

**Bond Market Clienteles, the Yield Curve and the
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Abstract

We propose a clientele-based model of the yield curve and optimal maturity structure of government debt. Clienteles are generations of agents at different life cycle stages in an overlapping-generations economy. An optimal maturity structure exists in the absence of distortionary taxes and induces efficient intergenerational risksharing. If agents are more risk-averse than log, then an increase in the long-horizon clientele raises the price and optimal supply of long-term bonds. But while a welfare-maximizing government caters to clienteles, it does not accommodate fully their demand, and limits issuance of long-term bonds to a level where these earn negative expected excess returns.

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1 Introduction

The government-bond market involves many distinct investor clienteles. For example, pension funds and insurance companies invest typically in long maturities as a way to hedge their long-term liabilities, while asset managers and banks' treasury departments hold shorter maturities. Clientele demands vary over time in response to demographic or regulatory changes. This time-variation can have important effects both on the yield curve and on the government's debt-issuance policy.

The UK pension reform provides a stark illustration of clientele effects. The Pensions Act of 2004 required pension funds to maintain their asset-liability ratio above a threshold. Moreover, pension liabilities, which are long-term and indexed to inflation, had to be marked to market using the prices of long-term inflation-indexed bonds. To minimize variation in their marked-to-market asset-liability ratio, pension funds bought large quantities of these bonds. The effect on the yield curve was dramatic. For example, in January 2006 the inflation-indexed bond maturing in 2011 was yielding 1.5%, while the 2055 bond was yielding only 0.5%. The 0.5% yield is very low relative to the 2.3% historical average of the long real interest rate in the UK in the 20th century.¹ Moreover, the downward-sloping yield curve is hard to attribute to an expectation of rates dropping below 0.5% after 2011. The steep decline in long rates induced the UK Treasury to tilt debt issuance towards long maturities. For example, bonds with maturities of fifteen years or longer constituted 58% of issuance during financial year 2006-7, compared to an average of 40% over the previous four years.²

While clientele considerations matter for the yield curve and the issuance of government debt, they are largely absent from theoretical models. Indeed most models of bond-price determination assume a representative agent, thus ruling out clienteles. Optimal issuance of government debt is also studied mostly in representative-agent models, which overcome the Ricardian equivalence result of Barro (1974) by assuming distortionary taxes.

In this paper, we develop a model of the yield curve and optimal issuance of government debt that emphasizes the role of clienteles. We focus on demographic rather than regulatory clienteles, and model them as generations of agents at different stages of their life cycle in an overlapping-generations economy. We show that an increase in the relative importance of the clientele with the longer investment horizon, i.e., the younger, renders long-term bonds more expensive if agents are more risk-averse than log. We also examine how changes in clientele demand affect optimal

¹The historical average is computed by Dimson, Marsh, and Staunton (2002), and concerns a maturity of approximately 20 years.

²The issuance numbers are from the website of the UK Debt Management Office. For a more detailed account of the UK pension reform, see Greenwood and Vayanos (2010). Another illustration of clientele-driven issuance is the French Treasury's first-time issuance of a 50-year bond in 2005, in response to strong demand by pension funds.

debt issuance by the government. This requires an analysis of the optimal maturity structure of government debt: we show that an optimal maturity structure exists even in the absence of distortionary taxes, and is the tool through which the government can induce efficient risksharing across generations. An increase in the relative importance of the long-horizon clientele increases the optimal supply of long-term bonds if agents are more risk averse than log. The government thus responds to an increase in demand in one segment of the yield curve by issuing more in that segment. Our model not only derives this empirically relevant behavior from welfare maximization, but also qualifies its extent. We show, in particular, that because the government takes into account the welfare of future generations, it does not accommodate fully changes in clientele demand. It also limits issuance of long-term bonds to a level where these earn negative expected returns relative to short-term bonds.

Section 2 describes our infinite-horizon overlapping-generations model. Generations live for three periods, receive an endowment in the first period of their lives, consume in the third period, and have CRRA preferences over consumption. They can invest their endowment in a linear technology with riskless return and in government bonds. For simplicity, we assume that the return of the linear technology is constant over time, except between a period j and $j + 1$ where it can take a high or a low value. This can be interpreted as a productivity shock, and its realization becomes known in period j . One-period non-contingent bonds are available in each period, and earn the same return as the linear technology because of no-arbitrage. Two-period non-contingent bonds are available in period $j - 1$, and they complete the market from the perspective of agents trading in that period since uncertainty in period j is described by two states. The government can levy non-distortionary income taxes on agents' endowments. We refer to one- and two-period bonds as short- and long-term, respectively.

Section 3 examines how the term structure of interest rates in period $j - 1$ depends on clientele demand and bond supply. An increase in the relative importance of the long-horizon clientele, consisting of the agents born in period $j - 1$, relative to the short-horizon clientele, consisting of the agents born in period $j - 2$, renders long-term bonds more expensive if agents are more risk-averse than log. Indeed, under these preferences, agents with a long horizon value more highly assets whose returns increase when interest rates decrease, and long-term bonds have this property. If instead agents are less risk averse than log, then an increase in the long-horizon clientele renders long-term bonds cheaper. Our model thus highlights that intuitive horizon-based clientele effects can arise only if agents are more risk-averse than log. Moreover, clientele effects do not arise under logarithmic utility, an assumption commonly used in term-structure models.

Section 3 shows additionally that a lengthening of the maturity structure, defined as an increase in the supply of long-term bonds holding total debt value constant, renders long-term bonds cheaper. Ricardian equivalence thus does not hold in our model, and this is because supply changes and the

accompanying tax changes concern different generations. Indeed, the supply changes must be absorbed by generations $j - 2$ and $j - 1$, who are the only ones that can trade in period $j - 1$. Since these generations acquire more long-term bonds, they become more exposed to the risk that interest rates increase. The accompanying tax changes represent exactly the opposite risk: with an increased supply of long-term bonds, the government can charge lower taxes when interest rates increase because the value of its debt is smaller. Tax changes, however, concern future generations, who do not trade in period $j - 1$. Therefore, prices of long-term bonds in that period reflect only the lower valuation of generations $j - 2$ and $j - 1$.

Section 4 determines the socially optimal maturity structure of government debt. In general, interest-rate risk is not shared optimally across generations because only generations $j - 2$ and $j - 1$ can trade in period $j - 1$. The government can affect risksharing through its choice of maturity structure. For example, lengthening the maturity structure raises the consumption of generations $j - 2$ and $j - 1$ when interest rates are low, but also the taxes of future generations in the same state. To determine the optimal quantity of long-term bonds, we consider a hypothetical “complete-participation” equilibrium, where there is no government but all generations can trade in period $j - 1$. We show that the quantity of long-term bonds that generations $j - 2$ and $j - 1$ buy from future generations in that equilibrium is also the optimal quantity that the government should issue in period $j - 1$. Thus, the government can raise welfare by replicating the trades of private agents not present in the market. Note that the government’s role is to trade the right quantity of an existing security rather than introduce a new security: long-term bonds can be available even in the government’s absence.

Section 5 derives properties of the optimal maturity structure. Recall that if agents are more risk-averse than log, then an increase in the size of the long-horizon clientele increases the demand for long-term bonds. Generations $j - 2$ and $j - 1$ thus buy a larger quantity of these bonds in the equilibrium with complete participation, and so the government should issue more such bonds. In lengthening the maturity structure of its debt in response to an increase in the size of the long-horizon clientele, the government is effectively catering to that clientele. It is also reducing interest payments on its debt since the increased demand for long-term bonds renders them more expensive and a cheaper financing option. We show, however, that the government does not accommodate fully changes in clientele demand, i.e., issuance of long-term bonds is smaller than the increased demand. This is because the government internalizes that the insurance it is offering to the long-horizon clientele by issuing more long-term bonds is costly for future generations: the risk that interest rates will be low is transferred to them through higher taxes. Note that when interest rates are low, fewer aggregate resources are available because productivity is low. Therefore, optimal risksharing requires that the long-horizon clientele consumes less in that state. In turn, optimal issuance of long-term bonds is limited to a level where these earn negative expected returns relative

to short-term bonds.

This paper is related to the literature on optimal public debt policy. The benchmark in that literature is the Ricardian equivalence result of Barro (1974): in a representative-agent model with non-distortionary taxes, the level and composition of government debt are irrelevant. Ricardian equivalence fails in the presence of distortionary taxes because debt can be used to smooth tax rates across time and states of nature. Distortionary taxes imply an optimal time path for the level of government debt, as shown in Barro (1979) and Aiyagari, Marcet, Sargent and Seppala (2002), and an optimal composition of the government debt portfolio. Lucas and Stokey (1983) derive the optimal portfolio in terms of Arrow-Debreu securities. Angeletos (2002) and Buera and Nicolini (2004) show how the optimal outcome can be implemented with non-contingent bonds of different maturities, provided that there are bonds of as many maturities as states of nature so that markets are complete. Nosbusch (2008) derives the optimal maturity structure under incomplete markets. Faraglia, Marcet and Scott (2008) extend the complete markets analysis of optimal maturity structure to a framework with capital accumulation.³ Optimal maturity structure in our model is determined by clienteles and intergenerational risksharing rather than distortionary taxes. This yields insights that differ from and complement those of the distortionary-tax literature. For example, we can examine how changes in demand in different segments of the yield curve affect optimal issuance, and the extent to which the government caters to clienteles. Moreover, as we explain in Section 3.2, changes in maturity structure in our model have typically opposite price effects than in the distortionary-tax literature.

It is well known that Ricardian equivalence fails in models with overlapping generations. Government debt shifts taxes to future generations, as shown in Diamond (1965) and Blanchard (1985), and this can be Pareto-improving when the economy is dynamically inefficient.⁴ Government debt can also improve intergenerational risksharing, as shown in Fischer (1983) and Gale (1990). In both papers generations live for two periods, and the introduction of one-period non-contingent government bonds can be Pareto-improving. Gale shows additionally that two-period bonds can yield even larger improvements.⁵ Our model differs because we allow for clienteles with different investment horizons, study how changes in clientele demand affect bond prices and the optimal maturity structure, and characterize the latter in terms of the complete-participation equilibrium.⁶

³A separate strand of the literature considers optimal debt policy when debt contracts are nominal and the government has control over inflation. The government can then use state-contingent inflation to smooth tax rates. Prominent examples of this approach include Lucas and Stokey (1983), Bohn (1988), Calvo and Guidotti (1990,1992), Barro (2002), Benigno and Woodford (2003), and Lustig, Sleet and Yeltekin (2008). Missale and Blanchard (1994) show that with nominal debt contracts, the optimal maturity can be decreasing in the total size of the debt.

⁴Dynamic inefficiency can also be alleviated through fiat money, as shown in Samuelson (1958).

⁵Blanchard and Weil (2001) provide general conditions under which the government can use bonds of different maturities to achieve Pareto improvements in a dynamically efficient economy. Weiss (1980) and Bhattacharya (1982) show that the government can use state-contingent inflation to improve intergenerational risksharing when debt contracts are nominal.

⁶Ball and Mankiw (2007) use the complete-participation equilibrium to characterize the optimal social security

The role of clienteles is emphasized in an early term-structure literature. According to the preferred-habitat hypothesis of Culbertson (1957) and Modigliani and Sutch (1966), there are investor clienteles with preferences for specific maturities and with limited ability to substitute across the yield curve. More recently, Vayanos and Vila (2009) develop a formal model of preferred habitat in which each maturity has its own clientele and substitution across maturities is performed by risk-averse arbitrageurs. Greenwood and Vayanos (2010) extend this model to accommodate time-variation in debt maturity structure, and find that a longer maturity structure predicts high excess returns of long-term bonds. In both papers clienteles are infinitely risk-averse over consumption at their desired maturity. We instead model clienteles through CRRA preferences, dispense with arbitrageurs, and perform a normative analysis of maturity structure.

Finally, this paper is related to a literature studying how private firms make capital-structure decisions to cater to investor clienteles or sentiment (e.g., Stein (1996), Baker and Wurgler (2002), Greenwood, Hanson and Stein (2010)). One difference between private firms in that literature and the government in our model is that the latter has price impact. An additional difference is that we study the intergenerational tradeoffs that arise because of the government's issuance decisions. We show, in particular, that because the government takes into account the welfare of future generations, it does not accommodate fully changes in clientele demand, nor does it minimize expected interest payments on its debt.

2 Model

Time t is discrete and goes from 0 to ∞ . There is a single good. In each period a new generation is born and lives for three periods. Generation t , born in period t , is young in that period, middle-aged in period $t + 1$, and old in period $t + 2$. It receives an endowment α_t of the good when young and consumes c_{t+2} when old. Utility over consumption is CRRA

$$u(c) \equiv \frac{c^{1-\gamma}}{1-\gamma}, \tag{2.1}$$

and the coefficient of relative risk aversion γ is equal across generations. Each generation derives its consumption by investing its net-of-tax endowment in a linear technology and in government bonds.

The linear technology yields a riskless return. This return is equal across periods, except for

system in a two-period overlapping-generations model. They show that an optimal social security system implements the equilibrium that would prevail if all generations could trade risksharing contracts ex-ante. Social security and debt maturity structure can be viewed as alternative mechanisms to improve intergenerational risksharing. We return to this issue in Section 4.2.

one specific period $j \geq 2$. One unit of the good invested in the technology in any period $t \neq j$ yields R units in period $t + 1$. The corresponding return between periods j and $j + 1$ takes the values R_h and $R_\ell < R_h$ with probabilities p and $1 - p$, respectively. Since the technology is riskless, the value of R_s in states $s = h, \ell$, is known in period j . It is not known in previous periods, however, and this constitutes the only uncertainty in the model. The endowment α_t can depend on t , but we restrict its asymptotic growth rate to be below R . Figure 1 summarizes endowments and consumption of generations around period j .

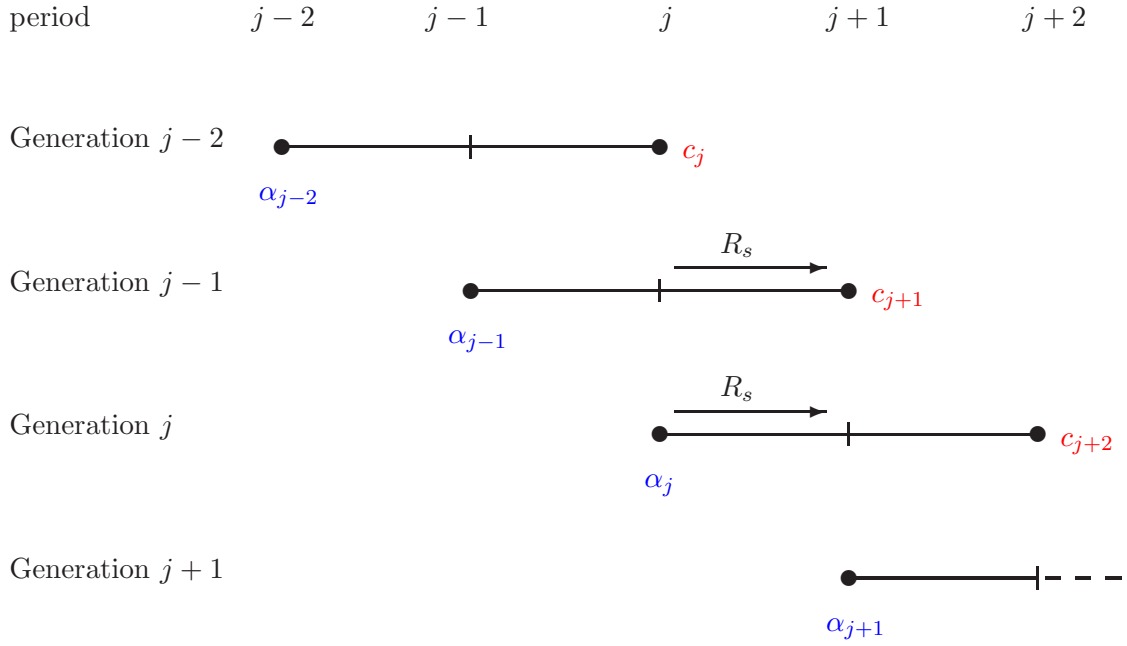


Figure 1: Life-cycles and return shock.

The government issues bonds and collects income taxes. Since only young agents receive an income (endowment), taxes are only on them. For simplicity, we set government spending to zero; assuming positive spending would not affect our results. Since government spending is zero, bond issuance and taxes can be negative, in which case they represent government investment in bonds and transfers towards the agents, respectively.

We assume that only one-period bonds are available in periods $t \neq j - 1$, while one- and two-period bonds are available in period $j - 1$. This entails no loss of generality relative to a more general maturity structure. Indeed, in periods $t \neq j - 1$, agents face no uncertainty one period ahead. Therefore, no-arbitrage implies that all bonds have the same deterministic return, which is also the return on the linear technology. Since bonds of multiple maturities are redundant, we can assume that only one maturity is available, and take it to be one period. In period $j - 1$, the uncertainty one period ahead is described by two states. Therefore, two maturities suffice to

complete the market from the perspective of agents trading in period $j - 1$, and we take these to be one and two periods.

We denote by τ_t the tax rate in period $t \leq j - 1$, by b_t the face value of one-period bonds issued by the government in the same period, and by B the face value of two-period bonds issued by the government in period $j - 1$. We denote by $\tau_{t,s}$ and $b_{t,s}$, respectively, the counterparts of τ_{j-1} and b_{j-1} for period $t \geq j$ and state $s = h, \ell$. We define $b_{j,s}$ to include both new one-period bonds issued by the government in period j , as well as two-period bonds issued in period $j - 1$ that become one-period in period j .

The choice of maturity structure concerns the mix (b_{j-1}, B) of one- and two-period bonds issued by the government in period $j - 1$. The generations investing in bonds in that period are $j - 2$ and $j - 1$, and maturity structure affects how risk is shared between them and generations $t \geq j$. We hence study the choice of maturity structure in conjunction with the taxes in periods $t \geq j$, which can be made state-contingent and can concern only the endowments of generations $t \geq j$. We treat taxes on generations $t \leq j - 1$ as exogenous, and for simplicity set them to zero. Since these taxes are zero, the face value b_t of one-period bonds issued by the government in periods $t \leq j - 2$ is also zero.

The government's budget constraint in period $j - 1$ is

$$\frac{b_{j-1}}{R} + \frac{B}{L^2} = 0. \quad (2.2)$$

Government revenue is the sum of taxes on generation $j - 1$, which are zero, plus the revenue b_{j-1}/R from issuance of one-period bonds and B/L^2 from issuance of two-period bonds. It is equal to the cost b_{j-2} of repaying the one-period bonds issued in period $j - 2$, which is zero. The one-period interest rate is equal to the return R of the linear technology because of no-arbitrage, and the two-period interest rate, denoted by L , is determined in equilibrium in the next section.

The government's budget constraint in period j and state s is

$$\alpha_j \tau_{j,s} + \frac{b_{j,s} - B}{R_s} = b_{j-1}. \quad (2.3)$$

Government revenue is the sum of taxes $\alpha_j \tau_{j,s}$ on generation j , plus the revenue $(b_{j,s} - B)/R_s$ from issuance of new one-period bonds. It is equal to the cost b_{j-1} of repaying the one-period bonds issued in period $j - 1$. The government's budget constraint in period $t > j$ and state s is similarly

$$\alpha_t \tau_{t,s} + \frac{b_{t,s}}{R} = b_{t-1,s}. \quad (2.4)$$

Note that the one-period interest rate is equal to R_s in period j and to R in period $t > j$.

3 Supply and Demand Effects

In this section we characterize equilibrium in period $j - 1$, i.e., just before the realization of the return shock. We derive the equilibrium term structure and examine how it depends on bond supply and clientele demand.

3.1 Equilibrium Term Structure

In period $j - 1$, only two generations can trade bonds: the middle-aged generation $j - 2$ and the young generation $j - 1$. We denote by B_{j-2} and B_{j-1} , respectively, the face values of two-period bonds that they hold. Generation $j - 2$'s consumption in period j is

$$c_{j,s} = R^2 \alpha_{j-2} + \left(\frac{L^2}{R_s} - R \right) \frac{B_{j-2}}{L^2}. \quad (3.1)$$

The first term in the right-hand side is the consumption achieved by investing the endowment α_{j-2} in the linear technology or in one-period bonds between periods $j - 2$ and j . The second term is the additional return from investing in two-period bonds, which become available in period $j - 1$. It is equal to the excess return of two- relative to one-period bonds between periods $j - 1$ and j , times the market value of two-period bonds held by generation $j - 2$. Generation $j - 2$ chooses its holdings B_{j-2} of two-period bonds to maximize expected utility $\mathbb{E}[u(c_{j,s})]$ subject to (3.1). The first-order condition is

$$\mathbb{E} \left[u'(c_{j,s}) \left(\frac{1}{R_s} - \frac{R}{L^2} \right) \right] = 0. \quad (3.2)$$

Since $u'(0) = \infty$, (3.2) has a solution B_{j-2} for any two-period interest rate L satisfying no-arbitrage, i.e., $L \in (RR_\ell, RR_h)$. Moreover, the solution is unique since utility is strictly concave.

Generation $j - 1$'s consumption in period $j + 1$ and state s is

$$c_{j+1,s} = RR_s \alpha_{j-1} + (L^2 - RR_s) \frac{B_{j-1}}{L^2}. \quad (3.3)$$

The first term in the right-hand side is the consumption achieved by investing the endowment α_{j-1} in the linear technology or in one-period bonds between periods $j - 1$ and $j + 1$. The second term is the additional return from investing in two-period bonds. It is equal to the excess return of two-

relative to one-period bonds between periods $j - 1$ and $j + 1$, times the market value of two-period bonds held by generation $j - 1$. Note that the relevant excess return for generation $j - 1$ is over two periods because this generation consumes one period later than generation $j - 2$. The first-order condition is

$$\mathbb{E} \left[u'(c_{j+1,s}) \left(1 - \frac{RR_s}{L^2} \right) \right] = 0. \quad (3.4)$$

The same argument as for (3.2) implies that (3.4) has a unique solution B_{j-1} .

The two-period interest rate L is determined from the market-clearing condition

$$B_{j-2} + B_{j-1} = B. \quad (3.5)$$

Proposition 3.1 shows that (3.5) has a unique solution L because the demand functions (B_{j-2}, B_{j-1}) are increasing in L , converge to $-\infty$ when L^2 goes to RR_ℓ , and converge to ∞ when L^2 goes to RR_h .

Proposition 3.1. *There exists a unique equilibrium two-period interest rate L in period $j - 1$.*

The equilibrium term structure in period $j - 1$ consists of the one-period interest rate R (equal to the return of the riskless technology because of no-arbitrage) and the two-period interest rate L . Since supply and demand do not affect R , their effects on L , the slope of the term structure, and the excess return of two- relative to one-period bonds are all in the same direction.

3.2 Bond Supply

Bond supply in period $j - 1$ can be measured by the face values b_{j-1} and B of one- and two-period bonds, respectively. We hold total supply constant in market value terms, and examine how a shift towards two-period bonds, i.e., a lengthening of the maturity structure, affects the two-period interest rate L . This amounts to increasing B holding $b_{j-1}/R + B/L^2$ constant.

Proposition 3.2. *A lengthening of the maturity structure in period $j - 1$ raises the two-period interest rate L .*

Intuitively, when B increases, generations $j - 2$ and $j - 1$ must absorb more two-period bonds, and this gives them larger exposure to the risk that the one-period interest rate will increase in period j . They are compensated for that risk through a decrease in the price of two-period bonds. The two-period interest rate thus increases, and so do the slope of the term structure and the excess return of two- over one-period bonds.

The effect of supply derived in Proposition 3.2 stands in contrast to the Ricardian equivalence result of Barro (1974), derived in a representative-agent model with non-distortionary taxes. In that model, supply changes can affect consumption and bond demand, but the effects are exactly offset by those of the accompanying tax changes.⁷ In our model, however, supply and tax changes concern different agents. Indeed, an increase in the supply of two-period bonds in period $j - 1$ is absorbed by generations $j - 2$ and $j - 1$. The accompanying tax changes, however, are contingent on the information revealed in period j , and hence affect generations $t \geq j$, who receive their endowments from period j onwards.

3.3 Clientele Demand

Clienteles in our model correspond to generations that are alive in any given period and differ in their investment horizons. For example, the clienteles in period $j - 1$ are the young generation $j - 1$, with a two-period investment horizon, and the middle-aged generation $j - 2$, with a one-period horizon. We refer to them as the long- and short-horizon clientele, respectively. To examine how the mix of these clienteles affects the term structure, we change the relative size of their endowments holding the value of their aggregate endowment measured as of period $j - 1$ constant. Thus, an increase in the size of the long-horizon clientele amounts to increasing α_{j-1} holding $R\alpha_{j-2} + \alpha_{j-1}$ constant.

Proposition 3.3. *If $\gamma > 1$, then an increase in the size of the long-horizon clientele in period $j - 1$ lowers the two-period interest rate L . The result is reversed if $\gamma < 1$.*

According to the preferred-habitat hypothesis of Culbertson (1957) and Modigliani and Sutch (1966), short-term bonds are demanded mainly by short-horizon investors, while long-term bonds are demanded by long-horizon investors. Therefore, when generation $j - 1$ commands more resources, two-period bonds should be in higher demand and thus more expensive. Proposition 3.3 confirms this intuition when agents' coefficient of relative risk aversion γ is larger than one. When $\gamma < 1$, however, the effect is reversed, and when $\gamma = 1$ (logarithmic utility) the clientele mix has no effect. Intuitively, when utility is logarithmic, agents behave myopically and their portfolio choice is independent of the time when they need to consume. When instead $\gamma \neq 1$, generation $j - 1$ has a hedging demand which depends on assets' covariance with the interest rate in period j . In the case $\gamma > 1$, the hedging demand favors assets whose returns increase when the interest rate decreases. Two-period bonds have this property, and hence generation $j - 1$ invests a larger share of its wealth

⁷Supply effects can arise in representative-agent models with distortionary taxes. The mechanism driving these effects, however, is different than in our model, and the effects themselves are typically in the opposite direction. In particular, an increase in the supply of two-period bonds can increase their price, holding the one-period interest rate constant. Indeed, when supply increases, the government must collect more taxes in states where the interest rate is low because more resources are required in these states to eventually repay the two-period bonds. Because taxes are distortionary, consumption is reduced in the low-interest rate states. As a consequence, two-period bonds, which are more valuable in these states, become more attractive to the representative agent and their price increases.

in these bonds than generation $j - 2$. In summary, our model can generate intuitive clientele effects, while also highlighting that such effects can arise only when $\gamma > 1$. Clientele effects do not arise, in particular, under logarithmic utility, an assumption commonly used in term-structure models.

4 Optimal Maturity Structure

The properties derived in Section 3 concern a general maturity structure, i.e., hold for any quantities of one- and two-period bonds that the government issues in period $j - 1$. In this section we show that there exists a socially optimal maturity structure. The government's choice of maturity structure affects social welfare because it induces reallocations of risk across generations.

Recall that risk in our model arises because the return of the linear technology between periods j and $j + 1$, revealed in period j , can be high or low. This risk affects generations $j - 1$ and j because they invest in period j . Risk cannot be shared with generations that consume before period j because their consumption cannot be made contingent on information revealed in that period. Moreover, in the absence of the government, risk cannot be shared with generations that are not available to trade in period $j - 1$. Thus, the only generations that can share risk in the government's absence are $j - 2$ and $j - 1$. Risksharing between these generations is optimal if the market in period $j - 1$ is complete, e.g., there exist one- and two-period bonds in zero net supply.

The government can improve on risksharing by issuing suitable quantities of one- and two-period bonds and by choosing suitable taxes in periods $t \geq j$. An optimal combination of maturity structure and taxes can effectively induce participation by generations $t \geq j$ in period $j - 1$, and hence result in optimal risksharing between these generations and generations $j - 2$ and $j - 1$.

To set the stage for our analysis of optimal maturity structure, we first consider the benchmark case of complete participation, where all generations $t \geq j - 2$ are available to trade in period $j - 1$. We then show that the government can use maturity structure and taxes to effectively induce complete participation.

4.1 Complete Participation

We assume that all generations $t \geq j - 2$ are available to trade one- and two-period bonds in period $j - 1$. We also remove the government, setting bond supply and taxes to zero. The optimization problems of generations $j - 2$ and $j - 1$ are as in Section 3.1, so we only need to consider those of generations $t \geq j$. Generation j 's consumption in period $j + 2$ and state s is

$$c_{j+2,s} = RR_s \alpha_j + R(L^2 - RR_s) \frac{B_j}{L^2}, \quad (4.1)$$

and generation $t > j$'s consumption in period $t + 2$ and state s is

$$c_{t+2,s} = R^2 \alpha_t + R^{t+1-j} (L^2 - RR_s) \frac{B_t}{L^2}, \quad (4.2)$$

where B_t denotes the face value of two-period bonds held by generation $t \geq j$. The interpretation of (4.1) and (4.2) is analogous to that of (3.1) and (3.3): the first term in the right-hand side is the consumption achieved by investing the endowment in the linear technology or in one-period bonds, and the second term is the additional return from investing in two-period bonds. The excess return of two- relative to one-period bonds is computed between periods $j - 1$ and $j + 1$, and brought forward to period t at the rate R . The first-order condition of generation $t \geq j$ is

$$\mathbb{E} \left[u'(c_{t+2,s}) \left(1 - \frac{RR_s}{L^2} \right) \right] = 0. \quad (4.3)$$

The same argument as in Section 3.1 implies that (4.3) has a unique solution B_t .

The two-period interest rate L under complete participation is determined by the market-clearing condition

$$\sum_{t=j-2}^{\infty} B_t = 0. \quad (4.4)$$

Eq. (4.4) differs from its counterpart (3.5) in Section 3.1 because the summation is over all generations $t \geq j - 2$, and the supply of two-period bonds is zero. Proposition 4.1 shows that (4.4) has a solution L .⁸

Proposition 4.1. *Under complete participation, there exists an equilibrium two-period interest rate L in period $j - 1$.*

Combining the first-order conditions (3.2), (3.4) and (4.3), and noting that there are two states $s = h, \ell$, we find

$$\frac{u'(c_{j,h})}{u'(c_{j,\ell})} = \frac{R_h u'(c_{t+2,h})}{R_\ell u'(c_{t+2,\ell})} \quad (4.5)$$

for all $t \geq j - 1$. Eq. (4.5) characterizes risksharing between generations $t \geq j - 2$ under complete

⁸The solution is unique for $\gamma \leq 1$, but multiplicity cannot be ruled out for high values of γ . The properties of consumption, expected returns and bond holdings, shown in Section 4.1, and the link with optimal maturity structure, shown in Sections 4.2 and 4.3, hold for any equilibrium with complete participation. In the numerical example used to draw Figures 2 and 3, the equilibrium is unique.

participation. Risk is shared optimally, and the condition for optimality is that the ratio across states of the marginal benefit from receiving the good in period j is identical for all generations. Since generation $j - 2$ consumes in period j , its marginal benefit coincides with its marginal utility of consumption. The marginal benefit of generation $t \geq j - 1$ is equal to the marginal utility of consumption times the return on investment from period j to period $t + 2$, which is when that generation consumes. This return is $R^{t+1-j}R_s$, and its ratio across states h and ℓ is R_h/R_ℓ . Since this ratio is identical for all generations $t \geq j - 1$, so is the ratio across states of the marginal utility of consumption. Thus, if one of generations $t \geq j - 1$ consumes more in state h than in state ℓ , so do all of them.

Since the return from the linear technology is higher in state h than in state ℓ , more aggregate resources are available in that state. As a consequence, the consumption of all generations $t \geq j - 1$ is higher in state h . The consumption of generation $j - 2$, however, can be lower. The reason why this is consistent with optimal risksharing is that while the marginal utility of generations $t \geq j - 1$ is lower in state h , their return from investment is higher. As a consequence, their marginal benefit from receiving the good in period j can be higher in state h , in which case optimal risksharing requires that the marginal utility of consumption of generation $j - 2$ is also higher. Put differently, it can be socially efficient that generation $j - 2$ foregoes consumption in state h since investment in that state has high return for future generations.

Whether or not generation $j - 2$ consumes more in state h than in state ℓ depends on the coefficient of relative risk aversion γ . If γ is high, then the marginal benefit of generations $t \geq j - 1$ from receiving the good in period j is driven more by their marginal utility of consumption than by their return on investment. Since their marginal utility is lower in state h , so is that of generation $j - 2$, meaning that generation $j - 2$ consumes more in state h . If instead $\gamma \leq 1$, then generation $j - 2$ consumes more in state ℓ .⁹

Proposition 4.2. *Under complete participation, generations $t \geq j - 1$ consume more in state h*

⁹A related intuition can be derived from the social planner's problem. Since the allocation under complete participation is Pareto efficient, it maximizes a weighted sum of utilities

$$\sum_{t=j-2}^{\infty} \lambda_t \mathbb{E}[u(c_{t+2,s})], \quad (4.6)$$

subject to the intertemporal budget constraint

$$c_{j,s} + \frac{1}{R_s} \sum_{t=j-1}^{\infty} \frac{c_{t+2,s}}{R^{t+1-j}} = R^2 \alpha_{j-2} + R \alpha_{j-1} + \alpha_j + \frac{1}{R_s} \sum_{t=j+1}^{\infty} \frac{\alpha_t}{R^{t-1-j}}.$$

The utility (4.6) can be interpreted as that of a representative agent who allocates consumption intertemporally. A high interest rate (state h) has three effects on the agent's consumption. The substitution effect induces the agent to reduce consumption in period j and substitute towards later periods. The income effect induces the agent to increase consumption in all periods, including period j , because the consumption stream is cheaper in present value terms. The wealth effect induces the agent to reduce consumption in all periods, including period j , because the endowment stream is cheaper in present value terms. If $\gamma \leq 1$, then the substitution effect dominates the income effect, and hence generation $j - 2$ consumes more in state ℓ . If instead γ is high, then the income effect dominates the combined substitution and wealth effects.

than in state ℓ . Generation $j - 2$ consumes more in state h than in state ℓ if γ is high, but the comparison is reversed if $\gamma \leq 1$.

Figure 2 plots the consumption $(c_{t+2,h}, c_{t+2,\ell})$ of generations $t \geq j - 2$ in states h and ℓ . We consider both the case of complete participation and that of autarky, where each generation invests all its endowment in the linear technology. We assume that endowments are constant ($\alpha_t = 1$ for $t \geq j - 2$), the two states are equally likely ($p = 0.5$), and $\gamma = 4$. We interpret one period to last for 15 years and set the return R of the linear technology to $(1 + 2.1\%)^{15}$. The annual rate 2.1% is the historical average of the long real interest rate in the US in the 20th century (Dimson, Marsh, and Staunton (2002)). We assume that the return of the linear technology is 10% above its mean in state h and 10% below in state ℓ . We plot consumption normalized by its expected value, R^2 . Under autarky, only generations $j - 1$ and j are exposed to the return shock. Under complete participation, by contrast, all generations $t \geq j - 2$ share the risk. Consistent with Proposition 4.2, the consumption of generations $t \geq j - 1$ is higher in state h . Moreover, the parameter value $\gamma = 4$ is in the region where the consumption of generation $j - 2$ is also higher in state h .

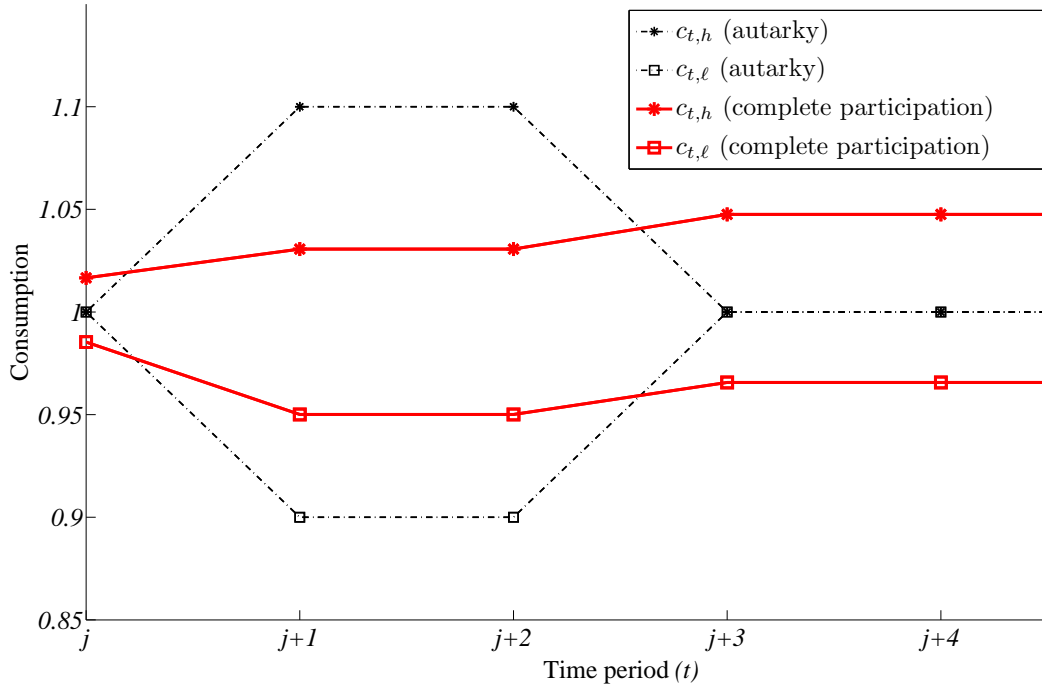


Figure 2: Complete participation vs. autarky.

The properties of consumption are reflected into those of excess bond returns. Consider first the return of two- relative to one-period bonds over a two-period horizon, i.e., investing in two-period bonds until maturity relative to investing in one-period bonds and rolling over. The two strategies

are available to generations $t \geq j - 1$. Since these generations consume more in state h than in state ℓ , two-period bonds, which dominate one-period bonds in state ℓ , are a valuable hedge and offer negative expected excess return. That is, investing in two-period bonds until maturity yields less in expectation than investing in one-period bonds and rolling over:

$$L^2 < \mathbb{E}(RR_s). \quad (4.7)$$

Consider next the return of two- relative to one-period bonds over a one-period horizon, i.e., investing in two-period bonds and selling after one period relative to investing in one-period bonds. The two strategies are available to generation $j - 2$. If γ is high, then this generation consumes more in state h than in state ℓ . Therefore, two-period bonds, which dominate in state ℓ , offer negative expected excess return. That is, investing in two-period bonds and selling after one period yields less in expectation than investing in one-period bonds:

$$\mathbb{E} \left[\frac{L^2}{R_s} \right] < R. \quad (4.8)$$

The opposite inequality holds, however, if $\gamma \leq 1$ since generation $j - 2$ then consumes more in state ℓ . Thus, two-period bonds can offer a positive expected excess return over a one-period horizon, even though their corresponding return over a two-period horizon is negative. Mathematically, this is possible because of Jensen's inequality; the economic intuition relates to clientele effects, as we explain next.

Corollary 4.1. *Under complete participation, the expected return of two- relative to one-period bonds over a two-period horizon is negative. The corresponding return over a one-period horizon is negative if γ is high, but positive if $\gamma \leq 1$.*

We next determine equilibrium bond holdings. The properties of bond holdings tie those of consumption and excess bond returns with the clientele effects derived in Section 3.3. They also connect to features of the optimal maturity structure derived in the following sections. Under complete participation, generations $j - 1$ and j seek to hedge against the risk that their return on investment in period j will be low (state ℓ). Generations $t > j$, who have no exposure to that risk, provide them with insurance by selling them two-period bonds. Since initial positions are zero, generations $t > j$ short-sell two-period bonds and use the proceeds to invest in one-period bonds. The investment in one-period bonds is the reason why the consumption of these generations is low in state ℓ (Proposition 4.2).

Generation $j - 2$ could be an additional seller of two-period bonds because it has no exposure to return risk. Generation $j - 2$, however, differs from generations $j - 1$ and j not only in risk

exposure, but also in investment horizon: that of generation $j - 2$ is one period, while that of generations $j - 1$ and j is two periods. If $\gamma > 1$, then a longer horizon implies larger investment in two-period bonds (Proposition 3.3). Therefore, generation $j - 2$ invests less in two-period bonds than generations $j - 1$ and j , which implies that generations $j - 1$ and j have to be buyers of these bonds. Generation $j - 2$ can be a buyer or a short-seller; it is a short-seller if γ is large since its consumption is higher in state h than in state ℓ (Proposition 4.2). If instead $\gamma \leq 1$, then a longer horizon implies smaller investment in two-period bonds. Therefore, generation $j - 2$ invests more in two-period bonds than generations $j - 1$ and j , which implies that generation $j - 2$ has to be a buyer of these bonds. This is another way to see why generation $j - 2$ consumes more in state ℓ (Proposition 4.2), and why one-period bonds offer positive expected excess return over a one-period horizon and negative return over two periods (Corollary 4.1). Generations $j - 1$ and j can be buyers or short-sellers of two-period bonds.

Corollary 4.2. *Under complete participation,*

- *Generations $t > j$ short-sell two-period bonds.*
- *Generations $j - 1$ and j buy two-period bonds if $\gamma \geq 1$.*
- *Generation $j - 2$ short-sells two-period bonds if γ is high, and buys them if $\gamma \leq 1$.*

4.2 Inducing Complete Participation through Debt Maturity Structure

We now return to the case where only generations $j - 2$ and $j - 1$ are available to trade in period $j - 1$. We show that although generations $t \geq j$ cannot trade in that period, there exists a choice of maturity structure and taxes that results in optimal risksharing between them and generations $j - 2$ and $j - 1$. Through this choice, the government can effectively induce complete participation.

Suppose that the government issues the aggregate quantity B^* of two-period bonds that generations $j - 2$ and $j - 1$ buy under complete participation. Then, the two-period interest rate coincides with that under complete participation. The same is true for the quantity of two-period bonds bought by each of generations $j - 2$ and $j - 1$, and for these generations' consumption. Generations $t \geq j$ can also consume the same as under complete participation through a choice of state-contingent taxes that meet the government's budget constraint. Such taxes are feasible: since generations $j - 2$ and $j - 1$ buy the same quantity of two-period bonds as under complete participation, government issuance of these bonds coincides with the aggregate quantity that generations $t \geq j$ sell when they can trade in period $j - 1$. Therefore, the government's capital gain or loss on its bond portfolio (which determines future taxes) is equal to the aggregate capital gain or loss of generations $t \geq j$ under complete participation.

Proposition 4.3. *Suppose that only generations $j-2$ and $j-1$ can trade in period $j-1$. Then, the government can achieve the same outcome as under complete participation by issuing the aggregate quantity B^* of two-period bonds that generations $j-2$ and $j-1$ buy under complete participation, and levying appropriate state-contingent taxes on generations $t \geq j$.*

Since the government issues the same quantity of two-period bonds as that sold by generations $t \geq j$ under complete participation, it is effectively trading on behalf of these generations. Thus, the government induces optimal risksharing by replicating the trades of private agents not present in the market. Note that the government raises welfare by supplying the right quantity of two-period bonds rather than by introducing these bonds into the market. Two-period bonds can exist even in the government's absence; the government's role is to trade the right quantity of them.

To illustrate the government's optimal policy, we return to the calibration used in Figure 2 of Section 4.1. Under complete participation, generations $j-1$ and j share their risk with generations $j-2$ and $t > j$ by buying two-period bonds from them.¹⁰ Moreover, generations $j-2$ and $j-1$ buy two-period bonds in the aggregate, and the quantity bought is 0.7. Therefore, the government can induce complete participation by issuing a quantity $B^* = 0.7$ of two-period bonds.

Figure 3 depicts government debt and taxes in states h and ℓ for $t \geq j$. If the state is h , then the government makes a capital gain on its bond portfolio since it has issued two-period bonds. It also taxes generation j ($\tau_{j,h} > 0$), which buys two-period bonds under complete participation and consumes less in state h than under autarky. The government then invests its trading profits and tax proceeds in one-period bonds, rolls them over, and pays out the interest that it earns as a transfer to generations $t > j$. Thus, transfers to generations $t > j$ are constant over time ($\tau_{t,h} \equiv \tau_h < 0$ for $t > j$). Constant transfers are optimal because generations $t > j$ have the same initial endowment, CRRA preferences and investment opportunity set, and hence the same consumption profile under complete participation. Conversely, if the state is ℓ , then the government makes a capital loss on its bond portfolio, and makes a transfer to generation j ($\tau_{j,\ell} < 0$). It then finances its trading losses and transfer payments by issuing one-period bonds, rolls them over, and pays the interest by levying a constant tax on generations $t \geq j$ ($\tau_{t,\ell} \equiv \tau_\ell > 0$ for $t > j$).

Proposition 4.3 shows that the government can achieve the complete-participation allocation through a combination of debt maturity structure in period $j-1$, and taxes on future generations $t \geq j$. An alternative mechanism that achieves the same allocation is a social security system, where taxes concern both the current generations $j-2$ and $j-1$, as well as future ones. For the mechanisms to be equivalent, taxes must be contingent on the uncertainty revealed in period j .

¹⁰That generations $j-1$ and j buy two-period bonds, while generations $t > j$ short-sell these bonds is implied from Corollary 4.2 because the calibration assumes $\gamma = 4$. That generation $j-2$ short-sells two-period bonds is consistent with the high- γ case of Corollary 4.2 and with the fact that generation $j-2$'s consumption in Figure 2 is higher in state h .

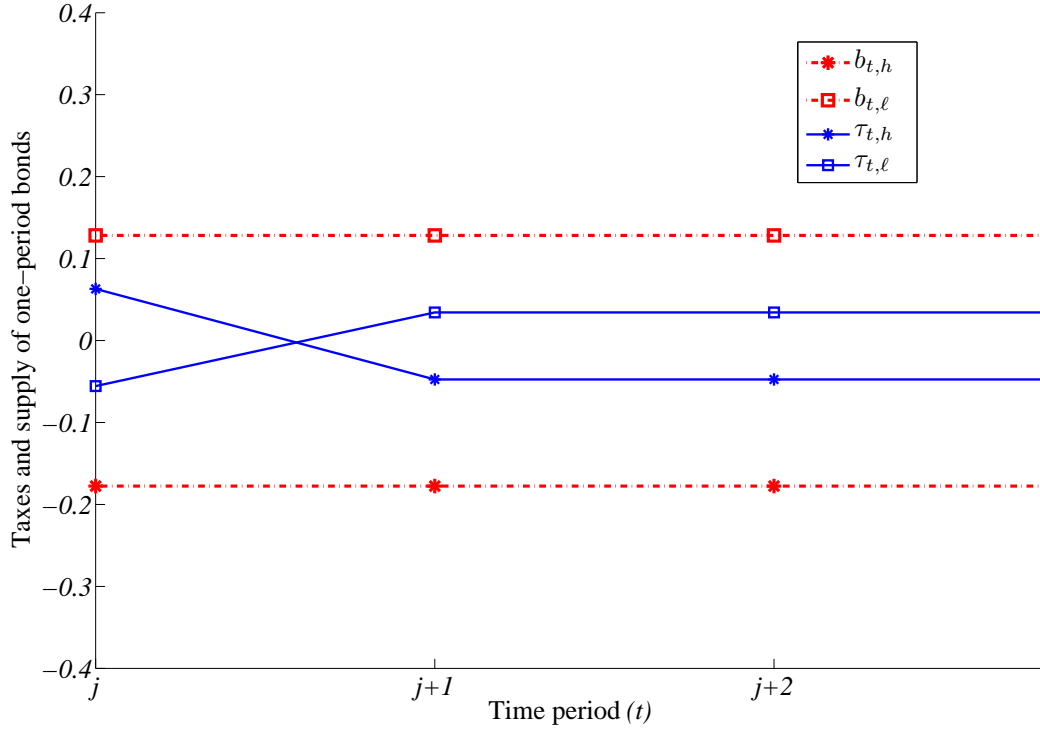


Figure 3: Taxes and supply of one-period bonds.

Hence, current generations must be taxed in the second or third period of their lives, and these taxes are not income (endowment) taxes as assumed in our model.

Debt maturity structure has the advantage over social security to not require taxes on the current generations. Indeed, the revenue is raised instead through the capital losses that these generations realize on their investments in government bonds. Collecting the revenue through capital losses on government bonds could be politically easier than imposing taxes since the investment decision is voluntary. Moreover, taxes could have the additional drawback of being distortionary. While both considerations are outside our model, understanding how debt maturity structure and social security compare in improving intergenerational risksharing is worthy of further investigation.¹¹

4.3 Uniqueness of Optimal Maturity Structure

Section 4.2 derives a maturity structure that induces complete participation and optimal risksharing. We next study more formally the optimization over maturity structures, and show that the

¹¹Rangel and Zeckhauser (2001) study the political economy of social security by modeling voting over taxes in a two-period overlapping-generations economy. Campbell and Nosbusch (2007) study how a social security system that shares risks across generations affects equilibrium asset prices.

maturity structure of Proposition 4.3 is the unique optimal one.¹²

A maturity structure is fully characterized by the quantity B of two-period bonds issued in period $j - 1$; the quantity b_{j-1} of one-period bonds issued in that period is determined by the government's budget constraint (2.2). We study the choice of B in conjunction with the state-contingent taxes $\{\tau_{t,h}, \tau_{t,\ell}\}_{t \geq j}$. The policy $(B, \{\tau_{t,h}, \tau_{t,\ell}\}_{t \geq j})$ must satisfy the government's budget constraints (2.2)-(2.4). Eliminating b_{j-1} and $\{b_{t,h}, b_{t,\ell}\}_{t \geq j}$, we can condense (2.2)-(2.4) into the intertemporal budget constraint

$$\alpha_j \tau_{j,s} + \frac{1}{R_s} \sum_{t=j+1}^{\infty} \frac{\alpha_t \tau_{t,s}}{R^{t-1-j}} = \left(\frac{1}{R_s} - \frac{R}{L^2} \right) B. \quad (4.9)$$

We refer to a policy $(B, \{\tau_{t,h}, \tau_{t,\ell}\}_{t \geq j})$ that satisfies (4.9) as admissible.

We denote by \mathcal{A} the set of equilibrium allocations generated by admissible policies, and by \mathcal{A}_0 the set of all feasible allocations. The set \mathcal{A}_0 is strictly larger than \mathcal{A} because it includes allocations that can be achieved through general redistributions across generations, and not only through changes in maturity structure and subsequent taxes. We define Pareto optimal policies as those corresponding to the Pareto frontier of \mathcal{A} . A welfare-maximizing policy must be Pareto optimal, otherwise the government could achieve a Pareto improvement by choosing a different policy. The set of Pareto optimal policies includes the policy of Proposition 4.3 which induces complete participation. Indeed, since complete participation corresponds to a standard competitive equilibrium, the policy of Proposition 4.3 belongs not only to the Pareto frontier of \mathcal{A} , but also to that of \mathcal{A}_0 .¹³

In addition to the policy of Proposition 4.3, there exist other Pareto optimal policies. These can be derived, for example, by leaving maturity structure unchanged and reallocating taxes across generations $t \geq j$. Even changes in maturity structure might fail to generate Pareto ranked allocations. To study uniqueness, we strengthen our optimality criterion by ruling out not only Pareto improvements but also aggregate welfare gains. That is, we define a policy to be optimal if changes to it cannot benefit winners more than they hurt losers. Using this criterion, we examine whether all optimal policies involve the same maturity structure.

To ensure that gains and losses are comparable across agents, we measure them in monetary terms as of period $j - 1$. Consider a policy change from $(B, \{\tau_{t,h}, \tau_{t,\ell}\}_{t \geq j})$ to $(\hat{B}, \{\hat{\tau}_{t,h}, \hat{\tau}_{t,\ell}\}_{t \geq j})$,

¹²If there exist multiple equilibria with complete participation, Proposition 4.3 yields multiple maturity structures. We still refer to “the maturity structure of Proposition 4.3” or “the maturity structure B^* ” for expositional convenience and because multiplicity is uncommon. We also use the statement “the maturity structure of Proposition 4.3 is the unique optimal one” to mean that the set of optimal maturity structures consists of those derived in Proposition 4.3.

¹³The proof of this statement is included in the proof of Proposition 4.4.

and denote by $c_{t+2,s}$ and $\hat{c}_{t+2,s}$ the consumption of generations $t \geq j - 2$ in state $s = h, \ell$ in the corresponding equilibria. Define the gain T_t of generation t as the investment in one-period bonds that generation t would forego in period $j - 1$ to remain with the same utility as before the policy change.¹⁴ The gain of generation $j - 2$ is given by

$$\mathbb{E}u(c_{j,s}) \equiv \mathbb{E}u(\hat{c}_{j,s} - RT_{j-2}) \quad (4.10)$$

since foregoing T_{j-2} one-period bonds in period $j - 1$ costs RT_{j-2} in terms of consumption in period j . Similarly, the gain of generation $t \geq j - 1$ is given by

$$\mathbb{E}u(c_{t+2,s}) \equiv \mathbb{E}u(\hat{c}_{t+2,s} - R^{t+2-j}R_s T_t) \quad (4.11)$$

since one-period bonds return $R^{t+2-j}R_s$ between periods $j - 1$ and $t + 2$.

Definition 1. *A maturity structure B is optimal if it is associated to an admissible policy $(B, \{\tau_{t,h}, \tau_{t,\ell}\}_{t \geq j})$ such that the sum $\sum_{t=j-2}^{\infty} T_t$ of gains from switching to any other admissible policy $(\hat{B}, \{\hat{\tau}_{t,h}, \hat{\tau}_{t,\ell}\}_{t \geq j})$ is non-positive.*

The maturity structure B^* satisfies Definition 1. Indeed, if it is not optimal, then an aggregate gain can be achieved by switching from the policy of Proposition 4.3 to an admissible policy $(\hat{B}, \{\hat{\tau}_{t,h}, \hat{\tau}_{t,\ell}\}_{t \geq j})$. Using appropriate transfers from winners to losers, we can modify the allocation under $(\hat{B}, \{\hat{\tau}_{t,h}, \hat{\tau}_{t,\ell}\}_{t \geq j})$ to one that Pareto dominates the allocation under the policy of Proposition 4.3. This is a contradiction because the latter allocation is in the Pareto frontier of \mathcal{A}_0 , i.e., is Pareto optimal among all feasible allocations. Proposition 4.4 shows that B^* is the unique optimal maturity structure. Thus, Pareto optimal policies involving other maturity structures are not in the Pareto frontier of \mathcal{A}_0 (while being in that of \mathcal{A}). Under such policies, the non-participation of generations $t \geq j$ in period $j - 1$ impairs risksharing. Changing the maturity structure to B^* renders non-participation inconsequential, and achieves aggregate gains.

Proposition 4.4. *The unique optimal maturity structure is $B = B^*$.*

5 Properties of Optimal Maturity Structure

In this section we derive properties of the optimal maturity structure by exploiting the link with the equilibrium under complete participation.

¹⁴The gain T_t is analogous to the concept of compensating variation (e.g., Varian (1992)). Note that while generations $t \geq j$ are not trading in period $j - 1$, we can interpret T_t as the present value of one-period bonds that they would be willing to forego in period t , where this decision is made in period $j - 1$ (i.e., from an ex-ante viewpoint).

5.1 Excess Returns of Long-Term Bonds

Since consumption under the optimal maturity structure is the same as under complete participation, the expected excess returns of two-period bonds are as in Corollary 4.1.

Proposition 5.1. *Under the optimal maturity structure, the expected return of two- relative to one-period bonds over a two-period horizon is negative. The corresponding return over a one-period horizon is negative if γ is high, but positive if $\gamma \leq 1$.*

Proposition 5.1 suggests that positive expected excess returns of long-term bonds could be a symptom of non-optimal maturity structures, in which these bonds are in excessively large supply. Intuitively, the holders of long-term bonds require positive expected excess returns if their future consumption is negatively correlated with interest rates. Since, however, interest rates are driven by the productivity of the linear technology, they correlate positively with future aggregate resources and with the consumption of future generations. Therefore, risksharing is inefficient, with future generations bearing a disproportionate part of the risk that interest rates will be low. The government can improve risksharing by replacing long-term with short-term bonds. This transfers part of the risk that interest rates will be low from future to current generations. The transfer occurs through the taxes that the government must raise on future generations to repay debt held by current generations: replacing long-term with short-term bonds raises these taxes when interest rates are high but reduces them when interest rates are low.¹⁵

An additional implication of Proposition 5.1 concerns the link between welfare maximization and funding-cost minimization. A government could seek to minimize its funding cost by exploiting differences in expected returns across maturities. Suppose, for example, that long-term bonds are expected to offer negative returns relative to investing in short-term bonds and rolling over. Then, the government could reduce expected interest payments on its debt by tilting issuance towards long-term bonds—and do so up to the point where expected excess returns reach zero. Funding-cost minimization, suitably extended to account for the variance of funding cost in addition to the mean, is viewed as a relevant objective by many practitioners. Bernaschi, Missale and Vergni (2009) discuss merits and limitations of this objective in light of the optimal taxation theory of maturity structure. Our model shows that an additional limitation of funding-cost minimization is to fail to account properly for the welfare of future generations. In particular, long-term bonds carry the

¹⁵The empirical evidence on excess returns of long-term bonds differs depending on whether bonds are nominal or real: it is well known that nominal long-term bonds offer positive returns relative to their short-term counterparts, but the evidence for real bonds is less clearcut. (For nominal bonds see, for example, Cochrane (1999), who reports that the average one-year return of a nominal five-year zero-coupon bond during 1953-1997 was 6.36%, while the corresponding return of a one-year bond was 5.83%. For real bonds see, for example, Ang, Bekaert, and Wei (2008), who find that the real term structure is approximately flat, and hence real long-term bonds offer no significant returns relative to their short-term counterparts.) Since our model is real, our results concern real bonds. Hence they should not be taken to imply that actual maturity structures are suboptimal, although this issue is worthy of further investigation.

implicit cost of imposing high taxes on future generations when interest rates are low, which is also when these generations' consumption is low. As a consequence, the government should issue long-term bonds below the level where their expected returns are the same as of short-term bonds, i.e., long-term bonds should be offering negative excess returns. In addition to this limitation, however, we uncover a similarity between funding-cost minimization and welfare maximization, as we explain in Section 5.2.

5.2 Clientele Effects

We next determine how the optimal quantity B^* of two-period bonds that the government should issue in period $j - 1$ depends on clientele demand. Recall that B^* is the quantity bought by generations $j - 2$ and $j - 1$ in the equilibrium with complete participation. In that equilibrium, generations $t > j$ insure generations $j - 1$ and j against the risk that interest rates will be low by selling two-period bonds to them. If $\gamma < 1$, then generation $j - 2$ invests more in two-period bonds than generation $j - 1$ because a shorter horizon implies larger investment in long-term bonds. Therefore, generations $j - 2$ and $j - 1$ buy two-period bonds in the aggregate, and B^* is positive. If instead γ is high, then generation $j - 2$ short-sells two-period bonds. Therefore, the sign of B^* depends on the size of generations $j - 2$ and $j - 1$: if generation $j - 2$ has a small endowment α_{j-2} relative to other generations, then B^* is mainly driven by the purchases of generation $j - 1$ and is positive; if instead generation $j - 1$ has a small endowment α_{j-1} , then B^* is mainly driven by the short-sales of generation $j - 2$ and is negative.

Proposition 5.2. *For high γ , the optimal supply B^* of two-period bonds is positive if generation $j - 2$ has a small endowment, and negative if generation $j - 1$ has a small endowment. For $\gamma \leq 1$, the optimal supply B^* of two-period bonds is positive.*

Proposition 5.2 suggests that if γ is high, then B^* is increasing in the size of generation $j - 1$ relative to $j - 2$: it is negative when generation $j - 2$ has the larger endowment but becomes positive when generation $j - 1$ has the larger endowment. We next examine more generally how the mix of the two clienteles affects the optimal maturity structure. As in Proposition 3.3, we change the relative size of the clienteles' endowments holding the value of their aggregate endowment measured as of period $j - 1$ constant. Thus, an increase in the size of the long-horizon clientele, i.e., generation $j - 1$, amounts to increasing α_{j-1} holding $R\alpha_{j-2} + \alpha_{j-1}$ constant. We additionally make the following technical assumption:

Assumption 1. *The demand functions of generations $t > j$ for two-period bonds under complete participation are increasing in the two-period interest rate L .*

Assumption 1 ensures that the equilibrium with complete participation is unique. This is

because the demand functions of generations $j - 2$, $j - 1$ and j for two-period bonds are increasing in L , as shown in the proof of Propositions 3.1 and 4.1. Assumption 1 holds under the sufficient condition $\gamma \leq 1$, and hence holds also for values of γ larger than one.¹⁶

Proposition 5.3. *Suppose that Assumption 1 holds. If $\gamma > 1$, then an increase in the size of the long-horizon clientele in period $j - 1$:*

- *Raises the optimal supply B^* of two-period bonds.*
- *Lowers the equilibrium two-period interest rate L that prevails when two-period bonds are in supply B^* .*

The results are reversed if $\gamma < 1$.

Proposition 5.3 is closely related to Proposition 3.3, which examines how an increase in the size of the long-horizon clientele affects the equilibrium two-period interest rate L . Proposition 3.3 takes the supply of two-period bonds as fixed, while Proposition 5.3 examines both how the optimal supply changes and how the equilibrium two-period interest rate changes taking into account the change in supply. The effects are closely related. For example, Proposition 3.3 shows that if $\gamma > 1$, then an increase in the size of the long-horizon clientele raises the aggregate demand of generations $j - 2$ and $j - 1$ for two-period bonds. Due to the higher demand, these generations buy a larger quantity of two-period bonds in the equilibrium with complete participation, and the two-period interest rate in that equilibrium decreases. This, in turn, implies Proposition 5.3: the optimal supply B^* of two-period bonds in the equilibrium with incomplete participation increases, and the two-period interest rate prevailing under B^* decreases.

Propositions 3.3 and 5.3 reveal a similarity between welfare maximization and funding-cost minimization. Suppose, for example, that $\gamma > 1$. Holding maturity structure constant, an increase in the size of the long-horizon clientele lowers the two-period interest rate L (Proposition 3.3). This prompts the government to change the maturity structure by increasing the supply B^* of two-period bonds (Proposition 5.3). Hence, a welfare-maximizing government responds to demand shocks in a way that appears consistent with minimizing expected funding costs. Note that the same is true when $\gamma < 1$: the effects of demand shocks on L and B^* are in the opposite direction than when $\gamma > 1$, implying again a negative correlation between changes to the long-term interest rate and the issuance of long-term bonds.

In lengthening maturity structure in response to an increase in the size of the long-horizon clientele (Proposition 5.3, case $\gamma > 1$), the government is effectively catering to that clientele. This is analogous to private firms making capital-structure decisions to cater to investor clienteles

¹⁶The proof of this statement is available upon request.

or sentiment. One difference between private firms in the investor-catering literature (e.g., Stein (1996), Baker and Wurgler (2002), Greenwood, Hanson and Stein (2010)) and the government in our model is that the latter has price impact. An additional difference is that we study the intergenerational tradeoffs that arise because of the government’s issuance decisions. For example, in issuing more long-term bonds in response to an increase in the size of the long-horizon clientele, the government is offering that clientele more insurance against the risk that interest rates will be low. Yet, it stops short from offering full insurance, allowing the two-period interest rate to drop. This is because it internalizes that the insurance will cost future generations through higher taxes when interest rates are low.

6 Conclusion

We propose a clientele-based model of the yield curve and optimal maturity structure of government debt. Clienteles are generations of agents at different stages of their life cycle in an overlapping-generations economy. An optimal maturity structure exists in the absence of distortionary taxes and induces efficient intergenerational risksharing. If agents are more risk-averse than log, then an increase in the relative importance of the long-horizon clientele raises the price and optimal supply of long-term bonds. But while a welfare-maximizing government caters to clienteles, it does not accommodate fully changes in their demand. It also limits issuance of long-term bonds to a level where these earn negative expected excess returns.

Our model emphasizes a new and empirically relevant determinant of maturity structure: changes in demand in different segments of the yield curve, generated by clienteles. The literature has emphasized that an additional important determinant is the government’s desire to manage its balance-sheet risk and smooth tax rates. Combining the two determinants, by introducing distortionary taxes in our model, could yield a richer theory of optimal maturity structure and be an interesting extension of our work. It could also be interesting to develop further our calibration exercise and give precise quantitative prescriptions as to how maturity structure should respond to changes in clientele demand. This would probably require introducing uncertainty in all periods and extending the lives of generations to more than three periods.

APPENDIX

Proof of Proposition 3.1: We first show that the demand functions (B_{j-2}, B_{j-1}) are increasing in L . Setting $\omega \equiv R/L^2$ and $y_s \equiv 1/R_s$, and using (3.1), we can write (3.2) as

$$\begin{aligned} \mathbb{E} \{ u' [R^2 \alpha_{j-2} + (y_s - \omega) B_{j-2}] (y_s - \omega) \} &= 0 \\ \Leftrightarrow p [R^2 \alpha_{j-2} + (y_h - \omega) B_{j-2}]^{-\gamma} (y_h - \omega) &+ (1-p) [R^2 \alpha_{j-2} + (y_\ell - \omega) B_{j-2}]^{-\gamma} (y_\ell - \omega) = 0, \end{aligned} \quad (\text{A.1})$$

where the second step follows because utility is CRRA. Solving (A.1), we find

$$B_{j-2} = R^2 \alpha_{j-2} \frac{1 - Y}{Y(y_\ell - \omega) + \omega - y_h}, \quad (\text{A.2})$$

where

$$Y \equiv \left[\frac{p(\omega - y_h)}{(1-p)(y_\ell - \omega)} \right]^{\frac{1}{\gamma}}.$$

Differentiating (A.2) with respect to ω , we find

$$\frac{\partial B_{j-2}}{\partial \omega} = R^2 \alpha_{j-2} \frac{(y_h - y_\ell) \frac{\partial Y}{\partial \omega} - (1 - Y)^2}{[Y(y_\ell - \omega) + \omega - y_h]^2}. \quad (\text{A.3})$$

Since $\partial Y / \partial \omega > 0$, (A.3) implies that $\partial B_{j-2} / \partial \omega < 0$. Therefore, B_{j-2} is increasing in L . When L^2 converges to RR_ℓ , and so ω converges to y_ℓ , (A.2) implies that B_{j-2} converges to $-\infty$. When L^2 converges to RR_h , and so ω converges to y_h , (A.2) implies that B_{j-2} converges to ∞ .

Using (3.3) and (3.4) we can similarly compute B_{j-1} :

$$B_{j-1} = R \alpha_{j-1} \frac{R_h - Z R_\ell}{Z(1 - \omega R_\ell) + \omega R_h - 1}, \quad (\text{A.4})$$

where

$$Z \equiv \left[\frac{p(\omega R_h - 1)}{(1-p)(1 - \omega R_\ell)} \right]^{\frac{1}{\gamma}}.$$

Differentiating (A.4) with respect to ω , we find

$$\frac{\partial B_{j-1}}{\partial \omega} = R\alpha_{j-1} \frac{(R_\ell - R_h) \frac{\partial Z}{\partial \omega} - (R_h - ZR_\ell)^2}{[Z(1 - \omega R_\ell) + \omega R_h - 1]^2}. \quad (\text{A.5})$$

Since $\partial Z/\partial \omega > 0$, (A.3) implies that $\partial B_{j-1}/\partial \omega < 0$. Therefore, B_{j-1} is increasing in L . When L^2 converges to RR_ℓ , and so ωR_ℓ converges to one, (A.2) implies that B_{j-1} converges to $-\infty$. When L^2 converges to RR_h , and so ωR_h converges to one, (A.2) implies that B_{j-1} converges to ∞ . Since $B_{j-2} + B_{j-1}$ is increasing in L , and takes values from $-\infty$ to ∞ , (3.5) has a unique solution L . ■

Proof of Proposition 3.2: An increase in B holding $b_{j-1}/R + B/L^2$ constant has no effect on tax rates. Therefore, (3.1)-(3.4) imply that the only effect on (B_{j-2}, B_{j-1}) is through L . Since $B_{j-2} + B_{j-1}$ is increasing in L , (3.5) implies that L is increasing in B . ■

Proof of Proposition 3.3: We first show that an increase in the size of the long-horizon clientele, i.e., an increase in α_{j-1} holding $R\alpha_{j-2} + \alpha_{j-1}$ constant, raises the aggregate demand for two-period bonds $B_{j-2} + B_{j-1}$ if $\gamma > 1$, and lowers it if $\gamma < 1$. Because of CRRA utility, the quantities

$$\begin{aligned} \phi_{j-2} &\equiv \frac{B_{j-2}/L^2}{R\alpha_{j-2}}, \\ \phi_{j-1} &\equiv \frac{B_{j-1}/L^2}{\alpha_{j-1}}, \end{aligned}$$

characterizing the fraction of wealth that generations $j-2$ and $j-1$ invest in two-period bonds in period $j-1$, are independent of α_{j-2} and α_{j-1} . Therefore, an increase in α_{j-1} holding $R\alpha_{j-2} + \alpha_{j-1}$ constant raises $B_{j-2} + B_{j-1}$ if and only if $\phi_{j-1} > \phi_{j-2}$. To compare ϕ_{j-1} and ϕ_{j-2} , we substitute (3.1) and (3.3) into (4.5), written for $t = j-1$. (Eq. (4.5) holds for generations $j-2$ and $j-1$ in Section 3, and for all generations $t \geq j-2$ in Section 4.1.) Because of CRRA utility,

$$\frac{1 + \phi_{j-2}(\frac{L^2}{RR_h} - 1)}{1 + \phi_{j-2}(\frac{L^2}{RR_\ell} - 1)} = \left(\frac{R_h}{R_\ell}\right)^{1-\frac{1}{\gamma}} \frac{1 + \phi_{j-1}(\frac{L^2}{RR_h} - 1)}{1 + \phi_{j-1}(\frac{L^2}{RR_\ell} - 1)}.$$

Therefore, the quantity

$$\frac{1 + \phi_{j-2}(\frac{L^2}{RR_h} - 1)}{1 + \phi_{j-2}(\frac{L^2}{RR_\ell} - 1)} \frac{1 + \phi_{j-1}(\frac{L^2}{RR_\ell} - 1)}{1 + \phi_{j-1}(\frac{L^2}{RR_h} - 1)} - 1 = \frac{(\phi_{j-1} - \phi_{j-2}) \left(\frac{L^2}{RR_\ell} - \frac{L^2}{RR_h} \right)}{\left[1 + \phi_{j-2}(\frac{L^2}{RR_\ell} - 1) \right] \left[1 + \phi_{j-1}(\frac{L^2}{RR_h} - 1) \right]}$$

has the same sign as

$$\left(\frac{R_h}{R_\ell}\right)^{1-\frac{1}{\gamma}} - 1.$$

Since $R_h > R_\ell$, the sign of $\phi_{j-1} - \phi_{j-2}$ is the same as of $\gamma - 1$. Since $B_{j-2} + B_{j-1}$ is increasing in L , (3.5) implies that an increase in α_{j-1} holding $R\alpha_{j-2} + \alpha_{j-1}$ constant lowers L if and only if it raises $B_{j-2} + B_{j-1}$, i.e., if and only if $\gamma > 1$. ■

Proof of Proposition 4.1: Eqs. (3.3), (3.4), (4.1), (4.3) for $t = j$, and CRRA utility imply that

$$\frac{B_j}{\alpha_j} = \frac{B_{j-1}}{R\alpha_{j-1}}. \quad (\text{A.6})$$

Therefore, B_j has the properties of B_{j-1} shown in the proof of Proposition 3.1: it is increasing in L , converges to $-\infty$ when L^2 converges to RR_ℓ , and to ∞ when L^2 converges to RR_h . Using (4.2) and (4.3), and proceeding as in the proof of Proposition 3.1, we can compute B_t for $t > j$:

$$B_{j-1} = R^{-(t-1-j)} \alpha_t \frac{1 - Z}{Z(1 - \omega R_\ell) + \omega R_h - 1}, \quad (\text{A.7})$$

where Z is defined in the proof of Proposition 3.1. The demand function B_t is not always increasing in L , but converges to $-\infty$ when L^2 converges to RR_ℓ , and to ∞ when L^2 converges to RR_h . Since $\sum_{j=t-2}^{\infty} B_t$ takes values from $-\infty$ to ∞ , (4.4) has a solution L . ■

Proof of Proposition 4.2: Since utility is CRRA, we can write (4.5) as

$$\frac{c_{j,h}}{c_{j,\ell}} = \left(\frac{R_h}{R_\ell}\right)^{-\frac{1}{\gamma}} \frac{c_{t+2,h}}{c_{t+2,\ell}}. \quad (\text{A.8})$$

Eq. (3.1) implies that

$$\frac{c_{j,h}}{c_{j,\ell}} = \frac{R^2 \alpha_{j-2} + \left(\frac{1}{R_h} - \frac{R}{L^2}\right) B_{j-2}}{R^2 \alpha_{j-2} + \left(\frac{1}{R_\ell} - \frac{R}{L^2}\right) B_{j-2}}. \quad (\text{A.9})$$

Moreover,

$$\begin{aligned} \frac{c_{t+2,h}}{c_{t+2,\ell}} &= \frac{\sum_{t=j-1}^{\infty} \frac{c_{t+2,h}}{R^{t+1-j}}}{\sum_{t=j-1}^{\infty} \frac{c_{t+2,\ell}}{R^{t+1-j}}} \\ &= \frac{RR_h\alpha_{j-1} + R_h\alpha_j + \sum_{t=j+1}^{\infty} \frac{\alpha_t}{R^{t+1-j}} - \left(1 - \frac{RR_h}{L^2}\right) B_{j-2}}{RR_\ell\alpha_{j-1} + R_\ell\alpha_j + \sum_{t=j+1}^{\infty} \frac{\alpha_t}{R^{t+1-j}} - \left(1 - \frac{RR_\ell}{L^2}\right) B_{j-2}}, \end{aligned} \quad (\text{A.10})$$

where the first step follows because $\frac{c_{t+2,h}}{c_{t+2,\ell}}$ is identical for all $t \geq j-1$, and the second from (3.3), (4.1), (4.2) and (4.4).

If $B_{j-2} \geq 0$, then (A.10) implies that $\frac{c_{t+2,h}}{c_{t+2,\ell}} > 1$ for $t \geq j-1$. If $B_{j-2} < 0$, then (A.9) implies that $\frac{c_{j,h}}{c_{j,\ell}} > 1$, and hence (A.8) implies that $\frac{c_{t+2,h}}{c_{t+2,\ell}} > 1$ for $t \geq j-1$.

Since $\frac{c_{t+2,h}}{c_{t+2,\ell}} > 1$ for $t \geq j-1$, (A.8) implies that $\frac{c_{j,h}}{c_{j,\ell}} > 1$ for γ sufficiently high. Suppose next that $\gamma \leq 1$. If $B_{j-2} > 0$, then (A.9) implies that $\frac{c_{j,h}}{c_{j,\ell}} < 1$. If $B_{j-2} \leq 0$, then we can use (A.10) to write the right-hand side of (A.8) as

$$\left(\frac{R_h}{R_\ell}\right)^{1-\frac{1}{\gamma}} \frac{\frac{c_{t+2,h}}{R_h}}{\frac{c_{t+2,\ell}}{R_\ell}} = \left(\frac{R_h}{R_\ell}\right)^{1-\frac{1}{\gamma}} \frac{R\alpha_{j-1} + \alpha_j + \frac{1}{R_h} \sum_{t=j+1}^{\infty} \frac{\alpha_t}{R^{t+1-j}} - \left(\frac{1}{R_h} - \frac{R}{L^2}\right) B_{j-2}}{R\alpha_{j-1} + \alpha_j + \frac{1}{R_\ell} \sum_{t=j+1}^{\infty} \frac{\alpha_t}{R^{t+1-j}} - \left(\frac{1}{R_\ell} - \frac{R}{L^2}\right) B_{j-2}}.$$

Since this is smaller than one, (A.8) implies that $\frac{c_{j,h}}{c_{j,\ell}} < 1$. ■

Proof of Corollary 4.1: Eqs. (3.4) and $c_{j+1,h} > c_{j+1,\ell}$ (which follows from Proposition 4.2) imply that $L^2 < \mathbb{E}(RR_s)$. If γ is high, then (3.2) and $c_{j,h} > c_{j,\ell}$ imply that $\mathbb{E}\left(\frac{L^2}{R_s}\right) < R$. If $\gamma \leq 1$, then (3.2) and $c_{j,h} < c_{j,\ell}$ imply that $\mathbb{E}\left(\frac{L^2}{R_s}\right) > R$. ■

Proof of Corollary 4.2: Eqs. (4.2) and $c_{t+2,h} > c_{t+2,\ell}$ imply that $B_t < 0$ for $t > j$. If γ is high, then (3.1) and $c_{j,h} > c_{j,\ell}$ imply that $B_{j-2} < 0$. If $\gamma \leq 1$, then (3.1) and $c_{j,h} < c_{j,\ell}$ imply that $B_{j-2} > 0$. Suppose finally that $\gamma \geq 1$ and one of (B_{j-1}, B_j) is non-positive. Eq. (A.6) implies that both of (B_{j-1}, B_j) are non-positive. Since $\gamma \geq 1$, generation $j-2$ invests a smaller fraction of its wealth in two-period bonds than generation $j-1$ (proof of Proposition 3.3), and hence $B_{j-2} \leq 0$. The non-positivity of (B_{j-2}, B_{j-1}, B_j) , together with the negativity of B_t for all $t > j$ is inconsistent with (4.4), a contradiction. Therefore, both of (B_{j-1}, B_j) are positive. ■

Proof of Proposition 4.3: Suppose that the government issues a quantity B^* of two-period

bonds. Since the demand functions of generations $j - 2$ and $j - 1$ are the same as under complete participation, the two-period interest rate is also the same. As a consequence, each of generations $j - 2$ and $j - 1$ has the same investment in two-period bonds and the same consumption as under complete participation. Generations $t \geq j$ can also have the same consumption as under complete participation through an appropriate choice of state-contingent taxes that we denote by $\{\tau_{t,h}^*, \tau_{t,\ell}^*\}_{t \geq j}$. Since the budget constraint of generation j under incomplete participation is

$$c_{j+2,s} = RR_s \alpha_j (1 - \tau_{j,s}), \quad (\text{A.11})$$

(4.1) implies that the taxes that leave that generation with the same consumption as under complete participation are given by

$$\begin{aligned} RR_s \alpha_j + R(L^2 - RR_s) \frac{B_j}{L^2} &= RR_s \alpha_j (1 - \tau_{j,s}^*) \\ \Leftrightarrow \alpha_j \tau_{j,s}^* &= \left(\frac{R}{L^2} - \frac{1}{R_s} \right) B_j. \end{aligned} \quad (\text{A.12})$$

Likewise, since the budget constraint of generation $t > j$ under incomplete participation is

$$c_{t+2,s} = R^2 \alpha_t (1 - \tau_{t,s}), \quad (\text{A.13})$$

(4.2) implies that the taxes that leave that generation with the same consumption as under complete participation are given by

$$\begin{aligned} R^2 \alpha_t + R^{t+1-j} (L^2 - RR_s) \frac{B_t}{L^2} &= R^2 \alpha_t (1 - \tau_{t,s}^*) \\ \Leftrightarrow \alpha_t \tau_{t,s}^* &= R^{t-1-j} R_s \left(\frac{R}{L^2} - \frac{1}{R_s} \right) B_t. \end{aligned} \quad (\text{A.14})$$

The policy $(B^*, \{\tau_{t,h}^*, \tau_{t,\ell}^*\}_{t \geq j})$ meets the government's budget constraints (2.2)-(2.4) if it meets the government's intertemporal budget constraint (4.9). Eqs. (A.12), (A.14) and $\sum_{t=j}^{\infty} B_t = -B^*$ (which follows from (4.4)) imply that (4.9) is met. \blacksquare

Proof of Proposition 4.4: We first note that any admissible policy $(B, \{\tau_{t,h}, \tau_{t,\ell}\}_{t \geq j})$ must satisfy the resource constraint

$$c_{j,s} + \frac{1}{R_s} \sum_{t=j-1}^{\infty} \frac{c_{t+2,s}}{R^{t+1-j}} = R^2 \alpha_{j-2} + R \alpha_{j-1} + \alpha_j + \frac{1}{R_s} \sum_{t=j+1}^{\infty} \frac{\alpha_t}{R^{t-1-j}}. \quad (\text{A.15})$$

Eq. (A.15) follows by combining (3.1), (3.3), (3.5), (4.9), (A.11) and (A.13).

We next show (formalizing the argument that precedes the statement of the proposition) that B^* is an optimal maturity structure. We proceed by contradiction, and suppose that there exists an admissible policy $(\hat{B}, \{\hat{\tau}_{t,h}, \hat{\tau}_{t,\ell}\}_{t \geq j})$ to which a switch from $(B^*, \{\tau_{t,h}^*, \tau_{t,\ell}^*\}_{t \geq j})$ generates a positive sum $\sum_{t=j-2}^{\infty} T_t$ of gains. Denote by $\hat{c}_{t+2,s}$ and $c_{t+2,s}^*$ the consumption of generations $t \geq j-2$ in state $s = h, \ell$ in the corresponding equilibria. The allocation $\{\check{c}_{t+2,h}, \check{c}_{t+2,\ell}\}_{t \geq j-2}$ defined by

$$\check{c}_{j,s} \equiv \hat{c}_{j,s} + R \sum_{t=j-1}^{\infty} T_t, \quad (\text{A.16})$$

$$\check{c}_{t+2,s} \equiv \hat{c}_{t+2,s} - R^{t+2-j} R_s T_t \quad \text{for } t \geq j-1, \quad (\text{A.17})$$

yields the same utility as $\{c_{t+2,h}^*, c_{t+2,\ell}^*\}_{t \geq j-2}$ for generations $t \geq j-1$ because of (4.11), and higher utility for generation $j-2$ because of (4.10) and $\sum_{t=j-2}^{\infty} T_t > 0$. This is a contradiction because $\{c_{t+2,h}^*, c_{t+2,\ell}^*\}_{t \geq j-2}$ is Pareto optimal among all feasible allocations, i.e., in the Pareto frontier of \mathcal{A}_0 . To show Pareto optimality and the contradiction, we denote the risk-neutral probabilities for states $s = h, \ell$ under complete participation by

$$\pi_h \equiv \frac{\frac{L^2}{R_\ell} - R}{\frac{L^2}{R_\ell} - \frac{L^2}{R_h}},$$

$$\pi_\ell \equiv 1 - \pi_h,$$

respectively. We also use (3.1), (3.3), (4.1) and (4.2) to write the optimization problem of generation $t \geq j-2$ under complete participation as maximizing utility within the budget set \mathcal{B}_t , where

$$\begin{aligned} \mathcal{B}_{j-2} &\equiv \left\{ \{c_{j,h}, c_{j,\ell}\} : \pi_h c_{j,h} + \pi_\ell c_{j,\ell} \leq R^2 \alpha_{j-2} \right\}, \\ \mathcal{B}_{j-1} &\equiv \left\{ \{c_{j+1,h}, c_{j+1,\ell}\} : \pi_h \frac{c_{j+1,h}}{R_h} + \pi_\ell \frac{c_{j+1,\ell}}{R_\ell} \leq R \alpha_{j-1} \right\}, \\ \mathcal{B}_j &\equiv \left\{ \{c_{j+2,h}, c_{j+2,\ell}\} : \pi_h \frac{c_{j+2,h}}{R_h} + \pi_\ell \frac{c_{j+2,\ell}}{R_\ell} \leq R \alpha_j \right\}, \\ \mathcal{B}_t &\equiv \left\{ \{c_{t+2,h}, c_{t+2,\ell}\} : \pi_h \frac{c_{t+2,h}}{R_h} + \pi_\ell \frac{c_{t+2,\ell}}{R_\ell} \leq R^2 \alpha_t \left(\frac{\pi_h}{R_h} + \frac{\pi_\ell}{R_\ell} \right) \right\} \quad \text{for } t > j. \end{aligned}$$

Since $\{c_{j,h}^*, c_{j,\ell}^*\}$ maximizes the utility of generation $j-2$ within \mathcal{B}_{j-2} , $\{\check{c}_{j,h}, \check{c}_{j,\ell}\}$, which yields higher utility, is outside \mathcal{B}_{j-2} . Likewise, since $\{c_{t+2,h}^*, c_{t+2,\ell}^*\}$ maximizes the expected utility of generation

$t \geq j - 1$ within \mathcal{B}_t , $\{\check{c}_{t+2,h}, \check{c}_{t+2,\ell}\}$, which yields the same utility, is weakly outside \mathcal{B}_t . Combining the corresponding inequalities, we find

$$\sum_{s=h,\ell} \pi_s \left(\check{c}_{j,s} + \frac{1}{R_s} \sum_{t=j-1}^{\infty} \frac{\check{c}_{t+2,s}}{R^{t+1-j}} \right) > \sum_{s=h,\ell} \pi_s \left(R^2 \alpha_{j-2} + R \alpha_{j-1} + \alpha_j + \frac{1}{R_s} \sum_{t=j+1}^{\infty} \frac{\alpha_t}{R^{t-1-j}} \right). \quad (\text{A.18})$$

Eq. (A.18) contradicts the resource constraint (A.15). Indeed, (A.15) holds for $\{\hat{c}_{t+2,h}, \hat{c}_{t+2,\ell}\}_{t \geq j-2}$, since this allocation corresponds to the admissible policy $(\hat{B}, \{\hat{\tau}_{t,h}, \hat{\tau}_{t,\ell}\}_{t \geq j})$. Moreover, because of (A.16) and (A.17), (A.15) holds also for $\{\check{c}_{t+2,h}, \check{c}_{t+2,\ell}\}_{t \geq j-2}$.

We finally show that B^* is the unique optimal maturity structure. Consider an optimal maturity structure B and an associated admissible policy $(B, \{\tau_{t,h}, \tau_{t,\ell}\}_{t \geq j})$. The ratio across states of marginal utilities must be equal across all generations $t \geq j$; otherwise, it would be possible to generate a positive sum of gains by reallocating taxes across two generations so that one is kept equally well off and the other becomes better off. Consider next an infinitesimal change $(dB, \{d\tau_{t,h}, d\tau_{t,\ell}\}_{t \geq j})$ that preserves admissibility. Eqs. (4.10) and (4.11) imply that

$$dT_{j-2} = \frac{\mathbb{E}[u'(c_{j,s})dc_{j,s}]}{R\mathbb{E}u'(c_{j,s})}, \quad (\text{A.19})$$

$$dT_t = \frac{\mathbb{E}[u'(c_{t+2,s})dc_{t+2,s}]}{R^{t+2-j}\mathbb{E}[u'(c_{t+2,s})R_s]} \quad \text{for } t \geq j-1. \quad (\text{A.20})$$

Using (3.1) and (A.19), and recalling the definition $\omega \equiv R/L^2$, we find

$$dT_{j-2} = \frac{\mathbb{E}\left\{u'(c_{j,s}) \left[\left(\frac{1}{R_s} - \omega\right) dB_{j-2} - d\omega B_{j-2}\right]\right\}}{R\mathbb{E}u'(c_{j,s})} = -\frac{d\omega B_{j-2}}{R}, \quad (\text{A.21})$$

where the second step follows from (3.2). Using (3.3) and (A.20) for $t = j - 1$, we find

$$dT_{j-1} = \frac{\mathbb{E}\{u'(c_{j+1,s}) [(1 - \omega R_s) dB_{j-1} - d\omega R_s B_{j-1}]\}}{R\mathbb{E}[u'(c_{j+1,s})R_s]} = -\frac{d\omega B_{j-1}}{R}, \quad (\text{A.22})$$

where the second step follows from (3.4). Using (A.11) and (A.20) for $t = j$, we find

$$dT_j = -\frac{\mathbb{E}[u'(c_{j+2,s})\alpha_j R_s d\tau_{j,s}]}{R\mathbb{E}[u'(c_{j+2,s})R_s]}, \quad (\text{A.23})$$

and using (A.13) and (A.20) for $t > j$, we find

$$dT_t = -\frac{\mathbb{E}[u'(c_{t+2,s})\alpha_t d\tau_{t,s}]}{R^{t-j}\mathbb{E}[u'(c_{t+2,s})R_s]}. \quad (\text{A.24})$$

Since B is an optimal maturity structure, the sum $\sum_{t=j-2}^{\infty} dT_t$ should be zero; if it were negative, the opposite infinitesimal change would make it positive. Therefore, (A.21)-(A.24) imply that

$$\begin{aligned} & -\frac{d\omega(B_{j-2} + B_{j-1})}{R} - \frac{\mathbb{E}[u'(c_{j+2,s})\alpha_j R_s d\tau_{j,s}]}{R\mathbb{E}[u'(c_{j+2,s})R_s]} - \sum_{t=j+1}^{\infty} \frac{\mathbb{E}[u'(c_{t+2,s})\alpha_t d\tau_{t,s}]}{R^{t-j}\mathbb{E}[u'(c_{t+2,s})R_s]} = 0 \\ \Leftrightarrow & -\frac{d\omega B}{R} - \frac{\mathbb{E}\left[u'(c_{t+2,s})R_s \left[\alpha_j d\tau_{j,s} + \frac{1}{R_s} \sum_{t=j+1}^{\infty} \frac{\alpha_t d\tau_{t,s}}{R^{t-1-j}}\right]\right]}{R\mathbb{E}[u'(c_{t+2,s})R_s]}, \end{aligned} \quad (\text{A.25})$$

for all $t \geq j$, where the second step follows from (3.5) and because the ratio across states of marginal utilities is equal across all generations $t \geq j$. Differentiating the government's intertemporal budget constraint (4.9), we find

$$\alpha_j d\tau_{j,s} + \frac{1}{R_s} \sum_{t=j+1}^{\infty} \frac{\alpha_t d\tau_{t,s}}{R^{t-1-j}} = \left(\frac{1}{R_s} - \omega\right) dB - \omega dB. \quad (\text{A.26})$$

Substituting (A.26) into (A.25), we find

$$\mathbb{E}[u'(c_{t+2,s})(1 - \omega R_s)] = 0 \quad (\text{A.27})$$

for all $t \geq j$. Eq. (A.27) can be interpreted as a first-order condition for generations $t \geq j$ under complete participation. To make this interpretation precise, we consider a representative agent who has CRRA utility with coefficient of relative risk aversion γ , and consumes in period $j+2$ the present value of the consumption stream of generations $t \geq j$:

$$C_s \equiv \sum_{t \geq j} \frac{c_{t+2,s}}{R^{t-j}} = R R_s \left(\alpha_j + \frac{1}{R_s} \sum_{t=j+1}^{\infty} \frac{\alpha_t}{R^{t-1-j}} \right) - R(1 - \omega R_s)B, \quad (\text{A.28})$$

where the second step in (A.28) follows from (4.9), (A.11) and (A.13). Because of CRRA utility, the common ratio across states of marginal utilities of generations $t \geq j$ is equal to the ratio of marginal utilities of the representative agent. Therefore, the first-order condition (A.27) holds for

the representative agent as

$$\mathbb{E} [u'(C_s)(1 - \omega R_s)] = 0,$$

and implies that if the representative agent were to invest in one- and two-period bonds in period $j - 1$ subject to the budget constraint

$$C_s = RR_s \left(\alpha_j + \frac{1}{R_s} \sum_{t=j+1}^{\infty} \frac{\alpha_t}{R^{t-1-j}} \right) + R(1 - \omega R_s) \hat{B}, \quad (\text{A.29})$$

then he would choose a quantity $\hat{B} = -B$ of two-period bonds. As a consequence, the two-period interest rate L clears the market with complete participation by generations $t \geq j - 2$, i.e., in the equilibrium of Section 4.1. Indeed, (4.3) implies (A.27), and (4.1) and (4.2) imply that (A.28) holds with $\sum_{t=j}^{\infty} B_t$ instead of $-B$. Since the representative agent's demand for two-period bonds is uniquely determined, $\sum_{t=j}^{\infty} B_t = -B \equiv -(B_{j-2} + B_{j-1})$. Therefore, (4.4) is met, and $B = B^*$. ■

Proof of Proposition 5.1: The proposition follows from Corollary 4.1 and the fact that consumption under the optimal maturity structure is the same as under complete participation (Propositions 4.3 and 4.4). ■

Proof of Proposition 5.2: We show the properties of B^* using those of consumption $c_{t,s}$ and demand B_t under complete participation. To show that $B^* > 0$ for $\gamma \leq 1$, we proceed by contradiction. If $B^* \leq 0$, then $B_{j-1} \leq 0$. Indeed, proceeding again by contradiction, suppose that $B_{j-1} > 0$. Since $\gamma \leq 1$, generation $j - 2$ invests a larger fraction of its wealth in two-period bonds than generation $j - 1$ (proof of Proposition 3.3). Therefore, $B_{j-2} > 0$ and $B^* \equiv B_{j-2} + B_{j-1} > 0$, a contradiction. Since $B_{j-1} \leq 0$, (A.6) implies that $B_j \leq 0$. Since, however, $B_t < 0$ for $t > j$ (Corollary 4.2), (4.4) is violated, a contradiction. Therefore, $B^* > 0$.

To show the properties of B^* for high γ , we consider the limit cases where α_{j-2} and α_{j-1} are equal to zero. When $\alpha_{j-1} = 0$, the argument in the proof of Proposition 4.2 that establishes $\frac{c_{j,h}}{c_{j,\ell}} > 1$ carries through, and implies that $B_{j-2} < 0$. Since, in addition, $B_{j-1} = 0$, $B^* \equiv B_{j-2} + B_{j-1} < 0$, and by continuity $B^* < 0$ for small positive α_{j-1} . When $\alpha_{j-2} = 0$, the argument in the proof of Proposition 4.2 that establishes $\frac{c_{t+2,h}}{c_{t+2,\ell}} > 1$ for $t \geq j - 1$ carries through (and is even simpler because $B_{j-2} = 0$). Therefore, the argument in the proof of Corollary 4.2 that establishes $B_{j-1} > 0$ also carries through. Since, in addition, $B_{j-2} = 0$, $B^* \equiv B_{j-2} + B_{j-1} > 0$, and by continuity $B^* > 0$ for small positive α_{j-2} . ■

Proof of Proposition 5.3: We consider a change from α_{j-1} to $\alpha_{j-1} + \epsilon$ holding $R\alpha_{j-2} + \alpha_{j-1}$ constant. Differentiating (4.4) with respect to ϵ and treating L as a function of ϵ , we find

$$\begin{aligned} \frac{\partial}{\partial \epsilon}(B_{j-2} + B_{j-1}) + \frac{dL}{d\epsilon} \frac{\partial}{\partial L} \sum_{t=j-2}^{\infty} B_t &= 0 \\ \Rightarrow \frac{dL}{d\epsilon} &= -\frac{\frac{\partial}{\partial \epsilon}(B_{j-2} + B_{j-1})}{\frac{\partial}{\partial L} \sum_{t=j-2}^{\infty} B_t}, \end{aligned} \quad (\text{A.30})$$

where the first step follows because ϵ does not affect B_t for $t \geq j$. The numerator in (A.30) has the same sign as $\gamma - 1$, as shown in the proof of Proposition 3.3. The denominator is positive because the demand functions B_t for $t \geq j - 2$ are increasing in L : for $t \in \{j - 2, j - 1\}$ this is shown in the proof of Proposition 3.1, for $t = j$ it is shown in the proof of Proposition 4.1, and for $t > j$ it follows from Assumption 1. Therefore, $\frac{dL}{d\epsilon}$ has the same sign as $1 - \gamma$.

Differentiating $B_{j-2} + B_{j-1}$ with respect to ϵ and treating L as a function of ϵ , we find

$$\begin{aligned} \frac{d}{d\epsilon}(B_{j-2} + B_{j-1}) &= \frac{\partial}{\partial \epsilon}(B_{j-2} + B_{j-1}) + \frac{dL}{d\epsilon} \frac{\partial}{\partial L}(B_{j-2} + B_{j-1}) \\ &= \frac{\partial}{\partial \epsilon}(B_{j-2} + B_{j-1}) \frac{\frac{\partial}{\partial L} \sum_{t=j}^{\infty} B_t}{\frac{\partial}{\partial L} \sum_{t=j-2}^{\infty} B_t}, \end{aligned} \quad (\text{A.31})$$

where the second step follows from (A.30). The numerator and denominator in (A.31) are positive because the demand functions B_t for $t \geq j - 2$ are increasing in L . Therefore, $\frac{d}{d\epsilon}(B_{j-2} + B_{j-1})$ has the same sign as $\frac{\partial}{\partial \epsilon}(B_{j-2} + B_{j-1})$ and hence as $\gamma - 1$. The comparative statics in the proposition follow from $B^* = B_{j-2} + B_{j-1}$, and because the equilibrium two-period interest rate L under incomplete participation and B^* coincides with that under complete participation. \blacksquare

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