ISSN 0956-8549-745

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FINANCIAL MARKETS GROUP
DISCUSSION PAPER No 745

June 2015

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Multi-Asset Noisy Rational Expectations Equilibrium with Contingent Claims*

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Abstract

We consider a noisy rational expectations equilibrium in a multi-asset economy populated by informed and uninformed investors, and noise traders. We relax the usual assumption of normally distributed asset payoffs and allow for assets with very general payoff distributions, including non-redundant contingent claims, such as options and other derivatives. We provide necessary and sufficient conditions under which contingent claims provide information about the source of uncertainty in the economy and, hence, reduce the asymmetry of information. We also apply our results to pricing risky debt and equity and demonstrate that firms cannot manipulate the information contained in debt and equity prices by changing the face value of debt. Our paper provides a new tractable framework for studying asset prices under asymmetric information. When the market is complete, we derive the equilibrium in closed form. When the market is incomplete, we derive it in terms of easily computable inverse functions.

Journal of Economic Literature Classification Numbers: D82, G12, G14.

Keywords: asymmetric information, rational expectations, learning from prices, contingent claims, derivative securities.

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*We are grateful to Ulf Axelson, Suleyman Basak, Bruno Biais, Margaret Bray, Adrian Buss, Amil Dasgupta, Bernard Dumas, Diego Garcia, Sergey Glebkin, Denis Gromb, Stephen Figlewski, Thierry Foucault, Joel Hasbrouck, Christian Julliard, Leonid Kogan, Igor Makarov, Semyon Malamud, David Martimort, Giovanna Nicodano, Romans Pancs, Anna Pavlova, Marcel Rindisbacher, Dimitri Vayanos, Wei Xiong, Liyan Yang and seminar participants at ESSFM in Gerzensee (evening session), European Winter Meeting of the Econometric Society, FIRS Conference, NYU Stern Microstructure Conference, Birkbeck University of London, HEC Paris, INSEAD, London School of Economics, and the University of Reading for helpful comments. We also thank the Paul Woolley Centre at the LSE for financial support. All errors are our own.

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1. Introduction

The informational role of prices has been in the forefront of the economic literature since the seminal work of Hayek (1945). Investors in financial markets use their private information to extract gains from trading financial securities. Their trades impound information into the prices of assets, from which the information can be partially recovered by other investors. Moreover, informed investors often trade in a multitude of correlated securities, which creates a diffusion of information across securities and makes their prices interdependent because the price of each security can assist in inferring the payoff distribution of any other. The economic literature typically studies the informational role of prices in restrictive settings with normally distributed asset payoffs, which do not allow studying markets for assets with positive and state-contingent payoffs, such as stock options. In this paper, we propose a multi-asset noisy rational expectations equilibrium (REE) model where private information can be contained in the prices of all securities and where payoffs of securities can be positive and contingent on the payoffs of other securities.

We consider a multi-asset economy with two dates and a finite but arbitrary number of discrete states. The probabilities of states are functions of an aggregate shock with a certain prior distribution. The asset payoffs can have strictly positive payoffs and can be derivative securities, such as options. The economy is populated by three groups of investors, informed and uninformed investors with constant relative risk aversion (CARA) preferences over terminal wealth, and noise traders with random exogenous asset demands. The informed investors observe the realization of the aggregate shock whereas the uninformed investors use asset prices to extract the information about the shock, which is obfuscated by the noise traders.

We solve for equilibrium asset prices and investor's portfolios in closed form when the number of assets equals the number of states of the economy, and in terms of easily computable inverse functions when there are fewer assets than states and certain additional conditions are satisfied, as elaborated below. We show that prices are non-linear functions of the shock and the noise trader demands, derive new conditions for the existence of rational expectations equilibrium (REE), and provide economic applications of our theory. The tractability of our analysis is facilitated by the structure of the probabilities of states, inspired by logit models in econometrics. This structure is such that the log-likelihood ratios are linear functions of the aggregate shock with coefficients interpreted as shock sensitivities. When the number of states equals the number of assets, we relax the distributional assumptions and solve the model in closed form for general probabilities and distributions

of shock and noise trader demands satisfying only mild conditions. Distributions of such generality have never been studied in the related literature.

Below, we discuss the main results. We start with identifying the economic channels through which the information is incorporated into prices and then derive a set of conditions under which asset prices reveal information about the aggregate shock. The analysis is aided by a tractable expression for the informed investor's portfolio of assets which we decompose into information-sensitive and information-insensitive components. This decomposition yields hitherto unexplored economic insight on how the informed investors extract gains from their superior information. In particular, our analysis reveals that the information-sensitive component of the portfolio can be regarded as a portfolio that replicates the sensitivities of probabilities of states to the aggregate shock. The intuition is that holding such a replicating portfolio allows the informed investor to have more wealth in more likely states.

The portfolio decomposition in our economy is achieved by introducing new *informational spanning condition* under which the probability sensitivities to the aggregate shock are spanned by the tradable assets. This condition is always satisfied in economies where the number of assets equals the number of states, which we label as complete-market economies, and also in many plausible incomplete-market economies with a single risky asset [e.g., economies in Grossman and Stiglitz (1980) and Breon-Drish (2014)], economies with firms that issue risky debt and equity, and various other economies with derivative securities. Therefore, all these economies are included as special cases of our model.[1] The informational spanning condition underscores the importance of the motive to replicate the probability sensitivities to shocks, discussed above.

Guided by our portfolio decomposition, we identify and disentangle two channels through which the information is impounded into asset prices: the information-sensitive demand of the informed investor and potential correlation of noise trader demands across different markets. The first channel directly incorporates the information about the shock into prices via the market clearing conditions. The second channel allows the uninformed investor to infer the structure of noise correlations from prices and use it for more efficient extraction of the aggregate shock from noisy prices.[2]

[1]Although some of these economies have continuous state-space whereas we use discrete state-space in our baseline analysis, we show that these economies can be obtained as special cases of ours when the number of states of the economy increases to infinity.

[2]The informational effects of the correlation of noises can be easily illustrated in an extreme case with identical noises across all assets. In such an economy, the market clearing conditions for securities can be interpreted as a system of multiple equations with only two unknowns: the shock and the noise. Therefore, under certain conditions, there is enough information to recover the shock and the noise from prices.

Our intuition on holding a portfolio that replicates shock sensitivities, elaborated above, yields a surprising result: only those assets that help span the probability sensitivities to the aggregate shock have trading volumes that depend on the realization of the shock. If noise trader demands are uncorrelated across different markets, then all other assets are *informationally redundant* in the sense that their trading volumes and prices do not reveal new information despite the fact that they are non-redundant from the perspective of completing the market. We identify precise conditions under which asset prices reveal information about the shock. We also show that informational redundancy is a generic property of certain economies with one risky asset [e.g., the economies in Grossman and Stiglitz (1980) and Breon-Drish (2014)] in the sense that adding any derivatives to these economies does not reveal more information about the economic states.

Our second set of results is on the interaction between the asymmetry of information and market incompleteness. We show that when the informational spanning condition is satisfied, the information-sensitive component of the informed investor's portfolio in incomplete-market economies looks similar to its complete-market counterpart. This is because the informational spanning condition alleviates the effects of market incompleteness on risk sharing by increasing the efficiency with which the informed investor can allocate more wealth to more likely states. As a result, the asset prices in incomplete and complete markets have similar properties. Therefore, our closed-form prices in the latter economy emerge as tractable alternatives to prices in the former.

Finally, we apply our theory to study the prices of risky debt and equity of a firm with an exogenous cash flow at the terminal date in the presence of asymmetric information. We provide closed-form expressions for the asset prices and explore the violation of the capital structure irrelevance result of Modigliani and Miller (1958) in our economy. We show that the violation is mainly due to the price pressure effect of noise traders which makes debt and equity prices nonlinear functions of the face value of debt. The asymmetry of information significantly affects the level of asset prices and amplifies the nonlinear dependence of prices on the face value of debt. However, we establish an informational analogue of the irrelevance result according to which firms cannot manipulate the amount of information about cash flows inferred from their debt and equity prices by manipulating the face value of debt. The intuition is that debt and equity replicate firm cash flows with equal portfolio weights that do not depend on the face value of debt, and hence the information-sensitive demand for these assets remains unchanged.

The paper makes several methodological contributions. First, we provide a unifying framework that includes as special cases some of the previous models with one or multiple

risky asset. Second, we obtain new closed-form prices and portfolios in a multi-asset economy with asymmetric information. When the market is complete, we derive these expressions for very general probability density functions of shocks, noise trader demands and asset cash flows that have never been studied in the related literature on asymmetric information. Third, when the market is complete, we extend the no-arbitrage valuation to economies with asymmetric information. As a result, the asset prices are given by expected discounted cash flows under the risk-neutral measure. Finally, we prove the existence of REE in an incomplete-market economy. The proof is significantly complicated by the presence of multiple assets which makes it impossible to employ the intermediate value theorem as in economies with one risky asset. Therefore, we devise a new method based on a global implicit function theorem. Overall, due to its tractability, our model emerges as a convenient building block for economic research on asymmetric information.

Our paper is related to large literature on noisy REE models, which were pioneered by Grossman (1976), Grossman and Stiglitz (1980) and Hellwig (1980). These early works typically consider so called CARA-normal economies with CARA investors and one risky asset with normally distributed payoffs. Admati (1985) extends these models to the case of multiple securities and shows that many of their insights cannot be extrapolated to the multi-asset case. Wang (1993) develops a dynamic REE model and shows that uniformed investors increase the risk premium. Both Admati (1985) and Wang (1993) demonstrate that asset demands can be upward sloping due to asymmetric information, which also happens in our model. Diamond and Verrecchia (1981), Marín and Rahi (2000), Vives (2008), García and Urošević (2013) and Kurlat and Veldkamp (2013) discuss further extensions of CARA-normal models. Banerjee and Green (2014) extend the analysis to economies with mean-variance preferences and an uncertain number of informed investors. In contrast to this literature, we allow for more general payoff distributions and derivative securities.

There is a growing literature that departs from CARA-normal frameworks. Yuan (2005) studies a two-state non-linear REE where cash flows are normally distributed but the states are endogenously determined by prices, leading to truncated normal payoffs. Breon-Drish (2010) and Breon-Drish (2014) study economies with CARA investors but without normality. The latter work derives prices in terms of tractable inverse functions and proves the existence and uniqueness of equilibrium, and the former demonstrates the violation of weak-form efficiency in certain markets without normality. Bernardo and Judd (2000) solve models numerically and demonstrate that the REE in Grossman and Stiglitz (1980) is not robust to parametric assumptions. Our work differs from this literature by allowing for more general distributions and derivative securities.

Barlevy and Veronesi (2000) study a model with risk-neutral investors facing position limits and non-normal payoffs, and demonstrate that learning is a strategic complement allowing for multiplicity of equilibria. Albagli, Hellwig, and Tsyvinski (2013) consider a noisy REE model under general distributions for the fundamentals and preferences for the traders but impose position limits. Position limits lead to an equilibrium which can be characterized by the behavior of one marginal investor but also enforce limits to arbitrage, and hence, their model leads to different predictions from our no-arbitrage setup. Hassan and Mertens (2014) introduce a standard Hellwig (1980) REE model into a real business cycle model. The interaction leads to non-linearity in prices, which the authors tackle by using market completeness and an asymptotically valid higher-order expansion in state variables. Peress (2004) solves a Grossman and Stiglitz (1980) model under general preferences using a "small risk" log-linear approximation. Our paper differs from the latter two papers in that all solutions are exact for both complete and incomplete markets.

Our paper is also related to the literature that studies the informational role of derivatives. Brennan and Cao (1996) consider a CARA-normal model with one risky underlying asset and a derivative asset with a quadratic payoff. Vanden (2008) extends the latter model to the case of gamma distributions and a derivative with logarithmic payoffs but uses non-standard distributions of noise trader demands. In the latter two works derivatives do not reveal information about the underlying, although the papers do not explore whether this result is specific to the restrictive assumptions on asset and derivative payoffs. In contrast to these papers, we allow for general derivative payoffs, more general distributions, and show that derivatives do reveal valuable information under certain conditions.

Back (1993) provides a micro-foundation on stochastic volatility in a dynamic Kyle (1985) model where a single informed investor trades in the stock and a single call option. Biais and Hillion (1994) have a static model with a stock and show that the introduction of a single option can have ambiguous effects on the dissemination of information. Malamud (2014) studies an REE with options in a continuous-space complete-market economy in a paper concurrent with ours. He characterizes REE in terms of fixed points of operators and finds conditions for price discovery under general preferences.

Our paper differs from the above literature in that we consider multi-asset economies both with complete and incomplete markets, provide closed-form solutions not only for options but also for general contingent claims, and our framework is easily extendable to economies with multiple shocks. We also provide new conditions for informational redundancy of assets in terms of shock sensitivities of the probabilities of states.

The structure of the paper is as follows. Section 2 describes the model, investors'

optimizations, and distributional assumptions. Section 3 solves for equilibria both in economies with complete and incomplete financial markets. Section 4 provides several economic applications of our model. Section 5 extends the model to the case of general distributions and probabilities. Section 6 concludes. The Appendix provides the proofs.

2. Model

2.1. Securities Markets and Information Structure

We consider a single-period exchange economy with two dates $t = 0$ and $t = T$, and N states $\omega_1, \ldots, \omega_N$ at the terminal date, where $N \geq 2$. The economy is populated by three representative investors, informed and uninformed investors, labeled I and U, and noise traders. Investors I and U have CARA preferences over terminal wealth and risk aversions γ_I and γ_U.[3] The investors can trade one riskless bond in perfectly elastic supply paying \$1 at T and $M-1 \geq 1$ zero net supply risky assets with state-contingent terminal payoff $C_m(\omega_n)$ in state ω_n, where $m = 1, \ldots, M-1$ and $n = 1, \ldots, N$. These assets can be Arrow-Debreu securities, options, or other derivative securities, and are assumed to be non-redundant in the sense that no asset has payoffs that can be replicated by trading other assets. The investors are competitive and do not have impact on prices.

We work with a discrete state-space because it yields the equilibrium in terms of easily solvable systems of equations instead of less tractable operator equations when the state-space is continuous. Using several examples, we later show that economies with a continuous state-space can be obtained in the limit of our economy when N goes to infinity.

The probabilities of states ω_n are functions of a shock $\varepsilon \in \mathbb{R}$, and are denoted by $\pi_n(\varepsilon)$. Shock ε has a prior probability density function (PDF) $\phi_\varepsilon(x)$. We think of ε as an aggregate shock that affects the probabilities of states ω and, hence, the payoff distributions of all assets in the economy. Before the markets open (i.e., at time $t = -1$), the informed investor observes ε. The uninformed investor observes only the asset prices at time $t = 0$. Noise traders have exogenous random demands $v = (v_1, \ldots, v_{M-1})^T$ with multivariate normal distribution $N(0, \Sigma_v)$, where Σ_v is a $(M-1) \times (M-1)$ symmetric positive-definite matrix. Random demands v prevent asset prices from being fully revealing.

We denote the vector of observed prices of the risky assets at time $t = 0$ by $p =$

[3]CARA preferences allow us to consider each class of investors with the same information as representative of many CARA investors with different wealths and risk aversions [see the aggregation theorem in Rubinstein (1974, p. 232)].

$(p_1, \ldots, p_{M-1})^T$, the vector of risky asset payoffs in state ω_n by $\Pi_n = (C_1(\omega_n), \ldots, C_{M-1}(\omega_n))^T$, and the vector of asset m's payoffs in different states by $C_m = (C_m(\omega_1), \ldots, C_m(\omega_N))^T$. The price of the riskless asset is set to $p_0 = e^{-rT}$, where r is an exogenous risk-free rate of return.[4] The prices of risky assets are determined in equilibrium. Finally, by $P(\varepsilon, v) \in \mathbb{R}^{M-1}$ we denote the vector of equilibrium prices as functions of shock ε and noise v.

2.2. Investors' Optimization and Definition of Equilibrium

Each investor $i = I, U$ is endowed with initial wealth $W_{i,0}$, and allocates it to buy α_i units of the riskless asset and $\theta_{i,m}$ units of risky asset m. By $\theta_i = (\theta_{i,1}, \ldots, \theta_{i,M-1})^T$ we denote the vector of units of risky assets purchased by investor i. The budget constraints of investors I and U at time $t = 0$ are given by $W_{i,0} = \alpha_i p_0 + p^T\theta_i$. Investor i's wealth at time $t = T$ and state n is then given by $W_{i,T,n} = \alpha_i + \Pi_n^T\theta_i$. In what follows, the subscript n will be dropped to denote a random variable in an uncertain state. Substituting out α_i we obtain the budget constraint in the following form: $W_{i,T} = W_{i,0}e^{rT} + (\Pi - e^{rT}p)^T\theta_i$. The informed and the uninformed investors solve the following utility maximization problems

$$\max_{\theta_I} \mathbb{E}\left[-e^{-\gamma_I W_{I,T}} \,\middle|\, \varepsilon, p\right], \tag{1}$$

$$\max_{\theta_U} \mathbb{E}\left[-e^{-\gamma_U W_{U,T}} \,\middle|\, P(\varepsilon, v) = p, p\right], \tag{2}$$

respectively, subject to each investors' self-financing budget constraint

$$W_{i,T} = W_{i,0}e^{rT} + (\Pi - e^{rT}p)^T\theta_i, \quad i = I, U. \tag{3}$$

The solutions to the above optimization problems give investors' optimal portfolios of risky assets $\theta_I^*(p;\varepsilon)$ and $\theta_U^*(p)$. The prices p should be such that all the markets for the risky securities clear. More formally, the definition of equilibrium is as follows.

Definition 1. *A competitive noisy rational expectations equilibrium is a set of asset prices $P(\varepsilon; v)$ and investor asset holdings $\theta_I^*(p;\varepsilon)$ and $\theta_U^*(p)$ such that θ_I^* and θ_U^* solve optimization problems* (1) *and* (2) *subject to self-financing budget constraints* (3)*, taking asset prices as given, and the market clearing conditions are satisfied:*

$$\theta_I^*(P(\varepsilon, v); \varepsilon) + \theta_U^*(P(\varepsilon, v)) + v = 0. \tag{4}$$

[4] In models with utility over terminal wealth risk-free rate r is indeterminate and is set exogenously.

2.3. Probability Distributions

To solve the model in closed form, we consider probabilities of states $\pi_n(\varepsilon)$ given by:

$$\pi_n(\varepsilon) = \frac{e^{a_n+b_n\varepsilon}}{\sum_{j=1}^{N} e^{a_j+b_j\varepsilon}}, \quad n = 1, \ldots, N. \tag{5}$$

The structure of probabilities is similar to that of probabilities in multinomial logit models, widely used in econometrics. When $\varepsilon = 0$, by properly choosing parameters a_n, states ω_n can have a multinomial distribution that approximates a particular desired continuous distribution in the limit as $N \to \infty$. We label vector $b = (b_1, \ldots, b_N)^T$ as the *probability sensitivities to the aggregate shock* ε, because it determines the deviations of probabilities $\pi_n(\varepsilon)$ from benchmark probabilities $\pi_n(0)$ in response to shock ε.

Shock ε is a scalar random variable with generalized normal distribution $\bar{N}(\mu_\varepsilon, \sigma_\varepsilon^2)$, mean $\mathbb{E}[\varepsilon] = \mu_\varepsilon$ and variance $\mathrm{var}[\varepsilon] = \sigma_\varepsilon^2$, which has PDF $\phi_\varepsilon(x)$ given by:

$$\phi_\varepsilon(x) = \frac{\left(\sum_{j=1}^{N} e^{a_j+b_jx}\right) e^{-0.5(x-\mu_0)^2/\sigma_0^2}}{\int_{-\infty}^{\infty}\left(\sum_{j=1}^{N} e^{a_j+b_jx}\right) e^{-0.5(x-\mu_0)^2/\sigma_0^2}dx}. \tag{6}$$

Distribution (6) allows us to obtain the equilibrium in terms of elementary functions. This distribution is given in terms of vectors $a = (a_1, \ldots, a_N)^T$ and $b = (b_1, \ldots, b_N)^T$, and scalars μ_0 and σ_0^2. PDF (6) can also be equivalently rewritten as a weighted average of PDFs of normal distributions. For fixed parameters a and b, we choose μ_0 and σ_0^2 so that ε has any desired mean μ_ε and variance σ_0^2. The relationship between (μ_0, σ_0^2) and $(\mu_\varepsilon, \sigma_\varepsilon^2)$ is given by Equations (A.1) and (A.2) in the Appendix. In Section 5, we extend the analysis to general probabilities $\pi_n(\varepsilon)$ and PDFs $\phi_\varepsilon(x)$ and $\phi_v(x)$ for shock ε and noisy demands v when the number of states equals the number of securities, that is, $N = M$.

Our economy includes as a special case a one-asset economy in Breon-Drish (2014) in which the informed investor receives signal ε about asset payoff C at the terminal date, payoff C has general unconditional PDF $\phi_C(x)$ and the signal is given by $\varepsilon = C + \delta$, where $\delta \sim N(\mu_0, \sigma_0^2)$. The latter economy is a limiting case (i.e., when $N \to \infty$) of ours. Consider a discretization in which the risky asset pays $C_n = \underline{C}_N + h(n-1)$ in states $n = 1, 2, \ldots, N$, where $h = (\bar{C}_N - \underline{C}_N)/(N-1)$, $\underline{C}_N$ and $\bar{C}_N$ are the minimum and the maximum payoffs that can be bounded or unbounded in the limit, and payoff C_n has unconditional probability $\phi_C(C_n)/(\phi_C(C_1) + \ldots + \phi_C(C_N))$. Then, using standard expressions for conditional distributions, it can be shown that conditional probabilities $\mathrm{Prob}(C_n|\varepsilon)$ correspond to probabilities $\pi_n(\varepsilon)$ given by Equation (5) and the PDF of signal ε is given by (6), where $a_n = -0.5C_n^2/\sigma_0^2 - C_n\mu_0/\sigma_0^2 + \ln\phi_C(C_n)$ and $b_n = C_n/\sigma_0^2$.

The structures of probabilities (5) and PDF (6) are reasonable for two reasons. First, as demonstrated above, these structures endogenously emerge in plausible economic settings in which payoff C has general PDF $\phi_C(x)$ and the informed investor receives signal $\varepsilon = C+\delta$, where $\delta \sim N(\mu_0, \sigma_0^2)$. Second, by varying vectors a and b, the distribution of cash flows conditional on observing ε has flexible shapes, and PDF $\phi_\varepsilon(x)$ is sufficiently close to a normal distribution in our calibrations, as elaborated below. Therefore, the fact that distribution $\phi_\varepsilon(x)$ depends on vectors a and b has only minor effect on our results.

Figure 1 shows the distribution of cash flows C_n conditional on knowing shock ε (i.e., probabilities $\pi_n(\varepsilon)$ plotted against payoffs C_n), and PDF $\phi_\varepsilon(x)$ for an example with $N = 100$ states. The asset payoffs are given by $C_n = 300(n-1)/(N-1)$. Vectors a and b are calibrated in such a way that $\pi_n(1)$ and $\pi_n(-1)$ are discrete approximations of gamma distributions with shape and scale parameter pairs (1,2) and (5,1), respectively, and parameters μ_0 and σ_0 are chosen in such a way that $\mu_\varepsilon = 0$, $\sigma_\varepsilon = 1$. Panel (b) shows function $\phi_\varepsilon(x)$ along with the PDF of a normal distribution $N(\mu_\varepsilon, \sigma_\varepsilon^2)$ for $\mu_\varepsilon = 0$, $\sigma_\varepsilon = 1$. From Panel (b) we observe that the two distributions are very close to each other.

One of the disadvantages of standard CARA-normal models in the literature is that asset payoffs can be negative. Moreover, it is very difficult to include assets with nonlinear payoffs, such as put and call options. Our model is free from these disadvantages, and allows extra flexibility in modeling probability distributions and asset payoffs. To the best of our knowledge ours is the first noisy REE model that admits closed form solutions in the multi-asset case and where joint normality of asset payoffs is not required.

Remark 1 (Grossman-Stiglitz economy as a special case). An important special limiting case (when $N \to \infty$) of our economy is the standard CARA-normal Grossman and Stiglitz (1980) economy with one risky asset with payoff $C(\omega) \sim N(\varepsilon, \sigma_C^2)$, an informed investor who observes mean $\mathbb{E}[C(\omega)] = \varepsilon$, an uninformed investor with prior distribution $o \sim N(\mu_\varepsilon, \sigma_\varepsilon^2)$, noise traders, and no costs of information acquisition. Such an economy can be approximated in our framework as follows. Let $M = 2$ and consider an asset with payoffs $C(\omega_n) = -A + nh$, where $h = 2A/N$, $a_n = -0.5C(\omega_n)^2/\sigma_C^2$ and $b_n = C(\omega_n)/\sigma_C^2$, for all $n = 1, \ldots, N$. For large A and N, we observe that $\left(\sum_{i=1}^N e^{a_i + b_i\varepsilon} h\right) e^{-0.5\varepsilon^2/\sigma_C^2} \approx \int_{-\infty}^{\infty} e^{-0.5(C-\varepsilon)^2/\sigma_C^2} dC = \sqrt{2\pi\sigma_C^2}$. Consequently, as $A, N \to \infty$, using Equations (5) and (6) and some algebra, we obtain point-wise convergences

$$\pi_n(\varepsilon) = \frac{e^{-0.5(C(\omega_n)-\varepsilon)^2/\sigma_C^2} h}{\sum_{j=1}^N e^{-0.5(C(\omega_j)-\varepsilon)^2/\sigma_C^2} h} \longrightarrow \frac{e^{-0.5(C-\varepsilon)^2/\sigma_C^2} dC}{\sqrt{2\pi\sigma_C^2}}, \quad \phi_\varepsilon(x) \longrightarrow \frac{e^{-0.5(x-\mu_\varepsilon)^2/\sigma_\varepsilon^2}}{\sqrt{2\pi\sigma_\varepsilon^2}}.$$

Therefore, our economy is CARA-normal in the limit for a particular choice of parameters.

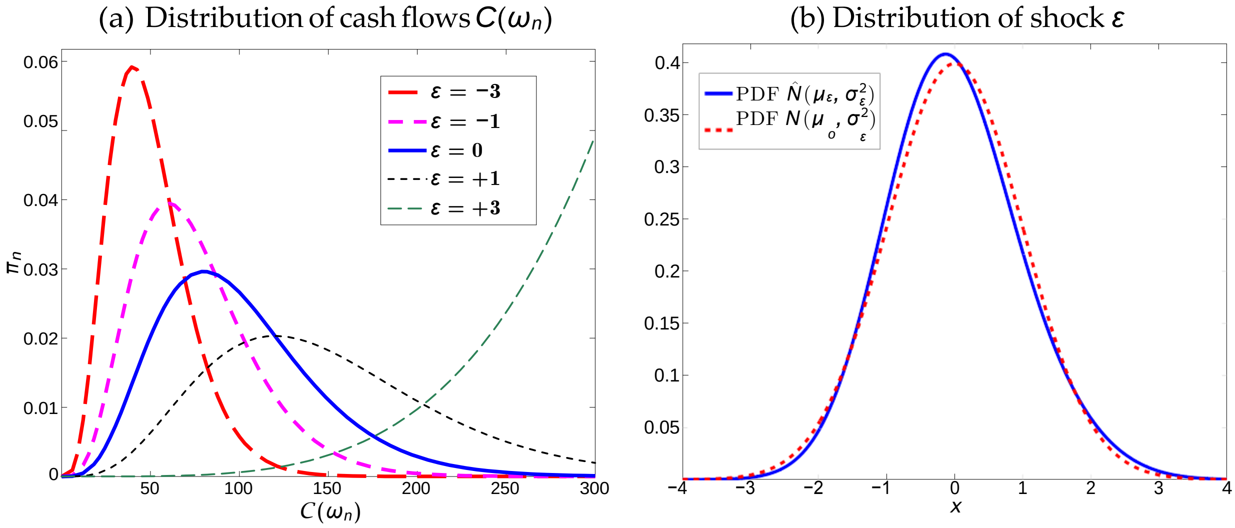

Figure 1: Distribution of cash flows and shock ε
Panel (a) shows probabilities $\pi_n(\varepsilon)$ for different ε for asset payoffs $C(\omega_n)$. Panel (b) shows the PDFs of $\bar{N}(\mu_\varepsilon, \sigma_o^2)$ and $N(\mu_\varepsilon, \sigma_\varepsilon^2)$ with $\mu_\varepsilon = 0$ and $\sigma_\varepsilon = 1$. We set $N = 100$, $C(\omega_n) =$ 1) and vectors a and b are such that $\pi_n(-$ 1) and $\pi_n(1)$ are discrete approximations of gamma distributions with shape and scale parameter pairs (1,2) and (5,1), respectively.

Remark 2 (Multi-dimensional shock ε). Our model can be generalized to the case of multi-dimensional shock ε. In this case, the probabilities of states ω_n are given by $\pi_n(\varepsilon) = \exp(a_n + b_n^T \varepsilon) / \sum_{j=1}^{N} \exp(a_j + b_j^T \varepsilon)$, where b_n are now vectors. This model includes a CARA-normal model with multiple correlated assets as a special case, which can be shown similarly to the case of a scalar shock ε. The model with multi-dimensional shock ε generalizes the multi-asset CARA-normal model with asset payoffs as in Admati (1985).

3. Characterization of Equilibrium

In this section, we first consider an economy with $M = N$ assets and find the equilibrium in closed form. Then, we consider a general economy with $M \leq N$ securities and, under an additional assumption, find asset prices in terms of easily computable inverse functions.

3.1. Economy with $M = N$ Securities

We start with an economy in which the number of assets equals the number of states, i.e., $M = N$, which we label as a *complete-market economy*. As demonstrated in Ross (1976),

markets can be completed by issuing a sufficient number of derivative securities.[5] In our model, derivative securities can reveal additional information about the underlying asset, which in turn can be used for more accurate pricing of derivatives. Therefore, the prices of all risky assets have to be found simultaneously.

Due to market completeness, we look for equilibrium prices p in the following form:

$$p_m = \left(\pi_1^{\text{RN}} C_m(\omega_1) + \pi_2^{\text{RN}} C_m(\omega_2) + \ldots + \pi_N^{\text{RN}} C_m(\omega_N) \right) e^{-rT}, \tag{7}$$

where $m = 1, \ldots, N-1$, and π_n^{RN} is the *risk-neutral* probability of state ω_n. The risk-neutral probabilities exist in equilibrium because the investors are unconstrained and can eliminate any arising arbitrage opportunities [e.g., Duffie (2001)]. Moreover, all investors agree on risk-neutral probabilities because these probabilities are uniquely determined from Equations (7) as functions of prices p.

Due to asymmetric information, investors I and U have different real probabilities of states ω_n. In particular, because investor U filters out shock ε from the market clearing condition (4), her real probabilities of states ω_n are given by conditional expectations of $\pi_n(\varepsilon)$. To demonstrate this, we rewrite the expected utility of investor U as follows:

$$\begin{aligned} \mathbb{E}\left[-e^{-\gamma_U W_{U,T}} \,\middle|\, P(\varepsilon, v) = p, p \right] &= -\sum_{n=1}^{N} \mathbb{E}\left[\pi_n(\varepsilon) \,\middle|\, P(\varepsilon, v) = p, p \right] e^{-\gamma_U W_{U,T,n}} \\ &= -\sum_{n=1}^{N} \pi_n^U(p; \theta_U^*(p)) e^{-\gamma_U W_{U,T,n}}, \end{aligned} \tag{8}$$

where $\pi_n^U(p; \theta_U^*(p)) = \mathbb{E}\left[\pi_n(\varepsilon) \,|\, P(\varepsilon, v) = p, p \right]$ is investor U's posterior probability of state ω_n. The latter probabilities, in general, depend on equilibrium portfolios $\theta_U^*(p)$ through the market clearing conditions, as shown below. Objective function (8) confirms that investor U's optimization can be solved as a complete-market problem with N states and N securities in which real probabilities $\pi_n^U(p; \theta_U^*(p))$ are taken as given.

The investors have different state price densities (SPDs) because they have different real probabilities, similar to models with heterogeneous beliefs [e.g., Basak (2005)]. The SPDs are given by the following equations [e.g., Duffie (2001)]:

$$\xi_I(\omega_n) = \frac{\pi_n^{\text{RN}} e^{-rT}}{\pi_n(\varepsilon)}, \qquad \xi_U(\omega_n) = \frac{\pi_n^{\text{RN}} e^{-rT}}{\pi_n^U(p; \theta_U^*(p))}. \tag{9}$$

The first order conditions (FOCs) of investors equate their marginal utilities and SPDs

[5] We note that the realizations of shock ε can be interpreted as a continuum of states of the economy in addition to states ω_n. However, N tradable non-redundant assets still suffice to replicate any contingent claim in our economy because the payoffs of such claims do not vary across ε for a fixed state ω_n. In other words, ε-states can be clumped together in such a way that only ω_n states matter for replication.

and are given by the following equations:

$$\gamma_I e^{-\gamma_I W_{I,T,n}} = f_{,I}\frac{\pi_n^{\mathrm{RN}} e^{-rT}}{\pi_n(\varepsilon)}, \quad \gamma_U e^{-\gamma_U W_{U,T,n}} = f_{,U}\frac{\pi_n^{\mathrm{RN}} e^{-rT}}{\pi_n^U(p;\theta_U^*(p))}, \tag{10}$$

where $f_{,i}$ denote Lagrange multipliers for investors' budget constraints. First order conditions (10) give us optimal wealths $W_{i,T}$, from which we can recover optimal trading strategies using investors' budget constraints (3). Lemma 1 below reports the results.

Lemma 1 (Investors' optimal portfolios).

1) *Suppose, probabilities $\pi_n(\varepsilon)$ and PDF $\phi_\varepsilon(x)$ are general functions (not necessarily as in Section 2.3) such that the equilibrium exists. Then, optimal portfolios of informed and uninformed investors, $\theta_I^*(p;\varepsilon)$ and $\theta_U^*(p)$, are given by:*

$$\theta_I^*(p;\varepsilon) = \frac{1}{\gamma_I}\Omega^{-1}\left(\left(\ln\frac{\pi_1(\varepsilon)}{\pi_N(\varepsilon)}, \ldots, \ln\frac{\pi_{N-1}(\varepsilon)}{\pi_N(\varepsilon)}\right)^T - v(p)\right), \tag{11}$$

$$\theta_U^*(p) = \frac{1}{\gamma_U}\Omega^{-1}\left(\left(\ln\left(\frac{\pi_1^U\left(p;\theta_U^*(p)\right)}{\pi_N^U\left(p;\theta_U^*(p)\right)}\right), \ldots, \ln\left(\frac{\pi_{N-1}^U\left(p;\theta_U^*(p)\right)}{\pi_N^U\left(p;\theta_U^*(p)\right)}\right)\right)^T - v(p)\right), \tag{12}$$

where probabilities $\pi_N^U(p;\theta_U^*(p))$ and function $v(p)$ are given by the following equations

$$\pi_n^U\left(p;\theta_U^*(p)\right) = \mathbb{E}[\pi_n(\varepsilon)\,|\,P(\varepsilon,v)=p,p], \tag{13}$$

$$v(p) = \left(\ln\frac{\pi_1^{\mathrm{RN}}}{\pi_N^{\mathrm{RN}}}, \ldots, \ln\frac{\pi_{N-1}^{\mathrm{RN}}}{\pi_N^{\mathrm{RN}}}\right)^T, \tag{14}$$

probabilities π_n^{RN} are functions of p that solve Equations (7), $\Omega \in \mathbb{R}^{(N-1)\times(N-1)}$ is a matrix with rows $(\Pi_n - \Pi_N)^T$ and elements $\Omega_{n,k} = C_k(\omega_n) - C_k(\omega_N)$, where $k, n = 1, \ldots, N-1$.

2) *If probabilities $\pi_n(\varepsilon)$ are given by Equation (5) and $\phi_\varepsilon(x)$ is an arbitrary PDF, then investor I's optimal portfolio is a linear function of shock ε, given by:*

$$\theta_I^*(p;\varepsilon) = \frac{\lambda\varepsilon}{\gamma_I} - \frac{1}{\gamma_I}\Omega^{-1}\left(v(p) - \tilde{a}\right), \tag{15}$$

where $\tilde{a} = (a_1 - a_N, \ldots, a_{N-1} - a_N)^T \in \mathbb{R}^{N-1}$ and $\lambda = \Omega^{-1}(b_1 - b_N, \ldots, b_{N-1} - b_N)^T \in \mathbb{R}^{N-1}$.

Lemma 1 determines optimal portfolios of investors in terms of real and risk-neutral probabilities for the case of complete markets. Its most important implication is that due to CARA preferences informed investor's portfolio $\theta_I^*(p;\varepsilon)$ is separable in shock ε and price p. Moreover, the latter portfolio is linear in shock ε in economies where probabilities $\pi_n(\varepsilon)$ are given by Equation (5), which can be easily demonstrated by substituting probabilities (5) into portfolio (11). We label the first terms of portfolios (11) and (15) as

information-sensitive demands because they depend on shock ε and the second terms as information-insensitive demands.

The intuition for the linear information-sensitive demand in Equation (15) is as follows. FOC of investor I in (10) demonstrates that, holding risk-neutral probabilities fixed, investor I has an incentive to allocate more wealth to states with higher real probabilities $\pi_n(\varepsilon)$, that is, to states with higher $a_n + b_n\varepsilon$. The definition of λ in Lemma 1 implies that λ solves a system of equations $b_n = \lambda_0 + \Pi_n^T\lambda$, where Π_n is the vector of risky asset payoffs in state ω_n, and λ_0 is a constant. Consequently, λ can be interpreted as a portfolio that replicates the vector of shock sensitivities b up to a constant λ_0, and hence $\lambda\varepsilon$ replicates $b\varepsilon$. Similarly, portfolio $\Omega^{-1}\tilde{a}$ replicates sensitivities a_n. Therefore, investing in the latter two portfolios gives investor I more wealth in states with high probabilities $\pi_n(\varepsilon)$. Furthermore, FOCs (10) reveal that investors have incentive to allocate less wealth to states with high risk-neutral probabilities π_n^{RN} because $\pi_n^{\text{RN}}e^{-rT}$ is the value of \$1 in state ω_n. The price effect gives rise to the information-insensitive term $\Omega^{-1}v(p)/\gamma_I$. Portfolio (15) then reflects the relative strength of the effects of probabilities $\pi_n(\varepsilon)$ and π_n^{RN}.

The linearity of portfolio (15) simplifies the filtering problem of investor U. In particular, substituting $\theta_I^*(p;\varepsilon)$ and $\theta_U^*(p)$ into the market clearing condition (4), we find that

$$\frac{\lambda\varepsilon}{\gamma_I} + v + H(p) = 0, \tag{16}$$

where $H(p)$ is a function of prices p, given by the following equation:

$$H(p) = \theta_U^*(p) - \frac{1}{\gamma_I}\Omega^{-1}\big(v(p) - \tilde{a}\big), \tag{17}$$

where $\tilde{a}$ and $v(p)$ are defined in Lemma 1. Equation (16) demonstrates that observing prices p allows investor U to infer a linear combination of shocks $\lambda\varepsilon/\gamma_I + v$. In this paper, we focus on equilibria in which asset prices are continuous functions of shock ε and noisy demand v, and $\lambda\varepsilon/\gamma_I + v$ is the only information revealed by asset prices.[6]

The posterior distribution of ε after observing $\lambda\varepsilon/\gamma_I + v$ is available in closed form when the shock has distribution $\varepsilon \sim N(\mu_\varepsilon, \sigma_\varepsilon^2)$, which allows us to compute investor U's posterior probabilities $\pi_n^U(p;\theta_U^*(p))$ also in closed form. Lemma 2 reports the results.

Lemma 2 (Conditional distributions). *Let probabilities $\pi_n(\varepsilon)$ be given by Equation* (5)*, and shock ε has PDF given by* (6)*. Let $\tilde{\varepsilon} = \lambda\varepsilon/\gamma_I + v + H(p)$, i.e. the left-hand side*

[6]Pálvölgyi and Venter (2014) demonstrate that there exist multiple discontinuous equilibria in Grossman and Stiglitz (1980) economy. Such equilibria may also exist in our model. The exact characterization of all equilibria is a challenging task in the case of multiple assets and is beyond the scope of our work.

of (16). *Then, the posterior PDF* $\phi_{\varepsilon|\tilde{\xi}}(x|y)$ *of shock* ε*, conditional on observing vector* $\tilde{\xi}$*, and posterior probabilities* $\pi_n^U(p;\theta_U^*(p))$ *of investor U are given by:*

$$\phi_{\varepsilon|\tilde{\xi}}(x|y) = \frac{\exp\Big\{-0.5\big(y - \lambda x/\gamma_I - H(p)\big)^T \Sigma_v^{-1}\big(y - \lambda x/\gamma_I - H(p)\big)\Big\}\phi_\varepsilon(x)}{G_1(y;p)}, \tag{18}$$

$$\pi_n^U(p;\theta_U^*(p)) = \frac{1}{G_2(p)}\exp\left(a_n + \frac{1}{2}\frac{b_n^2 - 2b_n\big(\lambda^T\Sigma_v^{-1}H(p)/\gamma_I - \mu_0/\sigma_0^2\big)}{\lambda^T\Sigma_v^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2}\right), \tag{19}$$

where function $H(p)$ *is given by Equation* (17)*, and* $G_1(y;p)$ *and* $G_2(p)$ *are certain functions that do not depend on state* ω_n*.*

Lemma 2 provides closed-form expressions for the posterior distribution of shock ε and posterior probabilities $\pi_n^U(p;\theta_U^*(p))$. We observe that when the prior precision of shock ε increases, that is, volatility parameter σ_0 of PDF $\phi_\varepsilon(x)$ in (6) converges to zero, then PDF (6) converges to a delta-function, mean parameter μ_0 converges to ε and probability $\pi_n^U(p;\theta_U^*(p))$ in (19) converges to conditional probability $\pi_n(\varepsilon)$. Therefore, the informed and uninformed investors have the same probabilities in the limit, and hence, the economy converges to the economy with full symmetric information.

From Lemma 2 we also observe that the log-likelihood ratio $\ln\big(\pi_n^U(p;\theta_U^*(p))/\pi_N^U(p;\theta_U^*(p))\big)$ is a linear function of $\theta_U^*(p)$, which allows us to solve the fixed point problem in Equation (12) in closed form and to obtain $\theta_U^*(p)$ as a function of vector $v(p)$. Then, vector $v(p)$ can be found from the market clearing condition (16), and the risk-neutral probabilities π_n^{RN} can be recovered from Equation (14) that expresses $v(p)$ in terms of log-likelihood ratios of probabilities π_n^{RN}. Proposition 1 reports the equilibrium in closed form.

Proposition 1 (Equilibrium with $M = N$ assets). *Suppose, probabilities* $\pi_n(\varepsilon)$ *are given by Equation* (5) *and shock* ε *has PDF* (6)*. Then, there exists unique equilibrium in which prices only reveal* $\lambda\varepsilon/\gamma_I + v$*. In this equilibrium, portfolios* $\theta_I^*(p;\varepsilon)$ *and* $\theta_U^*(p)$*, risk-neutral probabilities* π^{RN}*, and prices* $P(\varepsilon, v)$ *are given by:*

$$\theta_I^*(p;\varepsilon) = \frac{\lambda\varepsilon}{\gamma_I} - \frac{1}{\gamma_I}\Omega^{-1}\big(v(p) - \tilde{a}\big), \tag{20}$$

$$\theta_U^*(p) = \big(E + Q\big)^{-1}\left(\frac{1}{\gamma_I}Q\,\Omega^{-1}\big(v(p) - \tilde{a}\big) - \frac{1}{\gamma_U}\Omega^{-1}\big(v(p) - a\big) + \frac{(\mu_0/\sigma_0^2)\lambda}{\gamma_U\big(\lambda^T\Sigma_v^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2\big)}\right), \tag{21}$$

$$\pi_n^{RN} = \frac{e^{v_n}}{1 + \sum_{j=1}^{N-1} e^{v_j}}, \qquad \pi_N^{RN} = \frac{1}{1 + \sum_{j=1}^{N-1} e^{v_j}}, \tag{22}$$

$$P_m(\varepsilon, v) = \big(\pi_1^{RN}C_m(\omega_1) + \pi_2^{RN}C_m(\omega_2) + \ldots + \pi_N^{RN}C_m(\omega_N)\big)e^{-rT}, \tag{23}$$

where $m = 1, \dots, M-1$; $v(p)$ *is a function of observed prices* p *given by* (14) *and* $Q \in \mathbb{R}^{(N-1)\times(N-1)}$ *is a matrix, which in equilibrium are given by the following equations*

$$v(p) = \tilde{a} + \frac{1}{2}\frac{\gamma_I}{\gamma_I + \gamma_U}\frac{\tilde{b}^{(2)} + 2(\mu_0/\sigma_0^2)\Omega\lambda}{\lambda^T\Sigma_v^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2} + \frac{\gamma_I\gamma_U}{\gamma_I + \gamma_U}\Omega\big(E + Q\big)\Big(\frac{\lambda\varepsilon}{\gamma_I} + v\Big), \tag{24}$$

$$Q = \frac{\lambda\lambda^T\Sigma_v^{-1}}{\gamma_U\gamma_I\big(\lambda^T\Sigma_v^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2\big)}, \tag{25}$$

E *is the* $(N-1)\times(N-1)$ *identity matrix,* $\Omega \in \mathbb{R}^{(N-1)\times(N-1)}$ *is a matrix with rows* $(\Pi_n - \Pi_N)^T$, $\tilde{a} = (a_1 - a_N, \dots, a_{N-1} - a_N)^T$, $\hat{a} = \tilde{a} + 0.5\tilde{b}^{(2)}/(\lambda^T\Sigma_v^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2)$, $\tilde{b}^{(2)} = (b_1^2 - b_N^2, \dots, b_{N-1}^2 - b_N^2)^T$ *and* $\lambda = \Omega^{-1}(b_1 - b_N, \dots, b_{N-1} - b_N)^T$.[7]

Proposition 1 extends the no-arbitrage valuation approach to economies with asymmetric information and provides asset prices in terms of expected discounted cash flows under risk-neutral probabilities, familiar from the asset-pricing literature. The equilibrium prices are non-linear functions of shock ε and noise v, in contrast to CARA-normal noisy REE models [e.g., Grossman and Stiglitz (1980); Admati (1985), among others]. However, the linearity is preserved for the vector $v(p)$ of the log-likelihood ratios of risk-neutral probabilities, which determines optimal portfolios and prices. Furthermore, the tractability of our analysis allows us to study comparative statics for asset prices and investors' portfolios, which we report in Proposition 2 below.

Proposition 2 (Comparative statics). *The comparative statics for price* $P_m(\varepsilon, v)$ *of asset* m *with respect to shock* ε *and noisy demands* v *are as follows:*

$$\frac{\partial P_m(\varepsilon, v)}{\partial\varepsilon} = \frac{\gamma_U}{\gamma_U + \gamma_I}\Big(1 + \frac{\lambda^T\Sigma_v^{-1}\lambda/(\gamma_U\gamma_I)}{\lambda^T\Sigma_v^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2}\Big)\mathrm{cov}^{\mathrm{RN}}(b, C_m)e^{-rT}, \tag{26}$$

$$\frac{\partial P_m(\varepsilon, v)}{\partial v_l} = \frac{\gamma_U\gamma_I}{\gamma_U + \gamma_I}\Big(\mathrm{cov}^{\mathrm{RN}}(C_l, C_m) + \frac{\lambda^T\Sigma_v^{-1}e_l/(\gamma_U\gamma_I)}{\lambda^T\Sigma_v^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2}\mathrm{cov}^{\mathrm{RN}}(b, C_m)\Big)e^{-rT}. \tag{27}$$

The comparative statics for investors' portfolios with respect to prices p *are as follows:*

$$\frac{\partial\theta_I^*(p;\varepsilon)}{\partial p} = -\frac{1}{\gamma_I}\big(\mathrm{var}^{\mathrm{RN}}[\Pi]\big)^{-1}e^{rT}, \tag{28}$$

$$\frac{\partial\theta_U^*(p)}{\partial p} = -\Big(\frac{1}{\gamma_U}E + \frac{\gamma_U + \gamma_I}{\gamma_U\gamma_I}\frac{\lambda\lambda^T\Sigma_v^{-1}}{\gamma_U\gamma_I\big(\lambda^T\Sigma_v^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2\big) + \lambda^T\Sigma_v^{-1}\lambda}\Big)\big(\mathrm{var}^{\mathrm{RN}}[\Pi]\big)^{-1}e^{rT}, \tag{29}$$

[7] The inverse matrix $(E+Q)^{-1}$ in Equation (21) can be computed in closed form, and is given by:

$$(E+Q)^{-1} = E - \frac{\lambda\lambda^T\Sigma_v^{-1}}{\gamma_U\gamma_I\big(\lambda^T\Sigma_v^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2\big) + \lambda^T\Sigma_v^{-1}\lambda}.$$

The latter result can be directly verified by multiplying both sides of the above equality by matrix $(E+Q)$.

where $\text{cov}^{\text{RN}}(\cdot,\cdot)$ *and* $\text{var}^{\text{RN}}(\cdot)$ *are covariance and variance-covariance matrices under the risk-neutral probability measure, and* Π *is the vector of risky asset payoffs in random state* ω*. Furthermore, informed investor's demand for risky asset m is a downward-sloping function of that asset's price* p_m*, holding the prices of other assets fixed.*

The intuition for the effect of shock ε on asset prices is as follows. Positive shock ε increases the probabilities (both real and risk-neutral) of states with higher shock sensitivities b_n. Therefore, the prices of assets that pay more in states with higher b_n increase, and the opposite happens for a negative shock, which gives rise to the covariance term in (26). The asymmetry of information increases the sensitivity of asset prices to shock ε as captured by the second term in brackets in (26). Intuitively, the uninformed investors interpret high prices of assets that positively covary with b as the information that shock ε is positive, and hence, further increase the demand for such assets, which generates an amplifying spiral effect captured by the coefficient in front of $\text{cov}^{\text{RN}}(b, C_m)$ in (26).

The effect of noise traders' demand on asset prices can be decomposed into substitution and information effects, which correspond to the first and second terms in the brackets in (27), respectively. The first term demonstrates that positive demand shock v_l to asset l exerts positive pressure on the price of asset m if the cash flows of assets m and l have positive covariance. This is because following positive demand shock v_l and the resulting increase in the price of asset l the investors partially substitute asset l with asset m which positively covaries with the former. As a result, the price of asset m increases.[8]

The effects of v and ε on asset prices might be difficult to disentangle, as demonstrated by the second term in (27). For example, consider an asset m with payoffs that positively covary with shock sensitivities b. If demand shock v_l increases the price of asset m, the uninformed investors may partially attribute such an increase to a positive shock ε. The uninformed investors disentangle the effects of v and ε by looking at the (risk-neutral) covariances between assets to see whether the asset prices can be explained by payoff covariances with b or C_l, which gives rise to the second term in (27).

Equations (28) and (29) show the sensitivities of investors' asset demands to prices. The demand (28) of the informed investor is determined by the inverse risk-neutral variance-covariance matrix. Consequently, the informed investor's demand for risky asset m is a downward sloping function of that asset's own price p_m because the elements on the main

[8]Similar demand pressure effects also arise in the symmetric-information incomplete-market option pricing model of Gârleanu, Pedersen and Poteshman (2008).

diagonal of a positive-definite matrix $(\text{var}^{\text{RN}}[\Pi])^{-1}$ are all positive.[9]

The sensitivity of uninformed investor's portfolio $\theta^*_U(p)$ to prices p is determined by the two terms in brackets in Equation (29). The first term captures the substitution effects and is present even without asymmetric information. The second term, which depends on vector λ, captures the information effects. The latter term is absent in Equation (28) for $\partial\theta^*(p;\varepsilon)/\partial p$ because investor I has full information. In contrast to the demand of investor I, the demand of investor U for asset m can be an upward sloping function of p_m. This is because high asset prices can be interpreted as positive information about shock ε, in which case the demand for asset m may increase despite high price p_m. Admati (1985) finds a similar result in a multi-asset CARA-normal model with two dates. Our result extends the finding of Admati (1985) to the case of derivatives and assets with strictly positive cash flows. We note that the structure of sensitivities (28) and (29) is new to the literature, and it underscores how the information effects are determined by the replicating portfolio λ and the risk-neutral variance of the payoffs for determining the asset demands.

To demonstrate the latter result more formally, assume for simplicity that noisy demands are i.i.d., and hence $\Sigma_v = \sigma_v^2 E$. Consequently, matrix $\lambda\lambda^T\Sigma_v^{-1}$ is positive semi-definite, and hence has non-negative elements on the main diagonal. If these elements are positive and sufficiently large, the matrix on the right-hand side of Equation (29) may have positive elements on the diagonal.[10]

3.2. General Economy with $M \leq N$ Securities

In this section, we study a general economy with M securities, where $M \leq N$, which subsumes complete and incomplete market economies as special cases. For tractability, we impose the following assumption.

Assumption 1 (Informational spanning condition). *Shock sensitivity b is spanned by the traded assets in the economy. That is, there exist unique constant λ_0 and vector*

[9] Element i on the diagonal of matrix A is given by $e_i^T A e_i$, where $e_i = (0,0,\dots,1,\dots,0)^T$ is a vector with 1 on i's place and all other components equal to zero. If A is positive-definite, then $e_i^T A e_i > 0$, and hence the diagonal elements are positive.

[10] We verify the above intuition for the price and information effects in a simple economy with two risky assets in which we set $r = 0$, $T = 1$, $a = (-5.37, -4.11, -5.7)^T$ and $b = (1.19, 2.44, 3.69)^T$, where a and b are calibrated from gamma distributions, similarly to the example in Section 2. The risk aversions of investors are given by $\gamma_I = 0.004$ and $\gamma_U = 0.04$ [see the estimates in Paravisini, Rappoport, and Ravina (2010)]. We consider two risky assets with payoffs $C_1 = (0, 75, 300)^T$ and $C_2 = (0, 0, 225)^T$. In this economy, investor U's demand for the first asset increases with the increase in its price, whereas the demand for the second security decreases with the increase in its price, holding other prices fixed.

$\lambda = (\lambda_1, \ldots, \lambda_{M-1})^T \in \mathbb{R}^{M-1}$ such that

$$b = \lambda_0 \mathbb{I}_N + \lambda_1 C_1 + \ldots \lambda_{M-1} C_{M-1}, \tag{30}$$

or equivalently, $b_n = \lambda_0 + \Pi_n^T \lambda$, where $\mathbb{I}_N \in \mathbb{R}^N$ is a vector of ones, C_m are the payoffs of the risky assets, and Π_n is the vector of risky asset payoffs in state n.

We provide several plausible economies that satisfy informational spanning condition. First example is the complete-market economy, where $M = N$, and hence, there always exist constant λ_0 and vector λ satisfying Equation (30). Second example is an incomplete-market economy with only one risky asset with payoff $C_1 = b/\tilde{\lambda}$, where $\tilde{\lambda}$ is any positive constant. As discussed in Section 2.3, economies in Grossman and Stiglitz (1980) and Breon-Drish (2014) satisfy the latter condition. Third example is an economy with asset $C_1 = b/\tilde{\lambda}$, and call options with payoffs $C_2 = (C_1 - K_2)^+, \ldots, C_{M-1} = (C_1 - K_{M-1})^+$ written on asset 1, in which case $\lambda_0 = 0$, $\lambda_1 = \tilde{\lambda}$, $\lambda_2 = 0, \ldots, \lambda_{M-1} = 0$. In particular, a CARA-normal model with a risky asset C_1 and several call options written on it, is a special case of the latter economy. Fourth example is an economy with firms that have cash flows b and issue risky debt and equity with payoffs $\min(b, K)$ and $(b - K)^+$, where K is the face value of debt, and hence $\lambda_0 = 0$, $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = 0, \ldots, \lambda_{M-1} = 0$.

Assumption 1 implies that the information about the economy's sensitivity to shock ε, which is captured by vector b, is spanned by the payoffs of tradable assets in the economy. We note, that the informational spanning condition does not imply market completeness because the number of states of the economy is allowed to exceed the number of tradable assets. Lemma 3 below demonstrates that the informed investor's portfolio is a linear function of shock ε if the informational spanning condition is satisfied, which allows solving the updating problem of the uninformed investor in closed form.

Lemma 3 (Optimal portfolios with $M \le N$ assets). *Suppose Assumption 1 is satisfied, probabilities $\pi_n(\varepsilon)$ are given by Equation (5) and ε has a general PDF $\phi_\varepsilon(x)$. Then, if problem (1) has finite solution $\theta_I^*(p;\varepsilon)$, this solution is a linear function of ε, given by*

$$\theta_I^*(p;\varepsilon) = \frac{\lambda\varepsilon}{\gamma_I} - \frac{\hat{\theta}_I^*(p)}{\gamma_I}, \tag{31}$$

where vector λ is such that equation (30) is satisfied, and $\hat{\theta}_I^*(p)$ is a function of p.

Lemma 3 pinpoints a general condition which makes informed investor's portfolio linear in shock ε with general PDF $\phi_\varepsilon(x)$. Therefore, it provides new economic insight on why informed investors' portfolios are linear functions of signals in related models with one risky asset (e.g., Grossman and Stiglitz (1980); Breon-Drish (2014)), which are special cases of

our model. In particular, it turns out that the linearity in those models arises because the informational spanning condition is satisfied, and hence, the informed investors can replicate the economy's shock sensitivities b. Our paper is the first to demonstrate that the informed investor's portfolio and the informativeness of asset prices in REE economies are determined by the informed investor's incentive to replicate shock sensitivities b, which allows this investor to hold more wealth in more likely states, as elaborated in Section 3.1.

The linearity of $\theta_I^*(p;\varepsilon)$ simplifies the inference problem of the uninformed investor. In particular, substituting $\theta_I^*(p;\varepsilon)$ into the market clearing condition (4) we obtain

$$\frac{\lambda\varepsilon}{\gamma_I} + v + \tilde{H}(p) = 0, \tag{32}$$

where $\tilde{H}(p) = \theta_U^*(p) - \bar{\theta}_I^*(p)/\gamma_I$, which has a similar structure to the market clearing condition (16) in the complete-market economy. The derivation of equilibrium then proceeds similarly to the case of complete markets. Proposition 3 summarizes the main results.

Proposition 3 (Equilibrium with $M \le N$ assets). *Suppose probabilities $\pi_n(\varepsilon)$ are given by Equation* (5)*, shock ε has PDF given by* (6)*, and the informational spanning condition is satisfied. Then, there exists unique equilibrium in which prices only reveal $\lambda\varepsilon/\gamma_I + v$. In this equilibrium, the investors' optimal portfolios are given by*

$$\theta_I^*(p;\varepsilon) = \frac{\lambda\varepsilon}{\gamma_I} - \frac{1}{\gamma_I} f_I^{-1}\big(e^{rT}p\big), \tag{33}$$

$$\theta_U^*(p) = \big(E+Q\big)^{-1}\Big(\frac{1}{\gamma_I} f_I^{-1}\big(e^{rT}p\big) - \frac{1}{\gamma_U} f_U^{-1}\big(e^{rT}p\big)\Big) + \frac{(\mu_0/\sigma_0^2)\lambda}{\gamma_U(\lambda^T\Sigma_v^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2)}, \tag{34}$$

price vector $P(\varepsilon, v)$ is a continuous, injective[11] *and differentiable function of $\lambda\varepsilon/\gamma_I + v$, and is the unique solution of equation*

$$\frac{1}{\gamma_U} f_U^{-1}\big(e^{rT}P(\varepsilon,v)\big) + \frac{1}{\gamma_I} f_I^{-1}\big(e^{rT}P(\varepsilon,v)\big) = \big(E+Q\big)\Big(\frac{\lambda\varepsilon}{\gamma_I} + v + \frac{(\mu_0/\sigma_0^2)\lambda}{\gamma_U(\lambda^T\Sigma_v^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2)}\Big), \tag{35}$$

where $E \in \mathbb{R}^{(M-1)\times(M-1)}$ is the identity matrix, $Q \in \mathbb{R}^{(M-1)\times(M-1)}$ is a matrix given by Equation (25)*, and functions $f_I, f_U : \mathbb{R}^{M-1} \to \mathbb{R}^{M-1}$ are given by*

$$f_I(x) = \frac{\sum_{j=1}^{N} \Pi_j \exp\{a_j + \Pi_j^T x\}}{\sum_{j=1}^{N} \exp\{a_j + \Pi_j^T x\}}, \tag{36}$$

$$f_U(x) = \frac{\sum_{j=1}^{N} \Pi_j \exp\Big\{a_j + \frac{1}{2}\frac{b_j^2}{\lambda^T\Sigma_v^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2} + \Pi_j^T x\Big\}}{\sum_{j=1}^{N} \exp\Big\{a_j + \frac{1}{2}\frac{b_j^2}{\lambda^T\Sigma_v^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2} + \Pi_j^T x\Big\}}, \tag{37}$$

[11] That is, $P(\varepsilon_1, v_1) = P(\varepsilon_2, v_2)$ implies that $\lambda\varepsilon_1/\gamma_I + v_1 = \lambda\varepsilon_2/\gamma_I + v_2$ for all ε_1, v_1, ε_2 and v_2.

and $f_I^{-1}(y)$ *and* $f_U^{-1}(y)$ *are inverse functions defined on the ranges of* $f_I(x)$ *and* $f_U(x)$.

Optimal portfolios (33) and (34) have the same structure as portfolios (20) and (21) in the complete-market economy. However, asset prices are no longer available in closed form, and solve a system of non-linear algebraic equations (35). The latter system of equations reveals that asset prices are functions of a linear combination of shocks, $\lambda\varepsilon/\gamma_I + v$, similar to the complete-market case.

The inverse functions $x = f_i^{-1}(y)$ can be found by solving $M-1$ equations with $M-1$ unknowns $y = f_i(x)$. We observe that solving Equation (35) is equivalent to solving the following system of equations for x_I and x_U, which does not involve inverse functions:

$$\frac{x_I}{\gamma_I} + \frac{x_U}{\gamma_U} = \left(E + Q\left(\frac{\lambda\varepsilon}{\gamma_I} + v\right) + \frac{(\mu_0/\sigma_0^2)\lambda}{\gamma_U(\lambda^T\Sigma_v^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2)}\right), \tag{38}$$

$$p = e^{-rT} f_I(x_I), \quad p = e^{-rT} f_U(x_U). \tag{39}$$

Furthermore, solving the above system reduces to finding x_I, which satisfies equation $f_I(x_I) = f_U\big(\gamma_U R(\varepsilon, v) - (\gamma_U/\gamma_I)x_I\big)$, where $R(\varepsilon, v)$ denotes the right-hand side of Equation (38). The latter equation can be solved using Newton's method [e.g., Judd (1998)], and then the equilibrium prices can be found from Equations (39).

We note that functions $f_i(x)$ given by (36) and (37) can be considered as expected asset payoffs under certain probability measures. For example, $f_I(x) = \mathbb{E}^q[\Pi]$, where the expectation is under probability measure $\pi_n^q(x) = \exp\{a_n + \Pi_n^T x\}/\sum_{j=1}^N \exp\{a_j + \Pi_j^T x\}$. Consequently, equations (39) imply that asset prices in incomplete markets are given by discounted expected payoffs under certain probability measures.

An important result of our paper is the existence and uniqueness of the solution of the system (38)–(39), which implies the existence of equilibrium in our economy. The proof of existence is significantly complicated by the market incompleteness and the multiplicity of risky assets, which makes it impossible to apply the intermediate value theorem as in the related models with one risky asset [e.g., Breon-Drish (2014)]. Therefore, we devise a new approach that might be useful in various other multi-asset incomplete-market economies.[12]

[12] In the proof of Proposition 3 in the Appendix, we show that solving system (38)–(39) reduces to solving an equation $f(x;0) = f(\bar{x} - cx;t)$, where $f(x;t)$ is a smooth function of x and t such that $f(x;0) = f_I(x)$ and $f(x;t) = f_U(x)$ when $t = 0.5/(\lambda^T\Sigma_v^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2)$. Next, we consider t as a parameter and notice that the equation has a solution $x = \bar{x}/(1+c)$ when $t = 0$. Hence, by the implicit function theorem [e.g., Rudin (1976)], there exists $\bar{t}$ such that there exists unique and continuous solution $x(t)$ for all $t \in [0, \bar{t})$. Our methodological innovation is to show that $\bar{t} = +\infty$, and hence, the solution exists for all values of t.

4. Economic Applications

In this Section, we provide several economic applications of our analysis. First, we derive general conditions under which the prices of derivative securities provide information about the aggregate shock and define a new concept of informationally redundant securities. Next, we explore the economic role of incomplete markets and establish conditions under which the asset prices in economies with complete and incomplete markets coincide. Finally, we derive closed-form expressions for the prices of risky debt and equity of a firm and explore how the value of the firm depends on the face value of debt.

4.1. Information Revelation and Market Transparency

Here, we investigate the informational role of derivative securities and determine precise conditions under which their prices reveal information about the probabilities of states. We disentangle two sources of information revealed by asset prices: the demands of informed investors and the correlations of noisy demands across different assets. We first analyze the dependence of the informed investor's demand on shock ε. Equation (31) demonstrates that the informed investor's portfolio is given by $\theta_I^*(p;\varepsilon) = \lambda\varepsilon/\gamma_I - \bar{\theta}_I^*(p)/\gamma_I$ for general PDFs $\phi_\varepsilon(x)$ and any number of assets satisfying $M \leq N$. The latter equation demonstrates that the demand for asset m releases new information about shock ε only if λ_m (the m^{th} component of vector λ) is non-zero because otherwise the demand for asset m does not depend on shock ε, and hence such a security is *informationally redundant*.

The informational redundancy of derivatives is a generic property of M-asset economies in which cash flows of the underlying asset are spanned by shock sensitivities b in a linear way, i.e., $C_1 = b/\tilde{\lambda}$, and hence, the replicating portfolio λ is given by $\lambda = (\tilde{\lambda}, 0, \ldots, 0)^T$ in accordance with the informational spanning condition (30). As demonstrated in Section 2, condition $C_1 = b/\tilde{\lambda}$ is satisfied in single risky asset economies of Grossman and Stiglitz (1980) and Breon-Drish (2014). Consequently, adding any non-redundant securities to these economies does not help reveal more information about ε (provided that noises v are uncorrelated across assets, as elaborated below).

The intuition for the role of vector λ in determining the informational role of asset prices is as follows. As demonstrated in Section 3, vector $\lambda\varepsilon$ can be interpreted as a portfolio that replicates the economy's shock sensitivities $b\varepsilon$, which determine the probabilities of states ω_n. Investing in portfolio $\lambda\varepsilon$ allows the informed investor to exploit her informational advantage and hold more wealth in states with higher real probabilities $\pi_n(\varepsilon)$. Assets with

$\lambda_m = 0$ do not help the investor to replicate shock sensitivities $b\varepsilon$, and hence the demand for such assets is not sensitive to ε. However, we observe from Equation (31) for portfolio $\theta_I^*(p;\varepsilon)$ that assets with $\lambda_m = 0$ are still held by the informed investor because they help complete the market, and hence, are non-redundant from the perspective of risk sharing.

When the market is complete, a condition for informational redundancy can be derived for general probabilities $\pi_n(\varepsilon)$. In particular, Equation (11) for portfolio $\theta_I^*(p;\varepsilon)$ demonstrates that investor I's holding of asset m does not depend on shock ε if and only if vector $\Omega^{-1}\big(\ln\big(\pi_1(\varepsilon)/\pi_N(\varepsilon)\big),\dots,\ln\big(\pi_{N-1}(\varepsilon)/\pi_N(\varepsilon)\big)\big)^T$ has zero m^{th} element. The latter condition can be easily violated even with modest amount of nonlinearity in probabilities. However, in situations when log-likelihoods can be linearized in such a way that for small shocks o the approximation $\ln\big(\pi_n(\varepsilon)\big) \approx a_n + b_n\varepsilon$ is satisfied, our intuition on informational redundancy has a first-order effect even when the log-likelihood ratios are non-linear.

To quantify the informational transparency of financial markets, we suggest looking at the posterior precision of shock ε, as estimated by investor U, which is given by $1/\bar{\sigma}^2_\varepsilon$, where $\bar{\sigma}^2_\varepsilon = \text{var}[\varepsilon|p]$. To underscore the important economic role of precision $1/\bar{\sigma}^2_\varepsilon$ we label it as the *transparency index*. For simplicity, we derive the posterior variance $\bar{\sigma}^2_\varepsilon$ assuming that shock ε has normal prior distribution $\varepsilon \sim N(\mu_\varepsilon, \sigma^2_\varepsilon)$, which yields a more tractable expression for $\bar{\sigma}^2_\varepsilon$ than the distribution $\varepsilon \sim \bar{N}(\mu_\varepsilon, \sigma^2_\varepsilon)$ in (6). As argued in Section 2, the normal distribution $\varepsilon \sim N(\mu_\varepsilon, \sigma^2_\varepsilon)$ is a good approximation of distribution (6) under plausible parameters, and hence, the posterior variances of ε implied by the two distributions are close. Lemma 4 below provides the transparency index in closed-form.

Lemma 4 (Market transparency). *Let shock ε be normally distributed, $\varepsilon \sim N(\mu_\varepsilon, \sigma^2_\varepsilon)$. Then, the transparency index $1/\bar{\sigma}^2_\varepsilon$, where $\bar{\sigma}^2_\varepsilon = \text{var}[\varepsilon|p]$, is given by*

$$\frac{1}{\bar{\sigma}^2_\varepsilon} = \frac{1}{\sigma^2_o} + \frac{\lambda^T\Sigma_v^{-1}\lambda}{\gamma_I^2}. \tag{40}$$

Equation (40) implies that $1/\bar{\sigma}^2_\varepsilon > 1/\sigma^2_\varepsilon$. Intuitively, the presence of informed traders reduces the uncertainty about ε by releasing new information via asset prices. More formally, the quantity of the released information can be measured as the difference between the entropies of posterior and prior distributions of ε.[13] In the case of normally distributed o the latter measure is given by $\ln(\bar{\sigma}_\varepsilon) - \ln(\sigma_\varepsilon)$ and, hence, is an increasing function of $\bar{\sigma}_\varepsilon$.

Consistent with the above intuition, transparency $1/\bar{\sigma}^2_\varepsilon$ is determined only by assets that replicate the vector of shock sensitivities b. Assume, for simplicity, that $\Sigma_v = \sigma^2_v E$, where E is an identity matrix. If asset $C_1 = b$ is traded, we obtain that $\lambda = (1, 0, \dots, 0)^T$,

[13] Entropy ε is defined as $-\int_{-\infty}^{\infty} \phi_\varepsilon(x)\ln\phi_\varepsilon(x)dx$, and for $\varepsilon \sim N(0,\sigma^2_\varepsilon)$ is given by $0.5\ln(2\pi e\sigma^2_\varepsilon)$.

and hence $1/\bar{\sigma}_0^2 = 1/\sigma_\varepsilon^2 + 1/(\gamma_I^2\sigma_v^2)$. Therefore, $1/\bar{\sigma}_\varepsilon^2$ does not depend on the number of traded derivatives because they do not help span shock sensitivities b. If instead there are two risky assets with payoffs $(b-K)^+$ and $\min(b, K)$, interpreted as equity and debt, then $\lambda = (1, 1, 0, \ldots, 0)^T$. Therefore, $1/\bar{\sigma}_0^2 = 1/\sigma_\varepsilon^2 + 2/(\gamma_I^2\sigma_v^2)$, and transparency increases.[14]

Transparency also depends on the variance-covariance matrix of noise trader demands v. In particular, higher correlations make the market more transparent by allowing inferring more information by comparing the market clearing conditions across securities. For example, consider a model with two risky assets and assume that $v_1 = v_2$, so that noisy demands are perfectly correlated. Taking the difference of the market clearing conditions (16) for the two markets we find that $(\lambda_1 - \lambda_2)\varepsilon/\gamma_I + (v_1 - v_2) + (1, -1)^T H(p) = 0$. Therefore, shock ε can be perfectly learned from prices if $\lambda_1 \neq \lambda_2$. More formally, matrix Σ_v becomes close to singular when noisy demands cross-correlations become closer to one. Therefore, the determinant of Σ_v^{-1} becomes large, and hence transparency $1/\bar{\sigma}_\varepsilon^2$ increases.

The empirical literature on price discovery in financial markets finds that the prices of derivative securities reveal information about the payoffs of the underlying asset [e.g., Easley, O'Hara, and Srinivas (1998); Chakravarty, Gulen, and Mayhew (2004); Pan and Poteshman (2006)]. The empirical evidence is consistent with our model when the derivatives help span shock sensitivities b and/or noise trader demands are correlated across assets. However, to our best knowledge the previous literature does not disentangle the latter two sources of information that we identify in this paper.

Finally, we note that transparency $1/\bar{\sigma}_\varepsilon^2$ is a decreasing function of the informed investor's risk aversion γ_I. Intuitively, investors with higher risk aversions have smaller demands for risky assets. Therefore, their private information is more difficult to filter out from the market clearing conditions, and hence, the market becomes less transparent.

4.2. Economic Role of Market Incompleteness

In this subsection, we compare the complete and incomplete market equilibria derived in Propositions 1 and 3. Our main finding is that market incompleteness has only second-order effect on equilibrium as compared to the first-order effect of the asymmetry of information. Therefore, our complete-market equilibrium emerges as a tractable alternative to single risky asset incomplete-market economies with asymmetric information.

[14]The results on the informational role of derivatives explain why in Brennan and Cao (1996) investors do not learn from the derivative asset. In particular, they consider a CARA-normal framework with a stock and a derivative with quadratic payoff C_1^2. In our terminology, the stock's payoff linearly spans b. Therefore, $\lambda = (1, 0)^T$, and hence the derivative does not reveal any useful information.

We consider an asset with payoffs $C_n = hn$, where $h = A/N$, $n = 1, \ldots, N$, $A > 0$. The probabilities of states ω_n are given by equation (5) in which a_n and b_n are given by:

$$a_n = \ln(C_n) - \frac{C_n^2}{2\sigma_C^2}, \quad b_n = \frac{C_n}{\sigma_C^2}. \tag{41}$$

In the limit of $N \to \infty$ cash flow C and shock ε have the following PDFs:

$$\phi_C(x) = \frac{C e^{-0.5(C-\varepsilon)^2/\sigma_C^2}}{\int_0^{+\infty} C e^{-0.5(C-\varepsilon)^2/\sigma_C^2} dC}, \quad C > 0, \tag{42}$$

$$\phi_\varepsilon(x) = \frac{\left(\frac{1}{\sqrt{2\pi}} e^{-0.5(x/\sigma_C)^2} + \frac{x}{\sigma_C}\Phi\left(\frac{x}{\sigma_C}\right)\right) e^{-0.5(x-\mu_0)^2/\sigma_0^2+0.5x^2/\sigma_C^2}}{\int_{-\infty}^{+\infty} \left(\frac{1}{\sqrt{2\pi}} e^{-0.5(x/\sigma_C)^2} + \frac{x}{\sigma_C}\Phi\left(\frac{x}{\sigma_C}\right)\right) e^{-0.5(x-\mu_0)^2/\sigma_0^2+0.5x^2/\sigma_C^2} dx}, \tag{43}$$

where $\Phi(\cdot)$ is the cumulative density function (CDF) of a standard normal distribution, and parameters μ_0 and σ_0 of PDF (42) are chosen in such a way that shock ε has certain prespecified mean $\mathbb{E}[\varepsilon] = \mu_\varepsilon$ and variance $\mathrm{var}[\varepsilon] = \sigma_\varepsilon^2$. PDF (42) of cash flow C has positive support and is known as Rayleigh distribution.

We complete the market by adding arbitrary non-redundant securities to the economy. This economy satisfies the informational spanning condition of Section 3.2 with $\lambda = (1/\sigma_C^2, 0, \ldots, 0)^T$. Consistent with Section 4.1, the added securities do not reveal information about ε because their weight λ_m in portfolio λ is zero, and hence, their only role is to facilitate risk sharing. To make complete- and incomplete-market economies comparable, we assume that there are no noise traders in the markets for added securities $2, \ldots, M$, and the noise traders only trade in the market for the risky asset with cash flow C.[15] We compare the complete-market prices with those in an economy with an extreme form of incompleteness in which only one risky asset with cash flow C is traded, keeping the number of states N the same in both economies and greater than two. Proposition 4 characterizes the asset prices in these economies.

Proposition 4 (Comparison of complete and incomplete market equilibria). *In the $N \to \infty$ limit, the price of the risky asset with cash flow C in the complete and incomplete market economies is given by:*

$$P^{cm}(s) = \sigma_C^{\mathrm{RN}} \Phi\left(\frac{\mu_C^{\mathrm{RN}}(s)}{\sigma_C^{\mathrm{RN}}}\right) e^{-rT}, \tag{44}$$

$$P^{icm}(s) = \bar{\sigma}_C \Phi\left(\frac{\bar{\mu}_C(s)}{\bar{\sigma}_C}\right) e^{-rT}, \tag{45}$$

[15] We derive the equilibrium as a limiting case of the equilibrium in Section 3.1 in which noises ν_m are independent across all assets and the variances of $\nu_2, \ldots, \nu_M$ converge to zero. The resulting equilibrium is not fully revealing because the assets $2, \ldots, M$ do not reveal information about ε, similar to Section 4.1.

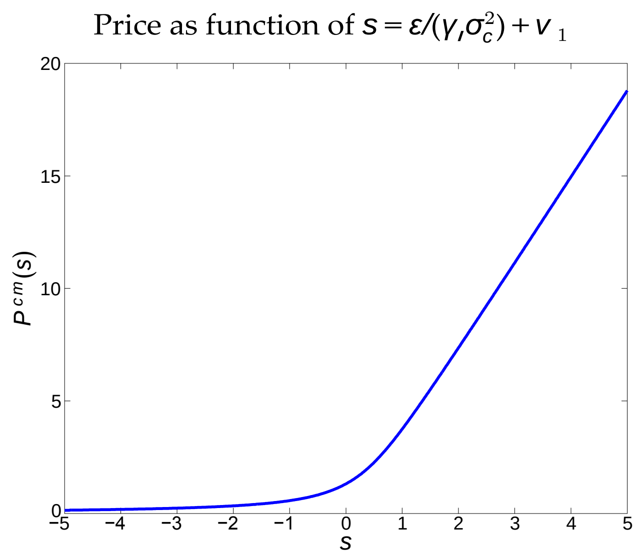

Figure 2: Asset price in complete-market economy
The figure shows asset prices $P^{cm}(s)$ when the market is complete for the following model parameters: $\mu_\varepsilon = 0$, $\sigma_\varepsilon = 1$, $\sigma_v = 1$, $\sigma_c = 1$, $\gamma_I = 5$, $\gamma_U = 5$ and $r = 0$.

respectively, where $s = \varepsilon/(\gamma_I\sigma_c^2) + v_1$, $\mu_c^{RN}(s)$ *and* σ_c^{RN} *are the risk-neutral measure parameters given by Equations* (A.42) *and* (A.43) *in the Appendix,* $\bar\mu_c(s)$ *and* $\bar\sigma_c$ *are parameters given by Equations* (A.48) and (A.47) *in the Appendix, and function* $\tilde\Phi(x)$ *is given by:*

$$\tilde\Phi(x) = \frac{x^2\Phi(x) + (2x/\sqrt{2\pi})e^{-0.5x^2} + 1_{\{x>0\}} - (\mathrm{sgn}(x)/\sqrt{\pi})\Gamma(1.5, 0.5x^2)}{x\Phi(x) + (1/\sqrt{2\pi})e^{-0.5x^2}}, \tag{46}$$

where $\Gamma(a, x) = \int_x^{+\infty} y^{a-1}e^{-y}dy$ *is the incomplete gamma-function,* $1_{\{x>0\}}$ *is an indicator function, and* $\mathrm{sgn}(x)$ *is the sign of* x.

Proposition 4 provides a fully closed-form asset price in the complete-market economy. Conveniently, the asset price does not depend on the type of securities which are used to complete the market. When the market is incomplete, the solution is explicit up to $\bar\mu_c(s)$, which is found by solving Equations (A.46) and (A.48) in the Appendix numerically. Therefore, the incomplete-market equilibrium appears to be less tractable than the complete-market equilibrium introduced in our paper. Figure 2 depicts the complete-market asset price $P^{cm}(s)$ as a function of $s = \varepsilon/(\gamma_I\sigma_c^2) + v_1$ for the following exogenous model parameters: $\mu_\varepsilon = 0$, $\sigma_\varepsilon = 1$, $\sigma_v = 1$, $\sigma_c = 1$, $\gamma_I = 5$, $\gamma_U = 5$ and $r = 0$. The asset price turns out to be a strictly positive and convex function of s.

We find a surprising result that despite severe market incompleteness in the economy, the incomplete-market price $P^{inc}(s)$ is very close to $P^{cm}(s)$ so that $\left(P^{inc}(s) - P^{cm}(s)\right)/P^{cm}(s) < 0.007$ for all s. Because the plots of the two prices are indistinguish-

able, we do not show $P^{inc}(s)$ in Figure 2. The closeness of prices is robust to perturbations of model parameters. The PDF (43) of shock ε is also very close to a standard normal PDF, and hence, is not shown for brevity.

We attribute the closeness of prices $P^{inc}(s)$ and $P^{cm}(s)$ to the similarity of information-sensitive demands in complete- and incomplete-market economies, given by the first terms of portfolios (20) and (31), when the informational spanning condition is satisfied. The similarity of portfolios demonstrates that the informational spanning condition alleviates the effects of market incompleteness, allows for efficient allocation of wealth to more likely states, and hence, leads to the similarity of prices in incomplete and complete markets.

Our results raise the question of finding conditions under which the asset prices in economies with complete and incomplete markets coincide. The following Proposition provides a sufficient condition.

Proposition 5 (CARA-normal models with complete and incomplete markets). *Consider a risky asset with cash flow $C \sim N(\varepsilon, \sigma_C^2)$, where ε has prior distribution $\varepsilon \sim N(\mu_\varepsilon, \sigma_\varepsilon^2)$. Then, the price of the latter asset is the same both in the economies with complete and incomplete markets, and is given by:*

$$P^{cm}(s) = P^{icm}(s) = \mu_C^{\mathrm{RN}}(s)e^{-rT}, \tag{47}$$

where $s = \varepsilon/(\gamma_I \sigma_C^2) + v_1$, and $\mu_C^{\mathrm{RN}}(s)$ is given by Equation (A.42) in the Appendix, in which we set $\sigma_0 = 1/\sqrt{1/\sigma_0^2 + 1/\sigma_C^2}$ and $\mu_0 = (\mu_\varepsilon/\sigma_\varepsilon^2)\sigma_0^2$.

As demonstrated in Section 3, the asset prices both in the economies with complete and incomplete markets can be represented as expected payoffs under certain probability measures. In the case of CARA-normal models it turns out that asset payoffs under these probability measures have the same mean but different variances. Consequently, the asset prices coincide despite the fact that distributions are different. The above results demonstrate the limited role of market incompleteness and suggest using complete-market equilibria as tractable alternatives to incomplete-market equilibria in economic research.

4.3. Pricing Debt and Equity under Asymmetric Information

In this section, we apply our results to study the pricing of risky debt and equity in REE with asymmetric information. We study a firm with cash flows C and face value of debt K. The firm issues risky debt with face value $K > 0$ and equity with payoffs $\min(C, K)$

and $(C-K)^+$ and prices P_D and P_E, respectively.[16] The firm value is then defined as the sum of the values of debt and equity, $V = P_D + P_E$.

As in Section 4.2, we consider a complete-market economy in the limit of $N \to +\infty$ in which cash flows C and shock ε have distributions (42) and (43), respectively. It is immediate to observe that the informational spanning condition is satisfied in our economy because the vector of shock sensitivities is given by $b = C/\sigma_C^2$ and is spanned by debt and equity so that $b = \left(\min(C,K) + (C-K)^+\right)/\sigma_C^2$. Therefore, the replicating portfolio λ is given by $\lambda = (1,1,0,\ldots,0)^T/\sigma_C^2$. Similar to the economy in Section 4.2, we allow for noise traders only in the debt and equity markets and denote their noisy demands by v_1 and v_2 respectively. Noisy demands have the same volatility σ_v and correlation ρ.[17] The equilibrium prices of debt and equity are reported in Proposition 6 below.

Proposition 6 (Debt and equity pricing).

1) *The price of debt and equity in the above complete-market economy are given by:*

$$P_D = \frac{\int_0^{+\infty} \min(C,K)\, C \exp\left(-\frac{(C-\tilde{\mu}_C)^2}{2(\sigma_C^{RN})^2} + \frac{\gamma_I\gamma_U}{\gamma_I+\gamma_U}\left(v_1 \min(C,K) + v_2 (C-K)^+\right)\right) dC}{\int_0^{+\infty} C \exp\left(-\frac{(C-\tilde{\mu}_C)^2}{2(\sigma_C^{RN})^2} + \frac{\gamma_I\gamma_U}{\gamma_I+\gamma_U}\left(v_1 \min(C,K) + v_2 (C-K)^+\right)\right) dC}, \tag{48}$$

$$P_E = \frac{\int_0^{+\infty} (C-K)^+ C \exp\left(-\frac{(C-\tilde{\mu}_C)^2}{2(\sigma_C^{RN})^2} + \frac{\gamma_I\gamma_U}{\gamma_I+\gamma_U}\left(v_1 \min(C,K) + v_2 (C-K)^+\right)\right) dC}{\int_0^{+\infty} C \exp\left(-\frac{(C-\tilde{\mu}_C)^2}{2(\sigma_C^{RN})^2} + \frac{\gamma_I\gamma_U}{\gamma_I+\gamma_U}\left(v_1 \min(C,K) + v_2 (C-K)^+\right)\right) dC}, \tag{49}$$

where volatility and drift parameters σ_C^{RN} and $\tilde{\mu}_C$ are independent of the face value of debt K, σ_C^{RN} is given by Equation (A.43) in the Appendix, and $\tilde{\mu}_C$ is given by:

$$\begin{aligned}\tilde{\mu}_C = \frac{\gamma_I\gamma_U}{\gamma_I+\gamma_U}\Bigg(& \frac{\mu_0/(\sigma_0^2\sigma_C^2)}{\gamma_U\left(2/(\gamma_I^2(1+\rho)\sigma_v^2\sigma_C^4) + 1/\sigma_0^2\right)} + \frac{\varepsilon}{\gamma_I\sigma_C^2} \\ & + \frac{2\varepsilon/(\gamma_I\sigma_C^2) + v_1 + v_2}{\gamma_U\gamma_I\left(2/\gamma_I^2 + (1+\rho)(\sigma_v^2\sigma_C^4/\sigma_0^2)\right)}\Bigg)(\sigma_C^{RN})^2.\end{aligned} \tag{50}$$

2) *The posterior distribution of shock ε does not depend on the face value of debt.*

Proposition 6 provides closed-form expressions for the prices of debt and equity in

[16] We make the standard assumptions that debt holders have priority of payment and that equity holders are residual claimants with limited liability.

[17] Consistent with the analyses in Sections 4.1 and 4.2, the equilibrium is not fully revealing despite the absence of noise in the other markets because the other assets are informationally redundant.

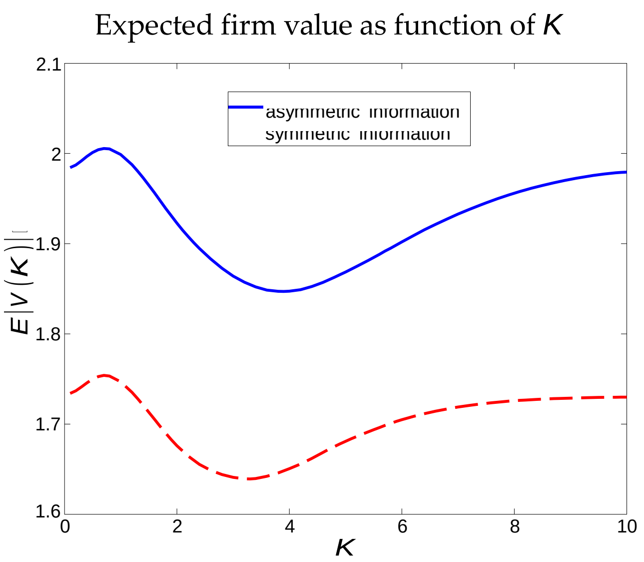

Figure 3: Expected firm value as a function of the face value of debt
The figure shows the expected firm value V as a function of the face value of debt K for markets with asymmetric and symmetric information (when both investors observe ε) for the following model parameters: $\mu_\varepsilon = 0$, $\sigma_\varepsilon = 1$, $\sigma_v = 1$, $\rho = 0$, $\sigma_C = 1$, $\gamma_I = 5$, $\gamma_U = 5$ and $r = 0$.

terms of easily computable integrals.[18] These expressions allow studying the asset prices and the firm value as functions of the face value of debt K. In particular, summing up the equations for debt and equity prices we find that the value of the firm is given by

$$V = \frac{\int_0^{+\infty} C^2 \exp\left(-\frac{(C-\tilde{\mu}_C)^2}{2(\sigma_C^{RN})^2} + \frac{\gamma_I\gamma_U}{\gamma_I+\gamma_U}\left(v_1\min(C,K)+v_2(C-K)^+\right)\right)dC}{\int_0^{+\infty} C \exp\left(-\frac{(C-\tilde{\mu}_C)^2}{2(\sigma_C^{RN})^2} + \frac{\gamma_I\gamma_U}{\gamma_I+\gamma_U}\left(v_1\min(C,K)+v_2(C-K)^+\right)\right)dC}. \qquad (51)$$

Equation (51) demonstrates that the value of the firm is affected by the face value of debt K, and hence the capital structure irrelevance theorem of Modigliani and Miller (1958), in general, is violated in our economy. One important result immediate from Equation (51) is that the face value of debt K does not affect distribution parameters σ_C^{RN} and $\tilde{\mu}_C$ that capture the effects of asymmetric information. The dependence of the firm value on K arises due to the price pressure of noise trader demands v_1 and v_2 and is present even in symmetric-information economies where all investors observe shock ε.

Figure 3 plots the unconditional expectation of the firm value with respect to shock ε and noises v_1 and v_2 for economies with symmetric and asymmetric information when noises are uncorrelated (i.e., $\rho = 0$) and the other parameters are as in Section 4.2.

[18] These integrals can be expressed in terms of the CDF of the standard normal distribution and incom-

plete gamma functions. However, expressions (48) and (49) appear to be more compact and intuitive.

In the symmetric information case, we assume that both investors observe shock ε.[19] The unconditional expectation $\mathbb{E}[V(K)]$ is then calculated using Monte-Carlo simulations. The results confirm the above economic intuition. In particular, the value of the firm appears to be a non-linear function of the face value of debt K. We also investigate the difference between the firm values in asymmetric- and symmetric-information economies (not reported for brevity). This difference is affected by the face value of K, and hence, the interaction between the asymmetry of information and noise trader demands contributes to the non-linear pattern on Figure 3, although the effect is small. Nevertheless, the asymmetry of information has a sizable effect on the level of the firm value.

It can be easily observed from (51) that the firm value is independent of K if noisy demands are zero or perfectly correlated with $\rho = 1$. In the latter case, because the noises have the same variance, we obtain that $\nu_1 = \nu_2$, and hence, the price-effect of shocks is given by $\nu_1 \min(C, K) + \nu_2(C - K)^+ = \nu_1 C$ and does not depend on the face value of debt K. Albagli, Hellwig, and Tsyvinski (2013) derive a similar condition in a model with risk-neutral investors and position-limit constraints. Our analysis demonstrates this result in a general setting with risk-averse investors. To our best knowledge this analysis is new even for economies with symmetric information.

Despite the fact that the capital structure irrelevance does not hold in our economy, the second part of Proposition 6 establishes an informational analogue of Modigliani-Miller theorem by showing that the face value of debt is irrelevant for the posterior distribution of ε. Therefore, slicing the firm cash flows into debt and equity in different way does not allow firms to manipulate the amount of information revealed by asset prices. The reason is that, as demonstrated above, the shock sensitivities b in the economy can be written as $b = \big(\min(C, K) + (C - K)^+\big)/\sigma_C^2$. Therefore, portfolio $\lambda = (1, 1, 0, \dots, 0)^T/\sigma_C^2$ that replicates sensitivities b does not depend on K. Then, consistent with Section 4.1, the face value of debt K does not affect the amount of information revealed by asset prices.

We note that the replicating portfolio λ should not be confused with information sensitivities of asset prices measured by partial derivatives $\partial P_D(\varepsilon, \nu)/\partial\varepsilon$ and $\partial P_E(\varepsilon, \nu)/\partial\varepsilon$. From Equation (26) for these sensitivities it is immediate to observe that they are affected by the face value of debt K. In particular, for low (high) K equity is more (less) sensitive to shock ε than debt. However, importantly for our analysis, summing up partial derivatives $\partial P_D(\varepsilon, \nu)/\partial\varepsilon$ and $\partial P_E(\varepsilon, \nu)/\partial\varepsilon$ it is immediate to observe that the sensitivity of the

[19] As argued in the discussion of Lemma 2 in Section 3.1, the symmetric-information economy is a limiting case of the asymmetric-information economy when the parameters σ_0 and μ_0 of PDF (6) converge to 0 and ε, respectively. Therefore, the symmetric-information equilibrium prices can be computed by taking limits $\sigma_0 \to 0$ and $\mu_0 \to \varepsilon$ in Equations (48) and (49).

firm value $V = P_D + P_E$ with respect to ε does not depend on K.

5. Extension to Economies with General Probabilities and Distributions

In this section, we consider an economy with $M = N$ assets and extend our analysis to an economy with general probabilities of states $\pi_n(\varepsilon)$, and also shocks ε and noises v drawn from distributions with general PDFs $\phi_\varepsilon(x)$ and $\phi_v(x)$, respectively. We show that despite the generality, the model preserves the structure of portfolios and prices derived in Proposition 1 for particular probabilities $\pi_n(\varepsilon)$ and PDFs $\phi_\varepsilon(x)$, given by Equations (5) and (6), respectively. We obtain prices $P(\varepsilon, v)$ and investor I's portfolio $\theta_I^*(p;\varepsilon)$ in closed form, and investor U's portfolio $\theta_U^*(p)$ in terms of a solution of a fixed-point problem.

In the economy with general distributions, portfolio $\theta_I^*(p;\varepsilon)$ is no longer a linear function of ε but remains separable in shock ε and prices p due to CARA preferences. Because the optimal portfolio (11) in Lemma 1 is derived for general probabilities $\pi_n(\varepsilon)$ and distributions of ε and v, from (11) we find that portfolio $\theta_I^*(p;\varepsilon)$ is given by $\theta_I^*(p;\varepsilon) = \eta(\varepsilon)/\gamma_I - 1/\gamma_I\, \Omega^{-1}v(p)$, where vector $\eta(\varepsilon) \in \mathbb{R}^{N-1}$ is defined as follows:

$$\eta(\varepsilon) = \Omega^{-1}\Big(\ln\Big(\frac{\pi_1(\varepsilon)}{\pi_N(\varepsilon)}\Big), \ldots, \ln\Big(\frac{\pi_{N-1}(\varepsilon)}{\pi_N(\varepsilon)}\Big)\Big)^T. \tag{52}$$

Substituting investors' portfolios into market clearing conditions we obtain:

$$\frac{\eta(\varepsilon)}{\gamma_I} + v + \hat{H}(p) = 0, \tag{53}$$

where $\hat{H}(p) = \theta_U^*(p) - \Omega^{-1}v(p)/\gamma_I$. Then, the derivation of equilibrium proceeds in the same way as in Section 3. Proposition 7 reports the results.

Proposition 7 (Equilibrium with $M = N$ and general distributions). *Let probabilities $\pi_n(\varepsilon)$ and PDFs $\phi_\varepsilon(x)$ and $\phi_v(x)$ of shock ε and noise v be continuous, bounded and positive functions on $\mathbb{R}$ and $\mathbb{R}^{M-1}$, respectively. Then, the following statements hold.*
1) *If there exists an REE, optimal portfolios $\theta_I^*(p;\varepsilon)$ and $\theta_U^*(p)$, risk-neutral probabilities π_n^{RN} and asset prices $P(\varepsilon, v)$ are given by:*

$$\theta_I^*(p;\varepsilon) = \frac{\eta(\varepsilon)}{\gamma_I} - \frac{1}{\gamma_I}\Omega^{-1}v(p), \tag{54}$$

$$\theta_U^*(p) = -\frac{1}{\gamma_U}\Omega^{-1}\Big(v(p) - \Psi\big(\hat{H}(p)\big)\Big), \tag{55}$$

$$\pi_n^{\text{RN}} = \frac{e^{v_n}}{1+\sum_{k=1}^{N-1} e^{v_k}}, \quad \pi_N^{\text{RN}} = \frac{1}{1+\sum_{k=1}^{N-1} e^{v_k}}, \tag{56}$$

$$P_m(\varepsilon, v) = \left(\pi_1^{\text{RN}} C_m(\omega_1) + \pi_2^{\text{RN}} C_m(\omega_2) + \ldots + \pi_N^{\text{RN}} C_m(\omega_N)\right) e^{-rT}, \tag{57}$$

where $m = 1, \ldots, M-1$, $\eta(\varepsilon)$ *is given by* (52), *matrix* Ω *is as in Proposition 1, and* $v(p) = (v_1, \ldots, v_{N-1})^T$ *is a function of* p *defined by* (14), *which in equilibrium is given by*

$$v(p) = \frac{\gamma_U \gamma_I}{\gamma_U + \gamma_I}\left(\frac{1}{\gamma_U}\Psi\left(\frac{\eta(\varepsilon)}{\gamma_I} + v\right) + \Omega\left(\frac{\eta(\varepsilon)}{\gamma_I} + v\right)\right), \tag{58}$$

functions $\hat{H}(p) : \mathbb{R}^{M-1} \to \mathbb{R}^{M-1}$ *and* $\Psi(\cdot) : \mathbb{R}^{M-1} \to \mathbb{R}^{M-1}$ *are defined by equations*

$$v(p) = \frac{\gamma_U \gamma_I}{\gamma_U + \gamma_I}\left(\frac{1}{\gamma_U}\Psi\left(\hat{H}(p)\right) - \Omega \hat{H}(p)\right), \tag{59}$$

$$\Psi(z) = \left(\Psi_1(z) - \Psi_N(z), \ldots, \Psi_{N-1}(z) - \Psi_N(z)\right)^T, \tag{60}$$

$$\Psi_n(z) = \ln\left(\int_{-\infty}^{+\infty} \pi_n(x)\phi_v\left(\frac{\eta(x)}{\gamma_I} - z\right)\phi_\varepsilon(x)dx\right). \tag{61}$$

2) *There exists an REE iff function* $\Psi(x)/\gamma_U - \Omega x : \mathbb{R}^{M-1} \to \mathbb{R}^{M-1}$ *is an injective map.*[20]

3) *If investor U observes both prices p and the residual demand* $\theta_I^*(p;\varepsilon) + v$, *then there always exists unique REE in which investor I's portfolio and asset prices are given by Equations* (54) *and* (57), *respectively, and investor U's portfolio is given by:*

$$\theta_U^*(p) = -\frac{1}{\gamma_U}\Omega^{-1}\left(v(p) - \Psi\left(-\frac{1}{\gamma_I}\Omega^{-1}v(p) - \theta_I^*(p;\varepsilon) - v\right)\right). \tag{62}$$

Proposition 7 provides closed-form expressions for asset prices as functions of $\eta(\varepsilon)/\gamma_I + v$ for general probabilities and distributions and derives necessary and sufficient conditions for the existence of REE. The equilibrium is derived in terms of function $\Psi(z)$ which, as shown in the proof of Proposition 7, can be interpreted as a vector of log-likelihood ratios of posterior probabilities of investor U. In the special case of normally distributed noises v and shocks ε with PDF (6) function $\Psi(z)$ becomes a linear function of z, and the equilibrium coincides with that in Proposition 1.

Proposition 7 demonstrates the existence of REE when function $\Psi(x)/\gamma_U - \Omega x$ is injective. If the latter function is not injective, Equations (57) and (58) imply that price $P(\varepsilon, v)$ is not an injective function. Therefore, there exist pairs (ε_1, v_1) and (ε_2, v_2) such

[20] That is, for all x_1 and x_2 such that $\Psi(x_1)/\gamma_U - \Omega x_1 = \Psi(x_2)/\gamma_U - \Omega x_2 \Rightarrow x_1 = x_2$. Gale and Nikaidô

(1965) show that an easily verifiable sufficient condition for a function to be injective is that all principal minors of its Jacobian matrix are positive.

that $P(\varepsilon_1, v_1) = P(\varepsilon_2, v_2)$ and yet $\eta(\varepsilon_1)/\gamma_I + v_1 \neq \eta(\varepsilon_2)/\gamma_I + v_2$. Consequently, the information contained in asset prices does not reveal the realization of $\eta(\varepsilon)/\gamma_I + v$, which contradicts the market clearing condition (53), and hence cannot happen in equilibrium.

In Proposition 7, similar to Breon-Drish (2010), we also demonstrate the existence of REE in economies in which uninformed investors in addition to prices p can observe the residual demand $\theta_I^*(p;\varepsilon) + v$, which is comprised of the demands of the informed and noise traders. The latter assumption is similar to that in Kyle (1985), where uninformed market makers observe the aggregate demand of informed and noise traders. Because $\theta_I^*(p;\varepsilon)+v = \eta(\varepsilon)/\gamma_I + v - \Omega^{-1}v(p)/\gamma_I$, observing price p and the residual demand $\theta_I^*(p;\varepsilon)+v$ allows investor U to infer $\eta(\varepsilon)/\gamma_I + v$ in our economy even in situations when the latter information is not revealed via asset prices, that is, when markets are not weak-form efficient. Then, if $\eta(\varepsilon)/\gamma_I + v$ is known to investor U, the derivation of equilibrium proceeds in the same way as in part 1 of Proposition 7.

6. Conclusion

We provide a general framework that can be used as a building block to study asset pricing and information asymmetry in an REE setting. In contrast to previous works, our model allows for general payoffs of assets, which do not need to be normally distributed. We provide a tractable closed-form characterization of equilibrium for a large class of probabilities of states of the economy and probability density functions of signals. We derive exact conditions under which asset prices reveal information about the signal and provide several economic applications of our theory. The tractability of the model allows us to obtain simple comparative statics for optimal portfolios and asset prices and to prove the existence of equilibrium.

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Appendix: Proofs

Lemma A.1 (Prior mean and prior variance of ε and prior probabilities). *Assume that ε has PDF* (6), *then its mean μ_ε and variance σ_ε^2 are given by the following expressions in terms of the parameters (μ_0, σ_0^2) and the vectors (a, b):*

$$\mu_\varepsilon = \frac{\sum_{j=1}^{N} \exp\left(a_j + \frac{\mu_j^2}{2\sigma_0^2}\right)\mu_j}{\sum_{j=1}^{N} \exp\left(a_j + \frac{\mu_j^2}{2\sigma_0^2}\right)}, \tag{A.1}$$

$$\sigma_\varepsilon^2 = \sigma_0^2 + \left[\frac{\sum_{j=1}^{N} \exp\left(a_j + \frac{\mu_j^2}{2\sigma_0^2}\right)\mu_j^2}{\sum_{j=1}^{N} \exp\left(a_j + \frac{\mu_j^2}{2\sigma_0^2}\right)} - \left(\frac{\sum_{j=1}^{N} \exp\left(a_j + \frac{\mu_j^2}{2\sigma_0^2}\right)\mu_j}{\sum_{j=1}^{N} \exp\left(a_j + \frac{\mu_j^2}{2\sigma_0^2}\right)}\right)^2\right], \tag{A.2}$$

where $\mu_j = (b_j + \mu_0/\sigma_0^2)/(1/\sigma_0^2)$.

Proof of Lemma A.1. The PDF of ε is given by

$$\phi_\varepsilon(x) = \frac{\sum_{j=1}^{N} e^{a_j+b_j x}\varphi(x;\mu_0,\sigma_0)}{\int_{-\infty}^{\infty} \sum_{j=1}^{N} e^{a_j+b_j x}\varphi(x;\mu_0,\sigma_0)\,dx},$$

where $\varphi(x;\mu,\sigma)$ denotes the pdf of a random variable distributed $N(\mu,\sigma^2)$,

$$\varphi(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}.$$

After some algebra, we rewrite $\phi_\varepsilon(x)$ as follows:

$$\phi_\varepsilon(x) = \frac{\sum_{j=1}^{N} e^{a_j+b_j\mu_0+\sigma_0^2 b_j^2/2}\varphi(x;\sigma_0^2 b_j+\mu_0,\sigma_0)}{\sum_{j=1}^{N} e^{a_j+b_j\mu_0+\sigma_0^2 b_j^2/2}},$$

$$= \frac{\sum_{j=1}^{N} \exp\left(a_j + \frac{\mu_j^2}{2\sigma_0^2}\right)\varphi(x;\mu_j,\sigma_0)}{\sum_{j=1}^{N} \exp\left(a_j + \frac{\mu_j^2}{2\sigma_0^2}\right)},$$

where by definition we set $\mu_j = \sigma_0^2 b_j + \mu_0 = (b_j + \mu_0/\sigma_0^2)/(1/\sigma_0^2)$. Computing μ_ε and σ_ε^2 with PDF $\phi_\varepsilon(x)$, after straightforward algebra, we obtain Equations (A.1) and (A.2).

Proof of Lemma 1. We derive portfolio (11) of the informed investor, whereas the proof for the uninformed is similar. Taking log on both sides of investor I's FOC (10),

and substituting wealth $W_{I,T,n}$ from the budget constraint (3), we obtain:

$$(\theta_I^*)^T(\Pi_n - e^{rT}p) = \frac{1}{\gamma_I}\Big(\ln\big(\pi_n(\varepsilon)\big) - \ln\big(\pi_n^{\mathrm{RN}}\big)\Big) + const, \quad n = 1, \ldots, N, \tag{A.3}$$

where $const$ is a constant that does not depend on n. Writing down Equation (A.3) for $n = N$ and subtracting it from the other equations in (A.3), we obtain the following system of $N-1$ equations with $N-1$ unknown components of vector θ_I^*:

$$(\theta_I^*)^T(\Pi_n - \Pi_N) = \frac{1}{\gamma_I}\Big(\ln\Big(\frac{\pi_n(\varepsilon)}{\pi_N(\varepsilon)}\Big) - \ln\Big(\frac{\pi_n^{\mathrm{RN}}}{\pi_N^{\mathrm{RN}}}\Big)\Big), \quad n = 1, \ldots, N-1, \tag{A.4}$$

where $\Pi_n - \Pi_N = (C_1(\omega_n) - C_1(\omega_N), \ldots, C_{N-1}(\omega_n) - C_{N-1}(\omega_N))^T$. Solving the system of equations (A.4), we obtain investor I's optimal portfolio

$$\theta_I^*(p;\varepsilon) = \frac{1}{\gamma_I}\Omega^{-1}\Big\{\Big(\ln\Big(\frac{\pi_1(\varepsilon)}{\pi_N(\varepsilon)}\Big), \ldots, \ln\Big(\frac{\pi_{N-1}(\varepsilon)}{\pi_N(\varepsilon)}\Big)\Big)^T - \Big(\ln\Big(\frac{\pi_1^{\mathrm{RN}}}{\pi_N^{\mathrm{RN}}}\Big), \ldots, \ln\Big(\frac{\pi_{N-1}^{\mathrm{RN}}}{\pi_N^{\mathrm{RN}}}\Big)\Big)^T\Big\}.$$

Finally, substituting probabilities $\pi_n(\varepsilon)$ from Equation (5) into the above equation, we obtain investor I's portfolio weight (15). $\blacksquare$

Proof of Lemma 2. From Bayes rule we have that

$$\phi_{\varepsilon|\tilde{\varepsilon}}(x|y) = \frac{\phi_{\tilde{\varepsilon}|\varepsilon}(y|x)\phi_\varepsilon(x)}{\int_{-\infty}^{\infty}\phi_{\tilde{\varepsilon}|\varepsilon}(y|x)\phi_\varepsilon(x)dx}.$$

Note that, since $v \sim N(0, \Sigma_v)$, $\tilde{\varepsilon} = \lambda\varepsilon/\gamma_I + v + H(p)$ conditional on ε has multivariate normal distribution $N(\lambda\varepsilon/\gamma_I + H(p), \Sigma_v)$. Hence substituting for $\phi_{\tilde{\varepsilon}|\varepsilon}$ above, we have

$$\phi_{\varepsilon|\tilde{\varepsilon}}(x|y) = \frac{\exp\Big\{-0.5\big(y - \lambda x/\gamma_I - H(p)\big)^T\Sigma_v^{-1}\big(y - \lambda x/\gamma_I - H(p)\big)\Big\}\phi_\varepsilon(x)}{G_1(y;p)}, \tag{A.5}$$

where $G_1(y;p)$ is a function that does not depend on state ω_n and normalizes the density. Next, to find probability π_n^U, from the market clearing condition (16), we note that by observing price p the uninformed investor can only learn that shock ε and noise trader demand v satisfy Equation (16). Therefore, from Equation (13) for π_n^U we obtain:

$$\begin{aligned}\pi_n^U &= \mathbb{E}[\pi_n(\varepsilon)\,|\,\lambda\varepsilon/\gamma_I + v + H(p) = 0] \\ &= \int_{-\infty}^{\infty}\frac{e^{a_n+b_nx}}{\sum_{j=1}^N e^{a_j+b_jx}}\phi_{\varepsilon|\tilde{\varepsilon}}(x|0)dx = \frac{1}{G_1(y;p)}\int_{-\infty}^{\infty}e^{d_n(x)}dx,\end{aligned} \tag{A.6}$$

where $d_n(x)$ is a quadratic function of x given by:

$$\begin{aligned} d_n(x) &= a_n + b_n x - 0.5\big(\lambda x/\gamma_I + H(p)\big)^T \Sigma_v^{-1}\big(\lambda x/\gamma_I + H(p)\big) - 0.5(x-\mu_0)^2/\sigma_0^2 \\ &= -\frac{\lambda^T\Sigma_v^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2}{2}\left(x - \frac{\mu_0/\sigma_0^2 + b_n - \lambda^T\Sigma_v^{-1}H(p)/\gamma_I}{\lambda^T\Sigma_v^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2}\right)^2 \\ &\quad + a_n + \frac{1}{2}\frac{b_n^2 - 2b_n\big(\lambda^T\Sigma_v^{-1}H(p)/\gamma_I - \mu_0/\sigma_0^2\big)}{\lambda^T\Sigma_v^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2} + g(p), \end{aligned} \tag{A.7}$$

where $g(p)$ is some function which only depends on p, and not on x or n. Substituting Equation (A.7) back into integral (A.6), after integrating, we obtain Equation (19) for π_n^U

$$\pi_n^U(p;\theta_U^*(p)) = \frac{1}{G_2(p)}\exp\left(a_n + \frac{1}{2}\frac{b_n^2 - 2b_n\big(\lambda^T\Sigma_v^{-1}H(p)/\gamma_I - \mu_0/\sigma_0^2\big)}{\lambda^T\Sigma_v^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2}\right),$$

where $G_2(p)$ is a function, which does not depend on ω_n and is not needed later. •

Proof of Proposition 1. Let $v = \big(\ln(\pi_1^{RN}/\pi_N^{RN}), \dots, \ln(\pi_{N-1}^{RN}/\pi_N^{RN})\big)^T$. Then, the risk-neutral probabilities are given by $\pi_n^{RN} = e^{v_n}/(1 + \sum_{j=1}^{N-1} e^{v_j})$ for $n = 1, \dots, N-1$ and $\pi_N^{RN} = 1/(1 + \sum_{j=1}^{N-1} e^{v_j})$. Therefore, from Equation (7) for prices p we obtain that the prices are given by Equation (23).

Investor I's portfolio (20) is the same as in Equation (15) in Lemma 1. To find investor U's portfolio $\theta_U^*(p)$, we use Equation (12) in Lemma 1, which gives $\theta_U^*(p)$ in terms of investor U's probabilities $\pi_n^U(p;\theta_U^*(p))$. Substituting probabilities $\pi_n^U(p;\theta_U^*(p))$ from (19) into portfolio (12) we obtain:

$$\begin{aligned} \theta_U^*(p) &= \frac{1}{\gamma_U}\Omega^{-1}\left(\left(\ln\frac{\pi_1^U(p;\theta_U^*(p))}{\pi_N^U(p;\theta_U^*(p))}, \dots, \ln\frac{\pi_{N-1}^U(p;\theta_U^*(p))}{\pi_N^U(p;\theta_U^*(p))}\right)^T - v\right) \\ &= \frac{1}{\gamma_U}\Omega^{-1}\left(\tilde{a} + \frac{1}{2}\frac{\tilde{b}^{(2)} - 2\tilde{b}\big(\lambda^T\Sigma_v^{-1}H(p)/\gamma_I - \mu_0/\sigma_0^2\big)}{\lambda^T\Sigma_v^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2} - v\right), \end{aligned} \tag{A.8}$$

where $\tilde{b}^{(2)} = (b_1^2 - b_N^2, \dots, b_{N-1}^2 - b_N^2)$ and $\tilde{b} = (b_1 - b_N, \dots, b_{N-1} - b_N)$. Recalling that $\lambda = \Omega^{-1}\tilde{b}$, and rearranging terms in Equation (A.8), we obtain:

$$\theta_U^*(p) = \frac{1}{\gamma_U}\Omega^{-1}\hat{a} + \frac{1}{\gamma_U}\frac{(\mu_0/\sigma_0^2)\lambda}{\lambda^T\Sigma_v^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2} - QH(p) - \frac{1}{\gamma_U}\Omega^{-1}v, \tag{A.9}$$

where $\hat{a}$ and matrix Q are are given by:

$$\hat{a} = \tilde{a} + \frac{1}{2}\frac{\tilde{b}^{(2)}}{\big(\lambda^T\Sigma_v^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2\big)}, \quad Q = \frac{\lambda\lambda^T\Sigma_v^{-1}}{\gamma_U\gamma_I\big(\lambda^T\Sigma_v^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2\big)}. \tag{A.10}$$

T_v^{-1} $\,^{2}$, $\qquad {}_0$

Next, we substitute $H(p) = \theta_U^* - \Omega^{-1}\big(v - \tilde{a}\big)/\gamma_I$, from Equation (17) into Equation (A.9), and after some algebra, we obtain a system of linear equations for portfolio $\theta_U^*(p)$:

$$\theta_U^*(p) = \frac{1}{\gamma_U}\Omega^{-1}\bar{a} + \frac{1}{\gamma_U}\frac{(\mu_0/\sigma_0^2)\lambda}{\lambda^T\Sigma_v^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2} - Q\theta_U^* + \frac{1}{\gamma_I}Q\Omega^{-1}\big(v - \tilde{a}\big) - \frac{1}{\gamma_U}\Omega^{-1}v.$$

Solving this system of equations, we obtain $\theta_U^*(p)$ in Proposition 1, given by

$$\theta_U^*(p) = \big(E + Q\big)^{-1}\left(\frac{1}{\gamma_I}Q\Omega^{-1}(v - \tilde{a}) - \frac{1}{\gamma_U}\Omega^{-1}(v - \bar{a}) + \frac{(\mu_0/\sigma_0^2)\lambda}{\gamma_U(\lambda^T\Sigma_v^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2)}\right).$$

Next, we find equilibrium prices. Substituting optimal portfolios $\theta_I^*(p;\varepsilon)$ and $\theta_U^*(p)$ from Equations (20) and (21) into the market clearing condition $\theta_I^*(p;\varepsilon) + \theta_U^*(p) + \nu = 0$, after rearranging terms, we obtain the following equation for vector v:

$$\begin{aligned}\big(E + Q\big)^{-1}&\left(\frac{1}{\gamma_I}Q\Omega^{-1}(v - \tilde{a}) - \frac{1}{\gamma_U}\Omega^{-1}(v - \bar{a}) + \frac{(\mu_0/\sigma_0^2)\lambda}{\gamma_U(\lambda^T\Sigma_v^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2)}\right)\\ &- \frac{1}{\gamma_I}\Omega^{-1}\big(v - \tilde{a}\big) + \frac{\lambda\varepsilon}{\gamma_I} + \nu = 0.\end{aligned} \tag{A.11}$$

We observe that the above equation can be further simplified by noting that

$$\begin{aligned}(E+Q)^{-1}\frac{1}{\gamma_I}Q\Omega^{-1}(v - \tilde{a}) &= (E+Q)^{-1}(E+Q-E)\frac{1}{\gamma_I}\Omega^{-1}(v - \tilde{a})\\ &= \frac{1}{\gamma_I}\Omega^{-1}(v-\tilde{a}) - (E+Q)^{-1}\frac{1}{\gamma_I}\Omega^{-1}(v-\tilde{a}).\end{aligned}$$

Substituting the latter expression into Equation (A.11), canceling like terms, substituting $\bar{a}$ from Equation (A.10) into Equation (A.11), and solving it for $v - \tilde{a}$ we obtain

$$v = \tilde{a} + \frac{1}{2}\frac{\gamma_I}{\gamma_I + \gamma_U}\frac{\tilde{b}^{(2)} + 2(\mu_0/\sigma_0^2)\Omega\lambda}{\lambda^T\Sigma_v^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2} + \frac{\gamma_I\gamma_U}{\gamma_I + \gamma_U}\Omega\big(E + Q\big)\left(\frac{\lambda\varepsilon}{\gamma_I} + \nu\right),$$

which gives v in Equation (24). The equilibrium asset prices are then given by Equation (23) in terms of vector v. Because v is a linear function of $\lambda\varepsilon/\gamma_I + \nu$, function $P(\varepsilon, \nu)$ is a one-to-one mapping between $\lambda\varepsilon/\gamma_I + \nu$ and prices p. Therefore, observing asset prices indeed reveals $\lambda\varepsilon/\gamma_I + \nu$, which completes the proof. ▪

Proof of Proposition 2. Although vector v in Proposition 1 is $(N-1)$-dimensional, for convenience we set $v_N = 0$. First, we find comparative statics for prices. Differentiating risk-neutral probability π_n^{RN} given by (22) with respect to ε we obtain:

$$\begin{aligned}\frac{\partial\pi_n^{\mathrm{RN}}}{\partial\varepsilon} &= \pi_n^{\mathrm{RN}}\frac{\partial v_n}{\partial\varepsilon} - \frac{\pi_n^{\mathrm{RN}}}{\sum_{k=1}^N e^{v_k}}\sum_{k=1}^N\frac{\partial v_k}{\partial\varepsilon}e^{v_k}\\ &= \pi_n^{\mathrm{RN}}\frac{\partial v_n}{\partial\varepsilon} - \pi_n^{\mathrm{RN}}\mathbb{E}^{\mathrm{RN}}\Big[\frac{\partial v(\omega)}{\partial\varepsilon}\Big],\end{aligned} \tag{A.12}$$

$$ {}_{n} \quad \partial\varepsilon \; {}^{-\pi_n \, \mathbb{E}} \qquad \partial\varepsilon $$

where $v(\omega)$ now denotes a random variable that takes value v_n in state ω_n. Next, differentiating price (23) with respect to ε, and using Equation (A.12), we obtain:

$$\begin{aligned}\frac{\partial P_m(\varepsilon, v)}{\partial \varepsilon} &= \mathbb{E}^{\mathrm{RN}}\left[\frac{\partial v(\omega)}{\partial \varepsilon} C_m(\omega)\right] - \mathbb{E}^{\mathrm{RN}}\left[\frac{\partial v(\omega)}{\partial \varepsilon}\right]\mathbb{E}^{\mathrm{RN}}\left[C_m(\omega)\right] \\ &= \mathrm{cov}^{\mathrm{RN}}\left(\frac{\partial v(\omega)}{\partial \varepsilon}, C_m(\omega)\right).\end{aligned} \tag{A.13}$$

Differentiating Equation (24) for vector v, substituting matrix Q from Equation (25), and denoting by $e_k = (0, \ldots, 1, \ldots 0)^T \in \mathbb{R}^{N-1}$ a vector with k^{th} element equal to 1 and other elements equal to 0, we obtain:

$$\begin{aligned}\frac{\partial v_k}{\partial \varepsilon} &= \frac{\gamma_U}{\gamma_U + \gamma_I} e_k^T\left(\Omega\lambda + \frac{\Omega\lambda\lambda^T\Sigma_v^{-1}\lambda/(\gamma_U\gamma_I)}{\lambda^T\Sigma_v^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2}\right) \\ &= \frac{\gamma_U}{\gamma_U + \gamma_I}(b_k - b_N)\left(1 + \frac{\lambda^T\Sigma_v^{-1}\lambda/(\gamma_U\gamma_I)}{\lambda^T\Sigma_v^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2}\right),\end{aligned} \tag{A.14}$$

where to derive the second line we used the fact that $e_k^T\Omega\lambda = e_k^T(b_1 - b_N, \ldots, b_{N-1} - b_N)^T = b_k - b_N$, when $k < N$. Equation (A.14) also holds for $k = N$, in which case $\partial v_N/\partial\varepsilon = 0$ because $b_k - b_N = 0$. Therefore, using Equation (A.14) we compute the covariance in Equation (A.13), and obtain:

$$\frac{\partial P_m(\varepsilon, v)}{\partial \varepsilon} = \frac{\gamma_U}{\gamma_U + \gamma_I}\left(1 + \frac{\lambda^T\Sigma_v^{-1}\lambda/(\gamma_U\gamma_I)}{\lambda^T\Sigma_v^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2}\right)\mathrm{cov}^{\mathrm{RN}}(b, C_m)e^{-rT}, \tag{A.15}$$

where we eliminated b_N because subtracting a constant does not affect covariances.

To find the derivative with respect to v_l, following the same steps as above, we obtain:

$$\frac{\partial P_m(\varepsilon, v)}{\partial v_l} = \mathrm{cov}^{\mathrm{RN}}\left(\frac{\partial v(\omega)}{\partial v_l}, C_m(\omega)\right), \tag{A.16}$$

where $l = 1, \ldots, M-1$. Then, differentiating Equation (24) for vector v and recalling that we additionally set $v_N = 0$, similarly to Equation (A.14) we obtain:

$$\begin{aligned}\frac{\partial v_k}{\partial v_l} &= \frac{\gamma_U}{\gamma_U + \gamma_I} e_k^T\Omega\left(E + Q\right)e_l \\ &= \frac{\gamma_U}{\gamma_U + \gamma_I}\left(e_k^T\Omega e_l + \frac{e_k^T\Omega\lambda\lambda^T\Sigma_v^{-1}e_l/(\gamma_U\gamma_I)}{\lambda^T\Sigma_v^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2}\right), \\ &= \frac{\gamma_U}{\gamma_U + \gamma_I}\left(C_k(\omega_l) - C_k(\omega_N) + \frac{(b_k - b_N)\lambda^T\Sigma_v^{-1}e_l/(\gamma_U\gamma_I)}{\lambda^T\Sigma_v^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2}\right),\end{aligned} \tag{A.17}$$

where to derive the last line we used the fact that by the definition of matrix Ω and vector λ, $e_k^T\Omega e_l = C_k(\omega_l) - C_k(\omega_N)$ and $e_k^T\Omega\lambda = b_k - b_N$. Clearly, Equation (A.17) also holds for

$k = N$, because then it implies that $\partial v_k/\partial v_l = 0$. Therefore, from Equations (A.17) and (A.16), we obtain the result in Proposition 2:

$$\frac{\partial P_m(\varepsilon, v)}{\partial v_l} = \frac{\gamma_U\gamma_I}{\gamma_U+\gamma_I}\left(\mathrm{cov}^{\mathrm{RN}}\left(C_l, C_m\right) + \lambda^T\frac{\Sigma_v^{-1}e_l/(\gamma_U\gamma_I)^2}{\Sigma_v\,\lambda/\gamma_I + 1/\sigma_0}\,\mathrm{cov}^{\mathrm{RN}}\left(b, C_m\right)\right)e^{-rT}.$$

Now, we find the derivatives of optimal portfolios with respect to prices p. First, we need to compute $\partial v/\partial p$. To do this, we find the Jacobian $J_p = \partial p/\partial v$ and then by the inverse function theorem we have $\partial v/\partial p = J_p^{-1}$. Substituting $\pi_N^{\mathrm{RN}} = 1 - \pi_1^{\mathrm{RN}} - \ldots - \pi_{N-1}^{\mathrm{RN}}$ into Equation (7) for prices p in terms of risk-neutral probabilities, we obtain:

$$p_m = \left(\pi_1^{\mathrm{RN}}\left(C_m(\omega_1) - C_m(\omega_N)\right) + \ldots + \pi_{N-1}^{\mathrm{RN}}\left(C_m(\omega_{N-1}) - C_m(\omega_N)\right) + C_m(\omega_N)\right)e^{-rT}, \quad \text{(A.18)}$$

where $m = 1, \ldots, N-1$. Let J_π be the Jacobian of vector $(\pi_1^{\mathrm{RN}}, \ldots, \pi_{N-1}^{\mathrm{RN}})^T$, that is, a matrix with (n, k) element given by $\partial\pi_n^{\mathrm{RN}}/\partial v_k$. Differentiating Equation (A.18) we find that $J_p = \Omega^T J_\pi e^{-rT}$, and hence

$$J_p\Omega e^{rT} = \Omega^T J_\pi\Omega. \quad \text{(A.19)}$$

To find J_π we first calculate $\partial\pi_n^{\mathrm{RN}}/\partial v_k$, where π_n^{RN} is given by the first equation in (22):

$$\frac{\partial\pi_n^{\mathrm{RN}}}{\partial v_k} = \begin{cases} -\pi_n^{\mathrm{RN}}\pi_k^{\mathrm{RN}}, & \text{if } n \neq k, \\ \pi_n^{\mathrm{RN}} - (\pi_n^{\mathrm{RN}})^2, & \text{if } n = k. \end{cases} \quad \text{(A.20)}$$

From Equation (A.20) we find $J_\pi = \mathrm{diag}\{\pi_1^{\mathrm{RN}}, \ldots, \pi_{N-1}^{\mathrm{RN}}\} - (\pi_1^{\mathrm{RN}}, \ldots, \pi_{N-1}^{\mathrm{RN}})^T(\pi_1^{\mathrm{RN}}, \ldots, \pi_{N-1}^{\mathrm{RN}})$, where $\mathrm{diag}\{\ldots\}$ is a diagonal matrix. Substituting J_π into Equation (A.19) we obtain:

$$J_p\Omega e^{rT} = \Omega^T\left(\mathrm{diag}\{\pi_1^{\mathrm{RN}}, \ldots, \pi_{N-1}^{\mathrm{RN}}\} - (\pi_1^{\mathrm{RN}}, \ldots, \pi_{N-1}^{\mathrm{RN}})^T(\pi_1^{\mathrm{RN}}, \ldots, \pi_{N-1}^{\mathrm{RN}})\right)\Omega. \quad \text{(A.21)}$$

Recalling that Ω is a matrix with rows $(\Pi_n - \Pi_N)^T$, where $\Pi_n = \left(C_1(\omega_n), \ldots, C_{M-1}(\omega_n)\right)^T$ and denoting $\tilde{C}_n = \left(C_n(\omega_1) - C_n(\omega_N), \ldots, C_n(\omega_{N-1}) - C_n(\omega_N)\right)^T$, we find that the (n, k) element of matrix $J_p\Omega e^{rT}$ is given by:

$$\begin{aligned}\{J_p\Omega e^{rT}\}_{n,k} &= \tilde{C}_n^T\mathrm{diag}\{\pi_1^{\mathrm{RN}}, \ldots, \pi_{N-1}^{\mathrm{RN}}\}\tilde{C}_k - \tilde{C}_n^T(\pi_1^{\mathrm{RN}}, \ldots, \pi_{N-1}^{\mathrm{RN}})^T(\pi_1^{\mathrm{RN}}, \ldots, \pi_{N-1}^{\mathrm{RN}})\tilde{C}_k \\ &= \sum_{i=1}^N\left(C_n(\omega_i) - C_n(\omega_N)\right)\left(C_k(\omega_i) - C_k(\omega_N)\right)\pi_i^{\mathrm{RN}} \\ &\quad - \left(\sum_{i=1}^N\left(C_n(\omega_i) - C_n(\omega_N)\right)\pi_i^{\mathrm{RN}}\right)\left(\sum_{i=1}^N\left(C_k(\omega_i) - C_k(\omega_N)\right)\pi_i^{\mathrm{RN}}\right) \\ &= \mathrm{cov}^{\mathrm{RN}}(C_n, C_k),\end{aligned}$$

where to derive the second equality we added zero terms $\left(C_n(\omega_N) - C_n(\omega_N)\right)\left(C_k(\omega_N) - C_k(\omega_N)\right)\pi_N^{\mathrm{RN}}$, $\left(C_n(\omega_N) - C_n(\omega_N)\right)\pi_N^{\mathrm{RN}}$ and $\left(C_k(\omega_N) - C_k(\omega_N)\right)\pi_N^{\mathrm{RN}}$ to summations, and then removed constants $C_n(\omega_N)$ and $C_k(\omega_N)$, because they do not affect covariances.

Therefore, we conclude that $J_p\Omega e^{rT} = \mathrm{var}^{\mathrm{RN}}[\Pi]$. Then, by the inverse function theorem, we now find that $\Omega^{-1}\partial v/\partial p = \big(\mathrm{var}^{\mathrm{RN}}[\Pi]\big)^{-1} e^{rT}$. Using the latter equality and differentiating optimal portfolios (20) and (21) with respect to p we obtain that the first of these two partial derivatives is given by (28) and the second is given by:

$$\frac{\partial\theta_U^*(p)}{\partial p} = \Big(\frac{1}{\gamma_I}E - \frac{\gamma_I+\gamma_U}{\gamma_I\gamma_U}(E+Q)^{-1}\Big)\big(\mathrm{var}^{\mathrm{RN}}[\Pi]\big)^{-1}e^{rT}. \tag{A.22}$$

We note the following equation for the inverse matrix $(E+Q)^{-1}$:

$$(E+Q)^{-1} = E - \frac{\lambda\lambda^T\Sigma_v^{-1}}{\gamma_U\gamma_I\big(\lambda^T\Sigma_v^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2\big) + \lambda^T\Sigma_v^{-1}\lambda},$$

which can be verified by multiplying both sides of the latter equation by $E+Q$. Substituting $(E+Q)^{-1}$ above into Equation (A.22), we obtain Equation (29) for $\partial\theta_U^*(p)/\partial p^T$.

Finally, we demonstrate that $\partial\theta^*_{I,m}(p;\varepsilon)/\partial p_m < 0$, i.e. investor I's demand for asset m is downward sloping in asset m^{th} price. This result follows from the fact that matrix $(\mathrm{var}^{\mathrm{RN}}[\Pi])^{-1}$ is positive-definite (as the inverse of a positive-definite matrix), and its element m of the diagonal is given by $e_m^T(\mathrm{var}^{\mathrm{RN}}[\Pi])^{-1}e_m > 0$, where $e_m = (0, 0, \ldots, 1, \ldots, 0)^T$ is a vector with m^{th} element equal to 1 and other elements equal to zero. Then, from Equation (28) it follows that $\partial\theta^*_{I,m}(p;\varepsilon)/\partial p_m < 0$. •

Proof of Lemma 3. From Assumption 1 we have that $b_n = \lambda_0 + \Pi_n^T\lambda$, which we substitute into the objective function (1) of the informed investor. After some algebra, we rewrite investor I's objective function as follows:

$$\begin{aligned}
\mathbb{E}\big[-e^{-\gamma_I W_{I,T}}\,|\,\varepsilon,p\big] &= -\frac{\sum_{j=1}^N \exp\{a_j + b_j\varepsilon - \gamma_I(\Pi_j - e^{rT}p)^T\theta_I\}}{\sum_{j=1}^N \exp\{a_j+b_j\varepsilon\}} \\
&= -\exp\{(\lambda_0 + e^{rT}p^T\lambda)\varepsilon - e^{rT}p^T(\lambda\varepsilon - \gamma_I\theta_I)\} \\
&\quad\times \frac{\sum_{j=1}^N \exp\{a_j + \Pi_j^T(\lambda\varepsilon - \gamma_I\theta_I)\}}{\sum_{j=1}^N \exp\{a_j+b_j\varepsilon\}} \\
&= -\exp\{(\lambda_0 + e^{rT}p^T\lambda)\varepsilon - e^{rT}p^T\tilde{\theta}_I\}\,\frac{\sum_{j=1}^N \exp\{a_j+\Pi_j^T\tilde{\theta}_I\}}{\sum_{j=1}^N \exp\{a_j+b_j\varepsilon\}},
\end{aligned} \tag{A.23}$$

where $\tilde{\theta}_I = \lambda\varepsilon - \gamma_I\theta_I$. From the last line in (A.23) we observe that finding optimal portfolio $\theta_I^*(p;\varepsilon)$ reduces to finding optimal $\tilde{\theta}_I^*$, which solves the optimization problem

$$\max_{\tilde{\theta}_I} e^{rT}p^T\tilde{\theta}_I - g_I(\tilde{\theta}_I), \tag{A.24}$$

where $g_I(\tilde{\theta}_I) = \ln\Big(\sum_{i=1}^N \exp\{a_i+\Pi_i^T\tilde{\theta}_I\}\Big)$. From the optimization problem (A.24), we see that $\tilde{\theta}_I^*$ does not depend on shock ε. Hence, portfolio $\theta_I^*(p;\varepsilon)$ is given by Equation (31). •

Proof of Proposition 3. Investor I's optimization problem (A.24) yields the FOC for the optimal $\tilde{\theta}_I^* = \lambda\varepsilon - \gamma_I\theta_I^*$:

$$g'_I\left(\tilde{\theta}_I^*\right) = e^{rT}p, \tag{A.25}$$

where $g_I(x) = \ln\left(\sum_{j=1}^{M}\exp\left\{a_j + \Pi_j^T x\right\}\right)$ and $g'_I(x) = \partial g_I(x)/\partial x^T$ is a column vector for $x \in \mathbb{R}^{M-1}$. Assuming that $g'_I(\cdot)$ is invertible (which we prove below), we find that $\tilde{\theta}_I^* = f_I^{-1}(e^{rT}p)$. Then, from equation $\tilde{\theta}_I^* = \lambda\varepsilon - \gamma_I\theta_I^*$, which defines $\tilde{\theta}_I^*$, we obtain portfolio $\theta_I^*(p,\varepsilon)$ in Equation (33).

Now, we find the portfolio of investor U. Let $\tilde{\varepsilon} = \lambda\varepsilon/\gamma_I + v + \tilde{H}(p)$, i.e., the left hand side of the market clearing condition (32), where $\tilde{H}(p) = -\tilde{\theta}_I^*(p)/\gamma_I + \theta_U^*(p)$. The inference problem of investor U is similar to that in the complete-market economy. Following exactly the same steps as in Lemma 1, we obtain:

$$\phi_{\varepsilon|\tilde{\varepsilon}}(x|y) = \exp\left\{-0.5\left(y - \lambda x/\gamma_I - \tilde{H}(p)\right)^T\Sigma_v^{-1}\left(y - \lambda x/\gamma_I - \tilde{H}(p)\right)\right\}\frac{\phi_\varepsilon(x)}{G_1(y;p)},$$

$$\pi_n^U(p;\theta_U^*(p)) = \exp\left\{a_n + \frac{1}{2}\frac{b_n^2 - 2b_n\left(\lambda^T\Sigma_v^{-1}\tilde{H}(p)/\gamma_I - \mu_0/\sigma_0^2\right)}{\lambda^T\Sigma_v^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2}\right\}\frac{1}{G_2(p)},$$

where $G_1(y;p)$ and $G_2(p)$ are some functions, irrelevant for subsequent derivations. Moreover, using that, by Assumption 1, $b_n = \lambda_0 + \Pi_n^T\lambda$, from the last equation we obtain:

$$\begin{aligned}
\pi_n^U(p;\theta_U^*(p)) &= \exp\left\{a_n + \frac{1}{2}\frac{b_n^2 + 2\Pi_n^T\left(\lambda\mu_0/\sigma_0^2 - \lambda\lambda^T\Sigma_v^{-1}\tilde{H}(p)/\gamma_I^2\right)}{\lambda^T\Sigma_v^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2}\right\} \times \\
&\quad \times \exp\left\{\lambda_0\frac{\left(\mu_0/\sigma_0^2 - \lambda^T\Sigma_v^{-1}\tilde{H}(p)/\gamma_I\right)}{\lambda^T\Sigma_v^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2}\right\}\frac{1}{G_2(p)} \\
&= \exp\left\{a_n + \frac{1}{2}\frac{b_n^2 + 2\Pi_n^T\left(\lambda\mu_0/\sigma_0^2 - \lambda\lambda^T\Sigma_v^{-1}\tilde{H}(p)/\gamma_I^2\right)}{\lambda^T\Sigma_v^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2}\right\}\frac{1}{G_3(p)},
\end{aligned}$$

where $G_3(p)$ is a function that does not depend on n and is not needed for the proofs. Using probabilities π_n^U we rewrite investor U's objective function (8) as follows:

$$\begin{aligned}
-\sum_{n=1}^{N}\pi_n^U(p;\theta_U)\exp\left\{-\gamma_U\left(W_{U,0}e^{rT} + \theta_U^T\left(\Pi_n - e^{rT}p\right)\right)\right\} &= -\frac{\exp\{-\gamma_U W_{U,0}e^{rT}\}}{G_3(p)} \times \\
\exp\{\gamma_U e^{rT}p^T\theta_U\}\sum_{n=1}^{N}\exp\left\{a_n + \frac{1}{2}\frac{b_n^2 + 2\Pi_n^T\left(\lambda\mu_0/\sigma_0^2 - \lambda\lambda^T\Sigma_v^{-1}\tilde{H}(p)/\gamma_I^2\right)}{\lambda^T\Sigma_v^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2} - \gamma_U\Pi_n^T\theta_U\right\}&.
\end{aligned} \tag{A.26}$$

Factoring out Π_n^T in the term in the curly brackets in the last line above we have

$$a_n + \frac{1}{2}\frac{b_n^2 + 2\Pi_n^T\left(\lambda\mu_0/\sigma_0^2 - \lambda\lambda^T\Sigma_v^{-1}\tilde{H}(p)/\gamma_I\right)}{\lambda^T\Sigma_v^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2} - \gamma_U\Pi_n^T\theta_U =$$

$$a_n + \frac{1}{2}\frac{b_n^2}{\lambda^T\Sigma_v^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2} + \Pi_n^T\left(\frac{\lambda\mu_0/\sigma_0^2 - \lambda\lambda^T\Sigma_v^{-1}\tilde{H}(p)/\gamma_I}{\lambda^T\Sigma_v^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2} - \gamma_U\theta_U\right).$$

Now, similarly to $g_I(\cdot)$, we define function $g_U : \mathbb{R}^{M-1} \to \mathbb{R}$ for $x \in \mathbb{R}^{M-1}$:

$$g_U(x) = \ln\left(\sum_{j=1}^{N} \exp\left\{a_j + \frac{1}{2}\frac{b_j^2}{\lambda^T\Sigma_v^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2} + \Pi_j^T x\right\}\right).$$

Then, investor U's optimization problem from Equation (A.26), becomes

$$\min_{\theta_U} \gamma_U e^{rT} p^T\theta_U + g_U\left(\frac{\lambda\mu_0/\sigma_0^2 - \lambda\lambda^T\Sigma_v^{-1}\tilde{H}(p)/\gamma_I}{\lambda^T\Sigma_v^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2} - \gamma_U\theta_U\right).$$

Let $f_U = g_U'$, then the FOC for the uniformed's optimal portfolio, θ_U^* is,

$$f_U\left(\frac{\lambda\mu_0/\sigma_0^2 - \lambda\lambda^T\Sigma_v^{-1}\tilde{H}(p)/\gamma_I}{\lambda^T\Sigma_v^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2} - \gamma_U\theta_U^*\right) = e^{rT}p.$$

Assuming that f_U is invertible, as shown below, and $e^{rT}p$ belongs to its range, we obtain

$$\frac{\lambda\mu_0/\sigma_0^2 - \lambda\lambda^T\Sigma_v^{-1}\tilde{H}(p)/\gamma_I}{\lambda^T\Sigma_v^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2} - \gamma_U\theta_U^* = f_U^{-1}(e^{rT}p).$$

Substituting for $\tilde{H}(p) = -f_I^{-1}(e^{rT})/\gamma_I + \theta_U^*$ and factoring out $\gamma_U\theta_U^*$ we have

$$\frac{\lambda\mu_0/\sigma_0^2 + \lambda\lambda^T\Sigma_v^{-1}f_I^{-1}(e^{rT}p)/\gamma_I^2}{\lambda^T\Sigma_v^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2} - \gamma_U\theta_U^*(E+Q) = f_U^{-1}(e^{rT}p),$$

where, as before, E is the $(M-1)\times(M-1)$ identity matrix and matrix Q is given by

$$Q = \frac{\lambda\lambda^T\Sigma_v^{-1}}{\gamma_U\gamma_I\left(\lambda^T\Sigma_v^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2\right)},$$

as in Proposition 1. Solving for θ_U^* yields

$$\begin{aligned}
\theta_U^*(p) &= \frac{1}{\gamma_U}(E+Q)^{-1}\left(\frac{\lambda\mu_0/\sigma_0^2 + \lambda\lambda^T\Sigma_v^{-1}f_I^{-1}(e^{rT}p)/\gamma_I^2}{\lambda^T\Sigma_v^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2} - f_U^{-1}(e^{rT}p)\right) \\
&= (E+Q)^{-1}\left(\frac{1}{\gamma_I}Qf_I^{-1}(e^{rT}p) - \frac{1}{\gamma_U}f_U^{-1}(e^{rT}p) + \frac{(\mu_0/\sigma_0^2)\lambda}{\gamma_U(\lambda^T\Sigma_v^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2)}\right).
\end{aligned} \tag{A.27}$$

Now, we verify that function $f_l(x)$ is invertible. The invertibility of $f_U(x)$ is demonstrated along the same lines. Recalling that by definition $\exp(g_l(x)) = \sum_{n=1}^N \exp\left(a_n + \Pi_n^T x\right)$ and then differentiating both sides of the latter equation twice, we obtain:

$$\frac{\partial g_l(x)}{\partial x^T}\exp(g_l(x)) = \sum_{j=1}^N \Pi_j \exp\left(a_j + \Pi_j^T x\right), \tag{A.28}$$

$$\left(\frac{\partial^2 g_l(x)}{\partial x^T x} + \frac{\partial g_l(x)}{\partial x^T}\frac{\partial g_l(x)}{\partial x}\right)\exp(g_l(x)) = \sum_{j=1}^N \Pi_j \Pi_j^T \exp\left(a_j + \Pi_j^T x\right). \tag{A.29}$$

Next, we introduce a new probability measure $\pi_n^q(x) = \exp\{a_n + \Pi_n^T x\}/\sum_{j=1}^N \exp\{a_j + \Pi_j^T x\}$. Dividing both sides of Equations (A.28) and (A.29) by $\exp(g_l(x)) = \sum_{j=1}^N \exp\left(a_j + \Pi_j^T x\right)$, respectively, we observe that the derivatives can be rewritten as

$$\frac{\partial g_l(x)}{\partial x^T} = \mathbb{E}^{q(x)}[\Pi], \qquad \frac{\partial^2 g_l(x)}{\partial x^T x} = \mathrm{var}^{q(x)}[\Pi],$$

where $\mathbb{E}^q(x)[\Pi]$ and $\mathrm{var}^{q(x)}[\Pi]$ are the mean and the variance of the risky assets payoffs vector Π under probability measure $q(x)$. Because all assets are non-redundant, matrix $\mathrm{var}^{q(x)}[\Pi]$ is positive-definite, and hence is invertible. Then, function $f_l(x) = \partial g_l(x)/\partial x^T$ is injective and invertible on its range by Lemma A.2. Similarly, $f_U(x) = \partial g_l(x)/\partial x^T$ is injective and invertible on its range.

Finally, we derive the equation for prices. Substituting θ_l^* and θ_U^* from Equations (33) and (34) into the market clearing condition $\theta_l^*(p;\varepsilon) + \theta_U^*(p) + v = 0$ yields, after some algebra, the following system of nonlinear algebraic equations for prices,

$$\frac{1}{\gamma_l} f_l^{-1}\left(e^{rT}P(\varepsilon, v)\right) + \frac{1}{\gamma_U} f_U^{-1}\left(e^{rT}P(\varepsilon, v)\right) = \left(E + Q\left(\frac{\lambda\varepsilon}{\gamma_l} + v\right) + \frac{(\mu_0/\sigma_0^2)\lambda}{\gamma_U(\lambda^T\Sigma_v^{-1}\lambda/\gamma_l^2 + 1/\sigma_0^2)}\right).$$

Next, we prove that the above equation has unique solution. Denote $x_l = f_l^{-1}\left(e^{rT}P(\varepsilon, v)\right)$ and $x_U = f_U^{-1}\left(e^{rT}P(\varepsilon, v)\right)$. Because functions $f_l(x)$ and $f_U(x)$ are injective, we obtain that if the equilibrium exists, then $f_l(x_l) = f_U(x_U) = e^{rT}P(\varepsilon, v)$. From the latter equation and the above equation for $P(\varepsilon, v)$ we obtain the following system of equations for x_l and x_U:

$$\frac{x_l}{\gamma_l} + \frac{x_U}{\gamma_U} = \left(E + Q\left(\frac{\lambda\varepsilon}{\gamma_l} + v\right) + \frac{(\mu_0/\sigma_0^2)\lambda}{\gamma_U(\lambda^T\Sigma_v^{-1}\lambda/\gamma_l^2 + 1/\sigma_0^2)}\right).$$

$$f_l(x_l) = f_U(x_U).$$

The existence and uniqueness of x_l and x_U solving the latter system of equation follows from Lemma A.5 below because this system is a special case of Equation (A.32), in which $x = x_l$, $c = \gamma_U/\gamma_l$, $t = 0.5/(\lambda^T\Sigma_v^{-1}\lambda/\gamma_l^2 + 1/\sigma_0^2)$, $f(x;0) = f_l(x)$, $f(x;t) = f_U(x)$ and

$$\bar{x} = \gamma_U\left(E + Q\left(\frac{\lambda\varepsilon}{\gamma_l} + v\right) + \frac{(\mu_0/\sigma_0^2)\lambda}{(\lambda^T\Sigma_v^{-1}\lambda/\gamma_l^2 + 1/\sigma_0^2)}\right).$$

By Lemma A.5, x_u and x_l are continuous and differentiable functions of ε and v. Therefore, there exists an equilibrium in which price $P(\varepsilon, v)$ exists, is unique and continuously differentiable in ε and v. •

Lemma A.2 (Gale and Nikaidô). *Let $f(x) : \mathbb{R}^{M-1} \to \mathbb{R}^{M-1}$ be a continuous differentiable function with a positive-definite Jacobian. Then, function $f(x)$ is injective. That is, $\forall x_1, x_2 \in \mathbb{R}^{M-1}$ such that $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.*

Proof of Lemma A.2. See the proof of Theorem 6 in Gale and Nikaidô (1965).

Lemma A.3. *Consider a sequence x_k such that $|x_k| \to \infty$ as $k \to \infty$. Then, there exists index m such that sequence $|\Pi_m^T x_k|$ is unbounded.*

Proof of Lemma A.3. Suppose, on the contrary, sequence $|\Pi_m^T x_k|$ is bounded for all m. Therefore, there exists constant A such that $|\Pi_m^T x_k| < A$ for all m and k. Because all securities are non-redundant, matrix Π with columns Π_n, $n = 1, \ldots, N$ has rank $M-1$. Therefore, vectors Π_n span $\mathbb{R}^{M-1}$. Without loss of generality, assume that $\Pi_1, \ldots, \Pi_{M-1}$ form basis in $\mathbb{R}^{M-1}$ (states ω_n can be always renumbered accordingly).

Consider vector $e_l = (0, \ldots, 0, 1, 0, \ldots, 0)^T \in R^{M-1}$ with l^{th} element equal to 1 and all other elements equal to 0. Then, there exist constants $\alpha_{m,l}$ such that $e_l = \alpha_{1,l}\Pi_1 + \ldots + \alpha_{M-1,l}\Pi_N$. Then, it can be easily observed that for all l

$$|e_l^T x_k| \leq |\alpha_{1,l}|\,|\Pi_1^T x_k| + \ldots + |\alpha_{M-1,l}|\,|\Pi_{M-1}^T x_k| \leq A(M-1) \max_{m,l} |\alpha_{ml}|.$$

Therefore, all elements of vector x_k are uniformly bounded, which contradicts the fact that $|x_k| \to \infty$. Hence, $|\Pi_m^T x_k|$ is unbounded for some m. •

Lemma A.4. *Consider a sequence x_k such that $|x_k| \to \infty$ as $k \to \infty$. Then, there exists index m such that sequence $|\Pi_m^T x_k - \Pi_{m+1}^T x_k|$ is unbounded.*

Proof of Lemma A.4. Suppose, on the contrary, sequence $|\Pi_m^T x_k - \Pi_{m+1}^T x_k|$ is bounded for all m. It can be easily observed that the latter fact implies that there exists constant A such that for all i and j we have inequality $|\Pi_i^T x_k - \Pi_j^T x_k| < A$. Because all risky assets and the riskless asset are non-redundant, column vectors $(\Pi_n^T, 1)^T \in \mathbb{R}^M$, where $n = 1, \ldots, N$, span $\mathbb{R}^M$. Without loss of generality, we assume that the first M vectors $(\Pi_n^T, 1)^T$, where $n = 1, \ldots, M$, form a basis in $\mathbb{R}^M$ (otherwise, states ω_n can be renumbered). Therefore,

there exists unique vector $(\alpha_1, \ldots, \alpha_M)^T$ which solves the following system of equation:

$$\begin{aligned} \alpha_1 \Pi_1 + \ldots + \alpha_M \Pi_M &= 0, \\ \alpha_1 + \ldots + \alpha_M &= 1. \end{aligned} \tag{A.30}$$

We pick the solution of system (A.30) and for an arbitrary index m we obtain:

$$\begin{aligned} |\Pi_m^T x_k| &= |(\alpha_1 + \ldots + \alpha_M)\Pi_m^T x_k - (\alpha_1 \Pi_1^T x_k + \ldots + \alpha_M \Pi_M^T x_k)| \\ &\leq |\alpha_1| \, |\Pi_m^T x_k - \Pi_1^T x_k| + \ldots + |\alpha_M| \, |\Pi_m^T x_k - \Pi_M^T x_k| \leq A \max_l |\alpha_l|, \end{aligned}$$

contradicting the fact that according to Lemma A.3 $|\Pi_m^T x_k|$ is unbounded for some m. •

Lemma A.5 *Consider function $f(x;t)$ with scalar parameter t, such that for a fixed t $f(\cdot;t) : \mathbb{R}^{M-1} \to \mathbb{R}^{M-1}$ and is given by the following expression*

$$f(x;t) = \frac{\sum_{j=1}^{N} \Pi_j \exp\{a_j + t b_j^2 + \Pi_j^T x\}}{\sum_{j=1}^{N} \exp\{a_j + t b_j^2 + \Pi_j^T x\}}. \tag{A.31}$$

Then, for all fixed $\bar{x} \in \mathbb{R}^{M-1}$, $c > 0$ and $t \in R$ there exists unique x which solves equation

$$f(x;0) = f(\bar{x} - cx;t). \tag{A.32}$$

Moreover, solution $x(\bar{x};t)$, considered as a function of $\bar{x}$ and t, is continuous and differentiable with respect to its variables.

Proof of Lemma A.5. It can be easily observed that for $t = 0$ Equation (A.32) has unique solution $x = \bar{x}/(1+c)$. Moreover, function $f(x;0) - f(\bar{x} - cx;t)$ is continuously differentiable with respect to x and t, and its derivative with respect to x has positive-definite Jacobian. In particular, we observe that $f(x;0) = f_l(x)$, where $f_l(x)$ is defined by Equation (36), and hence $f(x;0)$ has positive-definite Jacobian, as demonstrated in the proof of Proposition 3. Exactly in the same way, it can be shown that $-f(\bar{x} - cx;t)$ also has a positive-definite Jacobian with respect to x. Furthermore, it can be easily verified that $f(x;0) - f(\bar{x} - cx;t)$ is continuously differentiable with respect to t. Therefore, by the implicit function theorem (e.g., Rudin (1976)), there exists an open interval $(-\tilde{t}, \tilde{t})$ and a continuous and differentiable multi-variate function $x(t)$ that solves Equation (A.32) for all $t \in (-\tilde{t}, \tilde{t})$. Next, we show that the solution exists for all $t \in \mathbb{R}$. For brevity, we show the existence for $t \in \mathbb{R}^+$, and the existence for $t \in \mathbb{R}^-$ is demonstrated analogously.

Let $\bar{t}$ be the lowest upper bound of t to which solution $x(t)$ can be extended:

$$\bar{t} = \sup\{t : \forall \tau \in [0,t) \ \exists \text{ unique } x \text{ such that } f(x;0) = f(\bar{x} - cx;\tau)\}. \tag{A.33}$$

By the implicit function theorem, $\bar{t} > 0$. Next, we show by contradiction that $\bar{t} = +\infty$. Suppose, $\bar{t} < +\infty$, and consider two cases. First, suppose that Equation (A.32) has solution $x(\bar{t})$ for $t = \bar{t}$. Then, applying the implicit function theorem one more time at $t = \bar{t}$ we extend solution $x(t)$ to some $t > \bar{t}$, which contradicts the definition of $\bar{t}$ in (A.33). Second, suppose that (A.32) does not have a solution for $t = \bar{t}$. Next, we get a contradiction with the latter statement by showing that $x(t)$ can be extended to $\bar{t}$.

Consider a sequence $t_k \uparrow \bar{t}$. By the definition of $\bar{t}$ in (A.33) there exist x_k such that

$$f(x_k; 0) = f(\bar{x} - c x_k; t_k). \tag{A.34}$$

Suppose, x_k are bounded, so that there exists constant A such that $|x_k| < A$. Then, by Bolzano-Weierstrass Theorem, there exists a convergent subsequence such that $x_{k_n} \to x^*$ as $n \to +\infty$ (e.g., Rudin (1976)). To simplify the notation, throughout this proof the elements of subsequences are numbered by k rather than k_n, which can be achieved by renumbering the elements of the subsequence. Then, using the continuity of function $f(x; t)$ and taking the limit $k \to \infty$ in (A.34) we find that x^* is a solution of (A.32) for $t = \bar{t}$. Moreover, this solution is unique by Lemma A.2 because $f(x; 0) - f(\bar{x} - c x; t)$ has positive-definite Jacobian, which leads to a contradiction with the assumption that the solution does not exist for $t = \bar{t}$. Therefore, $\bar{t} = +\infty$.

It remains to show that x_k is indeed bounded. Next, we assume that x_k is unbounded and get the contradiction. If x_k is unbounded, there exists a subsequence such that $|x_{k_n}| \to \infty$. By relabeling elements k_n of the subsequence by k, we assume that $|x_k| \to \infty$. Let $j(k) = \arg\max_j \Pi_j^T x_k$. Because $j(k)$ takes only finite number of values, there exists index j^* such that $j^* = j(k_n)$ for an infinite sequence of $k_n \to \infty$. Without loss of generality, we assume that $j^* = 1$ (otherwise, we relabel states ω_n accordingly) and also focus on subsequence k_n and relabel its elements by k. Hence, for this subsequence $\Pi_1^T x_k \geq \Pi_j^T x_k$ for all $j = 1, \ldots, N$. Proceeding similarly, we obtain subsequence x_k such that

$$\Pi_1^T x_k \geq \ldots \geq \Pi_m^T x_k > \Pi_{m+1}^T x_k \geq \ldots \geq \Pi_N^T x_k, \tag{A.35}$$

for all k, where m is the fist index for which $\Pi_m^T x_k - \Pi_{m+1}^T x_k \to +\infty$ as $k \to \infty$. The existence of such an index m is guaranteed by Lemma A.4.

Next, define function $\pi_j(x; t)$ as follows:

$$\pi_j(x; t) = \frac{\exp\{a_j + t b_j^2 + \Pi_j^T x\}}{\sum_{j=1}^{N} \exp\{a_j + t b_j^2 + \Pi_j^T x\}}. \tag{A.36}$$

Consider sequences $\pi_j(x_k; t)$. Because $0 \leq \pi_j(x_k; t) \leq 1$, by Bolzano-Weierstrass theorem there exists a subsequence x_k such that $\pi_j(x_k; 0) \to \pi_j^+$ and $\pi_j(\bar{x} - c x_k; t_k) \to \pi_j^-$ for all

$j = 1, \ldots, N$, where $\sum_{j=1}^{N} \pi_j^+ = \sum_{j=1}^{N} \pi_j^- = 1$, $0 \leq \pi_j^+ \leq 1$ and $0 \leq \pi_j^- \leq 1$. Next, we demonstrate that

$$\pi_j^+ = 0, \text{ for } j = m+1, \ldots, N, \tag{A.37}$$

$$\pi_j^- = 0, \text{ for } j = 1, \ldots, m. \tag{A.38}$$

To derive equalities (A.37)–(A.38), by employing inequalities (A.35) and the fact that $\Pi_m^T x_k - \Pi_{m+1}^T x_k \to +\infty$ as $k \to \infty$, for all $j > m$ we obtain:

$$\pi_j^+ = \lim_{k\to+\infty} \pi_j(x_k; 0) \leq \lim_{k\to+\infty} \frac{\exp\{a_j + \Pi_j^T x_k\}}{\exp\{a_m + \Pi_m^T x_k\}} \leq \lim_{k\to+\infty} \frac{\exp\{a_j + \Pi_{m+1}^T x_k\}}{\exp\{a_m + \Pi_m^T x_k\}} = 0.$$

Similarly, for all $j \leq m$ we obtain:

$$\pi_j^- = \lim_{k\to+\infty} \pi_j(\bar{x} - c x_k; t_k) \leq \lim_{k\to+\infty} \frac{\exp\{a_j + t_k b_j^2 + \Pi_j^T \bar{x} - c\Pi_j^T x_k\}}{\exp\{a_{m+1} + t_k b_{m+1}^2 + \Pi_{m+1}^T \bar{x} - c\Pi_{m+1}^T x_k\}}$$
$$\leq \lim_{k\to+\infty} \frac{\exp\{a_j + t_k b_j^2 + \Pi_j^T \bar{x} - c\Pi_m^T x_k\}}{\exp\{a_m + t_k b_m^2 + \Pi_{m+1}^T \bar{x} - c\Pi_{m+1}^T x_k\}} = 0.$$

Using Equations (A.37)–(A.38) and taking limit $k \to +\infty$ in (A.34) we obtain:

$$\pi_{m+1}^+ \Pi_{m+1} + \ldots + \pi_N^+ \Pi_N = \pi_1^- \Pi_1 + \ldots + \pi_m^- \Pi_m.$$

The above equation implies that

$$\pi_{m+1}^+ \Pi_{m+1}^T x_k + \ldots + \pi_N^+ \Pi_N^T x_k = \pi_1^- \Pi_1^T x_k + \ldots + \pi_m^- \Pi_m^T x_k. \tag{A.39}$$

From the fact that $\sum_{j=1}^{N} \pi_j^+ = \sum_{j=1}^{N} \pi_j^- = 1$, demonstrated above, from Equations (A.37)–(A.39) and inequality (A.35) we obtain:

$$\pi_{m+1}^+ \Pi_{m+1}^T x_k + \ldots + \pi_N^+ \Pi_N^T x_k \leq \Pi_{m+1}^T x_k < \Pi_m^T x_k \leq \pi_1^- \Pi_1^T x_k + \ldots + \pi_m^- \Pi_m^T x_k.$$

The last inequality contradicts Equation (A.39). Consequently, x_k is bounded, and hence, as shown above, there exists unique solution $x(\bar{t})$ of Equation (A.32) for $t = \bar{t}$. Then, by the implicit function theorem, the solution $x(t)$ can be extended beyond $\bar{t}$, which contradicts the definition of $\bar{t}$ in (A.33). Therefore, $\bar{t} = +\infty$, which proves the global existence. Therefore, there exists unique $x(\bar{x}; t)$. The continuity and differentiability of $x(\bar{x}; t)$ w.r.t. $\bar{x}$ follow from the implicit function theorem applied to an arbitrary point $\bar{x}$, which completes the proof of Lemma A.5. •

Proof of Lemma 4. The proof is similar to that of Lemma 2. Assume that $\varepsilon \sim N(\mu_\varepsilon, \sigma_\varepsilon^2)$, and we observe vector $\tilde{\varepsilon} = \lambda\varepsilon + v + H(p)$. From Bayes' rule

$$\phi_{\varepsilon|\tilde{\varepsilon}}(x|y) = \frac{\phi_{\tilde{\varepsilon}|\varepsilon}(y|x)\phi_\varepsilon(x)}{\int_{-\infty}^{\infty}\phi_{\tilde{\varepsilon}|\varepsilon}(y|x)\phi_\varepsilon(x)dx}, \tag{A.40}$$

where now $\phi_\varepsilon(x) = (1/\sqrt{2\pi\sigma_0^2})\exp(-0.5(x-\mu_\varepsilon)^2/\sigma_\varepsilon^2)$. Since $v \sim N(0, \Sigma_v)$, $\tilde{\varepsilon} = \lambda\varepsilon/\gamma_I + v + H(p)$ conditional on ε has a multivariate normal distribution $N(\lambda\varepsilon/\gamma_I + H(p), \Sigma_v)$. Substituting for $\phi_{\tilde{\varepsilon}|\varepsilon}$ and ϕ_ε in the numerator above, we have

$$\begin{aligned}
\phi_{\varepsilon|\tilde{\varepsilon}}(x) &= \exp\left\{-0.5\left(y - \lambda x/\gamma_I - H(p)\right)^T\Sigma_v^{-1}\left(y - \lambda x/\gamma_I - H(p)\right) - 0.5(x-\mu_\varepsilon)^2/\sigma_\varepsilon^2\right\}\frac{1}{G_1(y;p)} \\
&= \exp\left\{-0.5\left(1/\sigma_0^2 + \lambda^T\Sigma_v^{-1}\lambda/\gamma_I^2\right)\left(x - \frac{\mu_\varepsilon/\sigma_\varepsilon^2 + (y-H(p))^T\Sigma_v^{-1}\lambda/\gamma_I}{1/\sigma_\varepsilon^2 + \lambda^T\Sigma_v^{-1}\lambda/\gamma_I^2}\right)^2\right\}\frac{1}{G_2(y;p)}
\end{aligned}$$

where $G_1(y;p)$ and $G_2(y;p)$ are some functions that do not depend on x. We observe that the above equation gives the PDF of a standard normal distribution with mean and precision parameters given by

$$\tilde{\mu}_\varepsilon = \frac{\mu_\varepsilon/\sigma_\varepsilon^2 + (y-H(p))^T\Sigma_v^{-1}\lambda/\gamma_I}{1/\sigma_0^2 + \lambda^T\Sigma_v^{-1}\lambda/\gamma_I^2}, \qquad \frac{1}{\tilde{\sigma}_\varepsilon^2} = \frac{1}{\sigma_\varepsilon^2} + \frac{\lambda^T\Sigma_v^{-1}\lambda}{\gamma_I^2},$$

which completes the proof. •

Proof of Proposition 4. 1) The expression for the asset price is given in closed form by Equation (23) in Proposition 1. To solve the model without noise trader demands in derivatives market, we first consider a diagonal volatility matrix of noise trader demands $\Sigma_v = \mathrm{diag}(\sigma_v^2, \tilde{\sigma}_v^2, \ldots, \tilde{\sigma}_v^2)$. Because in this economy $b = C/\sigma_C^2$, we obtain that $\lambda = (1, 0, \ldots, 0)^T/\sigma_C^2$. Given the structure of λ, the equilibrium does not depend on $\tilde{\sigma}_v$ because $\lambda^T\Sigma_v^{-1}\lambda = 1/(\sigma_v^2\sigma_C^2)$ and $\lambda^T\Sigma_v^{-1}v = v_1/(\sigma_v^2\sigma_C^2)$. After deriving the equilibrium prices (23) we take the limit $\tilde{\sigma}_v \to 0$, which allows us to set $v = (v_1, 0, \ldots, 0)^T$. Passing to $N \to \infty$ in the resulting equation for the asset price, taking into account the expressions for a_n and b_n in terms of asset cash flows in Equations (41), and after simple algebra, we obtain:

$$P^{cm}(s) = \frac{\int_0^{+\infty} C^2 \exp\{-0.5(C - \mu_C^{RN}(s))^2/(\sigma_C^{RN})^2\}dC}{\int_0^{+\infty} C\exp\{-0.5(C - \mu_C^{RN}(s))^2/(\sigma_C^{RN})^2\}dC}, \tag{A.41}$$

where $\mu_C^{RN}(s)$ and σ_C^{RN} are given by:

$$\mu_C^{RN}(s) = \left(\frac{\gamma_I\gamma_U}{\gamma_I+\gamma_U}\left(1 + \frac{1/(\sigma_v^2\sigma_C^4)}{\gamma_I\gamma_U\left(1/(\gamma_I^2\sigma_v^2\sigma_C^4) + 1/\sigma_0^2\right)}\right)\right)s \tag{A.42}$$

$$+\frac{\gamma_I}{\gamma_I+\gamma_U}\frac{\mu_0/(\sigma_0^2\sigma_C^2)}{1/(\gamma_I^2\sigma_V^2\sigma_C^4)+1/\sigma_0^2}\,(\sigma_C^{RN})^2,$$

$$\sigma_C^{RN}=\frac{\sigma_C}{\sqrt{1-\frac{\gamma_I}{\gamma_I+\gamma_U}\frac{1/\sigma_C^2}{1/(\gamma_I^2\sigma_V^2\sigma_C^4)+1/\sigma_0^2}}}. \tag{A.43}$$

Using a simple change of variable $y=(C-\mu_C^{RN})/\sigma_C^{RN}$ and calculating integrals in Equation (A.41) we obtain that the asset price is given by (44).

2) As demonstrated in Section 3.2, the asset price in the incomplete market can be found by solving the system of equations (38) and (39), which can be rewritten as follows:

$$P^{icm}=f_I\left(x^*(s)\right)e^{-rT}=f_U\left(\bar{x}(s)-(\gamma_U/\gamma_I)x^*(s)\right)e^{-rT}, \tag{A.44}$$

where we denote $x^*(s)=x_I$, and $\bar{x}(s)$ is given by:

$$\bar{x}(s)=\gamma_U\left[1+\frac{1/(\sigma_V^2\sigma_C^4)}{\gamma_I\gamma_U\left(1/(\gamma_I^2\sigma_V^2\sigma_C^4)+1/\sigma_0^2\right)}\right]s+\frac{\mu_0/(\sigma_0^2\sigma_C^2)}{1/(\gamma_I^2\sigma_V^2\sigma_C^4)+1/\sigma_0^2}, \tag{A.45}$$

Taking limit $N\to\infty$ in Equation (A.44) in which $f_I(x)$ and $f_U(x)$ are given by Equations (36) and (37), we obtain the following equation:

$$\sigma_C\Phi\left(x^*(s)\sigma_C\right)=\bar{\sigma}_C\Phi\left((\bar{x}(s)-(\gamma_U/\gamma_I)x^*(s))\bar{\sigma}_C\right), \tag{A.46}$$

where $\bar{x}(s)$ is given by (A.45) and $\bar{\sigma}_C$ is given by:

$$\bar{\sigma}_C=\frac{\sigma_C}{\sqrt{1-\frac{1/\sigma_C^2}{1/(\gamma_I^2\sigma_V^2\sigma_C^4)+1/\sigma_0^2}}}. \tag{A.47}$$

The price is then given by Equation (45) in which

$$\bar{\mu}_C(s)=\left(\bar{x}(s)-(\gamma_U/\gamma_I)x^*(s)\right)\bar{\sigma}_C^2. \quad \bullet \tag{A.48}$$

Proof of Proposition 5. The proof is analogous to the proof of Proposition 4, except that all the integrals can be evaluated in closed form due to the assumption that C and ε are normally distributed. In particular, in the limit of $N\to\infty$ functions $f_I(x)$ and $f_U(x)$ given by equations (36) and (37) converge to $f_I(x)=\sigma_C^2x$ and $f_U(x)=\bar{\sigma}_C^2x$, where $\bar{\sigma}_C$ is given by (A.47). Substituting the latter functions into Equation (35) for the asset price we recover expression (47). •

Proof of Proposition 6. 1) Similar to the proof of Proposition 4, we first derive the equilibrium in the market with the following volatility matrix of noise trader demands:

$$\Sigma_v = \begin{pmatrix} \sigma_v^2 & \rho\sigma_v^2 & 0 & 0 & \ldots & 0 \\ \rho\sigma_v^2 & \sigma_v^2 & 0 & 0 & \ldots & 0 \\ 0 & 0 & \tilde{\sigma}_v^2 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \tilde{\sigma}_v^2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & 0 & \tilde{\sigma}_v^2 \end{pmatrix}. \tag{A.49}$$

Because $\lambda = (1, 1, 0, \ldots, 0)^T/\sigma_c^2$, similar to the proof of Proposition 4, we observe that terms $\lambda^T\Sigma_v^{-1}\lambda$ and $\lambda^T\Sigma_v^{-1}v$ do not depend on $\tilde{\sigma}_v$. Therefore, after deriving the equilibrium with matrix (A.49) we then take the limit $\tilde{\sigma}_v \to 0$, which allows us to set $v_3 = v_4 = \ldots = 0$. The rest of the proof is analogous to the proof of Proposition 4 adjusted to the case of two risky assets. 2) The proof follows from the results of Lemma 4 and Section 4.2. •

Proof of Proposition 7. Let $\tilde{\varepsilon} = \eta(\varepsilon)/\gamma_I + v + \hat{H}(p)$, where $\hat{H}(p) = \theta^*_U(p) - (1/\gamma_I)\Omega^{-1}v$. Then, conditional density $\phi_{\varepsilon|\tilde{\varepsilon}}(x|y)$ is given by:

$$\phi_{\varepsilon|\tilde{\varepsilon}}(x|y) = \frac{\phi_v\big(y - \eta(x)/\gamma_I - \hat{H}(p)\big)\phi_\varepsilon(x)}{\int_{-\infty}^{+\infty}\phi_v\big(y - \eta(x)/\gamma_I - \hat{H}(p)\big)\phi_\varepsilon(x)dx}. \tag{A.50}$$

Suppose the equilibrium exists. Portfolio (54) of investor I remains exactly the same as in Equation (11) in Lemma 1. Next, we find investor U's portfolio. Similarly to Equation (A.6) in the Proof of Lemma 2, we note from the market clearing condition (53) that in equilibrium $\tilde{\varepsilon} = 0$, and hence find investor U's posterior probabilities π_n^U as follows:

$$\begin{aligned} \pi_n^U &= \int_{-\infty}^{+\infty} \pi_n(x)\phi_{\varepsilon|\tilde{\varepsilon}}(x|0)dx \\ &= \frac{1}{G_1(p)}\int_{-\infty}^{+\infty}\pi_n(x)\phi_v\Big(-\frac{\eta(x)}{\gamma_I} - \hat{H}(p)\Big)\phi_\varepsilon(x)dx = \frac{\exp\big\{\Psi_n\big(\hat{H}(p)\big)\big\}}{G_2(p)}, \end{aligned} \tag{A.51}$$

where $G_1(p)$ and $G_2(p)$ are some functions that do not depend on n. The integrals in (A.50) and (A.51) exist because $\phi_v(\cdot)$ and $\phi_\varepsilon(\cdot)$ are bounded and continuous PDFs. From Equation (A.51) we obtain that $\ln(\pi_n^U/\pi_N^U) = \Psi_n\big(\hat{H}(p)\big) - \Psi_N\big(\hat{H}(p)\big)$. Substituting the latter expression into (12) for investor U's portfolio we obtain portfolio $\theta_U^*(p)$ in (55).

Subtracting $(1/\gamma_I)\Omega^{-1}v$ from both sides of investor U's portfolio (55), multiplying both sides by Ω, using the definition of $\hat{H}(p)$, and rearranging terms we obtain Equation (59):

$$\frac{1}{\ldots}\Psi\big(\hat{H}(p)\big) - \Omega\hat{H}(p) = \frac{\gamma_U}{\gamma_U + \gamma_I\,\gamma_U\gamma_I}$$

$$v(p). \tag{A.52}$$

Next, we derive vector v. From the market clearing condition (53) we observe that in equilibrium $\hat{H}(p) = -\eta(\varepsilon)/\gamma_I - v$. Substituting the latter expression into Equation (A.52) and solving it for v, we obtain Equation (58). Then, risk-neutral probabilities and asset prices are given by Equations (56) and (57), respectively, which can be shown exactly in the same way as in Proposition 1. This completes the derivation of equilibrium.

Now we prove part 2 of the proposition. Suppose, there exists an REE but function $\Psi(x)/\gamma_U - \Omega x$ is not injective; that is, there exist x_1 and x_2 such that $\Psi(x_1)/\gamma_U - \Omega x_1 = \Psi(x_2)/\gamma_U - \Omega x_2$ but $x_1 \neq x_2$. Pick (ε_1, v_1) and (ε_2, v_2) such that $x_1 = \eta(\varepsilon_1)/\gamma_I + v_1$ and $x_2 = \eta(\varepsilon_2)/\gamma_I + v_2$. Then, Equations (57) and (58) for price $P(\varepsilon, v)$ and vector $v(p)$ imply that $P(\varepsilon_1, v_1) = P(\varepsilon_2, v_2)$ but $\eta(\varepsilon_1)/\gamma_I + v_1 \neq \eta(\varepsilon_2)/\gamma_I + v_2$. However, market clearing condition (53) clearly implies that price $P(\varepsilon, v)$ is injective, that is, $P(\varepsilon_1, v_1) = P(\varepsilon_2, v_2)$ $\Rightarrow \eta(\varepsilon_1)/\gamma_I + v_1 = \eta(\varepsilon_2)/\gamma_I + v_2$, which leads to contradiction.

Next we prove that if $\Psi(x)/\gamma_U - \Omega x$ is injective, then there exists an equilibrium. Portfolios and prices (54)–(57) are in equilibrium if price $P(\varepsilon, v)$ is injective. This is because the equilibrium is derived from the fact that price $p = P(\varepsilon, v)$ uniquely reveals $\eta(\varepsilon)/\gamma_I + v$ via the market clearing condition (53). Suppose, there exist pairs (ε_1, v_1) and (ε_2, v_2) such that $P(\varepsilon_1, v_1) = P(\varepsilon_2, v_2)$ but $\eta(\varepsilon_1)/\gamma_I + v_1 \neq \eta(\varepsilon_2)/\gamma_I + v_2$. Because there is one-to-one mapping between $P(\varepsilon, v)$ and v, from equation (58) we observe that $\Psi(x)/\gamma_U - \Omega x$ is not injective, which contradicts the assumption of the proposition.

Finally, we prove part 3. Suppose, investor U observes prices p and the residual demand

$$\theta^*_I(p;\varepsilon) + v = \frac{\eta(\varepsilon)}{\gamma_I} + v - \frac{1}{\gamma_I}\Omega^{-1}v(p).$$

The latter equation implies that investor U can now infer $\eta(\varepsilon)/\gamma_I + v$ even if prices $P(\varepsilon, v)$ are not injective. Then, investor U finds the posterior distribution of ε, equilibrium portfolios and prices in the same way as in part 1 of the proposition. •