)$ is an even cycle.



Circuits

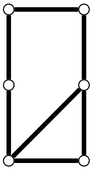
- Consider the fractional matching polytope:

$$\sum_{e \in \delta(v)} x_e \leq 1 \qquad \forall v \in V$$
$$x_e \geq 0 \qquad \forall e \in E$$

- Let g be a circuit.

What can we say about its support?

- $\sup(g)$ induces a connected graph.
- If $\sup(g)$ contains an even cycle, $\sup(g)$ is an even cycle.
- Any two odd cycles in $\sup(g)$ intersect in at most one vertex.



Circuits

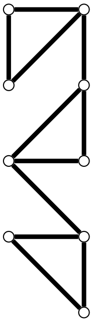
- Consider the fractional matching polytope:
- Let g be a circuit.

What can we say about its support?

$$\sum_{e \in \delta(v)} x_e \leq 1 \qquad \forall v \in V$$

$$x_e \geq 0 \qquad \forall e \in E$$

- $\sup(g)$ induces a connected graph.
- If $\sup(g)$ contains an even cycle, $\sup(g)$ is an even cycle.
- Any two odd cycles in $\sup(g)$ intersect in at most one vertex.
- $\sup(g)$ contains at most two odd cycles.



Circuits

- Consider the fractional matching polytope:

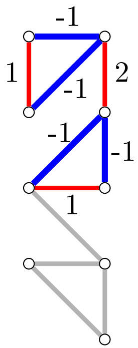
$$\sum_{e \in \delta(v)} x_e \leq 1 \qquad \forall v \in V$$

$$x_e \geq 0 \qquad \forall e \in E$$

- Let g be a circuit.

What can we say about its support?

- $\sup(g)$ induces a connected graph.
- If $\sup(g)$ contains an even cycle, $\sup(g)$ is an even cycle.
- Any two odd cycles in $\sup(g)$ intersect in at most one vertex.
- $\sup(g)$ contains at most two odd cycles.



Circuits

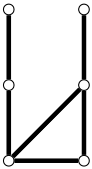
- Consider the fractional matching polytope:
- Let g be a circuit.

What can we say about its support?

$$\sum_{e \in \delta(v)} x_e \leq 1 \qquad \forall v \in V$$

$$x_e \geq 0 \qquad \forall e \in E$$

- $\sup(g)$ induces a connected graph.
- If $\sup(g)$ contains an even cycle, $\sup(g)$ is an even cycle.
- Any two odd cycles in $\sup(g)$ intersect in at most one vertex.
- $\sup(g)$ contains at most two odd cycles.
- If $\sup(g)$ contains one odd cycle, $\sup(g)$ has at most one vertex of degree 1



Circuits

- Consider the fractional matching polytope:

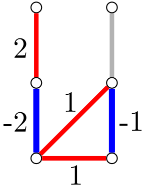
$$\sum_{e \in \delta(v)} x_e \leq 1 \qquad \forall v \in V$$

$$x_e \geq 0 \qquad \forall e \in E$$

- Let g be a circuit.

What can we say about its support?

- $\sup(g)$ induces a connected graph.
- If $\sup(g)$ contains an even cycle, $\sup(g)$ is an even cycle.
- Any two odd cycles in $\sup(g)$ intersect in at most one vertex.
- $\sup(g)$ contains at most two odd cycles.
- If $\sup(g)$ contains one odd cycle, $\sup(g)$ has at most one vertex of degree 1



Circuits

- Consider the fractional matching polytope:

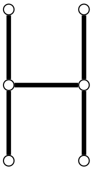
$$\sum_{e \in \delta(v)} x_e \leq 1 \qquad \forall v \in V$$

$$x_e \geq 0 \qquad \forall e \in E$$

- Let g be a circuit.

What can we say about its support?

- $\sup(g)$ induces a connected graph.
- If $\sup(g)$ contains an even cycle, $\sup(g)$ is an even cycle.
- Any two odd cycles in $\sup(g)$ intersect in at most one vertex.
- $\sup(g)$ contains at most two odd cycles.
- If $\sup(g)$ contains one odd cycle, $\sup(g)$ has at most one vertex of degree 1
- If $\sup(g)$ contains no cycles, $\sup(g)$ is a path



Circuits

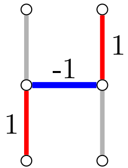
- Consider the fractional matching polytope:
- Let g be a circuit.

What can we say about its support?

$$\sum_{e \in \delta(v)} x_e \leq 1 \qquad \forall v \in V$$

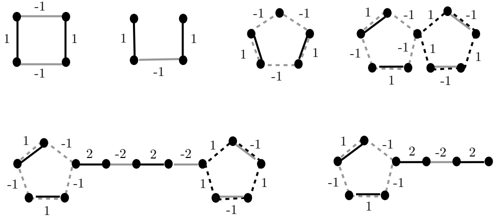
$$x_e \geq 0 \qquad \forall e \in E$$

- $\sup(g)$ induces a connected graph.
- If $\sup(g)$ contains an even cycle, $\sup(g)$ is an even cycle.
- Any two odd cycles in $\sup(g)$ intersect in at most one vertex.
- $\sup(g)$ contains at most two odd cycles.
- If $\sup(g)$ contains one odd cycle, $\sup(g)$ has at most one vertex of degree 1
- If $\sup(g)$ contains no cycles, $\sup(g)$ is a path



Circuits

- Hence, we get the following graphical characterization [De Loera,Kafer,S.'19]:



Algorithmic aspects

Circuit-augmentation algorithms

• [De Loera, Hemmecke, Lee'15] studied some augmentation algorithms for rational LPs in equality form based on circuits:

- moving maximally along the circuit that yields the greatest improvement, one reaches the optimum in (weakly) polynomially many steps!

Circuit-augmentation algorithms

• [De Loera, Hemmecke, Lee'15] studied some augmentation algorithms for rational LPs in equality form based on circuits:

- moving maximally along the circuit that yields the greatest improvement, one reaches the optimum in (weakly) polynomially many steps!

($\rightarrow$ in contrast w.r.t. the Simplex!)

Circuit-augmentation algorithms

• [De Loera, Hemmecke, Lee'15] studied some augmentation algorithms for rational LPs in equality form based on circuits:

- moving maximally along the circuit that yields the greatest improvement, one reaches the optimum in (weakly) polynomially many steps!
  ($\rightarrow$ in contrast w.r.t. the Simplex!)

• Formally, consider an LP $\max\{c^T x \;:\; Ax = b,\; u \geq x \geq \ell,\; x \in \mathbb{R}^n\}$ (Wlog, assume coefficents are integral).

Thm [De Loera, Hemmecke, Lee'15]

Using a greatest-improvement pivot rule, one can reach an optimal solution x^* from an initial one x_0 performing $O\big(n \log\left(\delta\, c^T(x^* - x_0)\right)\big)$ circuit augmentations.

$\rightarrow$ Here δ is the maximum determinant of any $n \times n$ submatrix of the constraint matrix.

Circuit-augmentation algorithms

• [De Loera, Hemmecke, Lee'15] studied some augmentation algorithms for rational LPs in equality form based on circuits:

- moving maximally along the circuit that yields the greatest improvement, one reaches the optimum in (weakly) polynomially many steps!
  ($\rightarrow$ in contrast w.r.t. the Simplex!)

• Formally, consider an LP $\max\{c^T x \ : \ Ax = b, \ u \geq x \geq \ell, \ x \in \mathbb{R}^n\}$ (Wlog, assume coefficents are integral).

Thm [De Loera, Hemmecke, Lee'15]

Using a greatest-improvement pivot rule, one can reach an optimal solution x^* from an initial one x_0 performing $\mathrm{O}\big(n \log\big(\delta\, c^T(x^* - x_0)\big)\big)$ circuit augmentations.

$\rightarrow$ Here δ is the maximum determinant of any $n \times n$ submatrix of the constraint matrix.

• **Obs.** Result extends to LPs of general form $\max\{c^T x \ : \ Ax = b, \ Bx \leq \ell\}$ (Details in [De Loera,Kafer,S.'19]).

Circuit-augmentation algorithms

Proof. Relies on the Sign-Compatible Representation Property of circuits:

Thm [Graver'75]

Let $v \in Kernel(A) \setminus \mathbf{0}$. Then $v = \sum_{i=1}^{n} \alpha_i g^i$ for some $\alpha_i \geq 0$ and circuits g^i

Circuit-augmentation algorithms

Proof. Relies on the Sign-Compatible Representation Property of circuits:

Thm [Graver'75]

Let $v \in Kernel(A) \setminus \mathbf{0}$. Then $v = \sum_{i=1}^{n} \alpha_i g^i$ for some $\alpha_i \geq 0$ and circuits g^i that are sign-compatible with v w.r.t. B (i.e., $(Bv)_j \geq 0$ iff $(Bg^i)_j \geq 0$)

Circuit-augmentation algorithms

Proof. Relies on the Sign-Compatible Representation Property of circuits:

Thm [Graver'75]

Let $v \in Kernel(A) \setminus \mathbf{0}$. Then $v = \sum_{i=1}^{n} \alpha_i g^i$ for some $\alpha_i \geq 0$ and circuits g^i that are sign-compatible with v w.r.t. B (i.e., $(Bv)_j \geq 0$ iff $(Bg^i)_j \geq 0$)

- Express $(x^* - x_0) = \sum_{i=1}^{n} \alpha_i g^i$.

Circuit-augmentation algorithms

Proof. Relies on the Sign-Compatible Representation Property of circuits:

Thm [Graver'75]

Let $v \in Kernel(A) \setminus \mathbf{0}$. Then $v = \sum_{i=1}^{n} \alpha_i g^i$ for some $\alpha_i \geq 0$ and circuits g^i that are sign-compatible with v w.r.t. B (i.e., $(Bv)_j \geq 0$ iff $(Bg^i)_j \geq 0$)

- Express $(x^* - x_0) = \sum_{i=1}^{n} \alpha_i g^i$. Then $c^T(x^* - x_0) = \sum_{i=1}^{n} c^T(\alpha_i g^i)$.

Circuit-augmentation algorithms

Proof. Relies on the Sign-Compatible Representation Property of circuits:

Thm [Graver'75]

Let $v \in Kernel(A) \setminus \mathbf{0}$. Then $v = \sum_{i=1}^{n} \alpha_i g^i$ for some $\alpha_i \geq 0$ and circuits g^i that are sign-compatible with v w.r.t. B (i.e., $(Bv)_j \geq 0$ iff $(Bg^i)_j \geq 0$)

- Express $(x^* - x_0) = \sum_{i=1}^{n} \alpha_i g^i$. Then $c^T(x^* - x_0) = \sum_{i=1}^{n} c^T(\alpha_i g^i)$.
- **Note**: for every i, $x_0 + \alpha_i g^i$ is feasible.

Circuit-augmentation algorithms

Proof. Relies on the Sign-Compatible Representation Property of circuits:

Thm [Graver'75]

Let $v \in \mathit{Kernel}(A) \setminus \mathbf{0}$. Then $v = \sum_{i=1}^{n} \alpha_i g^i$ for some $\alpha_i \geq 0$ and circuits g^i that are sign-compatible with v w.r.t. B (i.e., $(Bv)_j \geq 0$ iff $(Bg^i)_j \geq 0$)

- Express $(x^* - x_0) = \sum_{i=1}^{n} \alpha_i g^i$. Then $c^T(x^* - x_0) = \sum_{i=1}^{n} c^T(\alpha_i g^i)$.
- **Note**: for every i, $x_0 + \alpha_i g^i$ is feasible. Why?

Circuit-augmentation algorithms

Proof. Relies on the Sign-Compatible Representation Property of circuits:

Thm [Graver'75]

Let $v \in Kernel(A) \setminus \mathbf{0}$. Then $v = \sum_{i=1}^{n} \alpha_i g^i$ for some $\alpha_i \geq 0$ and circuits g^i that are sign-compatible with v w.r.t. B (i.e., $(Bv)_j \geq 0$ iff $(Bg^i)_j \geq 0$)

- Express $(x^* - x_0) = \sum_{i=1}^{n} \alpha_i g^i$. Then $c^T(x^* - x_0) = \sum_{i=1}^{n} c^T(\alpha_i g^i)$.
- **Note**: for every i, $x_0 + \alpha_i g^i$ is feasible. Why?
  - $A(x_0 + \alpha_i g^i) = b$

Circuit-augmentation algorithms

Proof. Relies on the Sign-Compatible Representation Property of circuits:

Thm [Graver'75]

Let $v \in Kernel(A) \setminus \mathbf{0}$. Then $v = \sum_{i=1}^{n} \alpha_i g^i$ for some $\alpha_i \geq 0$ and circuits g^i that are sign-compatible with v w.r.t. B (i.e., $(Bv)_j \geq 0$ iff $(Bg^i)_j \geq 0$)

- Express $(x^* - x_0) = \sum_{i=1}^{n} \alpha_i g^i$. Then $c^T(x^* - x_0) = \sum_{i=1}^{n} c^T(\alpha_i g^i)$.
- **Note**: for every i, $x_0 + \alpha_i g^i$ is feasible. Why?
  - $A(x_0 + \alpha_i g^i) = b \quad \rightarrow g^i$ is in $Kernel(A)$

Circuit-augmentation algorithms

Proof. Relies on the Sign-Compatible Representation Property of circuits:

Thm [Graver'75]

Let $v \in Kernel(A) \setminus \mathbf{0}$. Then $v = \sum_{i=1}^{n} \alpha_i g^i$ for some $\alpha_i \geq 0$ and circuits g^i that are sign-compatible with v w.r.t. B (i.e., $(Bv)_j \geq 0$ iff $(Bg^i)_j \geq 0$)

- Express $(x^* - x_0) = \sum_{i=1}^{n} \alpha_i g^i$. Then $c^T(x^* - x_0) = \sum_{i=1}^{n} c^T(\alpha_i g^i)$.
- **Note**: for every i, $x_0 + \alpha_i g^i$ is feasible. Why?
  - $A(x_0 + \alpha_i g^i) = b \quad \rightarrow g^i$ is in $Kernel(A)$
  - $\forall j: \ (B(x_0 + \alpha_i g^i))_j = (Bx_0)_j + \alpha_i (Bg^i)_j \leq \ell_j$

Circuit-augmentation algorithms

Proof. Relies on the Sign-Compatible Representation Property of circuits:

Thm [Graver'75]

Let $v \in Kernel(A) \setminus \mathbf{0}$. Then $v = \sum_{i=1}^{n} \alpha_i g^i$ for some $\alpha_i \geq 0$ and circuits g^i that are sign-compatible with v w.r.t. B (i.e., $(Bv)_j \geq 0$ iff $(Bg^i)_j \geq 0$)

- Express $(x^* - x_0) = \sum_{i=1}^{n} \alpha_i g^i$. Then $c^T(x^* - x_0) = \sum_{i=1}^{n} c^T(\alpha_i g^i)$.
- **Note**: for every i, $x_0 + \alpha_i g^i$ is feasible. Why?
  - $A(x_0 + \alpha_i g^i) = b \quad \rightarrow g^i$ is in $Kernel(A)$
  - $\forall j: \ (B(x_0 + \alpha_i g^i))_j = (Bx_0)_j + \alpha_i (Bg^i)_j \leq \ell_j \quad \rightarrow$ sign-compatibility!

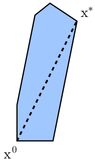
Circuit-augmentation algorithms

Proof. Relies on the Sign-Compatible Representation Property of circuits:

Thm [Graver'75]

Let $v \in Kernel(A) \setminus \mathbf{0}$. Then $v = \sum_{i=1}^{n} \alpha_i g^i$ for some $\alpha_i \geq 0$ and circuits g^i that are sign-compatible with v w.r.t. B (i.e., $(Bv)_j \geq 0$ iff $(Bg^i)_j \geq 0$)

- Express $(x^* - x_0) = \sum_{i=1}^{n} \alpha_i g^i$. Then $c^T(x^* - x_0) = \sum_{i=1}^{n} c^T(\alpha_i g^i)$.
- **Note**: for every i, $x_0 + \alpha_i g^i$ is feasible. Why?
  - $A(x_0 + \alpha_i g^i) = b \quad \rightarrow g^i$ is in $Kernel(A)$
  - $\forall j: \ (B(x_0 + \alpha_i g^i))_j = (Bx_0)_j + \alpha_i (Bg^i)_j \leq \ell_j \quad \rightarrow$ sign-compatibility!



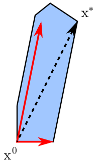
Circuit-augmentation algorithms

Proof. Relies on the Sign-Compatible Representation Property of circuits:

Thm [Graver'75]

Let $v \in Kernel(A) \setminus \mathbf{0}$. Then $v = \sum_{i=1}^{n} \alpha_i g^i$ for some $\alpha_i \geq 0$ and circuits g^i that are sign-compatible with v w.r.t. B (i.e., $(Bv)_j \geq 0$ iff $(Bg^i)_j \geq 0$)

- Express $(x^* - x_0) = \sum_{i=1}^{n} \alpha_i g^i$. Then $c^T(x^* - x_0) = \sum_{i=1}^{n} c^T(\alpha_i g^i)$.
- **Note**: for every i, $x_0 + \alpha_i g^i$ is feasible. Why?
  - $A(x_0 + \alpha_i g^i) = b \quad \rightarrow g^i$ is in $Kernel(A)$
  - $\forall j: \ (B(x_0 + \alpha_i g^i))_j = (Bx_0)_j + \alpha_i (Bg^i)_j \leq \ell_j \quad \rightarrow$ sign-compatibility!



Circuit-augmentation algorithms

Proof. Relies on the Sign-Compatible Representation Property of circuits:

Thm [Graver'75]

Let $v \in Kernel(A) \setminus \mathbf{0}$. Then $v = \sum_{i=1}^{n} \alpha_i g^i$ for some $\alpha_i \geq 0$ and circuits g^i that are sign-compatible with v w.r.t. B (i.e., $(Bv)_j \geq 0$ iff $(Bg^i)_j \geq 0$)

- Express $(x^* - x_0) = \sum_{i=1}^{n} \alpha_i g^i$. Then $c^T(x^* - x_0) = \sum_{i=1}^{n} c^T(\alpha_i g^i)$.
- **Note**: for every i, $x_0 + \alpha_i g^i$ is feasible. Why?
  - $A(x_0 + \alpha_i g^i) = b \quad \rightarrow g^i$ is in $Kernel(A)$
  - $\forall j: \ (B(x_0 + \alpha_i g^i))_j = (Bx_0)_j + \alpha_i (Bg^i)_j \leq \ell_j \quad \rightarrow$ sign-compatibility!
- Selecting the greatest improvement, yields $\approx 1/n$-factor decrease in the objective function difference.

Circuit-augmentation algorithms

Proof. Relies on the Sign-Compatible Representation Property of circuits:

Thm [Graver'75]

Let $v \in Kernel(A) \setminus \mathbf{0}$. Then $v = \sum_{i=1}^{n} \alpha_i g^i$ for some $\alpha_i \geq 0$ and circuits g^i that are sign-compatible with v w.r.t. B (i.e., $(Bv)_j \geq 0$ iff $(Bg^i)_j \geq 0$)

- Express $(x^* - x_0) = \sum_{i=1}^{n} \alpha_i g^i$. Then $c^T(x^* - x_0) = \sum_{i=1}^{n} c^T(\alpha_i g^i)$.
- **Note**: for every i, $x_0 + \alpha_i g^i$ is feasible. Why?
  - $A(x_0 + \alpha_i g^i) = b \quad \rightarrow g^i$ is in $Kernel(A)$
  - $\forall j: \ (B(x_0 + \alpha_i g^i))_j = (Bx_0)_j + \alpha_i (Bg^i)_j \leq \ell_j \quad \rightarrow$ sign-compatibility!
- Selecting the greatest improvement, yields $\approx 1/n$-factor decrease in the objective function difference.
- After $O\left(n \log\left(\frac{1}{\varepsilon} c^T(x^* - x_0)\right)\right)$ iterations, the current solution x_k satisfies $c^T(x^* - x_k) < \varepsilon$.

Circuit-augmentation algorithms

Proof. Relies on the Sign-Compatible Representation Property of circuits:

Thm [Graver'75]

Let $v \in Kernel(A) \setminus \mathbf{0}$. Then $v = \sum_{i=1}^{n} \alpha_i g^i$ for some $\alpha_i \geq 0$ and circuits g^i that are sign-compatible with v w.r.t. B (i.e., $(Bv)_j \geq 0$ iff $(Bg^i)_j \geq 0$)

- Express $(x^* - x_0) = \sum_{i=1}^{n} \alpha_i g^i$. Then $c^T(x^* - x_0) = \sum_{i=1}^{n} c^T(\alpha_i g^i)$.
- **Note**: for every i, $x_0 + \alpha_i g^i$ is feasible. Why?
  - $A(x_0 + \alpha_i g^i) = b \quad \rightarrow g^i$ is in $Kernel(A)$
  - $\forall j: \ (B(x_0 + \alpha_i g^i))_j = (Bx_0)_j + \alpha_i (Bg^i)_j \leq \ell_j \quad \rightarrow$ sign-compatibility!
- Selecting the greatest improvement, yields $\approx 1/n$-factor decrease in the objective function difference.
- After $\mathrm{O}\big(n \log\big(\frac{1}{\varepsilon} c^T(x^* - x_0)\big)\big)$ iterations, the current solution x_k satisfies $c^T(x^* - x_k) < \varepsilon$.
- How can we choose ε?

Circuit-augmentation algorithms

Proof. Relies on the Sign-Compatible Representation Property of circuits:

Thm [Graver'75]

Let $v \in Kernel(A) \setminus \mathbf{0}$. Then $v = \sum_{i=1}^{n} \alpha_i g^i$ for some $\alpha_i \geq 0$ and circuits g^i that are sign-compatible with v w.r.t. B (i.e., $(Bv)_j \geq 0$ iff $(Bg^i)_j \geq 0$)

- Express $(x^* - x_0) = \sum_{i=1}^{n} \alpha_i g^i$. Then $c^T(x^* - x_0) = \sum_{i=1}^{n} c^T(\alpha_i g^i)$.
- **Note**: for every i, $x_0 + \alpha_i g^i$ is feasible. Why?
    - $A(x_0 + \alpha_i g^i) = b \quad \rightarrow g^i$ is in $Kernel(A)$
    - $\forall j: \ (B(x_0 + \alpha_i g^i))_j = (Bx_0)_j + \alpha_i (Bg^i)_j \leq \ell_j \quad \rightarrow$ sign-compatibility!
- Selecting the greatest improvement, yields $\approx 1/n$-factor decrease in the objective function difference.
- After $O\left(n \log\left(\frac{1}{\varepsilon} c^T(x^* - x_0)\right)\right)$ iterations, the current solution x_k satisfies $c^T(x^* - x_k) < \varepsilon$.
- How can we choose ε? Set $\varepsilon = \frac{1}{\delta^2}$. At this point, move to any extreme point not worse than $x_k \rightarrow$ will be optimal! □

Circuit-augmentation algorithms

- As mentioned, these results imply that a greatest-improvement pivot rule yields an optimal solution in (weakly) polynomially many steps!

Circuit-augmentation algorithms

- As mentioned, these results imply that a greatest-improvement pivot rule yields an optimal solution in (weakly) polynomially many steps!
- **Question:** How hard is selecting the “greatest-improvement” circuit?

Circuit-augmentation algorithms

- As mentioned, these results imply that a greatest-improvement pivot rule yields an optimal solution in (weakly) polynomially many steps!
- **Question:** How hard is selecting the "greatest-improvement" circuit?

Thm [De Loera, Kafer, S.'19]

Selecting the circuit that yields the greatest improvement is NP-hard,

Circuit-augmentation algorithms

- As mentioned, these results imply that a greatest-improvement pivot rule yields an optimal solution in (weakly) polynomially many steps!
- **Question:** How hard is selecting the "greatest-improvement" circuit?

Thm [De Loera, Kafer, S.'19]

Selecting the circuit that yields the greatest improvement is NP-hard, already for the bipartite matching polytope.

Circuit-augmentation algorithms

- As mentioned, these results imply that a greatest-improvement pivot rule yields an optimal solution in (weakly) polynomially many steps!
- **Question:** How hard is selecting the "greatest-improvement" circuit?

Thm [De Loera, Kafer, S.'19]

Selecting the circuit that yields the greatest improvement is NP-hard, already for the bipartite matching polytope. However, any γ-approximation algorithm with γ polynomial in the input size, still guarantees convergence in poly-time.

Circuit-augmentation algorithms

- As mentioned, these results imply that a greatest-improvement pivot rule yields an optimal solution in (weakly) polynomially many steps!
- **Question:** How hard is selecting the "greatest-improvement" circuit?

Thm [De Loera, Kafer, S.'19]

Selecting the circuit that yields the greatest improvement is NP-hard, already for the bipartite matching polytope. However, any γ-approximation algorithm with γ polynomial in the input size, still guarantees convergence in poly-time.

Proof:

- *Approximation*: Straightforward extension of [DHL'15].
- *Hardness*: Follows from the hardness of determining whether a given extreme point has an optimal adjacent neighbor. $\square$

Circuit-augmentation algorithms

- As mentioned, these results imply that a greatest-improvement pivot rule yields an optimal solution in (weakly) polynomially many steps!
- **Question:** How hard is selecting the "greatest-improvement" circuit?

Thm [De Loera, Kafer, S.'19]

Selecting the circuit that yields the greatest improvement is NP-hard, already for the bipartite matching polytope. However, any γ-approximation algorithm with γ polynomial in the input size, still guarantees convergence in poly-time.

Corollary

Finding the shortest (monotone) circuit-path to an optimal solution is NP-hard, and hard-to-approximate within a factor better than 2.

Circuit-augmentation algorithms

- As mentioned, these results imply that a greatest-improvement pivot rule yields an optimal solution in (weakly) polynomially many steps!
- **Question:** How hard is selecting the "greatest-improvement" circuit?

Thm [De Loera, Kafer, S.'19]

Selecting the circuit that yields the greatest improvement is NP-hard, already for the bipartite matching polytope. However, any γ-approximation algorithm with γ polynomial in the input size, still guarantees convergence in poly-time.

Corollary

Finding the shortest (monotone) circuit-path to an optimal solution is NP-hard, and hard-to-approximate within a factor better than 2.

- **Consequences** (unless P=NP):
  - For any efficient pivoting rule, a circuit-augmentation algorithm can't reach the optimum with a min number of augmentations.

Circuit-augmentation algorithms

- The previous results raise a natural question:

Circuit-augmentation algorithms

- The previous results raise a natural question:

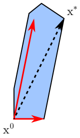
Can a (maximal) augmentation along an edge-direction be a 'good' approximation of a greatest-improvement circuit augmentation?

Circuit-augmentation algorithms

- The previous results raise a natural question:

Can a (maximal) augmentation along an edge-direction be a 'good' approximation of a greatest-improvement circuit augmentation?

- Recall previous example:

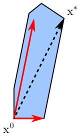


Circuit-augmentation algorithms

- The previous results raise a natural question:

  Can a (maximal) augmentation along an edge-direction be a 'good' approximation of a greatest-improvement circuit augmentation?

- Recall previous example:



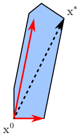
- Interestingly, the answer is 'yes' for 0/1-polytopes!

Circuit-augmentation algorithms

- The previous results raise a natural question:

Can a (maximal) augmentation along an edge-direction be a 'good' approximation of a greatest-improvement circuit augmentation?

- Recall previous example:



- Interestingly, the answer is 'yes' for 0/1-polytopes!

Def. For a given extreme point x of an LP and objective function vector c, a steepest-edge direction g is an edge-direction incident at x maximizing $\frac{c^T g}{||g||_1}$

0/1-Polytopes

0/1-Polytopes

- 0/1-Polytopes are of fundamental importance in optimization.

0/1-Polytopes

- 0/1-Polytopes are of fundamental importance in optimization.
- Recall: Their diameter is linear (Hirsch bound holds [Naddef'89]).

0/1-Polytopes

- 0/1-Polytopes are of fundamental importance in optimization.
- Recall: Their diameter is linear (Hirsch bound holds [Naddef'89]).

0/1-Polytopes

- 0/1-Polytopes are of fundamental importance in optimization.
- Recall: Their diameter is linear (Hirsch bound holds [Naddef'89]).
- **Note:** For 0/1-LPs, paths of strongly-polynomial length to an optimal solution (on the 1-skeleton) can be constructed from *any* augmentation oracle.

0/1-Polytopes

- 0/1-Polytopes are of fundamental importance in optimization.
- Recall: Their diameter is linear (Hirsch bound holds [Naddef'89]).
- **Note:** For 0/1-LPs, paths of strongly-polynomial length to an optimal solution (on the 1-skeleton) can be constructed from *any* augmentation oracle. (See [Schulz,Weismantel,Ziegler'95], [Del Pia,Michini'20]).

0/1-Polytopes

- 0/1-Polytopes are of fundamental importance in optimization.
- Recall: Their diameter is linear (Hirsch bound holds [Naddef'89]).
- **Note:** For 0/1-LPs, paths of strongly-polynomial length to an optimal solution (on the 1-skeleton) can be constructed from *any* augmentation oracle. (See [Schulz,Weismantel,Ziegler'95], [Del Pia,Michini'20]).

Question: Is there a 'natural' pivot rule for the Simplex algorithm that guarantees (strongly)poly-time convergence on 0/1-LPs?

0/1-Polytopes

- 0/1-Polytopes are of fundamental importance in optimization.
- Recall: Their diameter is linear (Hirsch bound holds [Naddef'89]).
- **Note:** For 0/1-LPs, paths of strongly-polynomial length to an optimal solution (on the 1-skeleton) can be constructed from *any* augmentation oracle. (See [Schulz,Weismantel,Ziegler'95], [Del Pia,Michini'20]).

Question: Is there a 'natural' pivot rule for the Simplex algorithm that guarantees (strongly)poly-time convergence on 0/1-LPs?

- For pivoting rules like Dantzig, Greatest-improvement, Steepest-edge:
  - Strongly-polynomial bounds on the # of distinct basic feasible solutions generated by Simplex are known for 0/1-LPs in Standard Equality Form [Kitahara&Mizuno'14][Kitahara,Matsui,Mizuno'12],[Blanchard,De Loera,Louveaux'20]

0/1-Polytopes

- 0/1-Polytopes are of fundamental importance in optimization.
- Recall: Their diameter is linear (Hirsch bound holds [Naddef'89]).
- **Note:** For 0/1-LPs, paths of strongly-polynomial length to an optimal solution (on the 1-skeleton) can be constructed from *any* augmentation oracle. (See [Schulz,Weismantel,Ziegler'95], [Del Pia,Michini'20]).

Question: Is there a 'natural' pivot rule for the Simplex algorithm that guarantees (strongly)poly-time convergence on 0/1-LPs?

- For pivoting rules like Dantzig, Greatest-improvement, Steepest-edge:
  - Strongly-polynomial bounds on the # of distinct basic feasible solutions generated by Simplex are known for 0/1-LPs in Standard Equality Form [Kitahara&Mizuno'14][Kitahara,Matsui,Mizuno'12],[Blanchard,De Loera,Louveaux'20]

...What do we get with the previous framework?

0/1-Polytopes

Thm [De Loera, Kafer, S.'19]

For a vertex of a 0/1-LP, a (maximal) augmentation along a steepest-edge direction is an n-approximation of a greatest-improvement circuit augmentation.

0/1-Polytopes

Thm [De Loera, Kafer, S.'19]

For a vertex of a 0/1-LP, a (maximal) augmentation along a steepest-edge direction is an n-approximation of a greatest-improvement circuit augmentation.

Proof: Let the 0/1-LP be $\max\{c^T y : y \in \mathcal{P}\}$ and x be a vertex.

0/1-Polytopes

Thm [De Loera, Kafer, S.'19]

For a vertex of a 0/1-LP, a (maximal) augmentation along a steepest-edge direction is an n-approximation of a greatest-improvement circuit augmentation.

Proof: Let the 0/1-LP be $\max\{c^T y : y \in \mathcal{P}\}$ and x be a vertex.
Consider the following optimization problem:

$$
\begin{aligned}
&\max c^T z && \\
&\|z\|_1 \leq 1 && (1)\\
&x + \varepsilon z \in \mathcal{P} \quad \text{for some } \varepsilon > 0 && (2)
\end{aligned}
$$

0/1-Polytopes

Thm [De Loera, Kafer, S.'19]

For a vertex of a 0/1-LP, a (maximal) augmentation along a steepest-edge direction is an n-approximation of a greatest-improvement circuit augmentation.

Proof: Let the 0/1-LP be $\max\{c^T y : y \in \mathcal{P}\}$ and x be a vertex.
Consider the following optimization problem:

$$\begin{aligned} &\max c^T z && \\ &\|z\|_1 \le 1 && (1) \\ &x + \varepsilon z \in \mathcal{P} \quad \text{for some } \varepsilon > 0 && (2) \end{aligned}$$

Obs 1: The feasible region is a polytope.

0/1-Polytopes

Thm [De Loera, Kafer, S.'19]

For a vertex of a 0/1-LP, a (maximal) augmentation along a steepest-edge direction is an n-approximation of a greatest-improvement circuit augmentation.

Proof: Let the 0/1-LP be $\max\{c^T y : y \in \mathcal{P}\}$ and x be a vertex.
Consider the following optimization problem:

$$
\begin{aligned}
&\max c^T z \\
&\|z\|_1 \leq 1 && (1) \\
&x + \varepsilon z \in \mathcal{P} \quad \text{for some } \varepsilon > 0 && (2)
\end{aligned}
$$

Obs 1: The feasible region is a polytope. Why?

0/1-Polytopes

Thm [De Loera, Kafer, S.'19]

For a vertex of a 0/1-LP, a (maximal) augmentation along a steepest-edge direction is an n-approximation of a greatest-improvement circuit augmentation.

Proof: Let the 0/1-LP be $\max\{c^T y : y \in \mathcal{P}\}$ and x be a vertex.
Consider the following optimization problem:

$$\begin{aligned} &\max c^T z && \\ &\|z\|_1 \leq 1 && (1) \\ &x + \varepsilon z \in \mathcal{P} \quad \text{for some } \varepsilon > 0 && (2) \end{aligned}$$

Obs 1: The feasible region is a polytope. Why?

- Constraint (2) is describing the feasible cone at x
- Constraint (1) corresponds to $\left(v^T z \leq 1 \ \ \forall v \in \{1, -1\}^n\right)$ – cross-polytope

0/1-Polytopes

Thm [De Loera, Kafer, S.'19]

For a vertex of a 0/1-LP, a (maximal) augmentation along a steepest-edge direction is an n-approximation of a greatest-improvement circuit augmentation.

Proof: Let the 0/1-LP be $\max\{c^T y : y \in \mathcal{P}\}$ and x be a vertex.
Consider the following optimization problem:

$$\begin{aligned} &\max c^T z && \\ &\|z\|_1 \leq 1 && (1) \\ &x + \varepsilon z \in \mathcal{P} \quad \text{for some } \varepsilon > 0 && (2) \end{aligned}$$

Obs 1: The feasible region is a polytope. Why?

- Constraint (2) is describing the feasible cone at x
- Constraint (1) corresponds to $\left(v^T z \leq 1 \ \ \forall v \in \{1, -1\}^n\right)$ – cross-polytope

Obs 2: A steepest-edge direction is an optimal solution.

0/1-Polytopes

Thm [De Loera, Kafer, S.'19]

For a vertex of a 0/1-LP, a (maximal) augmentation along a steepest-edge direction is an n-approximation of a greatest-improvement circuit augmentation.

Proof: Let the 0/1-LP be $\max\{c^T y : y \in \mathcal{P}\}$ and x be a vertex. Consider the following optimization problem:

$$\begin{aligned} &\max c^T z \\ &\|z\|_1 \leq 1 && (1) \\ &x + \varepsilon z \in \mathcal{P} \quad \text{for some } \varepsilon > 0 && (2) \end{aligned}$$

Obs 1: The feasible region is a polytope. Why?

- Constraint (2) is describing the feasible cone at x
- Constraint (1) corresponds to $\left(v^T z \leq 1 \;\; \forall v \in \{1, -1\}^n\right)$ – cross-polytope

Obs 2: A steepest-edge direction is an optimal solution. Why?

0/1-Polytopes

Thm [De Loera, Kafer, S.'19]

For a vertex of a 0/1-LP, a (maximal) augmentation along a steepest-edge direction is an n-approximation of a greatest-improvement circuit augmentation.

Proof: Let the 0/1-LP be $\max\{c^T y : y \in \mathcal{P}\}$ and x be a vertex.
Consider the following optimization problem:

$$\begin{aligned} &\max c^T z && \\ &\|z\|_1 \leq 1 && (1) \\ &x + \varepsilon z \in \mathcal{P} \quad \text{for some } \varepsilon > 0 && (2) \end{aligned}$$

Obs 1: The feasible region is a polytope. Why?

- Constraint (2) is describing the feasible cone at x
- Constraint (1) corresponds to $\left(v^T z \leq 1 \;\; \forall v \in \{1, -1\}^n\right)$ – cross-polytope

Obs 2: A steepest-edge direction is an optimal solution. Why?

- x is a 0/1 vector: only one constraint of the cross-polytope can be a facet.

0/1-Polytopes

Thm [De Loera, Kafer, S.'19]

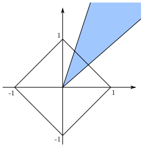
For a vertex of a 0/1-LP, a (maximal) augmentation along a steepest-edge direction is an n-approximation of a greatest-improvement circuit augmentation.

Proof: Let the 0/1-LP be $\max\{c^T y : y \in \mathcal{P}\}$ and x be a vertex.
Consider the following optimization problem:

$$\max c^T z$$

$$\|z\|_1 \leq 1 \qquad (1)$$

$$x + \varepsilon z \in \mathcal{P} \quad \text{for some } \varepsilon > 0 \qquad (2)$$



Obs 1: The feasible region is a polytope. Why?

- Constraint (2) is describing the feasible cone at x
- Constraint (1) corresponds to $\left(v^T z \leq 1 \ \ \forall v \in \{1, -1\}^n\right)$ – cross-polytope

Obs 2: A steepest-edge direction is an optimal solution. Why?

- x is a 0/1 vector: only one constraint of the cross-polytope can be a facet.

0/1-Polytopes

Thm [De Loera, Kafer, S.'19]

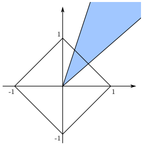
For a vertex of a 0/1-LP, a (maximal) augmentation along a steepest-edge direction is an n-approximation of a greatest-improvement circuit augmentation.

Proof: Let the 0/1-LP be $\max\{c^T y : y \in \mathcal{P}\}$ and x be a vertex. Consider the following optimization problem:

$$\max c^T z$$

$$\|z\|_1 \leq 1 \qquad (1)$$

$$x + \varepsilon z \in \mathcal{P} \quad \text{for some } \varepsilon > 0 \qquad (2)$$



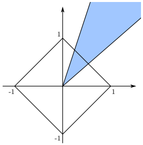
0/1-Polytopes

Thm [De Loera, Kafer, S.'19]

For a vertex of a 0/1-LP, a (maximal) augmentation along a steepest-edge direction is an n-approximation of a greatest-improvement circuit augmentation.

Proof: Let the 0/1-LP be $\max\{c^T y : y \in \mathcal{P}\}$ and x be a vertex. Consider the following optimization problem:

$$\max c^T z$$
$$\|z\|_1 \leq 1 \qquad (1)$$
$$x + \varepsilon z \in \mathcal{P} \quad \text{for some } \varepsilon > 0 \qquad (2)$$



Let

- $\alpha^* z^*$ be a steepest-edge augmentation at x (with $||z^*||_1 = 1$)
- $\alpha \tilde{z}$ be the greatest-improvement circuit-augmentation at x (with $||\tilde{z}||_1 = 1$)

Then:

- $\alpha^* c^T z^* \geq \alpha^* c^T \tilde{z} \geq \frac{\alpha^*}{\alpha} \alpha c^T \tilde{z} \geq \frac{1}{n} \alpha c^T \tilde{z}$ □

0/1-Polytopes

- Combining the previous theorems, we get the following:

Corollary 1

For $0/1$-LPs, moving along the steepest-edge yields an optimal solution from an initial extreme point in a strongly-polynomial number of steps.

0/1-Polytopes

- Combining the previous theorems, we get the following:

Corollary 1

For 0/1-LPs, moving along the steepest-edge yields an optimal solution from an initial extreme point in a strongly-polynomial number of steps.

Proof:

- One can reach an optimal solution in $\mathrm{O}\left(n^2 \log\left(\delta\, c^T (x^* - x_0)\right)\right)$ edge-augmentations.
- The analysis can be improved relying on the technique of [Frank,Tardos'87], to make the above number strongly polynomial. $\square$

0/1-Polytopes

- Combining the previous theorems, we get the following:

Corollary 1

For 0/1-LPs, moving along the steepest-edge yields an optimal solution from an initial extreme point in a strongly-polynomial number of steps.

Note:

- In the context of the Simplex Algorithm, moving to an adjacent vertex **does not** necessarily mean moving to an adjacent basis (because of degeneracy).

0/1-Polytopes

- Combining the previous theorems, we get the following:

Corollary 1

For $0/1$-LPs, moving along the steepest-edge yields an optimal solution from an initial extreme point in a strongly-polynomial number of steps.

Note:

- In the context of the Simplex Algorithm, moving to an adjacent vertex **does not** necessarily mean moving to an adjacent basis (because of degeneracy).

Corollary 2

For non degenerate $0/1$-LPs, the Simplex method with a steepest-edge pivot rule reaches an optimal solution in strongly-polynomial time.

Question: Can we get a similar result in presence of degeneracy?

Circuit-diameter

Circuit-Diameter

• As a corollary of the previous algorithmic results [DHL'15, DKS'19], we can get a (weakly) polynomial bound on the circuit-diameter of rational polyhedra.

Circuit-Diameter

- As a corollary of the previous algorithmic results [DHL'15, DKS'19], we can get a (weakly) polynomial bound on the circuit-diameter of rational polyhedra.

Corollary

There exists a polynomial function $f(m, \alpha)$ that bounds above the circuit-diameter of any rational polyhedron $\mathcal{P}$ with m facets and maximum encoding length of a coefficient in its description equal to α.

Circuit-Diameter

• As a corollary of the previous algorithmic results [DHL'15, DKS'19], we can get a (weakly) polynomial bound on the circuit-diameter of rational polyhedra.

Corollary

There exists a polynomial function $f(m, \alpha)$ that bounds above the circuit-diameter of any rational polyhedron $\mathcal{P}$ with m facets and maximum encoding length of a coefficient in its description equal to α.

Proof:

- Let y and z be two extreme points.
- Construct c by adding the rows of the tight constraints for z.
- Apply the bound on $\max\{c^T x : x \in \mathcal{P}\}$, with $x_0 = y$ and $x^* = z$. □

Circuit-Diameter

• As a corollary of the previous algorithmic results [DHL'15, DKS'19], we can get a (weakly) polynomial bound on the circuit-diameter of rational polyhedra.

Corollary

There exists a polynomial function $f(m, \alpha)$ that bounds above the circuit-diameter of any rational polyhedron $\mathcal{P}$ with m facets and maximum encoding length of a coefficient in its description equal to α.

Proof:

- Let y and z be two extreme points.
- Construct c by adding the rows of the tight constraints for z.
- Apply the bound on $\max\{c^T x : x \in \mathcal{P}\}$, with $x_0 = y$ and $x^* = z$. $\square$

Can we exploit circuits to get insights on other long-standing conjectures about diameters in the literature?

TSP Polytope

TSP Polytope

• The TSP polytope is given by the convex hull of characteristic vectors of Hamiltonian cycles in a complete **undirected** graph.

TSP Polytope

- The TSP polytope is given by the convex hull of characteristic vectors of Hamiltonian cycles in a complete **undirected** graph.
- The study of the diameter of the TSP polytope has a long history.
  - [Padberg&Rao'74] Showed that the asymmetric TSP polytope has diameter 2 (i.e. when considering complete **directed** graphs).

TSP Polytope

- The TSP polytope is given by the convex hull of characteristic vectors of Hamiltonian cycles in a complete **undirected** graph.
- The study of the diameter of the TSP polytope has a long history.
  - [Padberg&Rao'74] Showed that the asymmetric TSP polytope has diameter 2 (i.e. when considering complete **directed** graphs). Interestingly, their paper says:

> If we can indeed take the diameter of a polytope associated with a combinatorial problem as a measure of the computational complexity of such problems – a hypothesis that appears to be generally accepted, see e.g. [12], in particular the chapters written by V. Klee – our result seems to indicate that there may exist "good" algorithms for a large class of problems.

TSP Polytope

- The TSP polytope is given by the convex hull of characteristic vectors of Hamiltonian cycles in a complete **undirected** graph.
- The study of the diameter of the TSP polytope has a long history.
  - [Padberg&Rao'74] Showed that the asymmetric TSP polytope has diameter 2 (i.e. when considering complete **directed** graphs). Interestingly, their paper says:

    > If we can indeed take the diameter of a polytope associated with a combinatorial problem as a measure of the computational complexity of such problems – a hypothesis that appears to be generally accepted, see e.g. [12], in particular the chapters written by V. Klee – our result seems to indicate that there may exist "good" algorithms for a large class of problems.

  - [Grötschel&Padberg '86] conjectured that also for the TSP polytope the diameter is 2. **Still open!**

TSP Polytope

TSP Polytope

- The current best known value is due to [Rispoli&Cosares'98]:

Thm [Rispoli,Cosares'98]

The diameter of the TSP polytope is at most 4.

TSP Polytope

- The current best known value is due to [Rispoli&Cosares'98]:

Thm [Rispoli,Cosares'98]

The diameter of the TSP polytope is at most 4.

- Which adjacency relations do [RC'98] use?

TSP Polytope

- The current best known value is due to [Rispoli&Cosares'98]:

Thm [Rispoli,Cosares'98]

The diameter of the TSP polytope is at most 4.

- Which adjacency relations do [RC'98] use?
  - [Papadimitriou'78] proved that testing wether two TSP tours are adjacent is CO-NP-complete.

TSP Polytope

- The current best known value is due to [Rispoli&Cosares'98]:

Thm [Rispoli,Cosares'98]

The diameter of the TSP polytope is at most 4.

- Which adjacency relations do [RC'98] use?
  - [Papadimitriou'78] proved that testing wether two TSP tours are adjacent is CO-NP-complete.
  - [RC'98] exploit adjacencies for the perfect 2-matching polytope!

TSP Polytope

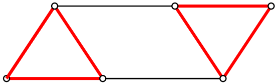
- The current best known value is due to [Rispoli&Cosares'98]:

Thm [Rispoli,Cosares'98]

The diameter of the TSP polytope is at most 4.

- Which adjacency relations do [RC'98] use?
  - [Papadimitriou'78] proved that testing wether two TSP tours are adjacent is CO-NP-complete.
  - [RC'98] exploit adjacencies for the perfect 2-matching polytope!

- A perfect 2-matching is a set F of edges such that each node is incident into exactly 2 edges of F.



- **Note:** A TSP tour is a perfect 2-matching!

TSP Polytope

- The current best known value is due to [Rispoli&Cosares'98]:

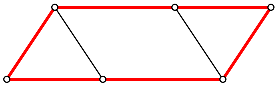
Thm [Rispoli,Cosares'98]

The diameter of the TSP polytope is at most 4.

- Which adjacency relations do [RC'98] use?
  - [Papadimitriou'78] proved that testing wether two TSP tours are adjacent is CO-NP-complete.
  - [RC'98] exploit adjacencies for the perfect 2-matching polytope!

- A perfect 2-matching is a set F of edges such that each node is incident into exactly 2 edges of F.

- **Note:** A TSP tour is a perfect 2-matching!



TSP Polytope

TSP Polytope

- Similarly to the matching polytope (see previous lecture) we have:

Two perfect 2-matchings M_1 and M_2 are adjacent $\Leftrightarrow$ $M_1 \Delta M_2$ contains a unique alternating cycle.

TSP Polytope

- Similarly to the matching polytope (see previous lecture) we have:

Two perfect 2-matchings M_1 and M_2 are adjacent $\Leftrightarrow$ $M_1 \Delta M_2$ contains a unique alternating cycle.

- The above yields a sufficient adjacency condition for two TSP tours!

TSP Polytope

- Similarly to the matching polytope (see previous lecture) we have:

Two perfect 2-matchings M_1 and M_2 are adjacent $\Leftrightarrow$ $M_1 \Delta M_2$ contains a unique alternating cycle.

- The above yields a sufficient adjacency condition for two TSP tours!

Proof sketch of [RC'98] (n even):

TSP Polytope

- Similarly to the matching polytope (see previous lecture) we have:

Two perfect 2-matchings M_1 and M_2 are adjacent $\Leftrightarrow$ $M_1 \Delta M_2$ contains a unique alternating cycle.

- The above yields a sufficient adjacency condition for two TSP tours!

Proof sketch of [RC'98] (n even):

- **Note:** A TSP tour T is the disjoint union of two perfect matchings.

TSP Polytope

- Similarly to the matching polytope (see previous lecture) we have:

Two perfect 2-matchings M_1 and M_2 are adjacent $\Leftrightarrow$ $M_1 \Delta M_2$ contains a unique alternating cycle.

- The above yields a sufficient adjacency condition for two TSP tours!

Proof sketch of [RC'98] (n even):

- **Note:** A TSP tour T is the disjoint union of two perfect matchings.
- Key Lemma: For every pair of tours having a perfect matching in common, the distance between their corresponding extreme points is at most 2.

TSP Polytope

- Similarly to the matching polytope (see previous lecture) we have:

Two perfect 2-matchings M_1 and M_2 are adjacent $\Leftrightarrow$ $M_1 \Delta M_2$ contains a unique alternating cycle.

- The above yields a sufficient adjacency condition for two TSP tours!

Proof sketch of [RC'98] (n even):

- **Note:** A TSP tour T is the disjoint union of two perfect matchings.
- Key Lemma: For every pair of tours having a perfect matching in common, the distance between their corresponding extreme points is at most 2.
- Let $T_1 = M_1 \cup M_2$, and $T_2 = M_3 \cup M_4$. There exists M such that both $M_1 \cup M$ and $M_3 \cup M$ are tours $\rightarrow$ bound of 6.

TSP Polytope

- Similarly to the matching polytope (see previous lecture) we have:

Two perfect 2-matchings M_1 and M_2 are adjacent $\Leftrightarrow$ $M_1 \Delta M_2$ contains a unique alternating cycle.

- The above yields a sufficient adjacency condition for two TSP tours!

Proof sketch of [RC'98] (n even):

- **Note:** A TSP tour T is the disjoint union of two perfect matchings.
- Key Lemma: For every pair of tours having a perfect matching in common, the distance between their corresponding extreme points is at most 2.
- Let $T_1 = M_1 \cup M_2$, and $T_2 = M_3 \cup M_4$. There exists M such that both $M_1 \cup M$ and $M_3 \cup M$ are tours $\rightarrow$ bound of 6.
- Improve to 4 by selecting M more carefully, as to have one simple cycle in the first and last step. $\square$

TSP Polytope

- Similarly to the matching polytope (see previous lecture) we have:

Two perfect 2-matchings M_1 and M_2 are adjacent $\Leftrightarrow$ $M_1 \Delta M_2$ contains a unique alternating cycle.

- The above yields a sufficient adjacency condition for two TSP tours!

Proof sketch of [RC'98] (n even):

- **Note:** A TSP tour T is the disjoint union of two perfect matchings.
- Key Lemma: For every pair of tours having a perfect matching in common, the distance between their corresponding extreme points is at most 2.
- Let $T_1 = M_1 \cup M_2$, and $T_2 = M_3 \cup M_4$. There exists M such that both $M_1 \cup M$ and $M_3 \cup M$ are tours $\rightarrow$ bound of 6.
- Improve to 4 by selecting M more carefully, as to have one simple cycle in the first and last step. $\square$

- They also state: 4 is best possible if you always exchange perfect matchings.

TSP Polytope

TSP Polytope

Question: *Does the bound of 2 hold for the circuit-diameter?*

TSP Polytope

Question: *Does the bound of 2 hold for the circuit-diameter?* Yes!

TSP Polytope

Question: *Does the bound of 2 hold for the circuit-diameter?* Yes!

Thm [Kafer, Pashkovich, S.'18]

The circuit-diameter of the TSP polytope is equal to

- *1 for* $|V| \neq 5$
- *2 for* $|V| = 5$

TSP Polytope

Question: *Does the bound of 2 hold for the circuit-diameter?* Yes!

Thm [Kafer, Pashkovich, S.'18]

The circuit-diameter of the TSP polytope is equal to

- *1 for* $|V| \neq 5$
- *2 for* $|V| = 5$

• For TSP tours T_1, T_2, let $\chi_1, \chi_2 \in \{0,1\}^E$ be the characteristic vectors of T_1, T_2.

TSP Polytope

Question: *Does the bound of 2 hold for the circuit-diameter?* Yes!

Thm [Kafer, Pashkovich, S.'18]

The circuit-diameter of the TSP polytope is equal to

- *1 for* $|V| \neq 5$
- *2 for* $|V| = 5$

• For TSP tours T_1, T_2, let $\chi_1, \chi_2 \in \{0,1\}^E$ be the characteristic vectors of T_1, T_2.

• Key point: For $n \neq 5$, $\chi_2 - \chi_1$ is a circuit of the TSP polytope!

TSP Polytope

Question: *Does the bound of 2 hold for the circuit-diameter?* Yes!

Thm [Kafer, Pashkovich, S.'18]

The circuit-diameter of the TSP polytope is equal to

- *1 for* $|V| \neq 5$
- *2 for* $|V| = 5$

• For TSP tours T_1, T_2, let $\chi_1, \chi_2 \in \{0,1\}^E$ be the characteristic vectors of T_1, T_2.

• Key point: For $n \neq 5$, $\chi_2 - \chi_1$ is a circuit of the TSP polytope!

$\rightarrow$ Which inequalities do we use?

TSP Polytope

TSP Polytope

• We consider the subtour relaxation [Dantzig,Fulkerson,Johnson'54] plus certain comb inequalities [Grötschel,Padberg'79]

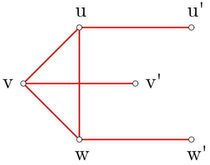
$$
\begin{aligned}
& x(E(S)) \leq |S| - 1 && \forall S \subset V,\ 2 \leq |S| \leq |V| - 2 \\
& x(\delta(v)) = 2 && \forall v \in V \\
& x_{uv} + x_{vw} + x_{wu} + x_{uu'} + x_{vv'} + x_{ww'} \leq 4 && \forall u, v, w, u', v', w' \in V \\
& x \geq 0
\end{aligned}
$$

TSP Polytope

- We consider the subtour relaxation [Dantzig,Fulkerson,Johnson'54] plus certain comb inequalities [Grötschel,Padberg'79]

$$\begin{aligned} & x(E(S)) \leq |S| - 1 && \forall S \subset V,\ 2 \leq |S| \leq |V| - 2 \\ & x(\delta(v)) = 2 && \forall v \in V \\ & x_{uv} + x_{vw} + x_{wu} + x_{uu'} + x_{vv'} + x_{ww'} \leq 4 && \forall u, v, w, u', v', w' \in V \\ & x \geq 0 \end{aligned}$$

- Example:

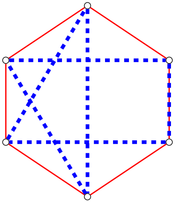


TSP Polytope

- We consider the subtour relaxation [Dantzig,Fulkerson,Johnson'54] plus certain comb inequalities [Grötschel,Padberg'79]

$$
\begin{aligned}
& x(E(S)) \leq |S| - 1 && \forall S \subset V,\ 2 \leq |S| \leq |V| - 2 \\
& x(\delta(v)) = 2 && \forall v \in V \\
& x_{uv} + x_{vw} + x_{wu} + x_{uu'} + x_{vv'} + x_{ww'} \leq 4 && \forall u, v, w, u', v', w' \in V \\
& x \geq 0
\end{aligned}
$$

- Example:



Further on circuit-diameters

Further on circuit-diameters

• The complexity of computing the circuit-diameter of a polytope is currently not known

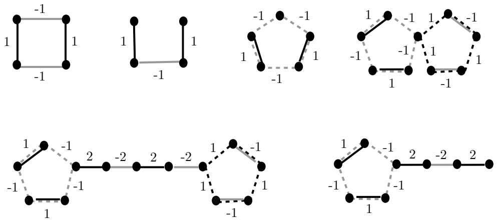
Further on circuit-diameters

• The complexity of computing the circuit-diameter of a polytope is currently not known ...even for the fractional matching polytope.

Further on circuit-diameters

• The complexity of computing the circuit-diameter of a polytope is currently not known ...even for the fractional matching polytope.

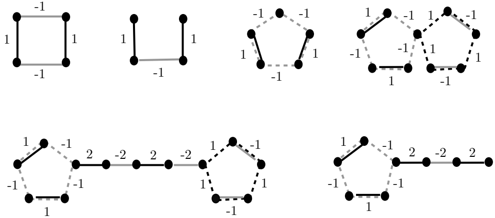
• Recall: There is a graphical characterization of the circuits of the fractional matching polytope.



Further on circuit-diameters

• The complexity of computing the circuit-diameter of a polytope is currently not known ...even for the fractional matching polytope.

• Recall: There is a graphical characterization of the circuits of the fractional matching polytope.



• **Note:** We can construct instances where the circuit-diameter is strictly smaller than the (standard) diameter value.

Final remarks

- **Main questions:**
  - Is the polynomial-Hirsch conjecture true?
  - Is there a polynomial pivoting rule for the Simplex algorithm?

Final remarks

- **Main questions:**
  - Is the polynomial-Hirsch conjecture true?
  - Is there a polynomial pivoting rule for the Simplex algorithm?

- (Mentioned in the survey of [Kaibel&Pfetsch'03]) What is the complexity of computing the diameter of a simple polytope?

Final remarks

- **Main questions:**
  - Is the polynomial-Hirsch conjecture true?
  - Is there a polynomial pivoting rule for the Simplex algorithm?

- (Mentioned in the survey of [Kaibel&Pfetsch'03]) What is the complexity of computing the diameter of a simple polytope?

- Diameter of TSP polytope?

Final remarks

- **Main questions:**
  - Is the polynomial-Hirsch conjecture true?
  - Is there a polynomial pivoting rule for the Simplex algorithm?
- (Mentioned in the survey of [Kaibel&Pfetsch'03]) What is the complexity of computing the diameter of a simple polytope?
- Diameter of TSP polytope?
- On circuits:
  - (Approximation) algorithms for selecting circuits?
  - What is the complexity of computing the circuit-diameter of a polytope?

Final remarks

- **Main questions:**
  - Is the polynomial-Hirsch conjecture true?
  - Is there a polynomial pivoting rule for the Simplex algorithm?

- (Mentioned in the survey of [Kaibel&Pfetsch'03]) What is the complexity of computing the diameter of a simple polytope?
- Diameter of TSP polytope?
- On circuits:
  - (Approximation) algorithms for selecting circuits?
  - What is the complexity of computing the circuit-diameter of a polytope?

Thank you!