

**Asset Pricing with Heterogeneous
Investors and Portfolio Constraints**

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Abstract

We study dynamic general equilibrium in one-tree and two-trees Lucas economies with one consumption good and two CRRA investors with heterogeneous risk aversions and portfolio constraints. We provide a tractable characterization of equilibrium without relying on the assumption of logarithmic constrained investors, popular in the literature, under which wealth-consumption ratios of these investors are unaffected by constraints. In one-tree economy we focus on the impact of limited stock market participation and margin constraints on market prices of risk, interest rates, stock return volatilities and price-dividend ratios. We demonstrate conditions under which constraints increase or decrease these equilibrium processes, and generate dynamic patterns consistent with empirical findings. In a two-trees economy we demonstrate that investor heterogeneity gives rise to large countercyclical excess stock return correlations, but margin constraints significantly reduce them by restricting the leverage in the economy, and give rise to rich saddle-type patterns. We also derive a new closed-form consumption CAPM that captures the impact of constraints on stock risk premia.

Journal of Economic Literature Classification Numbers: D52, G12.

Keywords: asset pricing, dynamic equilibrium, heterogeneous investors, portfolio constraints, stochastic correlations, stock return volatility, consumption CAPM with constraints.

Portfolio constraints have long been considered among key market frictions that affect investment decisions and asset prices. Consequently, equilibrium models with investors facing restricted participation, short-sale, leverage, and margin constraints have been widely employed by financial economists to explain a wide range of phenomena, such as the equity premium puzzle, mispricing of redundant assets, role of arbitrageurs, and comovement of asset returns [e.g., Detemple and Murthy (1997); Basak and Cuoco (1998); Basak and Croitoru (2000, 2006); Gallmeyer and Hollifield (2008); Pavlova and Rigobon (2008); Gârleanu and Pedersen (2011); among others]. Despite recent developments in portfolio optimization, tractable characterizations of equilibrium have only been obtained at the cost of assuming logarithmic constrained investors, which behave myopically due to the absence of hedging demands.

The myopia of logarithmic investors allows to study the implications of constraints for market prices of risk and interest rates, but impedes the evaluation of the impact of constraints on stock prices. As a result, the effects of constraints on stock price-dividend ratios, return volatilities, and correlations remain relatively unexplored. The reason is that wealth-consumption ratios of such investors remain constant, and hence unaffected by constraints, since income and substitution effects perfectly offset each other. Therefore, these ratios play no role in determining the impact of constraints on stock price-dividend ratios, which in equilibrium are given by the weighted average of wealth-consumption ratios of all investors in the economy. For example, in one-stock economies populated only by logarithmic investors, stock prices are unaffected by constraints [e.g., Detemple and Murthy (1997); Basak and Cuoco (1998); Basak and Croitoru (2000, 2006)].

In this paper we study the impact of portfolio constraints in both one-tree and two-trees dynamic general equilibrium Lucas (1978) economies with one consumption good, populated by one unconstrained and one constrained investors that have general constant relative risk aversion (CRRA) preferences. Our model provides a comprehensive analysis of the effects of constraints on market prices of risk, interest rates, stock price-dividend ratios, return volatilities and correlations. In particular, we demonstrate which constraints generate empirically observed dynamics of equilibrium processes, increase or decrease stock return volatilities and price-dividend ratios, and generate excess volatility. In a two-trees economy we derive new consumption CAPM with constraints, and study stock return correlations.

To provide the intuition for all involved economic forces, we start with a simple one-tree economy where both investors have identical risk aversions and one of them faces a limited participation constraint. This constraint restricts the investment in stocks only to a certain fraction of wealth, and is typical for pension funds.¹ Switching off the heterogeneity in risk aversions

¹Srinivas, Whitehouse and Yermo (2000) show that limits on both domestic and foreign equity holdings of pension funds are in place in a number of OECD countries such as Germany (30% on EU and 6% on non-EU equities), Switzerland (30% on domestic and 25% on foreign equities) and Japan (30% on domestic and 30% on foreign equities), among others. Similar constraint arises in models with passive investors that hold up to a fixed fraction of wealth in stocks [e.g., Chien, Cole, and Lustig (2011)], e.g., due to a “status quo bias,” documented in Samuelson and Zeckhauser (1988), or due to the lack of investment skills, as argued in Campbell (2006). Special case is restricted participation, when some investors do not participate in the stock market, which in year 2002 accounted for 50% of U.S. households [e.g., Basak and Cuoco (1998), Guvenen (2006, 2009)].

isolates the pure effect of portfolio constraints, not confounded by investor heterogeneity, which is not feasible in models with logarithmic investors. We derive all equilibrium processes as functions of constrained investor's share in the aggregate consumption and demonstrate that when the substitution effect dominates, the model can generate countercyclical market prices of risk and stock return volatilities, procyclical interest rates and price-dividend ratios, excess volatility, and negative correlation between risk premia and price-dividend ratios, consistently with the literature [e.g., Shiller (1981); Campbell and Shiller (1988); Schwert (1989); Ferson and Harvey (1991); Campbell and Cochrane (1999)]. We show that tighter constraints always increase market prices of risk and decrease interest rates, consistently with the previous studies [e.g., Basak and Cuoco (1998)]. Furthermore, they increase volatilities and decrease price-dividend ratios when the substitution effect dominates, and vice versa when the income effect is stronger.

Intuitively, tighter limited participation constraints increase market prices of risk to induce the unconstrained investors to hold more stocks in equilibrium, and decrease interest rates since constrained investors allocate more wealth to bonds. As a result, the investment opportunities of the constrained investor deteriorate because of falling interest rates, and because the benefits of higher market prices of risk are limited due to constraints. If the substitution effect dominates, the wealth-consumption ratio of this investor decreases due to low opportunity cost of consumption, pushing down the price-dividend ratio. The effect of constraints is stronger when the constrained investor accounts for larger fraction of the aggregate consumption. This happens in bad times, when the aggregate dividend is low, since the constrained investor is less exposed to negative dividend shocks. Consequently, the price-dividend ratio becomes procyclical. Moreover, since stock price is the product of the dividend and the price-dividend ratio, the procyclicality of the latter amplifies the dividend volatility, making stocks more volatile than dividends.

Next, we consider a richer setting where investors have heterogeneous risk aversions, and the less risk averse investor faces a margin constraint, i.e., is able to borrow only up to a certain fraction of wealth, using stocks as collateral. In this setting the constraint binds intermittently, depending on the amount of liquidity available for borrowing, which is provided by the more risk averse investor. Tighter constraints lead to deleveraging of the economy, which results in higher market prices of risk and lower interest rates, similarly to settings with limited participation. Furthermore, we demonstrate that tighter constraints decrease stock return volatilities by reducing the heterogeneity in portfolio strategies of the investors, consistently with the empirical evidence [e.g., Hardouvelis and Theodossiou (2002)]. We also find that stock return volatilities become less countercyclical, and spike around the point where constraints start to bind.

Then, we extend the analysis to the case of two-trees Lucas economy with heterogeneous investors, where the less risk averse investor faces portfolio constraints. We derive equilibrium processes as functions of the shares of the constrained investor and the first Lucas tree in the aggregate consumption. First, we show that heterogeneity in risk aversions significantly increases correlations relative to models with homogeneous investors [e.g., Cochrane, Longstaff, and Santa-Clara (2008); Martin (2011)]. In particular, the less risk averse investor scales the portfolio

weights up or down, depending on the amount of liquidity supplied by the more risk averse investor for borrowing, which increases the correlations. Consistently with the role of liquidity, we show that tighter constraints significantly decrease the correlations by reducing the ability of the less risk averse investor to lever up.

We also uncover rich saddle-type patterns in conditional stock return correlations under margin and leverage constraints. Specifically, in a calibration where two trees have the same mean and volatility of dividend growth rates, which are uncorrelated, the constraints lead to a larger fall in correlations when both trees have the same weight in the aggregate dividend. Intuitively, in the latter case, the stocks look symmetric, and hence investors invest equal amount of wealth in each stock. At the same time, the leverage constraint prohibits borrowing, and as a result induces investors to invest 50% of their wealth in each stock. Consequently, the heterogeneity in trading strategies is perfectly eliminated, which decreases the correlations towards the homogeneous investor benchmark. In contrast, when one tree accounts for a large share of aggregate dividend, the stocks have different risk and return characteristics, which generates considerable heterogeneity in trading strategies due to differences in risk aversions, and increases correlations.

In the two-trees economy we provide closed-form expressions for price-dividend ratios in the unconstrained benchmark, and when the investors face leverage constraints, and hence cannot borrow. Therefore, some of the effects of constraints, in particular the time-variation in correlations, can be studied using closed-form expressions. In contrast with one-tree economies with leverage constraint where market prices of risk and stock return volatilities are constant [e.g., Kogan, Makarov, and Uppal (2007)], in our two-trees economy all equilibrium processes are time-varying. We also derive a consumption CAPM with margin constraints in terms of observable parameters, which extends Breeden's (1979) consumption CAPM and Black's (1972) static mean-variance CAPM with leverage constraint. In the case of a leverage constraint, we obtain a closed-form expression for the deviation from Breeden's C-CAPM, which captures the interaction between the heterogeneity in preferences and constraints.

The methodological contribution of the paper is the tractable solution method that provides a laboratory for evaluating the interaction between investor heterogeneity and portfolio constraints without relying on investor myopia. The tractability comes from combining the duality approach of Cvitanić and Karatzas (1992) with dynamic programming. First, using the duality approach we derive equilibrium processes in terms of shadow costs of constraints from the market clearing conditions, as in the literature. Then, we depart from the literature, and instead of finding the shadow costs by solving a dual problem, which is tractable only for logarithmic preferences, we derive them from the complementary slackness conditions in Karatzas and Shreve (1998) in terms of price-dividend and wealth-consumption ratios of investors. Using the dynamic programming we then derive a system of differential equations for these ratios, which completely characterize the equilibrium. The equations are then solved via an iterative procedure with fast convergence.

There is a growing literature that studies dynamic equilibria with constraints. In particular, Detemple and Murthy (1997), Basak and Cuoco (1998), Basak and Croitoru (2000, 2006),

Kogan, Makarov, and Uppal (2007), Gallmeyer and Hollifield (2008), Prieto (2010), Gârleanu and Pedersen (2011), He and Krishnamurthy (2011), and Hugonnier (2012) consider models with heterogeneous investors where the constrained investor is logarithmic. These works provide a tractable settings to study the effect of constraints on market prices of risk and interest rates. However, since wealth-consumption ratios of logarithmic constrained investors are constant, they play only a limited role in determining stock prices. For example, in economies where all investors are logarithmic constraints do not affect stock prices [e.g., Detemple and Murthy (1997); Basak and Cuoco (1998); Basak and Croitoru (2000, 2006)]. In contrast to the previous literature, the constrained investors in this paper have general CRRA utility and significantly affect the formation of stock prices. The generality of our framework also allows us to evaluate the pure effect of constraints when both investors have identical risk aversions, as well as the interaction between investor heterogeneity and portfolio constraints.

This paper is also related to the literature that studies the equilibrium in economies with multiple stocks. In particular, Pavlova and Rigobon (2008), and Schornick (2009) study models with constrained logarithmic investors in international finance model with two Lucas trees. In contrast to their models, our model does not rely on heterogeneous home bias and logarithmic preferences. Ribeiro and Veronesi (2002), Menzly, Santos, and Veronesi (2004), Santos and Veronesi (2006), Cochrane, Longstaff, and Santa-Clara (2008), Buraschi, Trojani, and Vedolin (2010), Ehling and Heyerdahl-Larsen (2010), Chen and Joslin (2011) and Martin (2011) study the unconstrained equilibrium with multiple assets. We contribute to this literature by evaluating the impact of constraints, and by providing closed-form expressions for price-dividend ratios in the case of heterogeneity in risk aversions.

Related works also include Gromb and Vayanos (2002, 2009), Brunnermeier and Pedersen (2009), Fostel and Geanakoplos (2008), and Geanakoplos (2009), which study various implications of margin constraints in different settings. Gromb and Vayanos (2010) survey the related literature on limits to arbitrage. Heaton and Lucas (1996), Cuoco and He (2001), Coen-Pirani (2005), Guvenen (2006, 2009), Gomes and Michaelides (2008), Chien, Cole, and Lustig (2011), Buss and Dumas (2012), Dumas and Lyasoff (2012) solve for equilibrium in various discrete-time incomplete market settings. Other related works include Dumas and Maenhout (2002), Wu (2008), Danielson, Shin, and Zigrand (2009), and Rytchkov (2009) which study the implications of certain types of constraints, such as restricted participation, buy-and-hold, and various risk management constraints. Our paper also contributes to growing literature that studies the equilibrium in one-stock economies with heterogeneous unconstrained investors, such as Dumas (1989), Chan and Kogan (2002), Longstaff and Wang (2008), Bhamra and Uppal (2009, 2010), Gârleanu and Panageas (2010), Cvitanic and Malamud (2011). Relative to this literature, we demonstrate how the equilibria are affected by constraints, and provide new closed-form solutions in unconstrained equilibrium with heterogeneous investors.

The remainder of the paper is organized as follows. Section 1 discusses the economic setup and defines the equilibrium in one-stock economy. In Section 2, we provide the characterization of

equilibrium processes, discuss their properties, and describe the solution approach. In Section 3 we provide the analysis of equilibrium with limited participation and margin constraints, discuss the economic intuition and implications. Section 4 studies two-stock economies with constraints, derives a consumption CAPM, and evaluates the impact of constraints on stock return correlations. Section 5 concludes, Appendix A provides the proofs, Appendix B provides further details of our numerical method, and Appendix C discusses the sufficient conditions of optimality.

1. Economic Setup

We consider a continuous-time infinite horizon Lucas (1978) economy with one tree and one consumption good. The economy is populated by two heterogeneous investors that, in general, differ in risk aversions and portfolio constraints. In this Section we discuss the information structure of the economy, the investors' optimization, introduce notation, and define the equilibrium.

1.1. Information Structure and Securities Market

The uncertainty is represented by a filtered probability space $(\Omega, \{\mathcal{F}_t\}, \mathbb{P})$, on which is defined a Brownian motion w . The stochastic processes are adapted to the filtration $\{\mathcal{F}_t, t \in [0, \infty)\}$ generated by w . The economy is populated by two investors with constant relative risk aversion (CRRA) preferences, indexed by $i = A, B$, with risk aversions γ_A and γ_B , such that $\gamma_A \geq \gamma_B$. There is one tree in the economy that produces D_t units of consumption good at time t , and D_t follows a geometric Brownian motion (GBM)

$$dD_t = D_t[\mu_D dt + \sigma_D dw_t], \quad (1)$$

where $\mu_D \geq 0$ and $\sigma_D \geq 0$ are constants.

The investors continuously trade in two securities: a riskless bond in zero net supply with instantaneous interest rate r_t and a stock in positive net supply, normalized to one unit, which is a claim to the stream of output D_t , which we call dividends. We look for Markovian equilibria in which bond prices B_t and stock prices S follow dynamics:

$$dB_t = B_t r_t dt, \quad (2)$$

$$dS_t + D_t dt = S_t[\mu_t dt + \sigma_t dw_t], \quad (3)$$

where interest rate r , stock mean return μ , and volatility σ are stochastic processes determined in equilibrium, and bond price at time 0 is normalized to $B_0 = 1$.

1.2. Investors' Optimization and Portfolio Constraints

Each investor maximizes expected discounted utility of consumption with time discount $\rho > 0$:

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t} \frac{c_{it}^{1-\gamma_i}}{1-\gamma_i} dt\right], \quad i = A, B, \quad (4)$$

subject to a self-financing budget constraint, and for investor B subject to a portfolio constraint, given below. For $\gamma_i = 1$ the utility function in (4) is replaced by logarithmic utility $\ln(c_{it})$.

The investors maximize their utility by choosing optimal consumption c_{it} , and an investment policy $\{\alpha_{it}, \theta_{it}\}$, where α_{it} and θ_{it} denote the fractions of wealth invested in bonds and stocks, respectively. Investor i 's wealth process W_{it} satisfies a dynamic self-financing budget constraint:

$$dW_{it} = \left[W_{it}(r_t + \theta_{it}(\mu_t - r_t)) - c_{it} \right] dt + W_{it}\theta_{it}\sigma_t dw_t, \quad i = A, B. \quad (5)$$

The initial wealth is determined by investors' endowments at time $t = 0$: A is endowed with $1 - s$ units of stock and b units of bond, while B with s units of stock and $-b$ units of bond. These endowments are assumed to be consistent with portfolio constraints that the investors may face.

Investor A is unconstrained, while investor B faces the following constraint:

$$\theta_B \in \Theta_B = \{\theta : \theta_B m \leq 1\}, \quad (6)$$

where $m \geq 0$ is the *margin parameter*. For the simplicity of exposition we assume that margin m is constant, and discuss the case of stochastic margins in Remark 2 below. The special case of $m = 0$ corresponds to the unconstrained case, while $0 < m \leq 1$ to *margin requirements* for collateralized borrowing, when the investor can borrow only up to proportion $1 - m$ of the stock's value in the portfolio [e.g., Brunnermeier and Pedersen (2009); Gromb and Vayanos (2009)]. Special case $m = 1$ corresponds to a *leverage constraint*, when investor B is unable to borrow. We note, that imposing margin constraint with $m \leq 1$ also on investor A leaves the equilibrium unchanged since this constraint does not bind due to the fact that investor A is more risk averse than investor B .

Furthermore, for $1 < m < +\infty$ the constraint (6) is interpreted as *limited participation constraint*, when investor B is restricted to invest only a small fraction of wealth in stocks. The limiting case $m = +\infty$ corresponds to the restricted participation, when investor B does not invest in the stock market, while $m < 0$ to a short-sale constraint, discussed in Remark 3 below.

1.3. Equilibrium

In this paper we derive and study the equilibrium market price of risk $\kappa = (\mu - r)/\sigma$, interest rate r , volatility σ , *price-dividend* $\Psi = S/D$ and *wealth-consumption* $\Phi_i = W_i/c_i$ ratios, where $i = A, B$. Stock mean return is then given by $\mu = \sigma\kappa + r$. We derive all equilibrium processes as functions of constrained investor B 's *consumption share* in the aggregate consumption, $y = c_B^*/D$, which endogenously emerges as a state variable. We conjecture that y follows an Itô's process

$$dy_t = -y_t[\mu_{yt}dt + \sigma_{yt}dw_t], \quad (7)$$

where the drift μ_y and volatility σ_y are endogenously determined in equilibrium.

Definition 1. *An equilibrium is a set of processes $\{r_t, \mu_t, \sigma_t\}$ and of consumption and investment policies $\{c_{it}^*, \alpha_{it}^*, \theta_{it}^*\}_{i \in \{A, B\}}$ that maximize expected utility (4) for each investor, given processes*

$\{r_t, \mu_t, \sigma_t\}$, and consumption and financial markets clear, i.e.,

$$\begin{aligned} c_{At}^* + c_{Bt}^* &= D_t, \\ \theta_{At}^* W_{At} + \theta_{Bt}^* W_{Bt} &= S_t, \\ \alpha_{At}^* W_{At} + \alpha_{Bt}^* W_{Bt} &= 0, \end{aligned} \tag{8}$$

where W_{At} and W_{Bt} denote time- t wealths of investors A and B , respectively.

2. General Equilibrium with Constraints

In this Section we provide a characterization of equilibrium processes. First, in Section 2.1 we obtain optimal consumptions of investors in a partial equilibrium setting by employing the duality approach of Cvitanić and Karatzas (1992). Next, in Section 2.2, from the consumption clearing condition we obtain equilibrium processes for market prices of risk, interest rates, and stock return volatilities in terms of shadow costs of constraints. These shadow costs are then found from complementary slackness conditions in terms of wealth-consumption ratios and their derivatives. Finally, we derive differential equations for wealth-consumption ratios, which completes the characterization of equilibrium. We also provide the economic intuition for the impact of constraints, and in Section 2.3 discuss the computation of equilibrium.

2.1. Optimal Consumptions in Partial Equilibrium

We start by characterizing optimal consumptions of investors in a partial equilibrium setting, where stock and bond prices are taken as given. Solving the optimization with constraints is a challenging task even at a partial equilibrium level. Here, we follow the approach of Cvitanić and Karatzas (1992), and characterize constrained investors' optimal consumptions by embedding the partial equilibrium economy into an equivalent complete-market *fictitious economy*. We provide the economic intuition for the fictitious economy, and in Remark 1 demonstrate how it can be constructed via dynamic programming with constrained optimization. Then, we also show how the complementary slackness conditions emerge from Kuhn-Tucker conditions of optimality.

As demonstrated in Cvitanić and Karatzas (1992), the utility maximization subject to budget constraint (5) and portfolio constraint (6) can be solved as an unconstrained optimization in an economy with bond and stock prices following dynamics with adjustments:

$$dB_t = B_t[r_t + f(\tilde{\nu}_t)]dt, \tag{9}$$

$$dS_t + D_t dt = S_t[(\mu_t + \tilde{\nu}_t + f(\tilde{\nu}_t))dt + \sigma_t dw_t], \tag{10}$$

where *adjustment* $\tilde{\nu}$ can be interpreted as the shadow cost of portfolio constraint, and $f(\tilde{\nu})$ is the

support function for the set of portfolio constraints Θ_B , defined as:

$$f(\tilde{\nu}) = \sup_{\theta \in \Theta_B} (-\tilde{\nu}\theta). \quad (11)$$

The adjustments $\tilde{\nu}$ can be obtained either by solving a *dual optimization problem* or from *complementary slackness conditions*, and lie in the *effective domain* of function $f(\tilde{\nu})$, defined as $\Upsilon = \{\nu \in \mathbb{R} : f(\nu) < \infty\}$ [e.g., Cvitanić and Karatzas (1992); Karatzas and Shreve (1998)].

In the context of portfolio constraint (6) the intuition behind the fictitious economy is as follows. When portfolio constraint (6) binds, it reduces the share of wealth that investor B allocates to stocks, relative to the unconstrained case. This reduction in stock holding can be mimicked in an unconstrained economy with higher interest rates and lower risk premia than in the original economy. In such a fictitious economy the investor allocates less wealth to stocks than in the real unconstrained economy, consistently with the policy of the constrained investor. This intuition suggests that for constraint (6) adjustment $\tilde{\nu}$ is negative, while $f(\tilde{\nu})$ is positive, which can be confirmed by deriving the support function $f(\tilde{\nu})$ and its effective domain Υ :

$$f(\tilde{\nu}) = -\nu^*, \quad \tilde{\nu} = \nu^* m, \quad \Upsilon = \{\nu^* : \nu^* \leq 0\}. \quad (12)$$

In complete markets, the drift and volatility of the process for the state price density are given by $-r$ and $-\kappa$, respectively [e.g. Duffie (2001)], where κ denotes the market price of risk $(\mu - r)/\sigma$. Therefore, the state price densities in the unconstrained complete-market real and fictitious economies, ξ and ξ_{ν^*t} , evolve as follows:

$$d\xi_t = -\xi_t[r_t dt + \kappa_t dw_t], \quad d\xi_{\nu^*t} = -\xi_{\nu^*t}[(r_t - \nu_t^*)dt + (\kappa_t + \nu_t^* m/\sigma_t)dw_t]. \quad (13)$$

Next, we obtain optimal consumptions of investors from the first order conditions that equate their marginal utilities and state price densities [e.g., Huang and Pagés (1992); Cuoco (1997)]:

$$c_{At}^* = (\psi_A e^{\rho t} \xi_t)^{-\frac{1}{\gamma_A}}, \quad c_{Bt}^* = (\psi_B e^{\rho t} \xi_{\nu^*t})^{-\frac{1}{\gamma_B}}, \quad (14)$$

where ψ_i denote Lagrange multipliers for static budget constraints in the martingale approach.

Remark 1 (Fictitious Economy and Complementary Slackness Condition). The construction of the fictitious economy can be conveniently illustrated via dynamic programming. In particular, let J_{Bt} denote investor B 's time- t value function, which we conjecture to depend on wealth W , some state variable y , and time t . Let ℓ_t denote time- t Lagrange multiplier for portfolio constraint (6), and ν_t^* be the rescaled Lagrange multiplier, given by $\nu_t^* = \ell_t/(W_t \partial J_{Bt}/\partial W_t)$. Then, the value function satisfies the following Hamilton-Jacobi-Bellman (HJB) equation:

$$0 = \max_{c_{Bt}, \theta_{Bt}} \left\{ e^{-\rho t} \frac{c_{Bt}^{1-\gamma_B}}{1-\gamma_B} dt + \mathbb{E}_t[dJ_{Bt}] + \nu_t^* (\theta_{Bt} m - 1) W_t \frac{\partial J_{Bt}}{\partial W_t} dt \right\},$$

which can be further expanded as follows:

$$0 = \max_{c_{Bt}, \theta_{Bt}} \left\{ e^{-\rho t} \frac{c_{Bt}^{1-\gamma_B}}{1-\gamma_B} + \frac{\partial J_{Bt}}{\partial t} + \left[W_t (r_t - \nu_t^* + \theta_{Bt}(\mu_t - r_t + \nu_t^* m)) - c_{Bt} \right] \frac{\partial J_{Bt}}{\partial W_t} \right. \\ \left. - y_t \mu_{yt} \frac{\partial J_{Bt}}{\partial y_t} + \frac{1}{2} \left[W_t^2 \theta_{Bt}^2 \sigma_t^2 \frac{\partial^2 J_{Bt}}{\partial W_t^2} - 2W_t \theta_{Bt} \sigma_t y_t \sigma_{yt} \frac{\partial^2 J_{Bt}}{\partial W_t \partial y_t} + y_t^2 \sigma_{yt}^2 \frac{\partial^2 J_{Bt}}{\partial y_t^2} \right] \right\}, \quad (15)$$

subject to transversality condition $\mathbb{E}_t[J_{BT}] \rightarrow 0$, as $T \rightarrow \infty$. We observe, that equation (15) corresponds to an HJB equation in the unconstrained fictitious economy with bond and stock prices following processes (9)–(10), and adjustments $\tilde{\nu}$ and $f(\tilde{\nu})$ given by expressions (12). Furthermore, Kuhn-Tucker optimality conditions imply that $\nu_t^* \leq 0$, and the complementary slackness condition $\nu_t^*(\theta_{Bt}^* m - 1) = 0$ is satisfied.

2.2. Characterization of General Equilibrium

In this subsection we characterize the Markovian equilibrium and discuss its properties. Our solution approach does not rely on a widely used assumption of a logarithmic constrained investor [e.g., Detemple and Murthy (1997); Basak and Cuoco (1998); Basak and Croitoru (2000, 2006); Kogan, Makarov and Uppal (2003); Gallmeyer and Hollifield (2008); Pavlova and Rigobon (2008); Gârleanu and Pedersen (2011); among others] which allows for tractability at the cost of investor's myopia inherent in logarithmic preferences. We also explain the challenges arising in settings with non-logarithmic investors, and how these challenges are tackled in this paper. We proceed in three steps, outlined below, while further details are discussed in the proof of Proposition 1.

First, we derive the equilibrium market price of risk κ , interest rate r , volatility σ_y and drift μ_y as functions of investor B 's consumption share y and adjustment ν^* by substituting optimal consumptions (14) into the consumption clearing condition in (8), applying Itô's Lemma to both sides, and matching dt and dw terms. Then, we obtain stock return volatility σ in terms of volatility σ_y , stock price-dividend ratio Ψ , and its derivative Ψ' by applying Itô's Lemma to both sides of equality $S = \Psi D$ and matching dw terms. The first step also verifies that κ , r , σ_y and μ_y are Markovian in y and ν^* , which endogenously emerge as state variables since the fictitious economy in Section 2.1 can be constructed without assuming specific state variables. Furthermore, adjustment ν^* is not an independent variable since it can be determined as a function of consumption share y from Kuhn-Tucker optimality condition for portfolio weight θ_B^* , as discussed below. Therefore, our equilibrium turns out to be Markovian in y .

Next, in the second step, we derive differential equations for wealth-consumption ratios Φ_A and Φ_B , with coefficients dependant on adjustments ν^* . As demonstrated in Liu (2007), the wealth-consumption ratios satisfy linear PDEs in complete financial markets. Consequently, conditional on knowing the adjustment ν^* , ratios Φ_A and Φ_B satisfy linear differential equations, since the constrained investor's optimization is solved in the fictitious complete-market economy. Specifically, following Liu (2007) we conjecture the value functions in the form $J_i = W_i^{1-\gamma_i} \Phi_i^{\gamma_i} / (1-\gamma_i)$, where $i = A, B$, substitute them into HJB equations and after some algebra obtain equations

for $\Phi_A(y)$ and $\Phi_B(y)$. In Appendix C we discuss the sufficient conditions for optimality, and further justify the conjectured structure of value functions. The price-dividend ratio Ψ is then found from market clearing conditions.

In the third step, most challenging, we complete the characterization of equilibrium by finding adjustment ν^* from the complementary slackness condition $\nu^*(\theta_B^*m - 1) = 0$ [e.g., Karatzas and Shreve (1998); Remark 1 in Section 2.1], where portfolio policy θ_B^* is given by [e.g., Liu (2007)]:

$$\theta_{Bt}^* = \frac{\mu_t - r_t + \nu_t^*m}{\gamma_B \sigma_t^2} - \frac{y_t \sigma_{yt}}{\sigma_t} \frac{\Phi_B'(y_t)}{\Phi_B(y_t)}. \quad (16)$$

When the constraint does not bind, the complementary slackness condition implies that $\nu^* = 0$. When the constraint binds, ν^* is found from equation $\theta_B^*m = 1$, taking into account the expressions for volatilities σ_y and σ identified in the first step in terms of adjustment ν^* . Eventually, we obtain adjustment ν^* in terms of wealth-consumption ratios Φ_A and Φ_B , and their derivatives.

The tractability of logarithmic preferences, popular in the literature, is due to the fact that logarithmic investor's wealth-consumption ratio is given by $\Phi_B = 1/\rho$, and hence the hedging demand (second term) in expression (16) vanishes. The absence of the hedging demand facilitates finding adjustment ν^* from the complementary slackness condition. In particular, when both investors A and B are logarithmic, all equilibrium processes can be identified in closed form. However, in such a setting stock prices are unaffected by constraints, and volatility σ equals dividend volatility σ_D [e.g., Detemple and Murthy (1997); Basak and Cuoco (1998)]. The model is less tractable with non-logarithmic unconstrained investor A , even when investor B remains logarithmic. In such a setting, κ and r can be found in closed form only in a handful of special cases, such as restricted participation [e.g., Basak and Cuoco (1998)], short-sale constraint [e.g., Gallmeyer and Hollifield (2008)], and risk constraint [e.g., Prieto (2010)], while price-dividend ratios are obtained numerically by solving linear ODEs.

As demonstrated in Proposition 1 below, price-dividend ratio Ψ is a weighted average of wealth-consumption ratios of investors, and is given by $\Psi = (1 - y)\Phi_A + y\Phi_B$. Consequently, the assumption of a logarithmic constrained investor may distort the equilibrium by eliminating the effects of constraints on wealth-consumption ratio of investor B , who is the one most affected by constraints. In contrast to the literature, this paper provides a tractable unifying framework for studying interactions between the heterogeneity in preferences and portfolio constraints without relying on investor myopia. Relaxing the assumption of logarithmic preferences brings new economic insights via income and substitution effects that play no role for a logarithmic investor, as elaborated in Section 3. Proposition 1 below summarizes the structure of equilibrium.

Proposition 1. *The equilibrium market price of risk $\kappa = (\mu - r)/\sigma$, interest rate r , volatility σ_y and mean growth μ_y of consumption share y , and stock return volatility σ are given by:*

$$\kappa_t = \Gamma_t \sigma_D - \frac{\Gamma_t y_t}{\gamma_B} \frac{\nu_t^* m}{\sigma_t}, \quad (17)$$

$$r_t = \rho + \Gamma_t \mu_D - \frac{\Gamma_t \Pi_t}{2} \sigma_D^2 + \frac{\Gamma_t y_t \nu_t^*}{\gamma_B} + \frac{\nu_t^* m}{\sigma_t} \left(a_1(y_t) \sigma_D + a_2(y_t) \frac{\nu_t^* m}{\sigma_t} \right), \quad (18)$$

$$\sigma_{yt} = \frac{\Gamma_t (1 - y_t)}{\gamma_A \gamma_B} \left((\gamma_B - \gamma_A) \sigma_D - \frac{\nu_t^* m}{\sigma_t} \right), \quad (19)$$

$$\mu_{yt} = \mu_D - \sigma_D \sigma_{yt} - \frac{1 + \gamma_B}{2} (\sigma_D - \sigma_{yt})^2 - \frac{r_t - \nu_t^* - \rho}{\gamma_B}, \quad (20)$$

$$\sigma_t = \sigma_D - y_t \sigma_{yt} \frac{\Psi'(y_t)}{\Psi(y_t)}, \quad (21)$$

where $a_1(y)$ and $a_2(y)$ are functions given by equations (A2) in the Appendix, Γ and Π denote the risk aversion and prudence parameters of a representative investor,² and are given by

$$\Gamma = \frac{\gamma_A \gamma_B}{\gamma_A y + \gamma_B (1 - y)}, \quad \Pi = \Gamma^2 \left(\frac{1 + \gamma_A}{\gamma_A^2} (1 - y) + \frac{1 + \gamma_B}{\gamma_B^2} y \right), \quad (22)$$

ν^* is the adjustment given by equation (A4) in the Appendix as a function of price-dividend ratio Ψ , wealth-consumption ratio of constrained investor Φ_B , and their derivatives. The wealth-consumption ratios $\Phi_A(y)$ and $\Phi_B(y)$ satisfy ODEs:

$$\frac{y^2 \sigma_y^2}{2} \Phi_A'' - y \left(\mu_y + \frac{1 - \gamma_A}{\gamma_A} \kappa \sigma_y \right) \Phi_A' + \left(\frac{1 - \gamma_A}{2 \gamma_A} \kappa^2 + (1 - \gamma_A) r - \rho \right) \frac{\Phi_A}{\gamma_A} + 1 = 0, \quad (23)$$

$$\begin{aligned} \frac{y^2 \sigma_y^2}{2} \Phi_B'' - y \left(\mu_y + \frac{1 - \gamma_B}{\gamma_B} \left(\kappa + \frac{\nu^* m}{\sigma} \right) \sigma_y \right) \Phi_B' \\ + \left(\frac{1 - \gamma_B}{2 \gamma_B} \left(\kappa + \frac{\nu^* m}{\sigma} \right)^2 + (1 - \gamma_B) (r - \nu^*) - \rho \right) \frac{\Phi_B}{\gamma_B} + 1 = 0, \end{aligned} \quad (24)$$

and the price-dividend ratio is given by $\Psi(y) = (1 - y) \Phi_A(y) + y \Phi_B(y)$.

Proposition 1 provides equilibrium processes (17)–(21) in terms of adjustment parameter ν^* , given by expression (A4) in the Appendix in terms of wealth-consumption and price-dividend ratios, Φ_B and Ψ , respectively. Substituting adjustment ν^* into equations (23)–(24) we obtain a system of quasilinear ODEs which are solved numerically, as discussed in Section 2.3. Here, we provide the intuition for expressions (17)–(21) by noting that even though ν^* is not available in closed form, Kuhn-Tucker conditions imply that $\nu^* \leq 0$, and hence its sign is known [e.g., Cvitanić and Karatzas (1992); Karatzas and Shreve (1998); Remark 1 in Section 2.1].

Since the adjustment is such that $\nu^* \leq 0$, expression (17) implies that portfolio constraint (6) increases the market price of risk, if volatility σ is positive (which is confirmed by numerical computations). Intuitively, constrained investor B cannot invest in stocks as much as in the unconstrained economy. Consequently, the market price of risk increases to compensate the

²Similarly to Basak (2000, 2005) it can be demonstrated that the equilibrium in this economy is equivalent to the equilibrium in an economy with a *representative investor* with a utility function given by:

$$u(c; \lambda) = \max_{c_A + c_B = c} \frac{c_A^{1-\gamma_A}}{1-\gamma_A} + \lambda \frac{c_B^{1-\gamma_B}}{1-\gamma_B},$$

where $\lambda = \xi_{\nu^*} / \xi$. The expressions for the relative risk aversion Γ and prudence Π of the representative investor, given by (22), are special cases of those in Basak (2000, 2005), derived for general utility functions.

unconstrained investor A for holding more stocks to clear the market. Next, we observe that interest rate (18) is a quadratic function of ν^* , and hence the effect of constraint on r is ambiguous. Intuitively, on one hand, the interest rates tend to decrease since the constrained investors have higher demand for bonds, being unable to invest in stocks. On the other hand, due to the increase in the market price of risk, the unconstrained investors invest more in stocks and less in bonds, which tends to increase the interest rate. In our numerical analysis the former effect dominates because the unconstrained investors are risk averse, and hence the interest rate decreases.

We note that when the horizon is finite the equilibrium can be alternatively characterized in terms of forward and backward stochastic differential equations (FBSDE), following the same steps as in the partial equilibrium setting of Detemple and Rindisbacher (2005). In particular, from the expressions for optimal consumptions (14) and the fact that wealth can be represented as the present value of the optimal consumption [e.g., Huang and Pagés (1992); Cuoco (1997)], we observe that wealth-consumption ratios are given by:

$$\Phi_A(y_t) = \mathbb{E}_t \left[\int_t^T e^{-\frac{\rho}{\gamma_A}(\tau-t)} \left(\frac{\xi_\tau}{\xi_t} \right)^{1-\frac{1}{\gamma_A}} d\tau \right], \quad \Phi_B(y_t) = \mathbb{E}_t \left[\int_t^T e^{-\frac{\rho}{\gamma_B}(\tau-t)} \left(\frac{\xi_{\nu^*\tau}}{\xi_{\nu^*t}} \right)^{1-\frac{1}{\gamma_B}} d\tau \right], \quad (25)$$

where ξ and ξ_{ν^*} follow processes (13). Derivatives $\Phi'_i(y)$ can be evaluated via Malliavin calculus [e.g., Detemple, Garcia, and Rindisbacher (2003); Detemple and Rindisbacher (2005)]. Substituting $\Phi'_i(y)$ into portfolio weights θ_A^* and θ_B^* , and then using Kuhn-Tucker conditions [e.g., Karatzas and Shreve (1998); Remark 1 in Section 2.1], leads to a BSDE for adjustment ν^* , similar to equation (3.11) in Detemple and Rindisbacher (2005). Budget constraint (5) gives an additional forward equation. However, the resulting BSDE is difficult to solve numerically since, in contrast to a partial equilibrium, processes κ , r , σ , and σ_y also depend on Φ_A , Φ_B via ν^* .

The unconstrained economy is a convenient benchmark against which we compare our main results. We here provide an analytic solution in terms of familiar hypergeometric functions commonly employed in the literature [e.g., Cochrane, Longstaff, and Santa-Clara (2008); Longstaff and Wang (2008); Longstaff (2009); Martin (2011)]. Our closed-form expressions for equilibrium quantities generalize the results in Longstaff and Wang (2008), derived under a restrictive assumption that $\gamma_A = 2\gamma_B$. Proposition 2 reports our result.

Proposition 2. *In the unconstrained economy market price of risk κ , interest rate r , consumption share mean growth μ_y and volatility σ_y , and stock return volatility σ are given in closed form*

by expressions (17)–(21) in which $\nu^* = 0$, and the price-dividend ratio Ψ is given by:

$$\begin{aligned}\Psi(y) = \frac{1}{p} & \left[-\frac{1}{\gamma_A + \varphi_-} {}_2F_1\left(\left(1 - \frac{\gamma_B}{\gamma_A}\right)\varphi_- - \gamma_B, 1, 1 - \gamma_B - \frac{\gamma_B}{\gamma_A}\varphi_-; y\right) \right. \\ & + \left(1 - \frac{\gamma_B}{\gamma_A}\right) \frac{y}{1 - \gamma_B - \frac{\gamma_B}{\gamma_A}\varphi_-} {}_2F_1\left(\left(1 - \frac{\gamma_B}{\gamma_A}\right)\varphi_- + 1 - \gamma_B, 1, 2 - \gamma_B - \frac{\gamma_B}{\gamma_A}\varphi_-; y\right) \\ & + \frac{\gamma_B}{\gamma_A} \frac{1}{\varphi_+} {}_2F_1\left(\left(1 - \frac{\gamma_B}{\gamma_A}\right)\varphi_+ - \gamma_B, 1, 1 + \varphi_+; 1 - y\right) \\ & \left. + \left(1 - \frac{\gamma_B}{\gamma_A}\right) \frac{y}{\varphi_+} {}_2F_1\left(\left(1 - \frac{\gamma_B}{\gamma_A}\right)\varphi_+ + 1 - \gamma_B, 1, 1 + \varphi_+; 1 - y\right) \right],\end{aligned}\tag{26}$$

where constants p , φ_+ and φ_- are given in closed form by expressions (A22) in Appendix A, and ${}_2F_1(x_1, x_2, x_3; y)$ denotes a hypergeometric function, given in Appendix A.

Remark 2 (Time-varying Margins). Propositions 1 and Lemma A.1 in the Appendix remain valid without any changes for time-varying margins that depend on equilibrium processes, e.g., $m = m(y_t, \sigma_t)$. Such margins may arise in the case of constraints on portfolio volatility, or when margins tighten in periods of high volatility. In this work, for expositional simplicity, we assume that margins are constant, and focus on exploring the implications of the tightness of constraints on equilibrium. At the end of the proof of Lemma A.1 in Appendix A we elaborate on how adjustment ν^* can be computed when margin m is a function of volatility.

Remark 3 (Short-sale Constraints). We note that short-sale constraint $\theta_B \geq \underline{\theta}$, where $\underline{\theta} < 0$, is a special case of constraint (6) when $m < 0$. However, to make the constraint binding, the model requires the heterogeneity in beliefs about mean dividend growth μ_D arising, e.g., due to differences in time-0 priors [e.g., Basak (2000, 2005)]. The model can be easily solved when investors A and B believe that the mean dividend growth is constant, and equals $\mu_{A,D}$ and $\mu_{B,D}$, respectively, and do not update their beliefs. As argued in Abel (2002) and Colacito and Croce (2012), dogmatic beliefs and persistent disagreement can be rationalized when investors fear model misspecification and solve their optimisation using robust control [e.g., Hansen and Sargent (2007)]. Incorporating learning into this model is an interesting but challenging problem, which increases the dimensionality of equations by introducing extra state and time variables.

2.3. Boundary Conditions and Computation of Equilibrium

In this subsection we briefly discuss the computation of equilibrium, while further details are presented in Appendix B. First, we discuss the boundary conditions for ODEs (23)–(24). These conditions provide further insights on the dependence of wealth-consumption ratios on the tightness of constraints, and conditions for the constraint to be binding in the economy. Then, we describe the numerical method, based on finite differences approach. Here, we only consider the case of margin constraints with $m < 1$, while the case $m \geq 1$ is addressed in Appendix B.

Following the literature [e.g., Duffy (2006); Gârleanu and Pedersen (2011)] we obtain boundary conditions by passing to limits in ODEs (23)–(24) as $y \rightarrow 0$ and $y \rightarrow 1$ [see Appendix B]. Intuitively, these conditions coincide with wealth-consumption ratios in the limiting economies dominated by investor A (as $y \rightarrow 0$) and B (as $y \rightarrow 1$), respectively. We note, that the margin constraint does not bind when constrained investor B dominates (i.e., $y \approx 1$). The reason is that binding constraint $\theta_B = 1/m > 1$ would violate the market clearing in stocks in such an economy. Consequently, the market price of risk κ and the interest rate r in the limiting economies coincide with those in respective homogeneous-investor unconstrained economies.

The analysis of investor optimization in the limiting economies gives boundary conditions:

$$\begin{aligned}\Phi_A(0) &= \frac{\gamma_A}{\rho - \frac{1 - \gamma_A}{2\gamma_A}\kappa_A^2 - (1 - \gamma_A)r_A}, & \Phi_A(1) &= \frac{\gamma_A}{\rho - \frac{1 - \gamma_A}{2\gamma_A}\kappa_B^2 - (1 - \gamma_A)r_B}, \\ \Phi_B(0) &= \frac{\gamma_B}{\rho - \frac{1 - \gamma_B}{2\gamma_B}\left(\kappa_A + \frac{\bar{\nu}^* m}{\sigma_D}\right)^2 - (1 - \gamma_B)(r_A - \bar{\nu}^*)}, & \Phi_B(1) &= \frac{\gamma_B}{\rho - \frac{1 - \gamma_B}{2\gamma_B}\kappa_B^2 - (1 - \gamma_B)r_B},\end{aligned}\tag{27}$$

where κ_i and r_i denote the market price of risk and interest rate in the unconstrained homogeneous-agent economy populated by investor i , $\bar{\nu}^* = \nu^*(0)$, and these quantities are given by:

$$\kappa_i = \gamma_i \sigma_D, \quad r_i = \rho + \gamma_i \mu_D - \frac{\gamma_i(1 + \gamma_i)}{2} \sigma_D^2, \quad i = A, B,\tag{28}$$

$$\bar{\nu}^* = \left(\frac{\sigma_D}{m}\right)^2 \min(0, \gamma_B - \gamma_A m).\tag{29}$$

More formally, when we pass to the limit in ODEs (23)–(24), it can be shown that the coefficients in front of derivatives converge to zero, and hence, the boundary conditions (27) are determined by inverse coefficients in front of $\Phi_i(y)$, taken with negative sign. This approach assumes that Φ_i are sufficiently smooth at the boundaries, so that the first and second terms in ODEs (23)–(24) converge to zero as coefficients converge to zero. Later on, we verify that this assumption indeed holds in an equilibrium computed using finite-difference method with ten thousand grid points. We also note, that conditions $\Phi_i(0) > 0$ and $\Phi_i(1) > 0$ are necessary for the value functions of investors to be bounded in equilibrium. Therefore, we always choose parameters ρ , γ_i , μ_D and σ_D that satisfy these conditions. Moreover, after deriving the equilibrium numerically, we additionally verify that the value functions are indeed bounded.

The expression for $\Phi_B(0)$ gives the wealth-consumption ratio of a small constrained investor B in an economy dominated by investor A . It also provides valuable insights on the role of income and substitution effects in determining the effects of constraints on wealth-consumption ratios, and demonstrates how these effects disappear for logarithmic investors. In particular, it can be shown that, as constraint becomes tighter (i.e., m increases), wealth-consumption ratio $\Phi_B(0)$ decreases when the substitution effect dominates (i.e., $\gamma_B < 1$), increases when the income effect dominates (i.e., $\gamma_B > 1$), and is unchanged when these effects offset each other (i.e., $\gamma_B = 1$). We also note, that the expression for the adjustment parameter $\bar{\nu}^*$ in (29) provides a simple sufficient condition for the margin constraint to be binding in the economy, which requires $\bar{\nu}^* < 0$.

Next, we solve for equilibrium using finite difference method. The solution method is complicated by the quasilinearity of ODEs (23)–(24), since the adjustment ν^* is itself a function of Φ_i , Ψ , and their derivatives. We circumvent this difficulty by using two approaches suggested in the literature. The first one is the fixed point iteration method [e.g., Gomes and Michaelides (2008); Guvenen (2009); Chien, Cole and Lustig (2011); among others], in which we use candidate wealth-consumption ratios $\Phi_{i,k}(y)$ at step k to compute all the equilibrium processes, including the adjustment ν^* . Then, we note, that ODEs (23)–(24) become linear conditional on knowing the equilibrium processes and the adjustment. Accordingly, we obtain next-step wealth consumption ratios by solving these linear ODEs numerically. Then, we calculate the implied equilibrium processes again, and iterate until convergence. This method typically gives very fast convergence if the conjectured wealth-consumption ratios at step 0 are not very far from the equilibrium ones.

We note that this convergence is reminiscent of well-known *tâtonnement dynamics* [e.g., Mas-Colell, Whinston, and Green (1995)], which describes the transition of the economy from disequilibrium to equilibrium. In particular, similarly to *tâtonnement dynamics*, we start with disequilibrium processes, observe how they get incorporated into wealth-consumption ratios, which via the market clearing conditions translate into equilibrium processes at the next step. Consequently, the convergence of the numerical algorithm is an intuitive property of equilibria, which are resilient to perturbations in the equilibrium processes.

The second approach is inspired by the method of successive iterations for solving the equations of dynamic programming [e.g., Ljungqvist and Sargent (2004)], when the value function is set equal to a certain function at a distant horizon T and then the value functions at earlier dates are obtained by solving equations backwards. This approach is just a version of the first one, and in Appendix B we argue that it adds stability to the solution method. More specifically, we consider a finite-horizon problem with a large horizon parameter T , choose a terminal value for $\Phi_i(y, T)$ and then solve the equations backwards in time using a modification of Euler’s finite-difference method until the solutions converge to time-independent functions $\Phi_i(y)$.

To solve the differential equations we replace derivatives by their finite-difference analogues, then sitting at time t we compute the coefficients of finite-difference equations using the solutions obtained from the previous step $t + \Delta t$. As a result, at step t the coefficients of the equations for wealth-consumption ratios are known, and hence Φ_i can be found by solving a system of linear finite-difference equations with a three-diagonal matrix. In most of the cases studied in this paper we use fixed point iterations while in certain cases we use a combination of two methods to facilitate the convergence. Appendix B provides further details.

In Appendix C we also discuss some sufficient conditions for the optimality of investors’ consumption and portfolio strategies, which are easy to verify once the equilibrium processes are computed numerically. We show that these sufficient conditions are satisfied in economies with margin constraints, where all equilibrium processes are continuous and uniformly bounded [e.g., Figure 3 in Section 3.2]. However, the case of limited participation is not covered by our

verification result since the processes have a singularity at $y = 1$ [e.g., Figures 1 and 2 in Section 3.1], and may require more subtle sufficient conditions, which are not available in the literature.

3. Analysis of Equilibrium

In Sections 3.1 and 3.2 below, we explore the asset pricing implications of limited participation and margin constraints, respectively. The earlier literature primarily focused on the impact of various special cases of portfolio constraint (6) on market prices of risk κ and interest rates r in settings with a logarithmic constrained investor [e.g., Detemple and Murthy (1997); Basak and Cuoco (1998); Basak and Croitoru (2000, 2006); Gârleanu and Pedersen (2011); among others]. However, despite the importance of portfolio constraints, their impact on stock return volatilities and price-dividend ratios remained relatively unexplored.

Using the methodology of Section 2, we provide the full picture of the dependence of volatility σ and price-dividend ratio Ψ on margin m for general CRRA preferences. We establish conditions under which portfolio constraints increase or decrease volatility σ and ratio Ψ , make them procyclical or countercyclical, and generate excess volatility relative to the volatility of dividends. We also discuss the crucial role of classical income and substitution effects, absent for logarithmic investors, in determining the impact of constraints on equilibrium. We demonstrate how the substitution effect and the limited participation constraint can generate countercyclical market prices of risk, procyclical price-dividend ratios, countercyclical stock-return volatilities and risk premia, as well as excess volatility, consistently with empirical findings.

Our model provides a parsimonious framework for the qualitative exploration of the impact of constraints on return volatilities and price-dividend ratios, and matching their dynamic properties. While market price of risk κ and interest rate r are relatively easy to reconcile with empirically observed ones, volatility σ remains significantly lower than in the data. The difficulty of matching both first and second moments of asset returns in a single model has long been recognized in the literature, and is a feature shared by many asset pricing models, as argued in Heaton and Lucas (1996).

In the analysis of dynamic properties of equilibrium processes, following the literature, we call a stochastic Itô's process X_t *procyclical* if correlation $\text{corr}(dX_t, dD_t)$ is positive, and hence X_t increases (decreases) when dividend innovation dw_t is positive (negative) [e.g., Chan and Kogan (2002); Longstaff and Wang (2008); Gârleanu and Panageas (2010)]. Similarly, we will call process X_t *countercyclical* if $\text{corr}(dX_t, dD_t)$ is negative. In our calibrations we set $\mu_D = 1.8\%$ and $\sigma_D = 3.6\%$, which is within the ranges considered in the literature [e.g., Basak and Cuoco (1998); Mehra and Prescott (1985); Campbell (2003); Dumas and Lyasoff (2012); among others], and set the time discount parameter to $\rho = 0.01$.

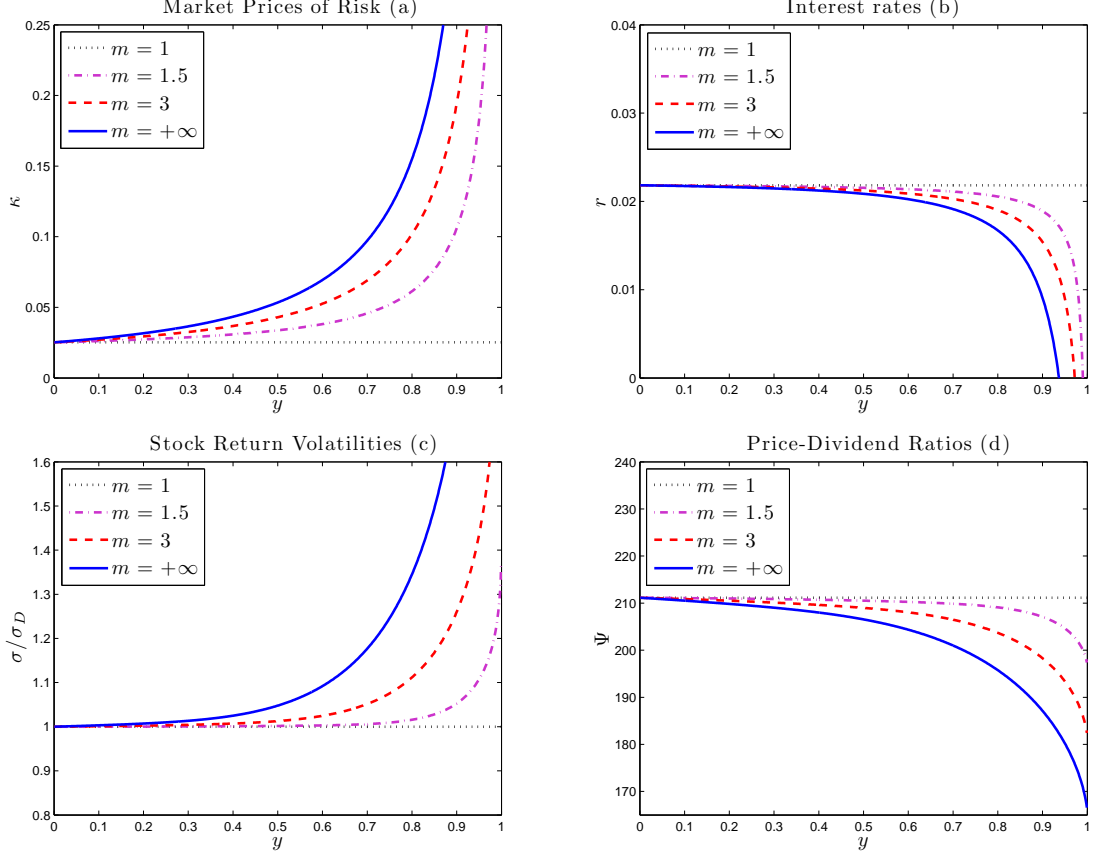


Figure 1: Equilibrium with Limited Participation, $\gamma < 1$.

Figure 1 presents the equilibrium processes for different margins m , where $m \geq 1$, when the substitution effect dominates. Consumption share $y = c_B^*/D$ is countercyclical, and model parameters are: $\gamma_A = 0.7$, $\gamma_B = 0.7$, $\rho = 0.01$, $\mu_D = 1.8\%$, and $\sigma_D = 3.6\%$.

3.1. Equilibrium with Limited Stock Market Participation

We start with an economy where both investors have the same risk aversion γ (i.e., $\gamma_A = \gamma_B = \gamma$), and investor B faces a limited participation constraint $\theta_B m \leq 1$, where $m > 1$. In this model to explore the role of income and substitution effects in transmitting the effects of constraints from market prices of risk and riskless rates into stock return volatilities and price-dividend ratios. Furthermore, this model allows us to evaluate the pure effects of constraints, not confounded by investor heterogeneity. We also note, that for the case $m = 1$ the equilibrium coincides with the equilibrium in the unconstrained economy, since in such an economy $\theta_B^* = \theta_A^* = 1$.

As discussed in the introduction, limited participation constraints are typical for pension funds [e.g., Srinivas, Whitehouse, and Yermo (2000)], and include the restricted participation, as a special case when $m = +\infty$ [e.g., Basak and Cuoco (1998); Guvenen (2006, 2009)]. We note, that absent the heterogeneity in preferences this constraint is identically binding at all times.

Intuitively, if investor B does not find the asset attractive enough to bind on the constraint, both investors would hold $\theta_A^* < 1/m$ and $\theta_B^* < 1/m$, since they have identical preferences. However, given that $m > 1$ the latter inequalities violate the market clearing in stocks, and hence the equilibrium processes will adjust to make the constraint binding.

Figures 1 and 2 present equilibrium processes as functions of investor B 's consumption share y for different levels of margin m for $\gamma = 0.7$ and $\gamma = 3$, respectively, and for calibrated parameters. Considering both $\gamma < 1$ and $\gamma > 1$ allows us to explore the role of income and substitution effects. We also note that in the limited participation case investor B 's consumption share y_t is countercyclical, and it can be shown that $\text{corr}(dy_t, dD_t) = -1$. Intuitively, the constrained investor is less exposed to stock market fluctuations, and hence negative (positive) dividend innovations shift relative consumption to investor B (investor A).

Figure 1 presents the results for the case $\gamma = 0.7$. Panels (a) and (b) show the market price of risk κ and interest rate r , respectively. As margin m increases and the constraint tightens, market price of risk increases while interest rate decreases, consistently with the intuition in Section 2.2. Furthermore, κ is an increasing function of consumption share y , and hence becomes countercyclical, consistently with the empirical evidence [e.g., Ferson and Harvey (1991)]. The intuition is that in states where investor B dominates, and hence her share y is high, unconstrained investor A possesses less wealth and requires higher κ to clear the market. In contrast to κ , the interest rate r is a decreasing function of y , and hence is procyclical, since in bad times investor B possesses more wealth, and is more willing to lend at low interest rates.

Panels (c) and (d) show the ratio of volatilities σ/σ_D and price-dividend ratio Ψ , respectively. It turns out that tighter constraints translate into higher volatility σ and lower price-dividend ratio Ψ . Moreover, volatility σ is countercyclical, and exceeds the volatility of dividends, so that $\sigma/\sigma_D > 1$, consistently with the empirical literature [e.g., Shiller (1981); Schwert (1989); Campbell and Cochrane (1999)]. The countercyclicity of κ and σ imply the countercyclicity of risk premia $\mu - r = \sigma\kappa$. Furthermore, price-dividend ratios turn out to be procyclical, and hence negatively correlated with risk premia, as in the historical data [e.g., Campbell and Shiller (1988); Schwert (1989); Campbell and Cochrane (1999)].

Next, we discuss the intuition for the effects of constraints on stock return volatility and the price-dividend ratio. We argue that these effects are driven by the relative strength of income and substitution effects. When the investment opportunities worsen, the income effect induces investors to decrease consumption and save more, while the substitution effect induces them to do the opposite, due to lower opportunity costs of current consumption. It is well known that for a CRRA investor the substitution effect dominates when $\gamma < 1$, income effect dominates when $\gamma > 1$, and the two effects cancel each other when $\gamma = 1$.

To understand the procyclicity of Ψ , we recall from Proposition 1 that $\Psi = (1 - y)\Phi_A + y\Phi_B$, where Φ_A and Φ_B are wealth-consumption ratios. Consequently, $\Psi \approx \Phi_B$ in bad times when y is close to 1. The investment opportunities for investor B worsen with tighter constraints and higher y since the interest rates decline and the investor is unable to benefit from the increase in

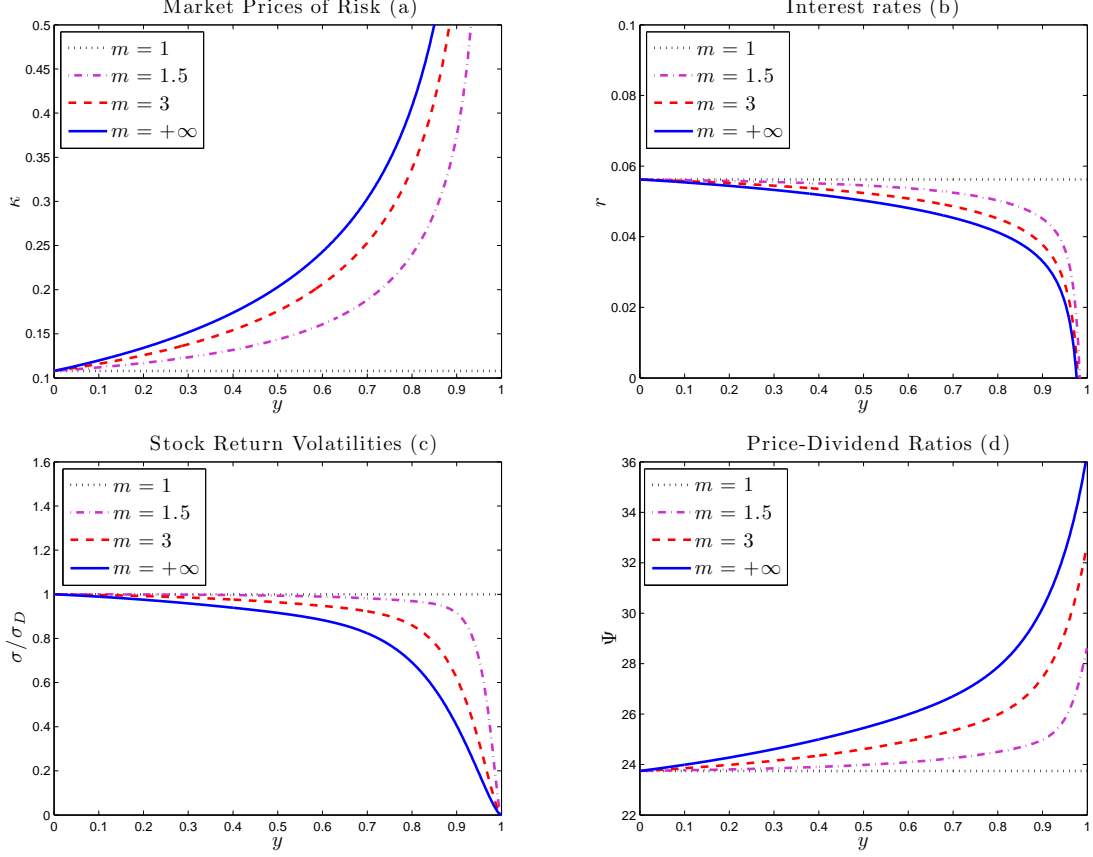


Figure 2: Equilibrium with Limited Participation, $\gamma > 1$.

Figure 2 presents the equilibrium processes for different margins m , where $m \geq 1$, when the income effect dominates. Consumption share $y = c_B^*/D$ is countercyclical, and model parameters are: $\gamma_A = 3$, $\gamma_B = 3$, $\rho = 0.01$, $\mu_D = 1.8\%$, and $\sigma_D = 3.6\%$.

market prices of risk, because of the portfolio constraint. Therefore, Φ_B , and hence Ψ , decrease via the substitution effect.³ Ψ decreases less when y is low, because the effects of constraints are weaker when the unconstrained investor dominates, which makes Ψ procyclical. The intuition for the excess volatility (i.e., $\sigma/\sigma_D > 1$) follows from the fact that stock price is given by $S = \Psi D$. Consequently, since Ψ is procyclical, the innovations to dividends change both Ψ and D in the same direction. As a result, the volatility of Ψ amplifies the volatility of dividends, and hence makes stocks more volatile than dividends. Furthermore, the concavity of Ψ illustrated on panel (d), gives rise to the countercyclicity of σ . Overall, the results on Figure 1 demonstrate that the model qualitatively replicates the dynamic patterns in equilibrium processes.

³The relation between wealth-consumption ratios and the attractiveness of investment opportunities can be conveniently illustrated by boundary conditions $\Phi_A(0)$ and $\Phi_B(0)$ in (27). These conditions give the wealth-consumption ratios of investors A and B in an economy where investor A dominates, and hence the impact of investor B is negligible. Consistently with our intuition, as m increases and constraint becomes tighter, $\Phi_B(0)$ decreases when the $\gamma < 1$, increases when $\gamma > 1$, and is unaffected when $\gamma = 1$.

Quantitatively, for the restricted participation case (i.e., $m = +\infty$) and the estimated level of consumption share $y = 0.7$ [e.g., Mankiw and Zeldes (1991); Basak and Cuoco (1998); Guvenen (2006)], the model generates 380% increase in market prices of risk κ , and 20% increase in volatilities σ , and low risk aversion r , relative to the unconstrained benchmark. Despite such a significant improvement, model implied $\kappa = 10\%$ and $\sigma = 4.2\%$ remain significantly lower than in the data because of the assumed low risk aversion $\gamma = 0.7$. Besides that, low volatility σ is a feature common to many general equilibrium asset pricing models, that has long been recognized in the literature [e.g., Heaton and Lucas (1996); among others].

Increasing the risk aversion to $\gamma = 3$ produces κ and r consistent with the data, as discussed below. The equilibrium processes for the case $\gamma = 3$ are shown on Figure 2. Qualitatively, the effects of constraints on market price of risk κ and the interest rate r remain the same as in the case of $\gamma = 0.7$. However, in contrast to the previous case, the stock return volatilities σ decrease, while the price-dividend ratios increase with tighter constraints. Moreover, the latter become countercyclical while the former become procyclical. The intuition for these results can be traced to the dominance of income effect when $\gamma > 1$, similarly to the previous case.

Under plausible parameters, $\gamma = 3$ and $y = 0.7$ [e.g., Mankiw and Zeldes (1991); Guvenen (2006)], for the restricted participation constraint (i.e., $m = +\infty$) we obtain $\kappa = 30\%$ and $r = 4.5\%$, which is close to the estimates in Campbell (2003): $\kappa = 36\%$ and $r = 2\%$. However, volatility σ decreases below the volatility of dividends σ_D , in contrast to the case of $\gamma = 0.7$. Consequently, our model requires $\gamma < 1$ to replicate the dynamic patterns in equilibrium processes, and $\gamma > 1$ to match the data quantitatively. The reason is that for CRRA preferences the risk aversion γ cannot be disentangled from the intertemporal elasticity of substitution $IES = 1/\gamma$, and hence it is not feasible to have the substitution effect (i.e., $IES > 1$) to match dynamic patterns, and the income effect (i.e., $IES < 1$) to match κ and r quantitatively in a single model.

3.2. Equilibrium with Margin and Leverage Constraints

We now turn to a setting where investors have heterogeneous risk aversions, and investor B faces margin constraint $\theta_{Bt}m \leq 1$, where $m < 1$. In this setting, we characterize the equilibrium for arbitrary margins m , which allows searching for m that achieves better fit with empirical findings. We assume, that $\gamma_B < \gamma_A$, since the heterogeneity in preferences is required to make the constraints binding. We also note, that constrained investor B 's consumption share y is now procyclical, in contrast to the case of limited participation. This is because the constrained investor is less risk averse and the constraint allows her to hold $\theta_B > 1$ in stocks. Consequently, investor B is more exposed to the stock market, and hence $\text{corr}(dy_t, dD_t) > 0$ since positive (negative) dividend innovations shift relative consumption to investor B (investor A).

In contrast with the case of limited participation, margin constraints are no longer identically binding. Whether the constraint is binding or not is determined by the amount of liquidity available for borrowing, which is supplied by the more risk averse investor A . In particular, when

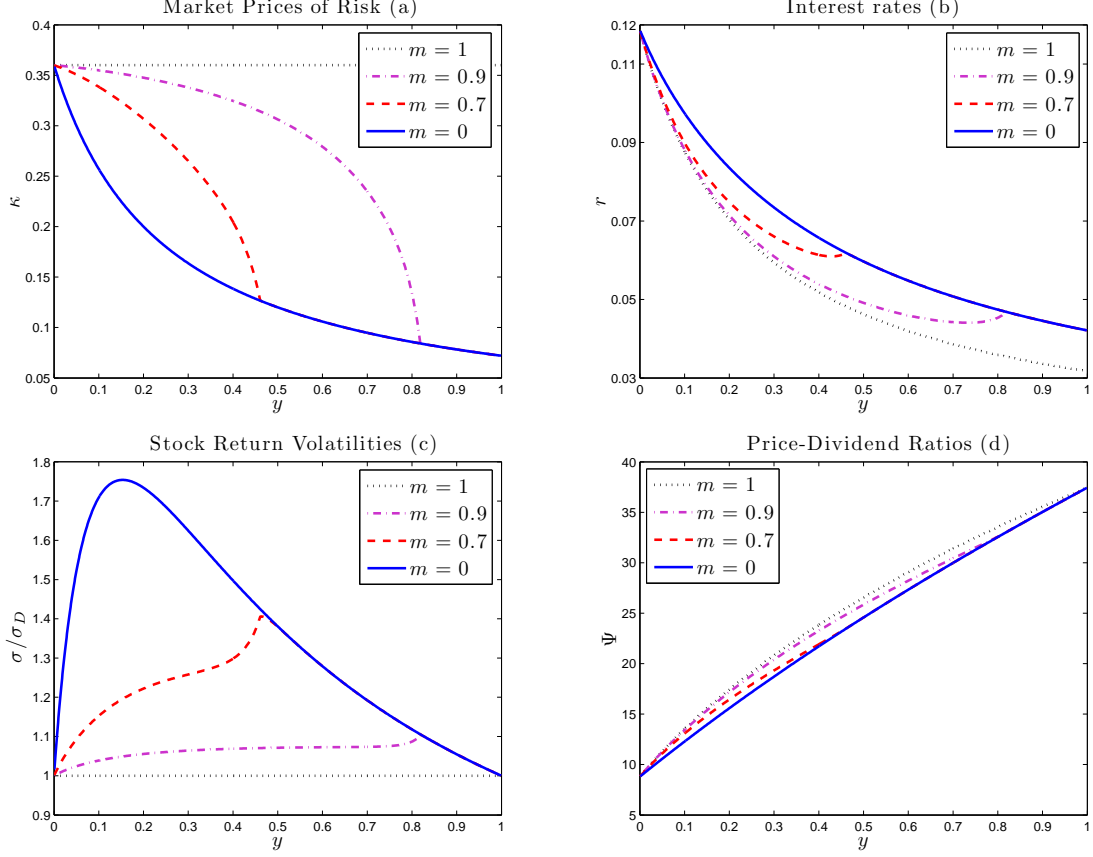


Figure 3: Equilibrium with Margin Constraints.

Figure 3 presents the equilibrium processes for different margins m , where $m \leq 1$. Consumption share $y_t = c_B^*/D$ is procyclical, and model parameters are: $\gamma_A = 10$, $\gamma_B = 2$, $\rho = 0.01$, $\mu_D = 1.8\%$, and $\sigma_D = 3.6\%$.

the economy is dominated by investor B (i.e., y is close to 1), and hence liquidity is scarce, this investor's leverage ratio declines, and the constraint does not bind. On the contrary, when the economy is dominated by investor A , investor B can easily lever up until the constraint binds.

Figure 3 shows the equilibrium processes as functions of consumption share y for different margins m for calibrated parameters. In our economy we set $\gamma_A = 10$ and $\gamma_B = 2$. While we need investor A to have high risk aversion, we note that the risk aversion of the representative agent, Γ , given in (22), remains low for a plausible range of consumption shares y . In our model, we interpret investor B as a representative constrained investor, which subsumes investors facing high margins, low margins, and leverage constraints. Accordingly, in the calibrations we set $m = 0.7$ and $m = 0.9$, which might be higher than the margins of some institutional investors.

Panels (a) and (b) of Figure 3 illustrate the impact of constraints on market prices of risk κ and interest rates r . Tighter constraints increase κ and decrease r in the region where the constraint is binding, consistently with the intuition in Section 2.2. Panel (b) also demonstrates

the non-monotonicity of interest rates when $0 < m < 1$, which has not been pointed out in the previous literature. Intuitively, interest rates go down around $y \approx 0$ because investor B 's demand for borrowing decreases due to the binding portfolio constraint. However, as noted above, the constraint does not bind for sufficiently high share y , and hence the interest rate reverts to the unconstrained case, giving rise a non-monotone pattern.

Panels (c) and (d) show stock return and dividend volatility ratios σ/σ_D , and price-dividend ratios Ψ , respectively. Previous literature has primarily studied volatilities σ only in the special cases of an unconstrained economy (i.e., $m = 0$) and an economy with the leverage constraint (i.e., $m = 1$). Consistently with the literature, in $m = 0$ case σ is countercyclical over large interval $[0.2, 1]$, and exceeds the volatility of dividends [e.g., Longstaff and Wang (2008); Bhamra and Uppal (2009, 2010)]. In $m = 1$ case, previously studied in a model with logarithmic investor B [e.g., Kogan, Makarov, and Uppal (2007)], the volatility σ equals the volatility of dividend σ_D . In contrast to the case with $m < 1$, the latter special case of $m = 1$ does not generate time-variation in volatilities, and hence leaves unanswered whether constraints decrease volatility in a way that preserves or destroys the countercyclicality observed in the unconstrained case.

Our general case with $m \in [0, 1]$ provides new insights relative to the special cases considered in the literature. In particular, Panel (c) demonstrates that constraints decrease volatility σ , reduce the countercyclicality region, and volatility spikes around the value of consumption share y at which constraint becomes binding. The decrease in σ is consistent with the empirical evidence in Hardouvelis and Theodossiou (2002) who study 22 episodes of changes in margin requirements by the Federal Reserve between 1934 and 1974 and demonstrate that tighter margins lead to lower stock market volatilities. Additionally, Panel (c) illustrates the sensitivity of σ with respect to changes in margin m and describes the tradeoff between achieving higher σ on one hand, and higher κ and low r on the other.

To provide the intuition for the effect of constraints on volatilities, from expressions (A3) for portfolio weights θ_A^* and θ_B^* in Appendix A, and expression (17) for the market price of risk κ , after simple algebra we obtain the following expression for consumption share volatility σ_y :

$$\sigma_{yt} = \frac{(\theta_{At}^* - \theta_{Bt}^*)\sigma_t}{\frac{1}{1 - y_t} - y_t \frac{\Phi_A'(y_t)}{\Phi_A(y_t)} + y_t \frac{\Phi_B'(y_t)}{\Phi_B(y_t)}}. \quad (30)$$

Intuitively, equation (30) demonstrates that the fluctuations in consumption share y arise due to the difference in portfolio strategies, θ_A^* and θ_B^* , which is driven by the ability of investor B to borrow from investor A . Tighter margin constraints limit the ability to borrow, make portfolio weights more homogeneous across investors, and hence decrease the magnitude of volatility σ_y . Expression (21) for volatility σ demonstrates that smaller σ_y translates into smaller difference between σ and σ_D . In the case of leverage constraint, $\sigma_y = 0$ since portfolio weights become homogeneous, $\theta_A^* = \theta_B^* = 1$, and hence $\sigma = \sigma_D$, as discussed above.

For $m = 0.9$ and $y = 0.7$ the model generates $\kappa = 24\%$, and $r = 4\%$, while the estimates in Campbell (2003) are $\kappa = 36\%$ and $r = 2\%$. We note, that the aggregate risk aversion required

to match κ and r is equal to $\Gamma(0.7) = 2.6$, and is reasonably low despite the fact that we have to assume that investor A has risk aversion $\gamma_A = 10$. The volatility σ remains stochastic in this calibration and is 7% higher than the volatility of dividends, in contrast to the case of leverage constraint. The level of volatility remains low relative to the data, which is a common feature of many equilibrium models, as pointed out above.

4. General Equilibrium with Two Trees and Constraints

In this Section we provide a characterization of equilibrium in a two-trees heterogeneous agents economy. Then, we apply the results to study the formation of stochastic stock return correlations in the presence of portfolio constraints. Specifically, we identify and disentangle different sources of the time-variation in asset return correlations, which is well documented in the empirical literature (e.g., Bekaert and Harvey, 1995; Moskowitz, 2003; Driessen, Maenhout, and Vilkov, 2009), and has long been argued to be an important source of risk, which increases the uncertainty about asset payoffs and lowers the diversification benefits. In particular, Driessen, Maenhout, and Vilkov (2009) document the economic significance of correlation risk premia, whereas Buraschi, Porchia and Trojani (2010) demonstrate that correlation risk generates large hedging demands.

The existing literature has explored the origin of correlation risk within multi-stock unconstrained Lucas economy [e.g., Ribeiro and Veronesi (2002); Menzly, Santos, and Veronesi (2004); Santos and Veronesi (2006); Cochrane, Longstaff, and Santa-Clara (2008); Buraschi, Trojani, and Vedolin (2010); Ehling and Heyerdahl-Larsen (2010); Martin (2011); among others]. In particular, Ribeiro and Veronesi (2002) study the impact of the uncertainty about the economy on the comovement of international market returns. Cochrane, Longstaff, and Santa-Clara (2008) and Martin (2011) demonstrate how the market clearing effects make stocks more correlated than fundamentals, thus explaining the excess correlation documented in Shiller (1989). Ehling and Heyerdahl-Larsen (2010) show the countercyclicality of correlations in a heterogeneous-agent economy by employing Monte Carlo simulations. Despite much work on the time-variation of correlations, the impact of portfolio constraints on them remains relatively unexplored. Pavlova and Rigobon (2008) study the comovement of stocks in a three-country model with multiple goods and portfolio constraints, logarithmic preferences, and heterogeneous home biases for domestic goods. In contrast to Pavlova and Rigobon (2008), our model does not require the heterogeneous home bias and logarithmic preferences, and hence can be used to study the correlations even in a one-country model.

In this Section, we focus on the impact of margin and leverage constraints on correlations and stock return volatilities in a setting where investors have heterogeneous CRRA utilities. In our model the less risk averse investor levers up by borrowing from the more risk averse one. The time-variation in the amount of liquidity available for borrowing causes the less risk averse investor to scale the portfolio weights up or down, depending on economic conditions, and hence generates an additional source of comovement in stock returns. This new source of correlation, which we label as the *leverage effect*, is absent in models with homogeneous investors [e.g., Ribeiro

and Veronesi (2002); Longstaff, Cochrane, and Santa-Clara (2008); Martin (2011)].

Given the prevalence of portfolio constraints in financial markets, studying their impact on equilibrium processes is important in its own right. However, our model yields an additional valuable insight by isolating and quantifying the impact of the leverage effect on asset prices, discussed above. In particular, by imposing the leverage constraint we completely eliminate the leverage effect, and hence separate it from the common *discount rate effect*, explored in the previous literature [e.g., Cochrane, Longstaff, and Santa-Clara (2008); Martin (2011)]. We find that the leverage effect accounts for a significant fraction of the total correlation. Furthermore, we demonstrate that margin and leverage constraints decrease correlations while preserving their empirically documented countercyclicality [e.g., Ribeiro and Veronesi (2002)]. Intuitively, the constraints reduce borrowing and make asset trading more homogeneous across investors, which reduces the correlations consistently with the role of the leverage channel, discussed above.

We derive closed-form equilibrium processes and price-dividend ratios in the unconstrained economy and the economy with leverage constraints when investors have heterogeneous general CRRA preferences. These closed-form expressions generalize those in Cochrane, Longstaff, and Santa-Clara (2008) and Martin (2011), derived for homogeneous agents. In contrast to single tree models with leverage constraint [e.g., Kogan, Makarov and Uppal (2007); Chabakauri (2009)], where stock return volatilities remain constant, our two-trees model generates time-variation in all equilibrium parameters, including stock return volatilities and correlations. The case of general margins we solve numerically, using the approach of Section 2. Finally, we derive consumption CAPM and extend Black's (1972) static mean-variance CAPM with leverage constraint to the dynamic economy with consumption. For the leverage constraint, we characterize the adjustment to Breeden's consumption CAPM in closed form.

4.1. Economic Setup

We consider an infinite horizon economy with two stocks and one consumption good, which is generated by two Lucas trees. The economy is populated by two CRRA investors, $i = A$ and $i = B$, with risk aversions γ_A and γ_B , where $\gamma_A \geq \gamma_B$. The uncertainty is generated by a two-dimensional Brownian motion $w = (w_1, w_2)^\top$. The Lucas trees produce streams of dividends D_{jt} that follow GBMs:

$$dD_{jt} = D_{jt}[\mu_{D_j}dt + \sigma_{D_j}dw_{jt}], \quad j = 1, 2, \quad (31)$$

where Brownian motions w_1 and w_2 are uncorrelated, and μ_{D_j} and σ_{D_j} are constants. The aggregate dividend $D = D_1 + D_2$ then, by Itô's Lemma, follows a process:

$$dD_t = D_t[\mu_{D_t}dt + \sigma_{D_t}^\top dw_t], \quad (32)$$

where $\mu_D = x\mu_{D_1} + (1-x)\mu_{D_2}$, $\sigma_D = (x\sigma_{D_1}, (1-x)\sigma_{D_2})^\top$, and $x = D_1/D$ is the share of the first tree in the aggregate dividend.

The investors continuously trade in three securities: a riskless bond in zero net supply with instantaneous interest rate r , and two stocks, each in net supply of one unit, which are claims to the output generated by Lucas trees (31). We consider Markovian equilibria in which bond prices, B , and stock prices, $S = (S_1, S_2)^\top$, follow dynamics:

$$dB_t = B_t r_t dt, \quad (33)$$

$$dS_{jt} + D_{jt} dt = S_{jt} [\mu_{jt} dt + \sigma_{jt}^\top dw_t], \quad j = 1, 2, \quad (34)$$

where $\sigma_j = (\sigma_{j1}, \sigma_{j2})^\top$, and we let $\mu = (\mu_1, \mu_2)^\top$ and $\sigma = (\sigma_1, \sigma_2)^\top$ denote the vector of mean returns and the volatility matrix of stock returns, respectively.

The investors maximize expected utility (4) subject to the self-financing budget constraint

$$dW_{it} = \left[W_{it} (r_t + \theta_{it}^\top (\mu_t - r_t)) - c_{it} \right] dt + W_{it} \theta_{it}^\top \sigma_t dw_t, \quad i = A, B, \quad (35)$$

where $\theta_i = (\theta_{i1}, \theta_{i2})^\top$ denotes the vector of portfolio weights, and subject to portfolio constraints. The fractions of wealth invested in bonds are given by $\alpha_i = 1 - \theta_{i1} - \theta_{i2}$. Investor A is unconstrained, while investor B faces the following margin constraint [e.g., Brunnermeier and Pedersen (2009), Gromb and Vayanos (2009), Gârleanu and Pedersen (2011); among others]:

$$\theta_{B1} m_1 + \theta_{B2} m_2 \leq 1, \quad (36)$$

where $0 \leq m_1 \leq 1$, and $0 \leq m_2 \leq 1$, and we let $m = (m_1, m_2)^\top$ denote the vector of margins.

Similarly to Gârleanu and Pedersen (2011), we note that inequality (36) can be rewritten as $\theta_{B1} + \theta_{B2} \leq 1 + \theta_{B1}(1 - m_1) + \theta_{B2}(1 - m_2)$, where $\theta_{B1}(1 - m_1) + \theta_{B2}(1 - m_2)$ is the fraction of wealth that can be borrowed using stocks as collateral. Consequently, margin m_j is interpreted as the proportion of asset j 's value against which the investor cannot borrow. The leverage constraint is a special case of constraint (36) when $m = (1, 1)^\top$. We do not consider the case of limited participation (i.e., $m_i \geq 1$), which is more difficult to analyze since the equilibrium processes have a singularity at the boundary $y = 1$, when the constrained investor dominates. Therefore, we leave this case for future research. The equilibrium is defined analogously to Section 1.3.

Definition 2. *An equilibrium is a set of processes $\{r_t, \mu_{jt}, \sigma_{jt}\}_{j \in \{1, 2\}}$ and of consumption and investment policies $\{c_{it}^*, \alpha_{it}^*, \theta_{it}^*\}_{i \in \{A, B\}}$ that maximize expected utility (4) for each investor, given processes $\{r_t, \mu_{jt}, \sigma_{jt}\}_{j \in \{1, 2\}}$, and market clearing conditions (8) are satisfied.*

The equilibrium processes are derived as functions of the first tree's and investor B 's consumption shares, $x = D_1/D$ and $y = c_B^*/D$, respectively, which follow the processes:

$$dx_t = x_t [\mu_{xt} + \sigma_{xt}^\top dw_t], \quad dy_t = -y_t [\mu_{yt} + \sigma_{yt}^\top dw_t], \quad (37)$$

where $\mu_x = (1 - x)(\mu_{D1} - \mu_{D2}) - \sigma_{D1}^2 x(1 - x) + \sigma_{D2}^2 (1 - x)^2$, $\sigma_x = ((1 - x)\sigma_{D1}, -(1 - x)\sigma_{D2})^\top$, while μ_y and $\sigma_y = (\sigma_{y1}, \sigma_{y2})^\top$ are conjectured to be Markovian and are determined in equilibrium.

4.2. Characterization of Equilibrium and Consumption CAPM

We now apply the methodology of Section 2 to study the impact of portfolio constraints on stock return correlations and volatilities in a two-trees economy. Similarly to Section 2, we characterize the equilibrium in three steps. First, from the consumption clearing conditions we recover market prices of risk $\kappa = \sigma^{-1}(\mu - r)$, interest rates r , mean growth μ_y and volatility σ_y of consumption share y in terms of adjustment parameter ν^* , and the volatility matrix σ . Second, we characterize the adjustment parameter ν^* in terms of price-dividend and wealth-consumption ratios of investors from the complementary slackness conditions. Finally, we derive PDEs for price-dividend and wealth-consumption ratios. Proposition 3 below summarizes our results.

Proposition 3. *The equilibrium market price of risk $\kappa = \sigma^{-1}(\mu - r)$, interest rate r , volatility and mean growth of consumption share y , μ_y and σ_y , and stock return volatilities $\sigma_j = (\sigma_{j1}, \sigma_{j2})^\top$ are given by:*

$$\kappa_t = \Gamma_t \sigma_{Dt} - \frac{\Gamma_t y_t}{\gamma_B} \nu_t^* \sigma_t^{-1} m, \quad (38)$$

$$r_t = \rho + \Gamma_t \mu_{Dt} - \frac{\Gamma_t \Pi_t}{2} \sigma_{Dt}^\top \sigma_{Dt} + \frac{\Gamma_t y_t \nu_t^*}{\gamma_B} + (\nu_t^* \sigma_t^{-1} m)^\top (a_1(y_t) \sigma_{Dt} + a_2(y_t) \nu_t^* \sigma_t^{-1} m), \quad (39)$$

$$\sigma_{yt} = \frac{\Gamma_t (1 - y_t)}{\gamma_A \gamma_B} \left((\gamma_B - \gamma_A) \sigma_{Dt} - \nu_t^* \sigma_t^{-1} m \right), \quad (40)$$

$$\mu_{yt} = \mu_{Dt} - \sigma_{Dt}^\top \sigma_{yt} - \frac{1 + \gamma_B}{2} (\sigma_{Dt} - \sigma_{yt})^\top (\sigma_{Dt} - \sigma_{yt}) - \frac{r_t - \nu_t^* - \rho}{\gamma_B}, \quad (41)$$

$$\sigma_{jt} = e_j \sigma_{Dj} + \sigma_{xt} \frac{\partial \Psi_{jt}}{\partial x_t} \frac{x_t}{\Psi_{jt}} - \sigma_{yt} \frac{\partial \Psi_{jt}}{\partial y_t} \frac{y_t}{\Psi_{jt}}, \quad j = 1, 2, \quad (42)$$

where Γ_t and Π_t are the representative agent's risk aversion and prudence parameters given by (22), $a_1(y)$ and $a_2(y)$ are functions given by equations (A2) in the Appendix, $e_1 = (1, 0)^\top$, $e_2 = (0, 1)^\top$, ν_t^* is the adjustment parameter given by equation (A31) in the Appendix as a function of price-dividend ratios Ψ_1 , Ψ_2 , wealth-consumption ratio Φ_B , and their derivatives.

The price-dividend ratios $\Psi_j(x, y)$ and wealth-consumption ratio $\Phi_B(x, y)$ satisfy PDEs:

$$\begin{aligned} \mathcal{D}\Psi_j + x(\mu_x - (\kappa - \sigma_{Dj} e_j)^\top \sigma_x) \frac{\partial \Psi_j}{\partial x} - y(\mu_y - (\kappa - \sigma_{Dj} e_j)^\top \sigma_y) \frac{\partial \Psi_j}{\partial y} \\ + (\mu_{Dj} - r - \sigma_{Dj} e_j^\top \kappa) \Psi_j + e_j^\top (x, 1 - x) = 0, \quad j = 1, 2, \end{aligned} \quad (43)$$

$$\begin{aligned} \mathcal{D}\Phi_B + x \left(\mu_x + \frac{1 - \gamma_B}{\gamma_B} (\kappa + \nu^* \sigma^{-1} m)^\top \sigma_x \right) \frac{\partial \Phi_B}{\partial x} - y \left(\mu_y + \frac{1 - \gamma_B}{\gamma_B} (\kappa + \nu^* \sigma^{-1} m)^\top \sigma_y \right) \frac{\partial \Phi_B}{\partial y} \\ + \left(\frac{1 - \gamma_B}{2\gamma_B^2} (\kappa + \nu^* \sigma^{-1} m)^\top (\kappa + \nu^* \sigma^{-1} m) + \frac{1 - \gamma_B}{\gamma_B} (r - \nu^*) - \frac{\rho}{\gamma_B} \right) \Phi_B + 1 = 0, \end{aligned} \quad (44)$$

where \mathcal{D} is a zero-drift Dynkin's operator.⁴ The wealth-consumption ratio $\Phi_A(x, y)$ is given by

⁴The zero-drift Dynkin operator \mathcal{D} is defined as follows:

$$\mathcal{D}F = \frac{1}{2} \left(x^2 \sigma_x^\top \sigma_x \frac{\partial^2 F}{\partial x^2} - 2xy \sigma_x^\top \sigma_y \frac{\partial^2 F}{\partial x \partial y} + y^2 \sigma_y^\top \sigma_y \frac{\partial^2 F}{\partial y^2} \right).$$

$$\Phi_A = (x\Psi_1 + (1-x)\Psi_2 - y\Phi_B)/(1-y).$$

The equilibrium processes in (38)–(42) capture new asset pricing effect stemming from the heterogeneity in preferences and portfolio constraints, which have not been explored in the literature. In particular, these new effects on stock return volatilities and correlations enter via the third term in the expressions for stock return volatilities in (42). Furthermore, the closed-form solutions are available in two important special cases, when investor B is unconstrained (i.e., $m = (0, 0)^\top$) or faces the leverage constraint (i.e., $m = (1, 1)^\top$). To the best of our knowledge, these solutions, which are reported in Proposition 4 below, are new to the literature.

Proposition 4.

- (i) *In the unconstrained economy equilibrium κ , r , μ_y , σ_y , σ_j are given in closed form by expressions (38)–(42) in which $\nu^* = 0$, and price-dividend ratios $\Psi_j(x, y)$ are given by:*

$$\Psi_1(x, y) = \Psi(x, y; \{\mu_{D_1}, \sigma_{D_1}\}, \{\mu_{D_2}, \sigma_{D_2}\}), \quad (45)$$

$$\Psi_2(x, y) = \Psi(1-x, y; \{\mu_{D_2}, \sigma_{D_2}\}, \{\mu_{D_1}, \sigma_{D_1}\}), \quad (46)$$

where

$$\begin{aligned} \Psi(x, y; \{\mu_{D_1}, \sigma_{D_1}\}, \{\mu_{D_2}, \sigma_{D_2}\}) = \\ \int_0^1 \int_0^1 \left(\frac{s}{x} \frac{y}{z} \right)^{\gamma_B} \frac{\gamma_B(1-z) + \gamma_A z}{s(1-s)z(1-z)} \frac{e^{q^\top \Sigma^{-1} u(s, z; x, y)} K_0(p \sqrt{u(s, z; x, y)^\top \Sigma^{-1} u(s, z; x, y)})}{\pi \gamma_A \sqrt{\det(\Sigma)}} ds dz, \end{aligned} \quad (47)$$

$$u(s, z; x, y) = \left(\ln\left(\frac{1-z}{1-y}\right) - \frac{\gamma_B}{\gamma_A} \ln\left(\frac{z}{y}\right) + \frac{\gamma_B - \gamma_A}{\gamma_A} \ln\left(\frac{s}{x}\right), \ln\left(\frac{1-s}{1-x} \frac{x}{s}\right) \right)^\top, \quad (48)$$

$K_0(\cdot)$ is a McDonald's function,⁵ while parameters $p \in \mathbb{R}$, $q \in \mathbb{R}^2$, and $\Sigma \in \mathbb{R}^{2 \times 2}$ are given by equations (A46)–(A47) in the Appendix as functions of dividend mean growths and volatilities, μ_{D_1} , μ_{D_2} , σ_{D_1} , and σ_{D_2} .

- (ii) *In the case of leverage constraint, with $m = (1, 1)^\top$, the equilibrium κ , r , μ_y and σ_y are given in closed form by expressions (38)–(41), where ν^* and $\nu^* \sigma^{-1} m$ are given by:*

$$\nu_t^* = \frac{\gamma_B - \gamma_A}{1/\sigma_{D_1}^2 + 1/\sigma_{D_2}^2}, \quad \nu_t^* \sigma_t^{-1} m = \frac{\gamma_B - \gamma_A}{1/\sigma_{D_1}^2 + 1/\sigma_{D_2}^2} \left(\frac{1}{\sigma_{D_1}}, \frac{1}{\sigma_{D_2}} \right)^\top. \quad (49)$$

Price-dividend ratios $\Psi_j(x, y)$ are given by equation (A58) in the Appendix in closed form.

⁵This function is available in Matlab and is given by (integral 8.432.1 in Gradshteyn and Ryzhik (2007)):

$$K_0(z) = \int_0^\infty e^{-z \cosh(s)} ds.$$

Proposition 4 provides tractable closed-form solutions, which allow to study the economic effects of investor heterogeneity, which are absent in homogeneous-investor two-trees economies studied in Cochrane, Longstaff, and Santa-Clara (2008) and Martin (2011). They are also useful for cross-checking with numerical solutions. In the case of leverage constraint Proposition 4 provides closed-form adjustment parameters (49), which, as we argue below, provide further insights on the effects of portfolio constraints. While the leverage constraint is less general than the margin constraint (36), Black (1972) and Heaton and Lucas (1996) advocate its economic importance based on the evidence that many investors face severe borrowing restrictions. The price-dividend ratios then can be obtained by solving two linear PDEs (43)–(44) or using closed-form solutions in the Appendix. The tractability of the leverage constraint is due to the fact that it restricts both investors to invest all their wealth in stocks, and hence the share of their aggregate stock investment is equal to 1, in contrast to models with margin constraints.

Propositions 3 and 4 allow us to obtain a tractable consumption CAPM with portfolio constraints in terms of empirically observable processes. In multiple-stocks economies consumption CAPMs with constraints have been primarily studied for the case of logarithmic constrained investors [e.g., Shapiro (2002); Pavlova and Rigobon (2008); Gârleanu and Pedersen (2011)]. Cuoco (1997) provides a constrained consumption CAPM for general preferences, but in terms of unobservable adjustments ν^* . First, by multiplying the expression for market price of risk (38) by σ we obtain the following C-CAPM in terms of adjustment ν^* :

$$\mu_t - r_t = \Gamma_t \sigma_t \sigma_D - \frac{\Gamma_t y_t}{\gamma_B} \nu_t^* m, \quad (50)$$

where $\mu - r = (\mu_1 - r, \mu_2 - r)^\top$, and $m = (m_1, m_2)^\top$.

Then, we note that multiplier $(\Gamma y / \gamma_B) \nu^*$ is the same for all stock risk premia in (50), and hence it can be obtained by looking at the cross-section of stocks. In particular, we obtain ν^* in terms of the risk premium of the market portfolio, $\mu_{Mt} - r_t$, and substituting ν^* into (50) obtain the consumption CAPM. For the case of the leverage constraint, ν^* is available in closed form in Proposition 4 and gives another consumption CAPM, which conveniently illustrates the interaction between constraints and investor heterogeneity. Corollary 1 reports the results.

Corollary 1 (Capital Asset Pricing Model).

(i) *In the economy with a margin constraint, stock risk premia $\mu - r$ are given by:*⁶

$$\mu_t - r_t = \left(I - \frac{m \theta_{Mt}^\top}{\theta_{Mt}^\top m} \right) \beta_{ct} + \frac{m}{\theta_{Mt}^\top m} (\mu_{Mt} - r_t), \quad (51)$$

where I is an identity matrix, $\theta_M = S / (S_1 + S_2)$ is the vector of market portfolio weights, μ_M is the market portfolio's mean return, and β_{ct} are consumption betas, given by:

$$\beta_{ct} = \Gamma_t \left(\frac{\text{cov}(dS_{1t}/S_{1t}, dD_t)}{dt}, \frac{\text{cov}(dS_{2t}/S_{2t}, dD_t)}{dt} \right)^\top. \quad (52)$$

⁶We note, that consumption CAPM (51) also holds for more general preferences, multiple assets, and time-varying margins.

(ii) *In the economy with a leverage constraint, stock risk premia $\mu - r$ are given by:*

$$\mu_t - r_t = \beta_{ct} - \frac{\Gamma_t(1 - y_t)}{\gamma_B} \frac{\gamma_B - \gamma_A}{1/\sigma_{D_1}^2 + 1/\sigma_{D_2}^2}. \quad (53)$$

Conveniently, consumption CAPM (51) is in terms of consumption beta β_c and empirically observable processes, such as the risk premia and weights of the market portfolio. Furthermore, consumption CAPM (53) extends Black's (1972) static mean-variance CAPM with leverage constraint to the dynamic economy, and characterizes the deviation from Breeden's (1979) consumption CAPM in closed form. We also note, that both terms in (53) are approximately of the same magnitude as the volatilities of dividends σ_{D_j} . Consequently, one implication of consumption CAPM (53) is that the portfolio constraint has the first-order effect on the risk premia. The second term in (53) is non-positive (since $\gamma_A \leq \gamma_B$), and hence the risk premia are higher than in the unconstrained case, analogously to the intuition in Section 2. Furthermore, the structure of the new term in (53) reveals that the effects of portfolio constraints and heterogeneity in preferences, quantified by the difference $\gamma_B - \gamma_A$, reinforce each other, and hence omitting one of these factors leads to underestimating the impact of the other. Interestingly, while Breeden's CAPM requires high risk aversion to generate sizeable risk premia, our model needs one of the investors to have a relatively small risk aversion to make the constraint binding.

4.3. Analysis of Equilibrium with Two Tress

Next, we apply the methodology discussed in Section 2 to solve for the equilibrium with margin constraints. Since the effects of constraints on κ , r , and σ have already been looked at in Section 2, here we focus on conditional stock return correlations $\text{corr}_t(dS_{1t}/S_{1t}, dS_{2t}/S_{2t})$. The numerical method remains essentially the same as in the one-tree economy, and therefore we skip the details. We just note that, as in Section 2, given the adjustments ν^* , the PDEs (43)–(44) are linear and can be solved by finite difference methods for two-dimensional equations [e.g., Duffy (2006)]. In particular, we first solve the PDEs for a conjectured $\nu^* = 0$, then calculate an updated adjustment ν^* using the expression (A31) in the Appendix, and iterate until convergence.

Three panels of Figure (4) present stock return correlations as functions of first tree's share x and consumption share y when investors are unconstrained (Panel (a)), investor B faces margin constraint with $m = (0.7, 0.7)^\top$ (Panel (b)), and leverage constraint with $m = (1, 1)^\top$ (Panel (c)), for calibrated parameters. In our calibration we set $\mu_{D_1} = \mu_{D_2} = 1.8\%$ and $\sigma_{D_1} = \sigma_{D_2} = 3.6\%$, and set the risk aversions to $\gamma_A = 10$ and $\gamma_B = 2$, consistently with Section 3.

Panel (a) demonstrates that the heterogeneity in preferences significantly increases correlations relative to one-investor economies, which correspond to limiting cases of $y = 0$ or $y = 1$. The ability of less risk averse investor B to lever up by borrowing from investor A increases investor B 's exposure to stock market fluctuations relative to a one-investor economy. In particular, investor B increases or decreases investment in stocks depending on the availability of liquidity

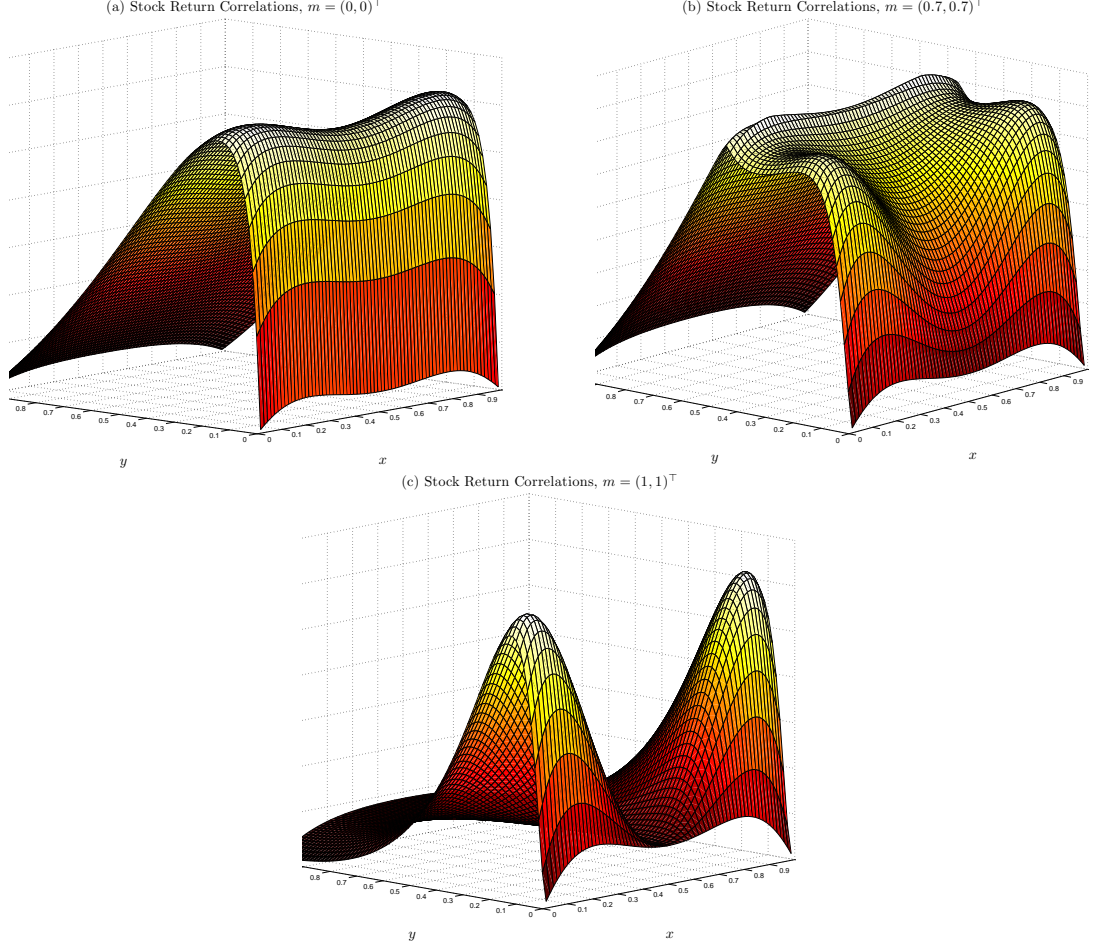


Figure 4: Conditional Stock Return Correlations.

Figure 4 presents the equilibrium stock return correlations when the investor is unconstrained (Panel (a)), faces margin constraint with $m = (0.7, 0.7)^\top$ (Panel (b)), and leverage constraint with $m = (1, 1)^\top$ (Panel (c)). Consumption share $y = c_B^*/D$ is countercyclical, and $x = D_1/D$. The parameters are: $\mu_{D_1} = \mu_{D_2} = 1.8\%$, $\sigma_{D_1} = \sigma_{D_2} = 3.6\%$, $\rho = 0.01$, $\gamma_A = 10$, and $\gamma_B = 2$.

for borrowing. Moreover, the portfolio positions in both stocks tend to be adjusted in the same direction in order to keep the portfolio diversified, which translates into higher correlations.

The increase in consumption share y of investor B has two opposing effects on the leverage effect, which gives rise to a hump-shaped pattern on Panel (a). On one hand, this increase makes the leverage effect more conspicuous by increasing the impact of the leveraged investor B on the economy, which pushes the correlations up. On the other hand, it reduces the consumption share $1 - y$ of the lender A , and hence the availability of the liquidity for borrowing, which pushes the correlations down. We further note that in the unconstrained case consumption share y of investor B is procyclical in the sense that $\text{corr}_t(dy_t, dD_t) = 1$. The procyclicality can be formally demonstrated by noting that $\text{corr}_t(dy_t, dD_t) = -\sigma_{yt}^\top \sigma_{Dt} / (|\sigma_{yt}| |\sigma_{Dt}|)$, substituting volatility σ_{yt}

from (40), and setting $\nu^* = 0$. The intuition is the same as in Section 3.2.

Panel (a) demonstrates that, consistently with the empirical evidence [e.g., Ribeiro and Veronesi (2002)], stock return correlation is decreasing in y , and hence countercyclical, over a large interval. Panels (b) and (c) demonstrate that tighter constraints decrease correlations. Higher margins m_1 and m_2 reduce the ability of the less risk averse investor B to borrow, which decreases correlations, consistently with the intuition on the role of leverage.

Imposing the leverage constraint helps isolate and quantify the impact of leverage on correlations. The comparison of Panels (a) and (c), i.e., with and without borrowing, reveals that the leverage effect accounts for a significant fraction of the magnitude of correlations. Panel (c) then demonstrates the impact of remaining effects on correlation, e.g. common discount factor effect, disentangled from the leverage effect. Panel (c) demonstrates that tight constraints give rise to a saddle-type pattern in correlations. In this case, the correlations appear to be higher when the economy is dominated by one tree, i.e. when share x is close to 0 or 1. Therefore, under portfolio constraints the correlations crucially depend on the relative size x of a tree in the economy.

The two humps shown on Panel (c) are due to a combination of two effects that reinforce each other. The first one is the common discount factor effect, which generates small humps even in the unconstrained case on Panel (a). When x is close to 0 or 1, the state price density ξ is mainly driven by one tree. Consequently, irrespective of whether the dividend innovations dD_1 and dD_2 move in the same or opposite directions, there will be extra comovement in stocks. When x is close to 0.5, ξ is less volatile since it is equally affected by both trees, and innovations dD_1 and dD_2 may partially offset each other, which decreases the discount-driven comovement.

The second effect is specific to constraints, and plays no role in the unconstrained case. We argue, that the leverage constraint homogenizes investors' asset holdings more strongly around $x = 0.5$ than around $x = 0$ and $x = 1$, which leads to stronger decline in correlations around $x = 0.5$. When $x = 0.5$ both trees and stocks are "symmetric" since the trees have the same drift and volatility parameters in the calibration. Therefore, in this case both stocks are equally attractive to investors, and hence they invest equal fractions of wealth in stocks, i.e., $\theta_{i1} = \theta_{i2}$. In the unconstrained case their asset holdings are still heterogeneous, since investor B can lever up. However, with leverage constraint, at $x = 0.5$ the shares of wealth invested in stocks have to be the same: $\theta_{11} = \theta_{12} = 0.5$, and $\theta_{21} = \theta_{22} = 0.5$. Therefore, the constraint perfectly homogenizes asset holdings at $x = 0.5$. This is no longer the case when x is around 0 or 1, where stocks have different riskiness and hence the portfolios remain different due to heterogeneity in risk aversions.

The intuition for the homogenization effect is similar to that in Section 3.2, where it was captured by equation (30). This intuition can be further verified by substituting $\nu^* \sigma^{-1} m$ from (49) into the expression for σ_y in (40). Then, for the case $\sigma_{D_1} = \sigma_{D_2}$ we obtain the following expression for σ_y , which quantifies the heterogeneity:

$$\sigma_{yt} = \frac{\Gamma_t(1 - y_t)(\gamma_B - \gamma_A)\sigma_{D_1}}{\gamma_A \gamma_B} (x - 0.5, 0.5 - x)^\top. \quad (54)$$

Consistently with the intuition above, σ_y in equation (54) vanishes when $x = 0.5$. Hence, the

third term in volatilities σ_j , given by equation (42), which captures the impact of heterogeneity on volatilities, also vanishes, leading to the reduction in correlations.

5. Conclusion

We study the effects of portfolio constraints in one-tree and two-trees Lucas economies with heterogeneous CRRA investors. The previous literature widely employed the assumption of myopic logarithmic investors, which makes the characterization of equilibrium incomplete by eliminating the effects of constraints on wealth-consumption ratios of investors. Our model does not rely on investor myopia, and hence provides a comprehensive analysis of the effects of constraints. In particular, we demonstrate under what conditions constraints increase or decrease stock return volatilities and price-dividend ratios, make them countercyclical or procyclical, and help quantitatively match market prices of risk and interest rates.

In one-tree economy we demonstrate that the limited participation constraint can generate equilibrium processes with dynamic characteristics consistent with empirical findings, and that margin constraints decrease stock return volatilities. In the economy with two trees we derive a generalization of Breeden's (1979) consumption CAPM, which in the special case of leverage constraints also generalizes Black's (1972) static mean-variance CAPM. We also derive and study conditional stock return correlations in unconstrained and constrained economies. We demonstrate how the availability of liquidity in heterogeneous investor economy generates excess correlations, relative to homogeneous investor economies. We further demonstrate that tighter margin constraints decrease correlations by restricting investors' ability to lever up.

Our model can be extended in various directions. In an ongoing work we study the impact of constraints in an economy with heterogeneous beliefs, where investors disagree on the aggregate consumption growth in the economy. This setting allows to explore the impact of short-sale constraints. Interesting direction for future research is the equilibrium with constraints in economies where investors are guided by Epstein-Zin recursive preferences, which help disentangle the effects of risk aversion and the intertemporal elasticity of substitution.

Appendix A: Proofs

Proof of Proposition 1. To derive market price of risk κ and interest rate r we substitute optimal consumptions (14) into consumption clearing condition in (8). Then, using dynamics (13) for state price densities in real and fictitious unconstrained economies, we apply Itô's Lemma to both sides of consumption clearing condition, and by matching dt and dw terms obtain:

$$\begin{aligned} \frac{r_t - \rho}{\Gamma_t} - \frac{y_t \nu_t^*}{\gamma_B} + \frac{1}{2} \left(\frac{1 + \gamma_A}{\gamma_A^2} (1 - y_t) \kappa_t^2 + \frac{1 + \gamma_B}{\gamma_B^2} y_t \left(\kappa_t + \frac{\nu_t^* m}{\sigma_t} \right)^2 \right) &= \mu_D, \\ \frac{1 - y_t}{\gamma_A} \kappa_t + \frac{y_t}{\gamma_B} \left(\kappa_t + \frac{\nu_t^* m}{\sigma_t} \right) &= \sigma_D. \end{aligned} \quad (\text{A1})$$

Solving equations (A1), we obtain κ and r in (17)–(18), where $a_1(y)$ and $a_2(y)$ are given by:

$$a_1(y_t) = \frac{\Gamma_t^3 y_t (1 - y_t) (\gamma_B - \gamma_A)}{\gamma_A^2 \gamma_B^2}, \quad a_2(y_t) = -\frac{\Gamma_t^3 y_t (1 - y_t)}{2 \gamma_A^2 \gamma_B^2} \left(1 + \frac{\gamma_A \gamma_B}{\Gamma_t} \right). \quad (\text{A2})$$

The expressions (19)–(20) for the volatility σ_y and drift μ_y of consumption share y , which is conjectured to follow process (7), are obtained by applying Itô's Lemma to $y_t = c_B^*/D_t$ and matching dt and dw terms. Stock return volatility σ is obtained by applying Itô's Lemma to both sides of $S = \Psi D$, and matching dw terms.

To obtain ODEs (23)–(24), we first derive the HJB equations for investors' value functions J_A and J_B . The HJB equation for investor B in fictitious unconstrained economy is given by equation (15) in Remark 1, while the HJB equation for investor A is similar, but with $\nu^* = 0$, since investor A is unconstrained. Following Liu (2007) we conjecture that $J_i = \exp(-\rho t) W^{1-\gamma_i} \Phi_i^{\gamma_i} / (1 - \gamma_i)$, where $i = A, B$. Then, the first order condition (F.O.C.) with respect to consumption, $\exp(-\rho t) c_i^{-\gamma_i} = \partial J_i / \partial W$, implies that $\Phi_i = c_i / W$, and hence Φ_i is the wealth-consumption ratio. The F.O.C. for portfolio weights gives the following expressions:

$$\theta_{At}^* = \frac{\mu_t - r_t}{\gamma_A \sigma_t^2} - \frac{y_t \sigma_{yt}}{\sigma_t} \frac{\Phi_A'(y_t)}{\Phi_A(y_t)}, \quad \theta_{Bt}^* = \frac{\mu_t - r_t + \nu_t^* m}{\gamma_B \sigma_t^2} - \frac{y_t \sigma_{yt}}{\sigma_t} \frac{\Phi_B'(y_t)}{\Phi_B(y_t)}. \quad (\text{A3})$$

Next, as standard in the portfolio choice literature [e.g., Liu (2007)], substituting (A3) back into HJB equations we obtain a PDE for the value function. Substituting conjectured J_i into this PDE and cancelling like terms we obtain ODEs (23)–(24). Since the optimization problems are solved in complete real and fictitious economies, the ODEs turn out to be linear, conditional on knowing the adjustment ν^* , consistently with Liu (2007). The adjustment ν^* is derived in Lemma A.1 below. Finally, to derive Ψ , by summing up bond and stock market clearing conditions in (8) we obtain: $W_A^* + W_B^* = S$. The latter equation can be rewritten in terms of wealth-consumption and price-dividend ratios as $(1 - y)\Phi_A + y\Phi_B = \Psi$. \square

Lemma A.1. *In a Markovian equilibrium the adjustment ν^* and ratio m/σ are given by:*

$$\nu_t^* = \begin{cases} \frac{1 - b_{2t} v_t}{b_{1t} v_t^2}, & \text{if } \theta_{Bt}^* m = 1, \quad \nu_t^* < 0, \\ 0, & \text{if } \theta_{Bt}^* m < 1, \end{cases} \quad (\text{A4})$$

$$\frac{m}{\sigma_t} = \begin{cases} v_t & , \quad \text{if } \theta_B^* m = 1, \quad \nu_t^* < 0, \\ \frac{m}{\sigma_D \left(1 - \frac{\Gamma_t y_t (1 - y_t) (\gamma_B - \gamma_A)}{\gamma_A \gamma_B} \frac{\Psi'(y_t)}{\Psi(y_t)} \right)} & , \quad \text{if } \theta_B^* m < 1, \end{cases} \quad (\text{A5})$$

where b_1 , b_2 , v , and $\theta_B^* m$ are given by:

$$b_{1t} = \frac{\Gamma_t (1 - y_t)}{\gamma_A \gamma_B} \left(1 + y_t \frac{\Phi'_B(y_t)}{\Phi_B(y_t)} \right), \quad b_{2t} = \sigma_D - (\gamma_B - \gamma_A) \sigma_D b_{1t}, \quad (\text{A6})$$

$$v_t = \frac{1}{\sigma_D} + \frac{1 + y_t \frac{\Phi'_B(y_t)}{\Phi_B(y_t)}}{1 + y_t \frac{\Phi'_B(y_t)}{\Phi_B(y_t)} - y_t \frac{\Psi'(y_t)}{\Psi(y_t)}} \frac{m - 1}{\sigma_D}, \quad (\text{A7})$$

$$\theta_B^* m = b_{2t} \frac{m}{\sigma_t} + b_{1t} \left(\frac{m}{\sigma_t} \right)^2 \nu_t^*, \quad (\text{A8})$$

and wealth-consumption ratio is given by $\Psi(y) = (1 - y)\Phi_A(y) + y\Phi_B(y)$.

Proof of Lemma A.1. The idea of the proof is to find adjustment ν^* and ratio m/σ from a system of two equations: the complementary slackness condition $\nu^*(\theta_B^* m - 1) = 0$ [e.g., Karatzas and Shreve (1998); Remark 1 in Section 2.1], and the equation for m/σ derived from the expression for stock return volatility (21). Consider investor B who takes equilibrium processes as given. From portfolio constraint $\theta_B^* m \leq 1$, where θ_B^* is given in (A3), we obtain an upper bound on ν^* :

$$\nu_t^* \leq \left(1 - \frac{\kappa_t}{\gamma_B} \frac{m}{\sigma_t} + y_t \sigma_{yt} \frac{m}{\sigma_t} \frac{\Phi'_B(y_t)}{\Phi_B(y_t)} \right) \frac{1}{\frac{1}{\gamma_B} \left(\frac{m}{\sigma_t} \right)^2}. \quad (\text{A9})$$

Inequality (A9) is satisfied as an equality when the constraint is binding. Furthermore, Kuhn-Tucker condition implies $\nu^* \leq 0$ [e.g., Karatzas and Shreve (1998); Remark 1 in Section 2.1]. Moreover, since either the latter or the former inequalities should be satisfied as an equality, depending on whether the constraint is binding or not, we obtain:

$$\nu_t^* = \min \left(0; 1 - \frac{\kappa_t}{\gamma_B} \frac{m}{\sigma_t} + y_t \sigma_{yt} \frac{m}{\sigma_t} \frac{\Phi'_B(y_t)}{\Phi_B(y_t)} \right) \frac{1}{\frac{1}{\gamma_B} \left(\frac{m}{\sigma_t} \right)^2}. \quad (\text{A10})$$

Multiplying both sides of equation (21) for σ by m/σ , we obtain that m/σ satisfies equation:

$$m = \frac{m}{\sigma_t} \left(\sigma_D - y_t \sigma_{yt} \frac{\Psi'(y_t)}{\Psi(y_t)} \right). \quad (\text{A11})$$

We observe that processes κ and σ_y depend on ν^* in equilibrium. Substituting κ and σ_y from (17) and (19) into equations (A10) and (A11) we obtain two equations for ν^* and m/σ , which we solve next. Also, substituting κ and σ_y from (17) and (19) into θ_B^* in (A3) after some algebra we obtain that $\theta_B^* m$ is given by (A8). When the constraint is binding, from equation (A10) or, equivalently, from equation $\theta_B^* m = 1$, where $\theta_B^* m$ is given by (A8), after some algebra we obtain:

$$\nu^* = \frac{1 - b_{2t}(m/\sigma_t)}{b_{1t}(m/\sigma_t)^2}, \quad (\text{A12})$$

where b_1 and b_2 are given by expressions (A6). When the constraint does not bind, i.e. $\theta_B^* m < 1$, Kuhn-Tucker condition $\nu^*(\theta_B^* m - 1) = 0$ implies that $\nu^* = 0$. Consequently, we obtain:

$$\nu_t^* = \begin{cases} \frac{1 - b_{2t}(m/\sigma_t)}{b_{1t}(m/\sigma_t)^2}, & \text{if constraint binds,} \\ 0 & \text{if constraint does not bind.} \end{cases} \quad (\text{A13})$$

Next, we obtain an equation for m/σ , which together with equation (A13) gives a system of equations for ν^* and m/σ . Substituting σ_y from (19) into the equation for volatility σ in (21), and then multiplying both sides of the resulting equation by m/σ we obtain:

$$m = \sigma_D \left(1 - \frac{\Gamma_t y_t (1 - y_t) (\gamma_B - \gamma_A)}{\gamma_A \gamma_B} \frac{\Psi'(y_t)}{\Psi(y_t)} \right) \frac{m}{\sigma_t} + \frac{\Gamma_t (1 - y_t)}{\gamma_A \gamma_B} \frac{\Psi'(y_t)}{\Psi(y_t)} \left(\frac{m}{\sigma_t} \right)^2 \nu_t^*. \quad (\text{A14})$$

If the constraint does not bind, and hence $\nu^* = 0$, the second term in (A14) vanishes, and m/σ can be easily found. If the constraint binds, substituting ν^* from (A13) into equation (A14) yields a simple linear equation for m/σ from which we obtain expression (A5) for m/σ . Therefore, $m/\sigma = v$ if the constraint binds, and hence from (A13) we obtain expression (A4) for ν^* .

We note, that the expressions in Lemma A.1 remain valid for the case of time-varying margins, e.g., $m = m(y_t, \sigma_t)$. If margin depends on volatility σ , the right-hand side of equation (A4) for ν^* depends on volatility σ via process v , given by (A7). To obtain ν^* and σ in terms of Ψ , Φ_B and their derivatives we note that ν^* and σ solve the following system of two equations. The first equation is the equation (A4) for ν^* , and the second equation is obtained by substituting σ_y given by (19) into expression (21) for σ . This system can be solved in closed form in special cases, e.g., when $m(y, \sigma)$ is linear, and numerically when $m(y, \sigma)$ is nonlinear. \square

Proof of Proposition 2. In the unconstrained economy both investors have the same state price density, i.e., $\xi = \xi_{\nu^*}$. Hence, from the expressions for optimal consumptions (14) we obtain: $c_{At}^{-\gamma_A} / c_{Bt}^{-\gamma_B} = \lambda$, where λ is a constant. Consumption clearing implies: $c_{At} + c_{Bt} = D_t$. The latter two equations imply that consumption share $y = c_B^* / D$ is given by

$$y_t = f\left(\lambda^{-\frac{1}{\gamma_A}} D_t^{\frac{\gamma_B - \gamma_A}{\gamma_A}}\right), \quad (\text{A15})$$

where $f(\cdot)$ is an implicit function satisfying equation:

$$Z f(Z)^{\frac{\gamma_B}{\gamma_A}} + f(Z) = 1. \quad (\text{A16})$$

The s.p.d. in terms of consumption share is then given by:

$$\xi_t = \tilde{\lambda} e^{-\rho t} (c_{Bt}^*)^{-\gamma_B} = \tilde{\lambda} e^{-\rho t} f\left(\lambda^{-\frac{1}{\gamma_A}} D_t^{\frac{\gamma_B - \gamma_A}{\gamma_A}}\right)^{-\gamma_B} D_t^{-\gamma_B}, \quad (\text{A17})$$

where $\tilde{\lambda}$ is a constant. Consequently, the price-dividend ratio is given by:

$$\begin{aligned}\Psi_t &= \frac{1}{\xi_t} \mathbb{E}_t \left[\int_t^{+\infty} \frac{\xi_\tau D_\tau}{D_t} d\tau \right] \\ &= \left(\frac{D_t}{c_{Bt}^*} \right)^{-\gamma_B} \mathbb{E}_t \left[\int_t^{+\infty} e^{-\rho(\tau-t)} \left(\frac{D_\tau}{D_t} \right)^{1-\gamma_B} f \left(Z_t \left(\frac{D_\tau}{D_t} \right)^{\frac{\gamma_B-\gamma_A}{\gamma_A}} \right)^{-\gamma_B} d\tau \right] \\ &= y_t^{\gamma_B} \mathbb{E}_t \left[\int_t^{+\infty} e^{-\hat{\rho}(\tau-t)} \frac{\eta_\tau}{\eta_t} f \left(Z_t \left(\frac{D_\tau}{D_t} \right)^{\frac{\gamma_B-\gamma_A}{\gamma_A}} \right)^{-\gamma_B} d\tau \right],\end{aligned}\tag{A18}$$

where $Z_t = \lambda^{-1/\gamma_A} D_t^{(\gamma_B-\gamma_A)/\gamma_A}$, $\hat{\rho} = \rho - (1-\gamma_B)\mu_D + 0.5(1-\gamma_B)\gamma_B\sigma_D^2$, and η_t is a GBM martingale:

$$d\eta_t = \eta_t(1-\gamma_B)\sigma_D dw_t.\tag{A19}$$

The third equality in (A18) uses the fact that D_t is a GBM, and hence it can be easily verified that $\exp(-\rho t)D_t^{1-\gamma_B} = \exp(-\hat{\rho}t)\eta_t$. To facilitate tractability, we switch to new probability measure $\hat{\mathbb{P}}$ with Radon-Nikodym derivative given by η_τ/η_t , where η follows GBM process (A19). By Girsanov's Theorem, $\hat{w}_t = w_t - (1-\gamma_B)\sigma_D t$ is a Brownian motion under measure $\hat{\mathbb{P}}$.

We let $\hat{\mathbb{E}}[\cdot]$ denote the expectation under the new measure, rewrite expression (A18) under the new measure, and then rewrite $\hat{\mathbb{E}}[\cdot]$ as an integral:

$$\begin{aligned}\Psi_t &= y_t^{\gamma_B} \hat{\mathbb{E}}_t \left[\int_t^{+\infty} e^{-\hat{\rho}(\tau-t)} f \left(Z_t \left(\frac{D_\tau}{D_t} \right)^{\frac{\gamma_B-\gamma_A}{\gamma_A}} \right)^{-\gamma_B} d\tau \right] \\ &= y_t^{\gamma_B} \hat{\mathbb{E}}_t \left[\int_t^{+\infty} e^{-\hat{\rho}(\tau-t)} f \left(Z_t e^{q(\tau-t) + \sqrt{\Sigma}(\hat{w}_\tau - \hat{w}_t)} \right)^{-\gamma_B} d\tau \right] \\ &= y_t^{\gamma_B} \int_{-\infty}^{+\infty} \left[\int_0^{+\infty} \frac{1}{\sqrt{2\pi\Sigma\tau}} e^{-\frac{(u-q\tau)^2}{2\Sigma\tau}} e^{-\hat{\rho}\tau} f \left(Z_t e^u \right)^{-\gamma_B} d\tau \right] du \\ &= y_t^{\gamma_B} \int_{-\infty}^{+\infty} \left[\int_0^{+\infty} \frac{1}{\sqrt{2\pi\Sigma\tau}} e^{-\left(\hat{\rho} + \frac{q^2}{2\Sigma}\right)\tau - \frac{u^2}{2\Sigma\tau}} d\tau \right] e^{\frac{qu}{\Sigma}} f \left(Z_t e^u \right)^{-\gamma_B} du \\ &= \frac{y_t^{\gamma_B}}{\sqrt{2\hat{\rho}\Sigma + q^2}} \int_{-\infty}^{+\infty} e^{\frac{q}{\Sigma}u - \frac{\sqrt{2\hat{\rho}\Sigma + q^2}}{\Sigma}|u|} f \left(Z_t e^u \right)^{-\gamma_B} du,\end{aligned}\tag{A20}$$

where $q = (\gamma_B - \gamma_A)(\mu_D + 0.5(1-2\gamma_B)\sigma_D^2)/\gamma_A$, $\Sigma = (\gamma_B - \gamma_A)^2\sigma_D^2/\gamma_A^2$, and similarly to Cochrane, Longstaff, and Santa-Clara (2008) the inner integral in the fourth equality is computed using formulas 3.471.9 and 8.469.3 in Gradshteyn and Ryzhik (2007).⁷

Next, we split the last integral in (A20) into two parts:

$$\Psi_t = \frac{y_t^{\gamma_B}}{p} \left[\int_0^{+\infty} e^{\varphi-u} f \left(Z_t e^u \right)^{-\gamma_B} du + \int_{-\infty}^0 e^{\varphi+u} f \left(Z_t e^u \right)^{-\gamma_B} du \right],\tag{A21}$$

⁷We use the following formula, implied by formulas 3.471.9 and 8.469.3 in Gradshteyn and Ryzhik (2007):

$$\int_0^{+\infty} \frac{1}{\sqrt{\tau}} e^{-a\tau - \frac{b}{\tau}} d\tau = \sqrt{\frac{\pi}{a}} e^{-2\sqrt{ab}}.$$

where constants p , φ_- and φ_+ are given by:

$$p = \sqrt{2\hat{\rho}\Sigma + q^2}, \quad \varphi_{\pm} = \frac{q \pm \sqrt{2\hat{\rho}\Sigma + q^2}}{\Sigma}, \quad (\text{A22})$$

and $\hat{\rho} = \rho - (1 - \gamma_B)\mu_D + 0.5(1 - \gamma_B)\gamma_B\sigma_D^2$, $q = (\gamma_B - \gamma_A)(\mu_D + 0.5(1 - 2\gamma_B)\sigma_D^2)/\gamma_A$, $\Sigma = (\gamma_B - \gamma_A)^2\sigma_D^2/\gamma_A^2$. Our methodological contribution is to calculate the two integrals in (A21) using the changes of variable $s = f(Z_te^u)/y_t$ and $s = (1 - f(Z_te^u))/(1 - y_t)$, respectively, by observing from equation (A16) for $f(Z)$ that inverse function $f^{-1}(y)$ is given by $f^{-1}(y) = (1 - y)y^{-\gamma_B/\gamma_A}$.

To perform the change of variable $s = f(Z_te^u)/y_t$ in the first integral, we note that $Z_te^u = f^{-1}(y_ts) = (1 - y_ts)(y_ts)^{-\gamma_B/\gamma_A}$. Differentiating the last expression, and dividing both sides by Z_te^u we obtain:

$$du = - \left(\frac{\gamma_B}{\gamma_A} + \left(1 - \frac{\gamma_B}{\gamma_A} \right) y_t s \right) \frac{1}{s(1 - y_ts)} ds. \quad (\text{A23})$$

Next, we perform the change of variable. We also note, that since $Z_t = \lambda^{-1/\gamma_A} D_t^{(\gamma_B - \gamma_A)/\gamma_A}$, equations (A15)–(A16) imply that $Z_t = (1 - y_t)y_t^{-\gamma_B/\gamma_A}$. After some algebra we obtain:⁸

$$\begin{aligned} I_1 &= y_t^{\gamma_B} \int_0^{+\infty} e^{\varphi_- u} f(Z_te^u)^{-\gamma_B} du \\ &= \int_0^1 \frac{(1 - y_ts)^{\varphi_- - 1}}{(1 - y_t)^{\varphi_-}} s^{-\gamma_B - \frac{\gamma_B}{\gamma_A}\varphi_- - 1} \left(\frac{\gamma_B}{\gamma_A} + \left(1 - \frac{\gamma_B}{\gamma_A} \right) y_t s \right) ds \\ &= - \frac{(1 - y_t)^{-\varphi_-}}{\gamma_A + \varphi_-} {}_2F_1 \left(1 - \varphi_-, -\gamma_B - \frac{\gamma_B}{\gamma_A}\varphi_-, 1 - \gamma_B - \frac{\gamma_B}{\gamma_A}\varphi_-; y_t \right) \\ &\quad + \left(1 - \frac{\gamma_B}{\gamma_A} \right) \frac{y_t(1 - y_t)^{-\varphi_-}}{1 - \gamma_B - \frac{\gamma_B}{\gamma_A}\varphi_-} {}_2F_1 \left(1 - \varphi_-, 1 - \gamma_B - \frac{\gamma_B}{\gamma_A}\varphi_-, 2 - \gamma_B - \frac{\gamma_B}{\gamma_A}\varphi_-; y_t \right), \end{aligned} \quad (\text{A24})$$

where ${}_2F_1(a, b, c; y)$ is a hypergeometric function [e.g., Gradshteyn and Ryzhik (2007)], given by:

$${}_2F_1(a, b, c; y) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{s^{b-1}(1-s)^{c-b-1}}{(1-ys)^a} ds.$$

Finally, we further simplify expression (A24) for integral I_1 by using the following formula 9.131.1 in Gradshteyn and Ryzhik (2007): ${}_2F_1(a, b, c; y)(1-y)^{a+b-c} = {}_2F_1(c-a, c-b, c; y)$. Then, along the same lines, we compute the second integral in (A21), and after some algebra we obtain expression (26) for price-dividend ratio Ψ . \square

Proof of Proposition 3. Similarly to Section 2, we embed the optimization of constrained investor B into a fictitious complete-market economy, in which bonds and stocks follow dynamics:

$$dB_t = B_t(r_t + f(\tilde{\nu}_t))dt, \quad (\text{A25})$$

$$dS_{jt} + D_{jt}dt = S_{jt}[(\mu_{jt} + \tilde{\nu}_{jt} + f(\tilde{\nu}))dt + \sigma_{jt}^\top dw_t], \quad (\text{A26})$$

⁸Bhamra and Uppal (2010) demonstrate how similar integrals can be evaluated using infinite series, and provide closed-form expansions for price-dividend ratios and other equilibrium processes.

where $\tilde{\nu} = (\tilde{\nu}_1, \tilde{\nu}_2) \in \mathbb{R}^2$, and $f(\tilde{\nu})$ is the support function, given by:

$$f(\tilde{\nu}) = \sup_{\theta_1 m_1 + \theta_2 m_2 \leq 1} (-\tilde{\nu}_1 \theta_1 - \tilde{\nu}_2 \theta_2), \quad (\text{A27})$$

where $\tilde{\nu} = (\tilde{\nu}_1, \tilde{\nu}_2)^\top$ are adjustments, proportional to Lagrange multipliers for constraints. As demonstrated in Karatzas and Shreve (1998), $\tilde{\nu} = (\tilde{\nu}_1, \tilde{\nu}_2)^\top \in \Upsilon$, where Υ is the effective domain for $f(\tilde{\nu})$, defined as $\Upsilon = \{\tilde{\nu} \in \mathbb{R}^2 : f(\tilde{\nu}) < +\infty\}$. Solving the optimization in (A27) we find that $\Upsilon = \{\tilde{\nu} = (m_1, m_2)^\top \nu : \nu \in \mathbb{R}, \nu \leq 0\}$. Noting the structure of effective domain Υ , we obtain:

$$\tilde{\nu}_t = (m_1, m_2)^\top \nu_t^*, \quad f(\tilde{\nu}) = -\nu_t^*, \quad (\text{A28})$$

where ν^* is such that $\nu^* \leq 0$, and is derived in Lemma A.2 below. Similarly to Section 2.1, s.p.d. in the unconstrained real and fictitious economies, ξ and ξ_{ν^*} , follow dynamics:

$$d\xi_t = -\xi_t[r_t dt + \kappa_t^\top dw_t], \quad d\xi_{\nu^*t} = -\xi_{\nu^*t}[(r_t - \nu_t^*)dt + (\kappa_t + \nu_t^* \sigma_t^{-1} m)^\top dw_t]. \quad (\text{A29})$$

Similarly to Section 2.1, the optimal consumptions of investors A and B are given by equations (14). Substituting these consumptions into consumption clearing condition in (8), applying Itô's Lemma to both sides of (8), after matching the dt and dw terms we obtain that κ and r are given by expressions (38)–(39), where $a_1(y)$ and $a_2(y)$ are given by (A2). We next derive expressions (40)–(41) for the volatility σ_y and drift μ_y of consumption share y by applying Itô's Lemma to $y = c_B^*/D$ and matching dt and dw terms. The expressions (42) for volatilities σ_j are obtained by applying Itô's Lemma to both sides of $S_j = \Psi_j D_j$ and matching dw terms.

To obtain the PDEs for the price-dividend ratios $\Psi_j(x, y)$ we apply Itô's Lemma to both sides of equation $\Psi_j = S_j/D_j$, and by matching dt terms we obtain:

$$\begin{aligned} \frac{1}{2} \left(x^2 \sigma_x^\top \sigma_x \frac{\partial^2 \Psi_j}{\partial x^2} - 2xy \sigma_y^\top \sigma_x \frac{\partial^2 \Psi_j}{\partial y \partial x} + y^2 \sigma_y^\top \sigma_y \frac{\partial^2 \Psi_j}{\partial y^2} \right) + x \mu_x \frac{\partial \Psi_j}{\partial x} - y \mu_y \frac{\partial \Psi_j}{\partial y} = \\ (-\mu_{D_j} + \sigma_{D_j}^2 + \mu_j - \sigma_{D_j} e_j^\top \sigma_j) \Psi_j - e_j^\top (x, 1 - x), \quad j = 1, 2, \end{aligned} \quad (\text{A30})$$

where $e_1 = (1, 0)^\top$, $e_2 = (0, 1)^\top$. From the definition of the market price of risk $\kappa = \sigma^{-1}(\mu - r)$, where $\mu - r = (\mu_1 - r, \mu_2 - r)^\top$, we observe that $\mu_j = r + e_j^\top \sigma \kappa = r + \sigma_j^\top \kappa$. Next, we substitute $\mu_j = r + \sigma_j^\top \kappa$ into the right-hand side of (A30), and then we replace σ_j by its expression (42) in terms of derivatives of Ψ_j . After some algebra we then obtain PDEs (43).

The PDE for wealth-consumption ratio Φ_B is obtained similarly to that in Section 2. We first set up the HJB equation in the fictitious unconstrained economy. Following the portfolio choice literature, e.g. Liu (2007), we conjecture that the value function takes the form $J_B = \exp(-\rho t) W^{1-\gamma_B} \Phi_B^{\gamma_B} / (1-\gamma_B)$. Substituting this conjectured value function into the HJB after some algebra, similarly to Liu (2007), we obtain PDE (44) which is linear in the fictitious complete-market economy if the adjustment ν^* is known. To derive Φ_A , by summing up bond and stock market clearing conditions in (8) we first obtain: $W_A^* + W_B^* = S_1 + S_2$. Then, we observe that the latter equality after some algebra can be rewritten as $(1-y)\Phi_A + y\Phi_B = x\Psi_1 + (1-x)\Psi_2$, from which we obtain Φ_A . The adjustment parameter ν^* is derived in the following Lemma A.2. \square

Lemma A.2. In a Markovian equilibrium the adjustment ν^* and vector $\sigma^{-1}m$ are given by:

$$\nu_t^* = \begin{cases} \frac{1 - b_{2t}^\top v_t}{b_{1t} v_t^\top v_t}, & \text{if } m^\top \theta_{Bt}^* = 1, \quad \nu_t^* < 0, \\ 0, & \text{if } m^\top \theta_{Bt}^* < 1, \end{cases} \quad (\text{A31})$$

$$\sigma_t^{-1}m = \begin{cases} v_t, & \text{if } m^\top \theta_{Bt}^* = 1, \quad \nu_t^* < 0, \\ (\tilde{\sigma}_t - (\gamma_B - \gamma_A)b_{3t}\sigma_{Dt}^\top)^{-1}m, & \text{if } m^\top \theta_{Bt}^* < 1, \end{cases} \quad (\text{A32})$$

where $b_1, b_2, b_3, \tilde{\sigma}, v$, and $m^\top \theta_B^*$ are given by:

$$b_{1t} = \frac{\Gamma_t(1 - y_t)}{\gamma_A \gamma_B} \left(1 + \frac{\partial \Phi_{Bt}}{\partial y_t} \frac{y_t}{\Phi_{Bt}}\right), \quad b_{2t} = \sigma_{Dt} + \sigma_{xt} \frac{\partial \Phi_{Bt}}{\partial x_t} \frac{x_t}{\Phi_{Bt}} - (\gamma_B - \gamma_A)\sigma_{Dt}b_{1t}, \quad (\text{A33})$$

$$b_{3t} = \frac{\Gamma_t(1 - y_t)}{\gamma_A \gamma_B} \left(\frac{\partial \Psi_{1t}}{\partial y_t} \frac{y_t}{\Psi_{1t}}, \frac{\partial \Psi_{2t}}{\partial y_t} \frac{y_t}{\Psi_{2t}}\right)^\top, \quad (\text{A34})$$

$$\tilde{\sigma}_t = \begin{pmatrix} \sigma_{D1} & 0 \\ 0 & \sigma_{D2} \end{pmatrix} + \left(\frac{\partial \Psi_{1t}}{\partial x_t} \frac{x_t}{\Psi_{1t}}, \frac{\partial \Psi_{2t}}{\partial x_t} \frac{x_t}{\Psi_{2t}}\right)^\top \sigma_{xt}^\top, \quad (\text{A35})$$

$$v_t = \left(\frac{1}{\sigma_{D1}}, \frac{1}{\sigma_{D2}}\right)^\top + Q^{-1}(m_1 - 1, m_2 - 1)^\top, \quad (\text{A36})$$

$$m^\top \theta_{Bt}^* = b_{2t}^\top (\sigma_t^{-1}m) + \nu_t^* b_{1t} (\sigma_t^{-1}m)^\top (\sigma_t^{-1}m), \quad (\text{A37})$$

and $Q \in \mathbb{R}^{2 \times 2}$ is a matrix with elements Q_{jk} given by:

$$(Q_{j1}, Q_{j2})^\top = e_j \sigma_{Dj} + \sigma_{xt} \frac{\partial \Psi_{jt}}{\partial x_t} \frac{x_t}{\Psi_{jt}} - \frac{\Gamma_t(1 - y_t)}{\gamma_A \gamma_B} \frac{1}{b_{1t}} \left(\sigma_{Dt} + \sigma_{xt} \frac{\partial \Phi_{Bt}}{\partial x_t} \frac{x_t}{\Phi_{Bt}}\right) \frac{\partial \Psi_{jt}}{\partial y_t} \frac{y_t}{\Psi_{jt}}, \quad (\text{A38})$$

where $j = 1, 2$, $e_1 = (1, 0)^\top$ and $e_2 = (0, 1)^\top$.

Proof of Lemma A.2. The idea of the proof is to demonstrate that ν^* and $\sigma^{-1}m$ satisfy a system of equations that can be solved in closed form in terms of Φ_B , Ψ_j , and their derivatives. The first equation is the complementary slackness condition $\nu^*(m^\top \theta_B^* - 1) = 0$ [e.g., Karatzas and Shreve (1998)], while the second is the equation for $\sigma^{-1}m$ which is derived from the expression for stock return volatilities in (42). First, we derive ν^* as a function of $\sigma^{-1}m$. It follows from the complementary slackness condition that $\nu^* = 0$, if the constraint does not bind. If the constraint binds, ν^* can be found from the condition $m^\top \theta_B^* = 1$, similarly to the proof of Lemma A.1.

In the fictitious complete-market economy with s.p.d. following (A29) the optimal θ_B^* can be found in terms of wealth-consumption ratio Φ_B in the same way as in complete-market portfolio choice literature [e.g., Liu (2007)], and is given by:

$$\theta_{Bt}^* = (\sigma_t^\top)^{-1} \left(\frac{\kappa_t + \nu_t^* \sigma_t^{-1}m}{\gamma_B} + \sigma_{xt} \frac{\partial \Phi_{Bt}}{\partial x_t} \frac{x_t}{\Phi_{Bt}} - \sigma_{yt} \frac{\partial \Phi_{Bt}}{\partial y_t} \frac{y_t}{\Phi_{Bt}} \right), \quad (\text{A39})$$

where $\kappa + \nu^* \sigma^{-1}m$ is the market price of risk in the fictitious unconstrained economy. Substituting κ from (38) and σ_y from (40) into θ_B^* in (A39), after some algebra we obtain expression for $m^\top \theta_B^*$

in (A37). Then, proceeding exactly as in the proof of Lemma A.1, we obtain:

$$\nu_t^* = \begin{cases} \frac{1 - b_{2t}^\top(\sigma_t^{-1}m)}{b_{1t}(\sigma_t^{-1}m)^\top \sigma_t^{-1}m}, & \text{if constraint binds,} \\ 0 & \text{if constraint does not bind.} \end{cases} \quad (\text{A40})$$

Next, we derive an equation for $\sigma^{-1}m$ in terms of ν^* . First, we substitute volatility σ_y from (40) into volatility σ_j in (42), which yields volatility matrix σ in terms of adjustment ν^* :

$$\sigma_t = \tilde{\sigma}_t - (\gamma_B - \gamma_A)b_{3t}\sigma_{Dt}^\top + \nu_t^*b_{3t}(\sigma_t^{-1}m)^\top, \quad (\text{A41})$$

where vector $b_3 \in \mathbb{R}^2$ and matrix $\tilde{\sigma} \in \mathbb{R}^{2 \times 2}$ are given by (A34) and (A35), respectively. Multiplying both sides of equation (A41) by $\sigma_t^{-1}m$ from the left, we obtain:

$$m = \tilde{\sigma}_t\sigma_t^{-1}m - (\gamma_B - \gamma_A)b_{3t}\sigma_{Dt}^\top\sigma_t^{-1}m + \nu_t^*b_{3t}(\sigma_t^{-1}m)^\top\sigma_t^{-1}m. \quad (\text{A42})$$

The adjustment ν^* and vector $\sigma^{-1}m$ satisfy the system of equations (A40) and (A42), which we next solve in closed form. When the constraint is binding, ν^* is given by the expression in the first line of (A40). Substituting this expression into equation (A42) we obtain a linear equation for $\sigma^{-1}m$, which has the following solution:

$$\sigma_t^{-1}m = Q^{-1}(m - b_{3t}/b_{1t}), \quad (\text{A43})$$

where $Q \in \mathbb{R}^{2 \times 2}$ is a matrix with elements given by (A38). To simplify the solution in (A43) we note that vector $v^* = (1/\sigma_{D_1}, 1/\sigma_{D_2})^\top$ solves equation $Qv^* = 1 - b_3/b_1$. This result can be verified by multiplying matrix Q and vector v^* , and using an easily verifiable fact that $\sigma_D^\top v^* = 1$ and $\sigma_x^\top v^* = 0$, where $\sigma_D = (x\sigma_{D_1}, (1-x)\sigma_{D_2})^\top$ and $\sigma_x = ((1-x)\sigma_{D_1}, -(1-x)\sigma_{D_2})^\top$ are defined in (32) and (37). Substituting $b_3/b_1 = 1 - Qv^*$ into (A43) we obtain $\sigma^{-1}m = (1/\sigma_{D_1}, 1/\sigma_{D_2})^\top + Q^{-1}(m_1 - 1, m_2 - 1)^\top$. Similarly, when the constraint does not bind, and hence $\nu^* = 0$, (A42) becomes a linear equation for $\sigma^{-1}m$, which can be solved in closed form. Consequently, we obtain expression (A32) for vector $\sigma^{-1}m$. Then, denoting $v = \sigma^{-1}m$ for the case when the constraint binds, we obtain expression (A32) for vector $\sigma^{-1}m$. \square

Proof of Proposition 4.

(i) We now prove the first part of the Proposition. Similarly to the proof of Proposition 2, consumption share y and s.p.d. ξ are given by (A15) and (A17), respectively, where $f(\cdot)$ is an implicit function satisfying equation (A16). Analogously to the proof of Proposition 2, the price-dividend ratio for the first stock is given by:

$$\begin{aligned}
\Psi_{1t} &= \frac{1}{D_{1t}(c_{Bt}^*)^{-\gamma_B}} \mathbb{E}_t \left[\int_t^{+\infty} e^{-\rho(\tau-t)} D_{1\tau} D_{\tau}^{-\gamma_B} f \left(\lambda^{-\frac{1}{\gamma_A}} D_{1\tau}^{\frac{\gamma_B-\gamma_A}{\gamma_A}} \right)^{-\gamma_B} d\tau \right] \\
&= \frac{1}{D_{1t}(c_{Bt}^*)^{-\gamma_B}} \mathbb{E}_t \left[\int_t^{+\infty} e^{-\rho(\tau-t)} D_{1\tau}^{1-\gamma_B} \left(1 + \frac{D_{2\tau}}{D_{1\tau}} \right)^{-\gamma_B} f \left(\lambda^{-\frac{1}{\gamma_A}} D_{1\tau}^{\frac{\gamma_B-\gamma_A}{\gamma_A}} \left(1 + \frac{D_{2\tau}}{D_{1\tau}} \right)^{\frac{\gamma_B-\gamma_A}{\gamma_A}} \right)^{-\gamma_B} d\tau \right] \\
&= \left(\frac{y_t}{x_t} \right)^{\gamma_B} \mathbb{E}_t \left[\int_t^{+\infty} e^{-\hat{\rho}(\tau-t)} \frac{\eta_\tau}{\eta_t} \left(1 + \frac{D_{2\tau}}{D_{1\tau}} \right)^{-\gamma_B} f \left(\lambda^{-\frac{1}{\gamma_A}} D_{1\tau}^{\frac{\gamma_B-\gamma_A}{\gamma_A}} \left(1 + \frac{D_{2\tau}}{D_{1\tau}} \right)^{\frac{\gamma_B-\gamma_A}{\gamma_A}} \right)^{-\gamma_B} d\tau \right] \\
&= \left(\frac{y_t}{x_t} \right)^{\gamma_B} \hat{\mathbb{E}}_t \left[\int_t^{+\infty} e^{-\hat{\rho}(\tau-t)} \left(1 + \frac{D_{2\tau}}{D_{1\tau}} \right)^{-\gamma_B} f \left(\lambda^{-\frac{1}{\gamma_A}} D_{1\tau}^{\frac{\gamma_B-\gamma_A}{\gamma_A}} \left(1 + \frac{D_{2\tau}}{D_{1\tau}} \right)^{\frac{\gamma_B-\gamma_A}{\gamma_A}} \right)^{-\gamma_B} d\tau \right],
\end{aligned} \tag{A44}$$

where $\hat{\rho} = \rho - (1 - \gamma_B)\mu_{D_1} + 0.5(1 - \gamma_B)\gamma_B\sigma_{D_1}^2$, η_t is a GBM martingale:

$$d\eta_t = \eta_t(1 - \gamma_B)\sigma_{D_1}dw_{1t},$$

and $\hat{\mathbb{E}}[\cdot]$ is the expectation under a new measure $\hat{\mathbb{P}}$ that has η_τ/η_t as its Radon-Nikodym derivative.

Next, rewriting the processes for D_1 and D_2 in (32) under the new measure, we obtain:

$$\lambda^{-\frac{1}{\gamma_A}} D_{1\tau}^{\frac{\gamma_B-\gamma_A}{\gamma_A}} = \lambda^{-\frac{1}{\gamma_A}} D_{1t}^{\frac{\gamma_B-\gamma_A}{\gamma_A}} e^{u_1}, \quad \frac{D_{2\tau}}{D_{1\tau}} = \frac{D_{2t}}{D_{1t}} e^{u_2}, \tag{A45}$$

where $u = (u_1, u_2)^\top$ has distribution $N(q(\tau-t), \Sigma(\tau-t))$, q and Σ are given by:

$$\begin{aligned}
q &= \left(\frac{\gamma_B - \gamma_A}{\gamma_A} (\mu_{D_1} + 0.5(1 - 2\gamma_B)\sigma_{D_1}^2), \mu_{D_2} - \mu_{D_1} + 0.5(2\gamma_B - 1)\sigma_{D_1}^2 - 0.5\sigma_{D_2}^2 \right)^\top, \\
\Sigma &= \begin{pmatrix} \left(\frac{\gamma_B - \gamma_A}{\gamma_A} \right)^2 \sigma_{D_1}^2 & -\frac{\gamma_B - \gamma_A}{\gamma_A} \sigma_{D_1}^2 \\ -\frac{\gamma_B - \gamma_A}{\gamma_A} \sigma_{D_1}^2 & \sigma_{D_1}^2 + \sigma_{D_2}^2 \end{pmatrix}
\end{aligned} \tag{A46}$$

Next, we also define parameter p as follows:

$$p = \sqrt{2 \left(\rho - (1 - \gamma_B)\mu_{D_1} + 0.5(1 - \gamma_B)\gamma_B\sigma_{D_1}^2 \right) + q^\top \Sigma^{-1} q}. \tag{A47}$$

Rewriting the expectation in (A44) as a double integral involving p.d.f. of the joint normal

distribution $N(q(\tau - t), \Sigma(\tau - t))$, we obtain:

$$\begin{aligned}
\Psi_{1t} &= \left(\frac{y_t}{x_t}\right)^{\gamma_B} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left(1 + \frac{D_{2t}}{D_{1t}} e^{u_2}\right)^{-\gamma_B} f\left(\lambda^{-\frac{1}{\gamma_A}} D_{1t}^{\frac{\gamma_B - \gamma_A}{\gamma_A}} e^{u_1} \left(1 + \frac{D_{2t}}{D_{1t}} e^{u_2}\right)^{\frac{\gamma_B - \gamma_A}{\gamma_A}}\right)^{-\gamma_B} \times \\
&\quad \frac{1}{2\pi\sqrt{\det(\Sigma)}} \left[\int_0^{+\infty} \frac{1}{\tau} e^{-\hat{\rho}\tau - \frac{1}{2\tau}(u - q\tau)^\top \Sigma^{-1}(u - q\tau)} d\tau \right] du_1 du_2, \\
&= \left(\frac{y_t}{x_t}\right)^{\gamma_B} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left(1 + \frac{D_{2t}}{D_{1t}} e^{u_2}\right)^{-\gamma_B} f\left(\lambda^{-\frac{1}{\gamma_A}} D_{1t}^{\frac{\gamma_B - \gamma_A}{\gamma_A}} e^{u_1} \left(1 + \frac{D_{2t}}{D_{1t}} e^{u_2}\right)^{\frac{\gamma_B - \gamma_A}{\gamma_A}}\right)^{-\gamma_B} \times \\
&\quad \frac{e^{q^\top \Sigma^{-1} u} K_0(p\sqrt{u^\top \Sigma^{-1} u})}{\pi\sqrt{\det(\Sigma)}} du_1 du_2,
\end{aligned} \tag{A48}$$

where the last equality is computed using integral 3.471.9 in Gradshteyn and Ryzhik (2007), and $K_0(\cdot)$ is McDonald's function, given in closed form by integral 8.432.1 in Gradshteyn and Ryzhik (2007) and also implemented in Matlab.⁹

To eliminate function $f(\cdot)$ from equation (A48) we perform the following change of variables:

$$z = f\left(\lambda^{-\frac{1}{\gamma_A}} D_{1t}^{\frac{\gamma_B - \gamma_A}{\gamma_A}} e^{u_1} \left(1 + \frac{D_{2t}}{D_{1t}} e^{u_2}\right)^{\frac{\gamma_B - \gamma_A}{\gamma_A}}\right), \quad s = \left(1 + \frac{D_{2t}}{D_{1t}} e^{u_2}\right)^{-1} \tag{A49}$$

From equation (A16) we observe that $f^{-1}(z) = (1 - z)z^{-\gamma_B/\gamma_A}$. Furthermore, from the definition of share $x = D_1/D$ we note that $D_2/D_1 = (1 - x)/x$, and from equation for share y in (A15) we obtain $\lambda^{-1/\gamma_A} D_1^{(\gamma_B - \gamma_A)/\gamma_A} = f^{-1}(y)/x^{(\gamma_A - \gamma_B)/\gamma_A} = (1 - y)y^{-\gamma_B/\gamma_A}/x^{(\gamma_A - \gamma_B)/\gamma_A}$. Using these expressions, we solve equations (A49), and obtain u_1 and u_2 as functions of s , z , x , and y given by (48). Finally, computing the partial derivatives of u_1 and u_2 in (48) w.r.t. s and z we obtain:

$$du_1 du_2 = \begin{vmatrix} \partial u_1 / \partial z & \partial u_1 / \partial s \\ \partial u_2 / \partial z & \partial u_2 / \partial s \end{vmatrix} ds dz = \frac{\gamma_B(1 - z) + \gamma_A z}{\gamma_A s(1 - s)z(1 - z)} ds dz. \tag{A50}$$

Using the expression for $du_1 du_2$ in (A50), after performing the change of variables, we obtain price-dividend ratio $\Psi_1(x, y)$ in (47). The expression for $\Psi_2(x, y)$ can be derived analogously.

(ii) We now prove the second part of Proposition 4. When the leverage constraint is binding, substituting $m_1 = 1$ and $m_2 = 1$ into expression (A32) we find that $\sigma^{-1}m = (1/\sigma_{D_1}, 1/\sigma_{D_2})^\top$. Next, we observe that the following easily verifiable equalities hold:

$$\sigma_{Dt}^\top (1/\sigma_{D_1}, 1/\sigma_{D_2})^\top = 1, \quad \sigma_{xt}^\top (1/\sigma_{D_1}, 1/\sigma_{D_2})^\top = 0, \tag{A51}$$

where $\sigma_{Dt} = (x_t \sigma_{D_1}, (1 - x_t) \sigma_{D_1})^\top$ and $\sigma_{xt} = ((1 - x_t) \sigma_{D_1}, -(1 - x_t) \sigma_{D_2})^\top$ are defined in (32) and (37). Substituting $\sigma^{-1}m$ into the expression for ν^* in (A40), using equalities (A51) we obtain the

⁹Integrals 3.471.9 and 8.432.1 in Gradshteyn and Ryzhik (2007) imply that

$$\int_0^{+\infty} \frac{1}{\tau} e^{-a\tau - \frac{b}{\tau}} d\tau = 2K_0(2\sqrt{ab}), \quad K_0(z) = \int_0^{+\infty} e^{-z \cosh(s)} ds.$$

expression for ν^* in (49). Using equalities (A51) one more time, we find that portfolio constraint is satisfied as an equality for all x and y , i.e., $(1, 1)^\top \theta_B^* = 1$. Intuitively, the constraint is identically binding since B always wants to borrow, but is prevented by constraints.

Then, we derive closed-form expressions for Ψ_j , $j = 1, 2$. Consider the ratio of marginal utilities: $\lambda_t = (c_{At}^*)^{-\gamma_A} / (c_{Bt}^*)^{-\gamma_B}$. From the F.O.C. in (14) we obtain that $\lambda_t = \tilde{\lambda} \xi_t / \xi_{\nu^*t}$, where $\tilde{\lambda}$ is a constant, and ξ_t and ξ_{ν^*t} follow processes (A29). Applying Itô's Lemma to λ_t we obtain:

$$d\lambda_t = -\lambda_t \left[\nu_t^* \left(1 - m^\top (\sigma_t^\top)^{-1} (\kappa_t + \nu_t^* \sigma_t^{-1} m) \right) dt - (\nu_t^* \sigma_t^{-1} m)^\top dw_t \right]. \quad (\text{A52})$$

Substituting κ_t from (38), $\sigma^{-1}m = (1/\sigma_{D_1}, 1/\sigma_{D_2})^\top$, and ν^* from (49) into process (A52), we find that λ_t follows a GBM:

$$d\lambda_t = \lambda_t \frac{\gamma_B - \gamma_A}{1/\sigma_{D_1}^2 + 1/\sigma_{D_2}^2} \left[(\gamma_B - 1)dt + \left(\frac{1}{\sigma_{D_1}}, \frac{1}{\sigma_{D_2}} \right)^\top dw_t \right]. \quad (\text{A53})$$

Similarly to the proof of the first part of Proposition 4, the consumption share of constrained investor B is given by $y_t = f(\lambda_t^{-1/\gamma_A} D_t^{(\gamma_B - \gamma_A)/\gamma_A})$, where $f(\cdot)$ solves equation (A16). The dividends should be priced using s.p.d. ξ_t , where from F.O.C. (14) for unconstrained investor A $\xi_t = (1/\psi_A) e^{-\rho t} ((1 - y_t) D_t)^{-\gamma_A}$. Then, proceeding similarly as in the unconstrained case, we obtain:¹⁰

$$\Psi_{1t} = \left(\frac{1 - y_t}{x_t} \right)^{\gamma_A} \tilde{\mathbb{E}}_t \left[\int_t^\infty e^{-\tilde{\rho}(\tau-t)} \left(1 + \frac{D_{2\tau}}{D_{1\tau}} \right)^{-\gamma_A} \left(1 - f \left(\lambda_\tau^{-\frac{1}{\gamma_A}} D_{1\tau}^{\frac{\gamma_B - \gamma_A}{\gamma_A}} \left(1 + \frac{D_{2\tau}}{D_{1\tau}} \right)^{\frac{\gamma_B - \gamma_A}{\gamma_A}} \right) \right)^{-\gamma_A} d\tau \right], \quad (\text{A54})$$

where $\tilde{\rho} = \rho - (1 - \gamma_A)\mu_{D_1} + 0.5(1 - \gamma_A)\gamma_A\sigma_{D_1}^2$, and $\tilde{\mathbb{E}}[\cdot]$ is an expectation under new measure $\tilde{\mathbb{P}}$, such that $\tilde{w}_t = w_t - (1 - \gamma_A)\sigma_{D_1}t$ is a Brownian motion under $\tilde{\mathbb{P}}$. From the fact that λ_t , D_{1t} and D_{2t} follow GBMs (A53) and (32), respectively, we obtain:

$$\lambda_\tau^{-\frac{1}{\gamma_A}} D_{1\tau}^{\frac{\gamma_B - \gamma_A}{\gamma_A}} = \lambda_t^{-\frac{1}{\gamma_A}} D_{1t}^{\frac{\gamma_B - \gamma_A}{\gamma_A}} e^{d_{11}(\tau-t) + d_{12}\varepsilon_t\sqrt{\tau-t}}, \quad \frac{D_{2\tau}}{D_{1\tau}} = \frac{D_{2t}}{D_{1t}} e^{d_{21}(\tau-t) + d_{22}\varepsilon_t\sqrt{\tau-t}}, \quad (\text{A55})$$

where $\varepsilon_t = (-\sigma_{D_1}(\tilde{w}_{1\tau} - \tilde{w}_{1t}) + \sigma_{D_2}(w_{2\tau} - w_{2t})) / \sqrt{(\sigma_{D_1}^2 + \sigma_{D_2}^2)(\tau - t)} \sim N(0, 1)$, and

$$\begin{aligned} d_{11} &= \frac{\gamma_B - \gamma_A}{\gamma_A} \left(\mu_{D_1} + \frac{1 - 2\gamma_A}{2} \sigma_{D_1}^2 \right) - \frac{(\gamma_B - \gamma_A)^2}{2\gamma_A} \frac{1}{1/\sigma_{D_1}^2 + 1/\sigma_{D_2}^2}, & d_{12} &= -\frac{\gamma_B - \gamma_A}{\gamma_A} \frac{\sigma_{D_1}^2}{\sqrt{\sigma_{D_1}^2 + \sigma_{D_2}^2}}, \\ d_{21} &= \mu_{D_2} - \mu_{D_1} - 0.5\sigma_{D_2}^2 - \frac{1 - 2\gamma_A}{2} \sigma_{D_1}^2, & d_{22} &= \sqrt{\sigma_{D_1}^2 + \sigma_{D_2}^2}. \end{aligned} \quad (\text{A56})$$

Finally, similarly to the unconstrained case, we rewrite the expectation operator in equation (A54) as an integral w.r.t. ε_t , and change variables ε and $\tau - t$ to the following ones:

$$\begin{aligned} z &= f \left(\lambda_t^{-\frac{1}{\gamma_A}} D_{1t}^{\frac{\gamma_B - \gamma_A}{\gamma_A}} e^{d_{11}(\tau-t) + d_{12}\varepsilon_t\sqrt{\tau-t}} \left(1 + \frac{D_{2t}}{D_{1t}} e^{d_{21}(\tau-t) + d_{22}\varepsilon_t\sqrt{\tau-t}} \right)^{\frac{\gamma_B - \gamma_A}{\gamma_A}} \right), \\ s &= \left(1 + \frac{D_{2t}}{D_{1t}} e^{d_{21}(\tau-t) + d_{22}\varepsilon_t\sqrt{\tau-t}} \right)^{-1}. \end{aligned} \quad (\text{A57})$$

¹⁰Since the equilibrium processes are given in closed form, are uniformly bounded and continuous, and the process for λ_t follows a GBM, it can be verified that $\xi_t S_{1t} + \int_0^t \xi_\tau D_{1\tau} d\tau$ is a martingale, there are no endogenous bubbles in the economy, and the price-dividend ratio is given by “present value” formula (A54).

After the change of variables in equation (A57), similarly to the unconstrained case, we obtain:

$$\Psi_{1t} = \int_0^1 \int_0^1 \left(\frac{s}{x} \frac{1-y}{1-z} \right)^{\gamma_A} \frac{\gamma_B(1-z) + \gamma_A z}{z(1-z)s(1-s)} F(s, z; x, y) ds dz, \quad (\text{A58})$$

where

$$F(s, z; x, y) = \begin{cases} \frac{e^{-\tilde{\rho}\tau(s, z; x, y) - 0.5\varepsilon(x, y)^2/\tau(s, z; x, y)}}{\gamma_A |d_{11}d_{22} - d_{12}d_{21}| \sqrt{2\pi\tau(s, z; x, y)}}, & \tau(s, z; x, y) > 0, \\ 0, & \tau(s, z; x, y) \leq 0, \end{cases} \quad (\text{A59})$$

$\tilde{\rho} = \rho - (1 - \gamma_A)\mu_{D_1} + 0.5(1 - \gamma_A)\gamma_A\sigma_{D_1}^2$, $\varepsilon(x, y)$ and $\tau(x, y)$ are given by:

$$\varepsilon(s, z; x, y) = \frac{1}{d_{12}d_{21} - d_{11}d_{22}} \left[d_{21} \left(\ln\left(\frac{1-z}{1-y}\right) - \frac{\gamma_B}{\gamma_A} \ln\left(\frac{z}{y}\right) + \frac{\gamma_B - \gamma_A}{\gamma_A} \ln\left(\frac{s}{x}\right) \right) - d_{11} \ln\left(\frac{1-s}{1-x} \frac{x}{s}\right) \right],$$

$$\tau(s, z; x, y) = \frac{1}{d_{11}d_{22} - d_{12}d_{21}} \left[d_{22} \left(\ln\left(\frac{1-z}{1-y}\right) - \frac{\gamma_B}{\gamma_A} \ln\left(\frac{z}{y}\right) + \frac{\gamma_B - \gamma_A}{\gamma_A} \ln\left(\frac{s}{x}\right) \right) - d_{12} \ln\left(\frac{1-s}{1-x} \frac{x}{s}\right) \right],$$

and d_{ij} are given by expressions (A56). Price-dividend ratio Ψ_{2t} is found analogously. \square

Proof of Corollary 1.

(i) The risk premium of the market portfolio is given by $\mu_M - r = \theta_M^\top (\mu - r)$. Multiplying both sides of the equation for excess returns (50) by θ_M we obtain:

$$\mu_M - r = \theta_M^\top \beta_C - \frac{\Gamma_t y_t}{\gamma_B} \nu^* \theta_M^\top m. \quad (\text{A60})$$

Next, from equation (A60) we find multiplier $(\Gamma y / \gamma_B) \nu^*$ in terms of $\mu_M - r$, and after substituting this multiplier back into equation (50) we obtain consumption CAPM (51).

(ii) Consumption CAPM (53) is derived by substituting ν^* given by (49) into equation (50). \square

Appendix B: Numerical Method

In this Appendix we first discuss in greater details how to obtain boundary conditions for ODEs (23)–(24). In particular, we derive the boundary conditions for the case of limited participation (i.e., $m > 1$), which is less tractable than the case of margin constraints (i.e., $m < 1$) discussed in Section 2.3. We also present an alternative way of dealing with boundary conditions. Then, we discuss the finite difference method, its numerical accuracy, and the speed of convergence. Finally, we discuss how our numerical approach is modified in the case of two trees.

B.1. Boundary Conditions

In this subsection we provide detailed derivations of boundary conditions at $y = 1$ for the case of limited participation, while the case of margin constraints is analogous. At $y = 0$ the conditions are the same as in Section 2.3. We remark here, that an alternative way of dealing with boundary conditions is to derive ODEs for functions $\tilde{\Phi}_i(y) = y(1 - y)\Phi_i$, which have boundary conditions $\tilde{\Phi}_i(0) = \tilde{\Phi}_i(1) = 0$, provided that Φ_i is sufficiently smooth at the boundaries (see discussion below). We check that this approach works equally well with proper boundary conditions. The only disadvantage of this approach is that it does not provide insights on the boundary behavior of solutions, which motivates us to look for proper boundary conditions.

In contrast to the case of margin constraints, in the case of limited participation the limit $y \rightarrow 1$ does not correspond to a homogeneous-investor economy populated by investor B only. The reason is that in such an economy investor B is constrained to hold $\theta_B^* = 1/m < 1$, which violates market clearing in stocks. Consequently, the state $y = 1$ is repulsive in a sense that it is never reached, and consumption share y is repulsed back after approaching 1 too closely. As a result, the boundary condition at $y = 1$ is not given by a wealth-consumption ratio in a limiting economy. Instead, as demonstrated below, this condition captures the admissible rate of change in wealth-consumption ratio Φ_B when the economy approaches $y = 1$.

We assume that $\Phi_i(y) \in C^1[0, 1]$ and are smooth enough at the boundaries, so that

$$\lim_{y \rightarrow 1} (1 - y)\Phi_A''(y) = 0, \quad \lim_{y \rightarrow 1} (1 - y)\Phi_B''(y) = 0. \quad (\text{B1})$$

After computing the equilibrium via finite difference method with a very fine grid of ten thousand points we check conditions (B1) numerically. We find that they are satisfied for the case of margin requirements when $m < 1$, and for the case of restricted participation when $m > 1$ and $\gamma_i > 1$. When $m > 1$ and $\gamma_i < 1$ the numerical analysis shows that assumptions (B1) are violated, in which case, to deal with boundary conditions we solve for smoother functions $\tilde{\Phi}_i(y) = y(1 - y)\Phi_i$, and then infer Φ_i . Lemma B.1 summarizes our result.

Lemma B.1. *If conditions (B1) are satisfied, the boundary conditions at $y = 1$ in the case of limited participation with $m > 1$ are given by:*

$$(\gamma_A - 1)\Phi_A(1) = 0, \quad (\gamma_B - 1)\Phi_B(1) = \Phi_B'(1), \quad (\text{B2})$$

while boundary conditions at $y = 0$ are the same as in equations (27).

Proof of Lemma B.1. First, we derive boundary conditions assuming that

$$\lim_{y \rightarrow 1} \nu^*(y) = l_1, \quad \lim_{y \rightarrow 1} (1 - y) \nu^*(y) v(y) = l_2, \quad (\text{B3})$$

where l_1 and l_2 are non-zero constants, and then show that assumptions (B3) are indeed satisfied. The adjustment ν^* and v are given by equations (A4) and (A7), respectively, and we recall that by definition $v = m/\sigma$ when the constraint is binding. We also note that the constraint has to be binding around $y = 1$, since investor B is less risk averse. Below we provide the analysis for Φ_B , while the derivation for Φ_A is analogous.

We write down ODE (24) for Φ_B around $y = 1$, substitute κ from (17) and $v = m/\sigma$, and after multiplying this ODE by $1 - y$ we obtain:

$$\begin{aligned} & \frac{y^2 \sigma_y^2}{2} (1 - y) \Phi_B'' - y(1 - y) \left(\mu_y + \frac{1 - \gamma_B}{\gamma_B \gamma_A} \left(\Gamma \gamma_A \sigma_D + (1 - y) \nu^* v \Gamma \right) \sigma_y \right) \Phi_B' \\ & + (1 - y) \left(\frac{1 - \gamma_B}{2 \gamma_B \gamma_A^2} \left(\Gamma \gamma_A \sigma_D + (1 - y) \nu^* v \Gamma \right)^2 + (1 - \gamma_B)(r - \nu^*) - \rho \right) \frac{\Phi_B}{\gamma_B} + 1 - y = 0, \end{aligned} \quad (\text{B4})$$

Expression (19) for σ_y , assumptions (B1) and (B3) imply that the first term in (B4) converges to zero in the limit. Similarly, using expressions (17)–(20) for equilibrium parameters we eliminate other terms that converge to zero, and obtain:

$$\left(\lim_{y \rightarrow 1} \frac{(1 - y)r}{\gamma_B} \right) \Phi_B'(1) + \left(\lim_{y \rightarrow 1} \frac{(1 - y)r}{\gamma_B} \right) (1 - \gamma_B) \Phi_B(1) = 0, \quad (\text{B5})$$

where $(1 - y)r/\gamma_B$ comes from μ_y in equation (B4), which is given by (20). It turns out that $(1 - y)r$ does not converge to zero due to term $a_2(y)(\nu^* m/\sigma)^2$ in the expression for r in (18), where function $a_2(y)$ is given by (A2). Indeed, from the expression for $a_2(y)$ we obtain that $(1 - y)a_2(y)\nu^*(m/\sigma)^2 \sim (1 - y)^2(\nu^* v)^2$, which converges to a non-zero limit in accordance with assumption (B3). Canceling like terms in (B5) we obtain the boundary condition for Φ_B .

Now it remains to prove assumptions (B3). Substituting ν^* and v from (A4) and (A7) into (B3), after taking the limit we obtain:

$$\begin{aligned} \lim_{y \rightarrow 1} (1 - y) \nu^*(y) v(y) &= \lim_{y \rightarrow 1} (1 - y) \frac{1 - b_2(y)v(y)}{b_1(y)v(y)} = -\gamma_A \sigma_D, \\ \lim_{y \rightarrow 1} (1 - y) v(y) &= \frac{1 + \frac{\Phi_B'(1)}{\Phi_B(1)}}{\left[1 + y_t \frac{\Phi_B'(y_t)}{\Phi_B(y_t)} - y_t \frac{\Psi'(y_t)}{\Psi(y_t)} \right]_{y=1}'} \frac{m - 1}{\sigma_D}, \end{aligned} \quad (\text{B6})$$

where the last limit uses the fact that $1 + y\Phi_B'(y)/\Phi_B(y) - y\Psi'(y)/\Psi(y) \rightarrow 0$, which can be verified by substituting $\Psi = (1 - y)\Phi_A + y\Phi_B$. It can be shown that the derivative of the expression in square brackets in (B6), evaluated at $y = 1$, is non-zero. The limits in (B6) imply (B3). \square

B.2. Finite Difference Method

Now, we turn to the computation of equilibrium. One way of computing the equilibrium is by fixed-point iterations. Namely, we conjecture ODE (23)–(24) solutions, and calculate ν^* , and equilibrium processes κ , r , σ_y , μ_y , σ . Then, we solve the ODEs again, with updated coefficients, find new solutions, and iterate the process until convergence. The second way is to fix large horizon T , find the differential equations for a model with finite horizon and then solve the model backwards until reaching a time-independent solution [e.g., Ljungqvist and Sargent (2004)].

Below, we argue that the second approach adds stability to the numerical algorithm but has lower speed of convergence. We use the fixed point iterations method to solve for the equilibria with margin constraints when $m < 1$, and use the combination of the two methods for the limited participation $m \geq 1$ where the performance of fixed point method deteriorates unless the initial conjecture is close enough to the actual equilibrium. Each step of our algorithms reduces to solving a system of linear equations.

For simplicity, we omit subscript i . Next, we fix horizon T , denote the time and state variable increments by $\Delta t \equiv T/M$ and $\Delta y \equiv 1/N$, respectively, where M and N are integer numbers, and index time and state variables by $t = 0, \Delta t, 2\Delta t, \dots, T$ and $y = 0, \Delta y, 2\Delta y, \dots, 1$, respectively. Then, we derive the following discrete-time analogues of ODEs (23)–(24) and boundary conditions by replacing derivatives with their finite-difference analogues:

$$d \frac{\Phi_{n,k+1} - \Phi_{n,k}}{\Delta t} + a_{n,k+1} \frac{\Phi_{n+1,k} - 2\Phi_{n,k} + \Phi_{n-1,k}}{\Delta y^2} + b_{n,k+1} \frac{\Phi_{n+1,k} - \Phi_{n-1,k}}{2\Delta y} + c_{n,k+1} \Phi_{n,k} + 1 = 0, \quad (\text{B7})$$

$$\Phi_{n,M} = h_n, \quad \Phi_{0,k} = e_{0,k}, \quad \Phi_{N,k} = \hat{e}_{N,k} \Phi_{N-1,k} + \tilde{e}_{N,k}, \quad (\text{B8})$$

where $n = 1, 2, \dots, N-1$, $k = 1, 2, \dots, M-1$, $\Phi_{n,k} = \Phi(n\Delta y, k\Delta t)$. The coefficients in (B7) correspond to coefficients in ODEs (23)–(24) and are computed using the solution $\Phi_{n,k+1}$ from a preceding step, while coefficients in (B8) are obtained by replacing terminal and boundary conditions at $y = 0$ and $y = 1$ by their finite-difference analogues. Parameter d in (B7) specifies the solution method. The cases $d = 0$ and $d = 1$ correspond to pure fixed point and backwards in time iterations, respectively, while $0 < d < 1$ describes a method in between. The system of equations in (B7)–(B8) is solved backwards, starting at $k = M-1$. Step- k coefficients in (B7) are computed using step- $(k+1)$ solution $\Phi_{n,k+1}$, and hence step- k function $\Phi_{n,k}$ solves a system of linear algebraic equations for fixed k . We then iterate until convergence.

When $d = 0$, the first term in (B7) vanishes, and the resulting numerical scheme becomes an implicit finite-differences scheme for ODEs (23)–(24), where k indexes the iteration. If $d = 1$, the first term in equation (B7) corresponds to $\partial\Phi(y, t)/\partial t$, which is a standard term in finite-horizon HJB equations. The algorithms with and without the first term in (B7) converge to the same result. However, sometimes methods with $0 < d \leq 1$ improve performance for the following reason. For a fixed k system of linear equations (B7) is a system with a three-diagonal matrix. Numerical methods with such matrices are known to be more stable when the absolute values of

the elements on the main diagonal exceed the sum of the absolute values of elements on upper and lower diagonals. It can be shown that the first term in (B7) increases the absolute value of the elements on the main diagonal, and hence increases the computational stability.

The convergence is assessed by computing the maximum weighted difference between wealth-consumption ratios 10 years (or iterations, when $d = 0$) apart: $\varepsilon_1 = 0.5 \max_y |\Phi_B(y, t) - \Phi_B(y, t + 10)| + 0.5 \max_y |\Phi_A(y, t) - \Phi_A(y, t + 10)|$. In the backwards iterations case we also looked at another measure given by $\varepsilon_2 = 0.5 \max_y |(\Phi_B(y, t) - \Phi_B(y, t + \Delta t)) / \Delta t| + 0.5 \max_y |(\Phi_A(y, t) - \Phi_A(y, t + \Delta t)) / \Delta t|$. We solve the model setting $N = 1000$ and for both convergence measures get typical precisions around $\varepsilon \sim 10^{-9}$ after a couple of seconds of Matlab calculations on a PC. We also cross-check the numerical and closed-form solutions, where possible.

Our methodology remains the same for solving the model with two trees. We use the method of iterations, described above, and at each point in time solve a system of linear algebraic equations. However, the problem becomes more computationally intensive since PDEs for wealth-consumption and price-dividend ratios are two-dimensional. We solve PDEs using fine-differences method, which involves solving a system of linear algebraic equations with matrix Ω given by:

$$\Omega = \begin{pmatrix} \Omega_{c,1} & \Omega_{u,1} & 0 & 0 & \dots & 0 & 0 & 0 \\ \Omega_{l,2} & \Omega_{c,2} & \Omega_{u,2} & 0 & \dots & 0 & 0 & 0 \\ 0 & \Omega_{l,3} & \Omega_{c,3} & \Omega_{u,3} & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & \Omega_{l,N-2} & \Omega_{c,N-2} & \Omega_{u,N-2} \\ 0 & 0 & 0 & 0 & 0 & 0 & \Omega_{l,N-1} & \Omega_{c,N-1} \end{pmatrix},$$

where elements $\Omega_{l,n}$, $\Omega_{c,n}$, and $\Omega_{u,n}$ are $(N - 1) \times (N - 1)$ three-diagonal matrices, and N is the number of grid points. The number of grid points is the same for $x \in [0, 1]$ and $y \in [0, 1]$. For simplicity of dealing with boundary conditions we solve for functions $\tilde{\Phi} = x(1 - x)y(1 - y)\Phi$, which have zero boundary conditions, and then recover function Φ . Other computational aspects of the algorithm remain the same as for the one-dimensional case.

Appendix C: Verification of Optimality

In this Appendix we derive easily verifiable sufficient conditions of optimality. The investors solve their optimization problem taking equilibrium processes as given. Consequently, the verification of optimality is studied holding these processes fixed. Our dynamic programming solution approach allows us to provide the verification results by employing the techniques available in the literature [e.g., Fleming and Soner (2005)], modified to take into account Kuhn-Tucker conditions. Our results also justify the conjectured functional form of the value function, that we used to derive ODEs (23)–(24) for wealth-consumption ratios.

For brevity, we consider only the optimization of constrained investor B , and hence restrict ourselves to considering portfolio weights that satisfy investor's portfolio constraint. We consider self-financing *admissible consumption and portfolio strategies* c_t and θ_t , such that

$$\left| \mathbb{E}_t \left[\int_t^{+\infty} \frac{c_\tau^{1-\gamma_B}}{1-\gamma_B} d\tau \right] \right| < +\infty, \quad |\theta_t| m \leq 1. \quad (\text{C1})$$

We also note, that in the economy with margin constraints (i.e., $m < 1$), these constraints never bind for investor A in equilibrium. Hence, margin constraints, and conditions (C1), can be imposed on investor A without loss of generality. Lemma C.1 below summarizes our results.

Lemma C.1 (Verification of Optimality). *Let $\Phi_B(y_t)$ be a twice continuously differentiable solution of ODE (24). Suppose, that $|\sigma_t| < C_1$, $|(\mu_t + \nu_t^* - r_t)/\sigma_t| < C_1$, where C_1 is a constant, and consider function $J_B(W_t, y_t, t)$, given by:*

$$J_B(W_t, y_t, t) = \frac{W_t^{1-\gamma_B} \Phi_B(y_t)^{\gamma_B}}{1-\gamma_B}. \quad (\text{C2})$$

(i) *Let strategies c_t and θ_t be such that conditions (C1) are satisfied, and*

$$\mathbb{E}_t \left[\int_0^T J_B(W_\tau, y_\tau, \tau)^2 d\tau \right] < +\infty, \quad \text{for all } T > 0, \quad (\text{C3})$$

$$\limsup_{T \rightarrow +\infty} \mathbb{E}_t [J_B(W_T, y_T, T)] \geq 0, \quad (\text{C4})$$

where W_τ is wealth, generated by strategies c_t and θ_t . Then, the following inequality holds:

$$J_B(W_t, y_t, t) \geq \mathbb{E}_t \left[\int_0^{+\infty} \frac{c_\tau^{1-\gamma_B}}{1-\gamma_B} d\tau \right]. \quad (\text{C5})$$

(ii) *Suppose, that $0 < \Phi_B(y) \leq C_1$. Then, $c_t^* = W_t/\Phi_B(y_t)$ and θ_{Bt}^* in (A3) are optimal, and*

$$J_B(W_t, y_t, t) = \mathbb{E}_t \left[\int_0^{+\infty} \frac{(c_\tau^*)^{1-\gamma_B}}{1-\gamma_B} d\tau \right]. \quad (\text{C6})$$

The proof of Lemma C.1 is given below, and is similar to the proofs of verification theorems in the literature [e.g., Fleming and Soner (2005)]. We also note that condition (C4) is trivially satisfied when $0 < \gamma_B < 1$, since in this case value function (C2) is positive. This condition is an analogue of terminal conditions in finite-horizon models, and is widely employed in the literature [e.g., Fleming and Soner (2005)]. We also require volatility σ and adjusted market price of risk $(\mu + \nu^* - r)/\sigma$ to be bounded, and wealth-consumption ratio Φ_B to be twice continuously differentiable, which can be verified numerically.

In particular, in the case of margin constraints, studied in Section 3.2, it can be shown that price-dividend ratio $\Phi_A(y)$, volatility σ , market price of risk κ , and adjustment ν^* are all continuous and bounded functions. Similarly, using very fine grid, we check that derivatives $\Phi'_B(y)$ and $\Phi''_B(y)$ also appear to be continuous. Consequently, Lemma C.1 applies to the case of margin constraints. In the case of limited participation in Section 3.1, market price of risk κ is unbounded, and hence this case requires a more subtle verification theorem, which is beyond our scope. We only note that one potential way of obtaining a more powerful verification result might be to determine the asymptotic rate of growth of equilibrium processes near $y = 1$, and to demonstrate that the singularity at this point is integrable.

Proof of Lemma C.1.

(i) First, we construct the following process:

$$U_t = \int_0^t \frac{c_\tau^{1-\gamma_B}}{1-\gamma_B} d\tau + J_B(W_t, y_t, t). \quad (\text{C7})$$

Next, we derive a stochastic differential equation (SDE) for U_t in the following form:

$$dU_t = U_t[\mu_{U_t}dt + \sigma_{U_t}dw_t]. \quad (\text{C8})$$

By applying Itô's Lemma to process (C7), adding and subtracting $\nu^*(\theta_t m - 1)W \partial J_B / \partial W$ from the drift, we find that μ_U and σ_U are given by:

$$\begin{aligned} \mu_{U_t} = \frac{1}{U_t} & \left(e^{-\rho t} \frac{c_t^{1-\gamma_B}}{1-\gamma_B} + \frac{\partial J_{Bt}}{\partial t} + \left[W_t \left(r_t - \nu_t^* + \theta_t (\mu_t - r_t + \nu_t^* m) \right) - c_t \right] \frac{\partial J_{Bt}}{\partial W_t} \right. \\ & \left. - y_t \mu_{y_t} \frac{\partial J_{Bt}}{\partial y_t} + \frac{1}{2} \left[W_t^2 \theta_t^2 \sigma_t^2 \frac{\partial^2 J_{Bt}}{\partial W_t^2} - 2W_t \theta_t \sigma_t y_t \sigma_{y_t} \frac{\partial^2 J_{Bt}}{\partial W_t \partial y_t} + y_t^2 \sigma_{y_t}^2 \frac{\partial^2 J_{Bt}}{\partial y_t^2} \right] \right) \\ & - \frac{\nu^*(\theta_t m - 1)}{U_t} W_t \frac{\partial J_{Bt}}{\partial W_t}. \end{aligned} \quad (\text{C9})$$

$$\begin{aligned} \sigma_{U_t} &= \frac{J_t}{U_t} \left((1 - \gamma_B) \theta_t \sigma_t - \gamma_B y_t \sigma_{y_t} \frac{\Phi'_B(y)}{\Phi_B(y)} \right) \\ &= \frac{J_t}{U_t} \left((1 - \gamma_B) \theta_t \sigma_t + \gamma_B \theta_{Bt}^* \sigma_t - \frac{\mu_t - r_t + \nu_t^*}{\sigma_t} \right), \end{aligned} \quad (\text{C10})$$

where the second equality in (C10) is derived by expressing $y\sigma_y\Phi'_B(y)/\Phi_B(y)$ in terms of θ_B^* and $(\mu-r+\nu^*)/\sigma$ from the expression for portfolio weight θ_B^* in (A3). We note that the first component of μ_U is non-positive because J_{Bt} satisfies HJB equation (15) with “max” operator, while the first term in (C9) is evaluated at a sub-optimal strategy (c, θ) . The second term in (C9) is non-positive because $\nu^* \leq 0$, $\theta_t m - 1 \leq 0$ by assumption (C1), and $\partial J_B / \partial W_t \geq 0$. Consequently, $\mu_U \leq 0$.

From conditions (C1) and (C3) it follows that $\mathbb{E}_t[\int_0^T (U_\tau \sigma_{U\tau})^2 d\tau] < +\infty$, and hence we obtain that $\mathbb{E}_t[\int_0^T U_\tau \sigma_{U\tau} dw_\tau] = 0$, for all T . Consequently, integrating SDE (C8) from t to T , and taking expectations on both sides, we find that $U_t \geq \mathbb{E}_t[U_T]$, which can be rewritten as follows:

$$J_B(W_t, y_t, t) \geq \mathbb{E}_t\left[\int_t^T \frac{c_\tau^{1-\gamma_B}}{1-\gamma_B} d\tau\right] + \mathbb{E}_t[J_B(W_T, y_T, T)]. \quad (\text{C11})$$

From condition (C4) it follows that there exists a subsequence T_n such that $\lim \mathbb{E}_t[J(W_{T_n}, y_{T_n}, T_n)] \geq 0$, as $n \rightarrow \infty$. We also note that $\int_t^{T_n} c_\tau^{1-\gamma_B} / (1-\gamma_B) d\tau$ is a monotone sequence of random variables, and hence its limit can be calculated using monotone convergence theorem [Shiryaev (1996)]. Consequently, taking the limit in (C11) we obtain inequality (C5).

(ii) First, we demonstrate that transversality condition $\mathbb{E}_t[J_B(W_T, y_T, T)] \rightarrow 0$ is satisfied as $T \rightarrow 0$, where wealth W_T is under strategies c^* and θ^* . To this end, we apply Itô's Lemma to $J_B(W_t, y_t, t)$. Then, we add and subtract $e^{-\rho t} (c_{Bt}^*)^{1-\gamma_B} / (1-\gamma_B)$ and $\nu^*(\theta_{Bt}^* m - 1) W_t \partial J_{Bt} / \partial W_t$ in the drift term of the process, taking into account condition $\nu^*(\theta_{Bt}^* m - 1) = 0$. Noting that $J_B(W_t, y_t, t)$ satisfies HJB equation (15) we obtain:

$$dJ_{Bt} = J_{Bt}[\mu_{Jt}dt + \sigma_{Jt}dw_t], \quad (\text{C12})$$

where drift μ_J and volatility σ_J are given by:

$$\mu_{Jt} = -e^{-\rho t} \frac{(c_{Bt}^*)^{1-\gamma_B}}{1-\gamma_B} \frac{1}{J_{Bt}}, \quad \sigma_{Jt} = (1-\gamma_B)\theta_{Bt}^* \sigma_t - \gamma_B y_t \sigma_{yt} \frac{\Phi'_B(y)}{\Phi_B(y)}. \quad (\text{C13})$$

From the F.O.C. for consumption we obtain that $e^{-\rho t} (c_B^*)^{1-\gamma_B} / (1-\gamma_B) = J_B / \Phi_B(y)$, and from the expression for the optimal portfolio weight (A3) we obtain that $y\sigma_y\Phi'_B(y)/\Phi_B(y) = (\mu-r+\nu^*)/\sigma - \gamma_B\theta_B^*\sigma$. Substituting these expressions back into expressions (C13), we obtain:

$$\mu_{Jt} = -\frac{1}{\Phi_B(y)}, \quad \sigma_{Jt} = \theta_{Bt}^* - \frac{\mu_t - r_t + \nu_t^*}{\sigma_t}. \quad (\text{C14})$$

By assumptions of Lemma 4 σ_J satisfies Novikov's condition, and hence process $d\eta_t = \eta_t \sigma_{Jt} dw_t$ is an exponential martingale. Using the martingality of η_t from (C12) and (C14) we obtain:

$$\begin{aligned} |\mathbb{E}_t[J(W_T, y_T, T)]| &= \mathbb{E}_t\left[\frac{1}{|1-\gamma_B|} \exp\left(-\int_t^T \frac{1}{\Phi_B(y_\tau)} d\tau\right) \frac{\eta_T}{\eta_t}\right] \\ &\leq \frac{e^{-(T-t)/C_1}}{|1-\gamma_B|} \mathbb{E}_t\left[\frac{\eta_T}{\eta_t}\right] = \frac{e^{-(T-t)/C_1}}{|1-\gamma_B|}. \end{aligned} \quad (\text{C15})$$

From inequality (C15) it easily follows that $\mathbb{E}_t[J(W_T, y_T, T)] \rightarrow 0$, as $T \rightarrow \infty$.

Next, we consider the following process:

$$U_t^* = \int_0^t \frac{c_\tau^{*1-\gamma_B}}{1-\gamma_B} d\tau + J_B(W_t, y_t, t). \quad (\text{C16})$$

Applying Itô's Lemma to process U_t^* we find its drift and volatility. We note, that the drift is given by expression (C9) evaluated at $c_t = c_t^*$, and $\theta_t = \theta_{Bt}^*$. Substituting c_t^* and θ_{Bt}^* into this expression, from the fact that J_{Bt} satisfies HJB equation (15) and the complementary slackness condition $\nu^*(\theta_{Bt}^* m - 1) = 0$ we observe that drift (C9) is zero. Consequently, U_t^* follows a process:

$$dU_t^* = U_t^* \frac{J_{Bt}}{U_t^*} \left(\theta_{Bt}^* - \frac{\mu_t - r_t + \nu_t^*}{\sigma_t} \right) dw_t. \quad (\text{C17})$$

Since $0 < J_{Bt}/U_t^* < 1$, and given the assumptions of Lemma 4, the volatility of process U_t^* satisfies Novikov's condition, and hence U_t^* is an exponential martingale. Consequently, integrating (C17) from t to T and taking the expectations on both sides, after some algebra we obtain:

$$J_B(W_t, y_t, t) = \mathbb{E}_t \left[\int_t^T \frac{(c_\tau^*)^{1-\gamma_B}}{1-\gamma_B} d\tau \right] + \mathbb{E}_t \left[J_B(W_T, y_T, T) \right]. \quad (\text{C18})$$

Next, we pass to the limit $T \rightarrow +\infty$ in (C18). In the limit, the last term in (C18) vanishes due to inequality (C15), while the first term converges to (C6) by monotone convergence theorem. \square

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