ISSN 0956-8549-658

Value of Information in Competitive Economies with Incomplete Markets

By

Piero Gottardi
Rohit Rahi

DISCUSSION PAPER 658

LSE FINANCIAL MARKETS GROUP

DISCUSSION PAPER SERIES

August 2010

Piero Gottardi is the Professor of Economics at the European University Institute. Since September 2008, he has been on leave from Department of Economics at Universita' di Venezia. His research interests include General Equilibrium Theory, Financial Economics; and Competitive Equilibrium Models with Asymmetric Information. Rohit Rahi is a Reader in Finance and Economics with the Department of Finance at the London School of Economics and Political Science (LSE). His research interests are Arbitrage in Segmented Markets; Financial Innovation and Security Design; Asset Pricing with Asymmetric Information; General Equilibrium Theory; and Incomplete Markets.

Value of Information in Competitive Economies with Incomplete Markets*

by

Piero Gottardi

Department of Economics

European University Institute

Villa San Paolo

Via della Piazzuola 43

50133 Florence, Italy

piero.gottardi@eui.eu

http://www.eui.eu/Personal/Gottardi/

and

Rohit Rahi

Department of Finance

Department of Economics

and Financial Markets Group

London School of Economics

Houghton Street

London WC2A 2AE, U.K.

r.rahi@lse.ac.uk

http://vishnu.lse.ac.uk/

August 19, 2010.

*We would like to thank Alex Citanna, Andy Postlewaite, Antonio Villanacci, and especially Atsushi Kajii and Tito Pietra, for helpful comments.

Abstract

We study the value of information in a competitive economy in which agents trade in asset markets to reallocate risk. We characterize the kinds of information that allow a welfare improvement when portfolios can be freely reallocated. We then compare competitive equilibria before and after a change in information. We show that generically, if markets are sufficiently incomplete, the welfare effects are completely arbitrary: there typically exist changes in information that make all agents better off, or all agents worse off.

Journal of Economic Literature Classification Numbers: D52, D60, D80.

Keywords: Competitive Equilibrium, Incomplete Markets, Value of Information.

1 Introduction

The objective of this paper is to analyze the value of information in the setup of a competitive exchange economy under uncertainty in which agents trade in asset markets to reallocate risk. It is well known that while information cannot reduce welfare in a single-agent decision-making context, this is not necessarily the case in a market setting. In a competitive economy with complete markets, the arrival of information prior to trading cannot improve upon the equilibrium allocation of risk. Such information can in fact impair risk sharing, and this is true whether or not markets are complete. Indeed, if the true state of the world is revealed before markets open, no mutually beneficial risk sharing trade is possible.

The negative effect on welfare of an increase in the information available to market participants has come to be known as the *Hirshleifer effect*, after Hirshleifer (1971) who produced an early example of it. In general, the Hirshleifer effect is due to changes in equilibrium prices, induced by a change in information, that alter the budget sets of agents (see Gottardi and Rahi (2001)).

If markets are incomplete, a second welfare effect arises. With additional information agents can achieve a larger set of state-contingent payoffs by conditioning their portfolios on this information. We refer to this as the *Blackwell effect*, after Blackwell (1951) who compared the value of different information structures in single-agent decision problems. Roughly speaking, we can think of the value of information in a competitive market economy as having a negative component due to the Hirshleifer effect, and a positive component due to the Blackwell effect.

There is an extensive literature on the value of information in a competitive pure exchange setting. A long line of papers has followed Hirshleifer's lead in comparing competitive equilibrium allocations associated with differing levels of information. Assuming complete markets, Schlee (2001) derives conditions under which more information is Pareto worsening. Green (1981) and Hakansson et al. (1982) provide (quite restrictive) conditions under which better information leads to a Pareto improvement when markets are incomplete. Milne and Shefrin (1987) show, by way of examples, that better information can lead to any pattern of welfare changes in an incomplete markets economy.

In this paper we provide a general characterization of the effect of changes in information on welfare, starting from a competitive equilibrium allocation, in the context of a two-period exchange economy with a single consumption good and a single round of asset trade. We provide two sets of results, the first in a situation where a hypothetical planner can freely reallocate portfolios after a change in information, and the second where we consider only equilibrium allocations corresponding to the new information. We refer to these as results relating to feasible changes in welfare and equilibrium changes in welfare, respectively.

Feasible welfare changes can be attributed entirely to the Blackwell effect. Since prices do not constrain attainable allocations, there is no Hirshleifer effect. Just as in a single-agent decision problem, information cannot reduce welfare, as it can

simply be disregarded. We characterize the set of informational changes for which a Pareto improvement can be achieved. In doing so, we obtain a characterization of risk sharing with incomplete markets that is of independent interest. The state space can be partitioned into a collection of "insurable events" such that, in equilibrium for a generic economy, all gains from trade in state-contingent consumption are exhausted between these events, but not within them. While a change in information that affects only the relative probabilities of insurable events has no value, a Pareto improvement can typically be attained for informational changes that alter the relative probabilities of states within an insurable event.

The information structures that allow a Pareto improvement in our model with a single round of trade are precisely those that lead to retrade in a setting where asset markets open both before and after the change in information under consideration. As such our results are related to those of Blume et al. (2006). In particular, one of our results is a generalization of their main theorem (see Section 4 for further details).

Next we consider equilibrium welfare changes, comparing agents' welfare at a competitive equilibrium before and after a change in information. We show that generically, if markets are sufficiently incomplete, equilibrium welfare effects are completely arbitrary: there typically exist informational changes that make all agents better off, or all agents worse off, or indeed any subset of agents better (or worse) off. Thus pecuniary externalities arising from price changes can outweigh the value that a change in information might otherwise have for any individual agent. To put it differently, when both the Hirshleifer and Blackwell effects are present, the net effect can go in any direction.

Our welfare analysis is in the spirit of the literature on constrained inefficiency in an incomplete markets economy, where welfare comparisons are made between competitive equilibrium allocations and allocations attainable subject to appropriately specified constraints. Diamond (1967) allows arbitrary reallocations of portfolios and shows that competitive equilibria are constrained efficient. In our results on feasible welfare changes, we identify conditions under which a welfare improvement can be achieved when information is modified as well as portfolios. Geanakoplos and Polemarchakis (1986) and Greenwald and Stiglitz (1986) consider the effect of portfolio reallocations on spot commodity prices, and establish a constrained inefficiency result; the first paper shows that constrained inefficiency is, in fact, a generic property. In order to establish our equilibrium welfare result, we utilize the analytical apparatus developed by Geanakoplos and Polemarchakis (1986), and later generalized by Citanna et al. (1998). This approach has been employed in several papers in the incomplete markets literature including, in particular, Cass and Citanna (1998) and Elul (1995), who show that generically the welfare effects of the introduction of a new asset are arbitrary. We provide such a result with respect to changes in public information.[1]

[1]We are able to exploit differential techniques by employing a smooth parametrization of changes in information, just as Cass and Citanna (1998) and Elul (1995) are able to use these techniques

The closest result to ours on equilibrium welfare changes is in Citanna and Villanacci (2000), who study the value of information in an asymmetric information economy with nominal assets. In their model, there is a continuum of equilibria which can be parametrized by the (state-contingent) price level. The authors show that generically there is an arbitrary change in welfare when moving from a non-revealing equilibrium to a nearby equilibrium at which asset prices reveal some information. In contrast, we show that there is an arbitrary change in welfare when moving from any equilibrium of a generic economy to a nearby equilibrium associated with a new information structure. Also, the welfare effects in Citanna and Villanacci (2000) involve changes in real asset payoffs (through changes in the price level), in asset prices, and in relative spot commodity prices, while in our one-good model, asset payoffs and spot commodity prices are fixed.

The paper is organized as follows. We describe the economy in Section 2, and analyze competitive equilibria in Section 3. In Section 4, we consider feasible welfare changes and characterize the set of (potentially) welfare-improving changes in information. In Section 5, we study equilibrium welfare effects, and show that they are typically arbitrary. Some of the more technical proofs are collected in the Appendix.

2 The Economy

There are two periods, 0 and 1, and a single physical consumption good. The economy is populated by $H \geq 2$ agents, with typical agent $h \in H$ (here, and elsewhere, we use the same symbol for a set and its cardinality). No consumption takes place at date 0 and agents have no endowment in that period. Uncertainty, which is resolved at date 1, is described by S states of the world.

Agent $h \in H$ has an endowment at date 1 given by $\omega^h \in \mathbb{R}^S_{++}$, and preferences over date 1 consumption described by a twice continuously differentiable von Neumann-Morgenstern utility function $u^h : \mathbb{R}_{++} \to \mathbb{R}$, satisfying $u^{h\prime} > 0$, $u^{h\prime\prime} < 0$, and $\lim_{c\to 0} u^{h\prime}[c] = \infty$. We denote the set of utility functions with these properties by $\mathcal{U}$.

Asset markets, in which $J \geq 2$ securities are traded, open at date 0. At date 1 assets pay off, and agents consume. The payoff of asset j in state s is denoted by r^j_s, and the vector of asset payoffs in state s by $r_s \in \mathbb{R}^J$. By default all vectors are column vectors, unless transposed. Thus $r_s^\top = (r^1_s \dots r^J_s)$. Let R be the $S \times J$ matrix whose s'th row is $r_s^\top$. We assume that $r_s \neq 0$ for all $s \in S$, and R has full column rank J. These assumptions are without loss of generality as the results depend only on the asset span (the column space of R), and states in which no asset pays off are irrelevant when considering the welfare effect of changes in information. We also assume that there is an asset, say asset J, whose payoff is nonnegative in every state. This condition, together with the monotonicity assumption on utility

by modeling the introduction of an asset in a way that avoids discontinuities. While the effect of increasing the information of agents on their trading possibilities has some analogies with the effect of introducing new securities, these are really two distinct problems (see Milne and Shefrin (1987)).

functions, ensures that the equilibrium price of asset J is positive. It also guarantees that budget constraints are satisfied with equality. Markets are complete if $S = J$, and incomplete if $S > J$.

We model the information of agents as a public signal, observed prior to trading, correlated with the state of the world s. This signal has support Σ, $\#\Sigma \geq 2$, with a typical element of Σ denoted by σ. Having fixed the spaces S and Σ, the information of agents can be completely described by the probabilities $\pi := \{\pi_{s\sigma}\}_{s\in S,\sigma\in\Sigma} \in \mathbb{R}^{S\Sigma}_{++}$, where $\pi_{s\sigma}$ denotes $\text{Prob}(s,\sigma)$. We will identify a signal by the vector π associated with it. The space of signals is thus $\Pi := \{\pi \in \mathbb{R}^{S\Sigma}_{++} \mid \sum_{s,\sigma} \pi_{s\sigma} = 1\}$. Let $\pi_{s|\sigma} := \text{Prob}(s|\sigma)$, $\pi_s := \text{Prob}(s)$, and $\pi_\sigma := \text{Prob}(\sigma)$. A signal π is *uninformative* about s if it satisfies the independence condition $\pi_{s\sigma} = \pi_s \pi_\sigma$, for all $s \in S$, $\sigma \in \Sigma$. We will often refer to a signal as an "information structure."

Let $\omega := \{\omega^h\}_{h\in H} \in \Omega := \mathbb{R}^{SH}_{++}$, and $u := \{u^h\}_{h\in H} \in \mathcal{U}^H$. An economy is described by the tuple $(\omega, u, \pi) \in \mathcal{E} := \Omega \times \mathcal{U}^H \times \Pi$. We formalize our notion of genericity as follows. The sets Ω and Π are endowed with the usual (Euclidean) topology. The set $\mathcal{U}$ is endowed with the $\mathcal{C}^2$ uniform convergence topology on compact sets, i.e. the sequence u^h_n in $\mathcal{U}$ converges to u^h if and only if $u^h_n, u^{h\prime}_n$ and $u^{h\prime\prime}_n$ converge uniformly to $u^h, u^{h\prime}$ and $u^{h\prime\prime}$ respectively, on any compact subset of $\mathbb{R}_{++}$. The set of economies $\mathcal{E}$ is endowed with the product topology. By "generic subset of Ω," we mean "for an open, dense subset of Ω," and likewise for $\mathcal{U}^H$ and Π. By "generically" we mean "for an open, dense subset of $\mathcal{E}$."[2]

Transversality. We use the transversality theorem to establish our genericity results. Since we employ the same argument at several different points in the paper, it is useful to summarize it here. Consider a function $\Psi : Z \times \mathcal{E} \to \mathbb{R}^{n+1}$, where Z is an open subset of $\mathbb{R}^n$. For $e \in \mathcal{E}$, let Ψ_e be the function $\Psi(\cdot, e)$. The argument involves identifying such a function Ψ, such that the desired result can be formulated as $\Psi_e^{-1}(0) = \varnothing$, for every e in a generic subset of $\mathcal{E}$. We show that the Jacobian $D_{z,e}\Psi$ has full row rank at all zeros of Ψ, i.e. Ψ is transverse to zero. By the transversality theorem, there is then a generic subset of $\mathcal{E}$ such that, for each e in this set, $\Psi_e : Z \to \mathbb{R}^{n+1}$ is transverse to zero.[3] It follows that $\Psi_e^{-1}(0) = \varnothing$. In other words, the equation system $\Psi_e(z) = 0$ has no solution since the number of (locally) independent equations exceeds the number of unknowns.

Notation. In our analysis we use the following shorthand notation for matrices. Given an index set $\mathcal{N}$ with typical element n, and a collection $\{z_n\}_{n\in\mathcal{N}}$ of vectors or matrices, we denote by $\text{diag}_{n\in\mathcal{N}}[z_n]$ the (block) diagonal matrix with typical entry z_n, where n varies across all elements of $\mathcal{N}$. For a given vector or matrix z, $\text{diag}_{n\in\mathcal{N}}[z]$ is then the diagonal matrix with the term z repeated $\#\mathcal{N}$ times. In similar fashion, we

[2] Only one of our main results, Theorem 5.1, requires perturbations of utility functions and the (initial) information structure. The other results hold for a generic subset of endowments, and for all $(u, \pi) \in \mathcal{U}^H \times \Pi$.

[3] Openness follows from a standard argument; see, for example, Citanna et al. (1998).

write $[\ldots z_n \ldots_{n\in\mathcal{N}}]$ to denote the row block with typical element z_n, and analogously for column blocks. We drop reference to the index set if it is obvious from the context: for example $\text{diag}_{h\in H}$ is shortened to diag_h, and $[\ldots z_s \ldots_{s\in S, s\neq s_1}]$ to $[\ldots z_s \ldots_{s\neq s_1}]$. We use the same symbol 0 for the zero scalar and the zero matrix; in the latter case we occasionally indicate the dimension in order to clarify the argument. We denote by I_N the $N \times N$ identity matrix, and by $\hat{I}$ the $(J-1)\times J$ matrix $(I_{J-1}\ 0)$. A "$*$" stands for any term whose value is immaterial to the analysis. The symbols $\sim_R$ and $\sim_C$ denote row and column equivalence, respectively.

We will sometimes need to order the set S (and similarly the sets Σ and H) as $\{s_1, s_2, \ldots\}$, s_1 being the first state, and so on.

3 Competitive Equilibrium

Consider an economy $(\omega, u, \pi) \in \mathcal{E}$. Let $y^h_\sigma \in \mathbb{R}^J$ denote the portfolio of agent h when the signal realization is σ. Since portfolios uniquely determine consumption (the consumption of agent h for state s and signal σ is given by $\omega^h_s + r_s \cdot y^h_\sigma$), an allocation is completely specified by a collection of portfolios, one for each agent h, and each signal σ. For each σ, asset prices are given by a vector $p_\sigma \in \mathbb{R}^J$.

Let $y_\sigma := \{y^h_\sigma\}_{h\in H}$, $y := \{y_\sigma\}_{\sigma\in\Sigma}$, and $p := \{p_\sigma\}_{\sigma\in\Sigma}$. A competitive equilibrium is defined as follows:

Definition 3.1 *Given an economy $(\omega, u, \pi) \in \mathcal{E}$, a competitive equilibrium consists of an allocation y, and prices p, satisfying the following two conditions:*

(a) *Agent optimization: $\forall h \in H$ and $\sigma \in \Sigma$, y^h_σ solves*

$$\begin{aligned} &\max\nolimits_{x\in\mathbb{R}^J} \textstyle\sum_s \pi_{s|\sigma}\, u^h\left[\omega^h_s + r_s \cdot x\right] \\ &\textit{subject to} \qquad p_\sigma \cdot x = 0. \end{aligned} \tag{1}$$

(b) *Market clearing: $\forall \sigma \in \Sigma$,*

$$\sum_h y^h_\sigma = 0. \tag{2}$$

We will often refer to an equilibrium (y, p) of the economy (ω, u, π) as a π-equilibrium in order to emphasize the signal structure under consideration. Since asset J has a nonnegative nonzero payoff, we can choose it as the numeraire, setting $(p_\sigma)_J = 1$. Let $\hat{p}_\sigma$ denote the vector consisting of the first $J-1$ elements of the price vector p_σ. Likewise, for given $\hat{p}_\sigma$, the corresponding p_σ is given by $\{\hat{p}_\sigma, 1\}$. Let $\hat{p} := \{\hat{p}_\sigma\}_{\sigma\in\Sigma}$.

The Kuhn-Tucker first-order conditions for the utility-maximization program (1) are:

$$\sum_s \pi_{s\sigma}\left(u^{h\prime}\left[\omega^h_s + r_s \cdot y^h_\sigma\right] r_s - \lambda^h_\sigma\, p_\sigma\right) = 0, \qquad \forall h \in H, \sigma \in \Sigma \tag{3}$$

$$p_\sigma \cdot y^h_\sigma = 0, \qquad \forall h \in H, \sigma \in \Sigma, \tag{4}$$

where λ^h_σ is the (positive) Lagrange multiplier associated with the budget constraint of agent h for signal σ. By Walras' law, the market-clearing equation for one asset is redundant, for each σ. Hence, the market-clearing condition (2) reduces to

$$\sum_h \hat{y}^h_\sigma = 0, \qquad \forall \sigma \in \Sigma, \tag{5}$$

where $\hat{y}^h_\sigma$ denotes the vector consisting of the first $J-1$ elements of the portfolio y^h_σ.

Let $\lambda_\sigma := \{\lambda^h_\sigma\}_{h\in H}$, and $\lambda := \{\lambda_\sigma\}_{\sigma\in\Sigma}$. A competitive equilibrium (y,p), together with the associated Lagrange multipliers λ, must satisfy the equation system (3)–(5). Indeed, (y,p) is a competitive equilibrium if and only if it satisfies the system (3)–(5) for some $\lambda \in \mathbb{R}^{H\Sigma}_{++}$. Let

$$\xi_\sigma := (y_\sigma, \hat{p}_\sigma, \lambda_\sigma) \in \mathbb{R}^{JH} \times \mathbb{R}^{J-1} \times \mathbb{R}^H_{++}$$

and

$$\begin{aligned} f_\sigma(\xi_\sigma) &:= \sum_s \pi_{s\sigma} \left(u^{h\prime}\left[\omega^h_s + r_s \cdot y^h_\sigma\right] r_s - \lambda^h_\sigma \, p_\sigma \right), \qquad \forall h \in H \\ g_\sigma(\xi_\sigma) &:= \begin{pmatrix} p_\sigma \cdot y^h_\sigma \\ \sum_h \hat{y}^h_\sigma \end{pmatrix}, \qquad h \in H. \end{aligned}$$

Then the equations that characterize a competitive equilibrium, (3)–(5), can be written as

$$F_\sigma(\xi_\sigma) := \begin{pmatrix} f_\sigma(\xi_\sigma) \\ g_\sigma(\xi_\sigma) \end{pmatrix} = 0, \qquad \forall \sigma \in \Sigma, \tag{6}$$

or more compactly as

$$F(\xi) := \begin{pmatrix} f(\xi) \\ g(\xi) \end{pmatrix} = 0, \tag{7}$$

where $\xi := \{\xi_\sigma\}_{\sigma\in\Sigma} = (y, \hat{p}, \lambda)$, $f := \{f_\sigma\}_{\sigma\in\Sigma}$, and $g := \{g_\sigma\}_{\sigma\in\Sigma}$. Henceforth, we identify a competitive equilibrium by ξ. If π is uninformative, we restrict attention to equilibria that are σ-invariant.[4]

Consider the equilibrium system (6) for a given value of σ. These are $(J+1)H + (J-1)$ equations, equal to the number of unknowns. Taking ω to be a parameter of this equation system, the Jacobian can be written as follows:

$$D_{\xi_\sigma,\omega} F_\sigma(\xi_\sigma, \omega) = \begin{pmatrix} D_{\xi_\sigma} f_\sigma & D_\omega f_\sigma \\ D_{\xi_\sigma} g_\sigma & 0 \end{pmatrix},$$

with

$$D_\omega f_\sigma = \mathrm{diag}_h \left[\ldots \pi_{s\sigma} \, u^{h\prime\prime}_{s\sigma} \left[\omega^h_s + r_s \cdot y^h_\sigma\right] \; r_s \ldots_s \right]$$

[4]In other words, we ignore equilibria in which dependence on σ arises merely from randomization across different equilibria associated with the same information.

and

$$D_{\xi_\sigma} g_\sigma = \left(\begin{array}{c|c|c} \mathrm{diag}_h [p_\sigma^\top] & [\ldots \hat{y}_\sigma^h \ldots_h]^\top & 0 \\ \hline \ldots \hat{I} \ldots_h & 0 & 0 \end{array} \right).$$

The matrix $D_\omega f_\sigma$ has full row rank since R has full column rank and the term $\pi_{s\sigma} u_{s\sigma}^{h\prime\prime}[\omega_s^h + r_s \cdot y_\sigma^h]$ is nonzero, for all h, s, σ. Also, $D_{y_\sigma} g_\sigma$ has full row rank, since $\hat{I} = (I_{J-1} \ 0)$ and $(p_\sigma)_J = 1$, and therefore so does $D_{\xi_\sigma} g_\sigma$. By a standard argument, for a generic subset of endowments $\Omega_{R\sigma}$, the number of equilibria conditional on σ (zeros of F_σ) is finite (and positive). Hence, for endowments in the generic subset $\Omega_R := \bigcap_{\sigma \in \Sigma} \Omega_{R\sigma}$, the number of equilibria (zeros of F) is finite as well.

Notice that $\pi^\sigma := \{\pi_{s\sigma}\}_{s \in S}$ is a parameter of F_σ. We will write $F_\sigma(\xi_\sigma; \pi^\sigma)$ when we need to make this explicit, or when we wish to consider F_σ for a particular choice of π^σ.

4 Feasible Changes in Welfare

From the first welfare theorem it follows that competitive equilibria in a complete markets economy are ex-post Pareto efficient, i.e. Pareto efficient conditional on each realization of σ. If markets are incomplete, on the other hand, competitive equilibria are ex-post Pareto inefficient for a generic subset of endowments, while being always ex-post constrained Pareto efficient, where the set of feasible allocations are those that can be achieved with the available assets. These results are well known (see, for example, Magill and Quinzii (1996)), and will serve as a benchmark for our welfare analysis.

Given an initial information structure, and a corresponding competitive equilibrium, we wish to investigate whether an increase in information, or more generally a change in information, can lead to an ex-post (and hence also ex-ante) welfare improvement via a reallocation of the available assets. It is convenient to formulate this in terms of whether the allocation under consideration is ex-post inefficient relative to the new information. We say that a portfolio allocation y is *feasible* if it satisfies (2).

Definition 4.1 *A $\overline{\pi}$-equilibrium allocation $\overline{y}$ is $\underline{\pi\text{-efficient}}$ if there does not exist a feasible allocation y, such that, given the information π, y ex-post Pareto dominates $\overline{y}$, i.e.*

$$\sum_s \pi_{s\sigma} \Big(u^h \big[\omega_s^h + r_s \cdot y_\sigma^h\big] - u^h \big[\omega_s^h + r_s \cdot \overline{y}_\sigma^h\big] \Big) \geq 0, \qquad \forall h \in H, \, \sigma \in \Sigma,$$

where at least one of these inequalities is strict.

In this definition, a feasible allocation is one that can be achieved with the existing assets, as in the above notion of constrained efficiency. In order to interpret the

change in the signal structure from $\overline{\pi}$ to π as a purely informational change, π must leave the marginal distribution over S invariant, i.e. it must belong to the set

$$\Pi_{\overline{\pi}} := \left\{ \pi \in \mathbb{R}^{S\Sigma}_{++} \;\middle|\; \sum_{\sigma \in \Sigma} \pi_{s\sigma} = \overline{\pi}_s, \forall s \in S \right\}.$$

Clearly $\Pi_{\overline{\pi}} \subset \Pi$. Notice that both the allocations $\overline{y}$ and y are evaluated at the same odds, given by π. It would not be sensible to evaluate $\overline{y}$ at the initial information structure $\overline{\pi}$, and y at the new information structure π (doing so can lead to the possibility of a "Pareto improvement" with no change in the allocation).

While a $\overline{\pi}$-equilibrium is $\overline{\pi}$-efficient (this being just a restatement of the fact that it is constrained ex-post Pareto efficient), it is not in general π-efficient. In other words, while a competitive equilibrium makes efficient use of the available information, there is in general a change in information that admits a Pareto improvement. We now study such informational changes.

We fix an initial information structure $\overline{\pi} \in \Pi$, and characterize the set of alternative information structures $\pi \in \Pi_{\overline{\pi}}$ such that $\overline{\pi}$-equilibria are π-inefficient. It is straightforward to check that this set is empty if markets are complete (see Lemma 4.1 below). On the other hand, if markets are incomplete, we show that generically an ex-post Pareto improvement can be attained for a large set of π's. The main results of this section, Theorems 4.1, 4.2 and 4.3, formalize this statement in different ways. We prove Theorems 4.1 and 4.2 for an arbitrary initial information structure $\overline{\pi} \in \Pi$. They can, of course, be specialized to the case where $\overline{\pi}$ is uninformative, as is typically assumed in the literature.

We use the shorthand $u^{h'}_{s\sigma} := u^{h'}_s\left[\omega^h_s + r_s \cdot y^h_\sigma\right]$ and $\overline{u}^{h'}_{s\sigma} := u^{h'}_s\left[\omega^h_s + r_s \cdot \overline{y}^h_\sigma\right]$, and similarly for the second derivatives, $u^{h''}_{s\sigma}$ and $\overline{u}^{h''}_{s\sigma}$.

Lemma 4.1 *Suppose markets are complete. Then a $\overline{\pi}$-equilibrium is π-efficient for all $\pi \in \Pi$.*

Proof:

Consider a $\overline{\pi}$-equilibrium allocation $\overline{y}$. Since markets are complete, $\overline{y}$ is ex-post Pareto efficient and this property, characterized by the equality of agents' marginal rates of substitution across states,

$$\frac{\overline{u}^{h'}_{s\sigma}}{\overline{u}^{h'}_{\hat{s}\sigma}} = \frac{\overline{u}^{\hat{h}'}_{s\sigma}}{\overline{u}^{\hat{h}'}_{\hat{s}\sigma}}, \qquad \forall h, \hat{h} \in H;\; s, \hat{s} \in S;\; \sigma \in \Sigma,$$

is independent of the value of $\overline{\pi}$. Hence, there cannot be an allocation which ex-post Pareto dominates $\overline{y}$, for any π.[5] $\square$

If markets are complete, all states are insurable (an *insurable state* is a state $s \in S$

[5] Notice that Lemma 4.1 applies for all $\pi \in \Pi$, and not just for $\pi \in \Pi_{\overline{\pi}}$.

for which the corresponding Arrow security can be replicated with the existing assets). Changing the relative probabilities of these states does not admit an ex-post Pareto improvement as all gains from trade (conditional on any σ) have already been exhausted in equilibrium.

When markets are incomplete, risk sharing and the possibility of a welfare improvement with a change in information can be characterized in terms of what we call insurable events. An insurable event is a generalization of the concept of an insurable state. We will show that gains from trade are exhausted in equilibrium across these events, while this is generically not the case within such events. While a change in information that affects only the relative probabilities of insurable events does not admit an ex-post Pareto improvement, such an improvement can typically be found for informational changes that affect the relative probabilities of states within an insurable event.

We formalize the notion of an insurable event as follows.[6] Consider a partition of S given by $\mathcal{S}(R) := \{S_1, \dots, S_K\}$. For each $k \in K := \{1, \dots, K\}$, let L_k be the subspace of $\mathbb{R}^J$ spanned by the vectors $\{r_s\}_{s \in S_k}$. We say that the subspaces $L_1, \dots, L_K$ are linearly independent if $\sum_{k \in K} \ell_k = 0$, $\ell_k \in L_k$, implies $\ell_k = 0$ for all k. Henceforth, we choose $\mathcal{S}(R)$ to be the partition for which $L_1, \dots, L_K$ are linearly independent, and K is maximal (it is easy to check that there is a unique such partition). We call $S_k \in \mathcal{S}(R)$ an *insurable event*. We justify this choice of terminology below, and show that this is indeed a generalization of the notion of an insurable state.

We denote the dimension of L_k by J_k. Thus we have $\sum_{k \in K} J_k = J$. Without loss of generality we can order the states in S so that the first S_1 states correspond to the event S_1, the following S_2 states correspond to the event S_2, and so on. The partition $\mathcal{S}(R)$ is invariant to changes in asset payoffs that do not affect the column span of R. Moreover, R is column-equivalent to a block-diagonal matrix, with each block corresponding to an insurable event S_k:[7]

Lemma 4.2 *Suppose the asset payoff matrices R and R' are column-equivalent. Then $\mathcal{S}(R) = \mathcal{S}(R')$. Furthermore, R is column-equivalent to* $\mathrm{diag}_{k \in K}[R_k]$*, where R_k is an $S_k \times J_k$ matrix with* $\mathrm{rank}(R_k) = J_k$.

The proof is in the Appendix. It follows from this result that, if agents' endowments are measurable with respect to $\mathcal{S}(R)$, i.e. constant in each cell S_k, then competitive equilibria are ex-post Pareto efficient. For general endowments, if an equilibrium allocation is not ex-post Pareto efficient, it must be because of unexploited gains from trade within an insurable event. All gains from trade across insurable events are exhausted in equilibrium.

We say that an insurable event S_k is *trivial* if it is a singleton, and *nontrivial* otherwise. A trivial insurable event consists of a single insurable state, while a

[6] Here we draw on Geanakoplos and Mas-Colell (1989). See, in particular, the analysis at the beginning of Section III of that paper.

[7] If there is only one insurable event, this block-diagonal matrix has only one diagonal block.

nontrivial insurable event (which exists if and only if markets are incomplete) consists of two or more states, none of which is insurable. An insurable event S_k is nontrivial if and only if $J_k < S_k$. We say that the asset payoff matrix R is in *general position* if every $J \times J$ submatrix of R is nonsingular. If markets are incomplete, and R is in general position, there is only one insurable event, i.e. $\mathcal{S}(R) = \{S\}$.[8]

Given an allocation y, and information structure π, let

$$\mu^{h\hat{h}}_{\sigma}(y_\sigma, \pi_\sigma) := \frac{\sum_{s \in S} \pi_{s\sigma}\, u^{h\prime}_{s\sigma} r^J_s}{\sum_{s \in S} \pi_{s\sigma}\, u^{\hat{h}\prime}_{s\sigma} r^J_s}$$

denote the ratio of marginal utilities of asset J for agents h and $\hat{h}$, conditional on σ. Since asset J has a nonnegative nonzero payoff, this is a positive scalar. A $\overline{\pi}$-equilibrium allocation $\overline{y}$ is π-efficient if and only if the marginal rates of substitution between assets, evaluated at π, are equal across agents, for every σ, i.e.

$$\frac{\sum_{s \in S} \pi_{s\sigma}\, \overline{u}^{h\prime}_{s\sigma} r^j_s}{\sum_{s \in S} \pi_{s\sigma}\, \overline{u}^{h\prime}_{s\sigma} r^J_s} = \frac{\sum_{s \in S} \pi_{s\sigma}\, \overline{u}^{\hat{h}\prime}_{s\sigma} r^j_s}{\sum_{s \in S} \pi_{s\sigma}\, \overline{u}^{\hat{h}\prime}_{s\sigma} r^J_s}, \qquad \forall h, \hat{h} \in H;\ j \in J;\ \sigma \in \Sigma,$$

which can be written as

$$\sum_{s \in S} \pi_{s\sigma} \left(\overline{u}^{h\prime}_{s\sigma} - \overline{\mu}^{h\hat{h}}_{\sigma}(\pi^\sigma)\, \overline{u}^{\hat{h}\prime}_{s\sigma} \right) r_s = 0, \qquad \forall h, \hat{h} \in H;\ \sigma \in \Sigma,$$

where $\overline{\mu}^{h\hat{h}}_{\sigma}(\pi_\sigma) := \mu^{h\hat{h}}_{\sigma}(\overline{y}_\sigma, \pi_\sigma)$. Since the subspaces $L_1, \ldots, L_K$ are linearly independent, this condition is equivalent to

$$\sum_{s \in S_k} \pi_{s\sigma} \left(\overline{u}^{h\prime}_{s\sigma} - \overline{\mu}^{h\hat{h}}_{\sigma}(\pi^\sigma)\, \overline{u}^{\hat{h}\prime}_{s\sigma} \right) r_s = 0, \qquad \forall h, \hat{h} \in H;\ S_k \in \mathcal{S}(R);\ \sigma \in \Sigma.$$

For $\mu \in \mathbb{R}$, let

$$\Delta^{h\hat{h}}_{s\sigma}(y_\sigma, \mu) := u^{h\prime}_{s\sigma} - \mu\, u^{\hat{h}\prime}_{s\sigma},$$

and

$$\bar{\Delta}^{h\hat{h}}_{s\sigma}(\mu) := \Delta^{h\hat{h}}_{s\sigma}(\overline{y}_\sigma, \mu) = \overline{u}^{h\prime}_{s\sigma} - \mu\, \overline{u}^{\hat{h}\prime}_{s\sigma}.$$

Then we can state the above result as follows:

Lemma 4.3 *A $\overline{\pi}$-equilibrium allocation $\overline{y}$ is π-efficient if and only if*

$$\sum_{s \in S_k} \pi_{s\sigma}\, \bar{\Delta}^{h\hat{h}}_{s\sigma}\big(\overline{\mu}^{h\hat{h}}_{\sigma}(\pi^\sigma)\big)\, r_s = 0, \qquad \forall h, \hat{h} \in H;\ S_k \in \mathcal{S}(R);\ \sigma \in \Sigma. \tag{8}$$

Since a $\overline{\pi}$-equilibrium is $\overline{\pi}$-efficient, condition (8) must hold at $\pi = \overline{\pi}$. In other words, at $\overline{\pi}$, agents' marginal utilities for assets are collinear in each insurable event. The allocation $\overline{y}$ is π-inefficient for all π's that violate (8). The possibility of finding such π's arises from the following result:

[8]The converse is not true, however: $\mathcal{S}(R) = \{S\}$ does not imply that R is in general position. It is also worth noting that if R is in general position, so is any R' that is column-equivalent to R.

Lemma 4.4 *Suppose markets are incomplete. Let $\hat{S}$ be a subset of S that contains a nontrivial insurable event, and suppose that there exists a portfolio with a nonnegative nonzero payoff in $\hat{S}$. Then, for a generic subset of Ω, at any $\overline{\pi}$-equilibrium allocation $\overline{y}$,*

$$\left\{\bar{\Delta}_{s\sigma}^{h\hat{h}}(\mu)\right\}_{s\in\hat{S}} \neq 0, \qquad \forall \mu \in \mathbb{R};\ h,\hat{h} \in H;\ \sigma \in \Sigma. \tag{9}$$

The proof is in the Appendix.[9] In the special case where $S = \hat{S}$, Lemma 4.4 says that agents' marginal utility vectors for state-contingent consumption, conditional on σ, are generically not collinear (the condition on asset payoffs is satisfied since asset J has a nonnegative nonzero payoff in S). This is just the standard result that competitive equilibria are generically ex-post Pareto inefficient if markets are incomplete. Lemma 4.4 strengthens this result by showing that agents' marginal utility vectors are generically not collinear in any nontrivial insurable event, subject to a mild condition on asset payoffs. Combining this with our discussion of Lemma 4.2, we can conclude that in equilibrium, while there are no unexploited gains from trade between insurable events, generically such gains do exist within each nontrivial insurable event.

We now use Lemma 4.4 to show that, for a generic subset of endowments, and for a generic choice of the information structure π, the π-efficiency condition (8) is violated at any competitive equilibrium:

Theorem 4.1 *Suppose markets are incomplete, and there are two assets whose payoffs are not collinear in any pair of states. Then, for any $\overline{\pi}$, there is a generic subset $\hat{\Omega} \times \hat{\Pi}_{\overline{\pi}}$ of $\Omega \times \Pi_{\overline{\pi}}$ such that every $\overline{\pi}$-equilibrium is π-inefficient, for all $\pi \in \hat{\Pi}_{\overline{\pi}}$.*

Proof:

Consider a $\overline{\pi} \in \Pi$. It suffices to establish that condition (8) is violated for one value of σ. Accordingly, fix a value of σ. Recall from Section 3 that there is a generic subset $\Omega_{R\sigma}$ of Ω for which the Jacobian $D_{\xi_\sigma} F_\sigma(\xi_\sigma; \overline{\pi}^\sigma)$ has full row rank, at all zeros of $F_\sigma(\xi_\sigma; \overline{\pi}^\sigma)$. Let $\bar{\Omega}$ be the generic subset of Ω for which Lemma 4.4 holds. For the rest of the proof, we restrict endowments to lie in the generic subset $\hat{\Omega} := \Omega_{R\sigma} \cap \bar{\Omega}$.

We consider a π^σ in the set

$$\Pi_{\overline{\pi}}^{\sigma} := \left\{\pi^\sigma \in \mathbb{R}_{++}^S \,\middle|\, \pi_{s\sigma} < \overline{\pi}_s, \forall s \in S\right\},$$

which is the projection of $\Pi_{\overline{\pi}}$ onto the S-dimensional subspace of $\mathbb{R}^{S\Sigma}$ corresponding to the value of σ that we have fixed. Let $\hat{r}_s \in \mathbb{R}^2$ be the payoff in state s of two assets that satisfy the non-collinearity condition of the theorem. We will show that, for π^σ in a generic subset of $\Pi_{\overline{\pi}}^{\sigma}$, at every $\overline{\pi}$-equilibrium allocation y, there is no solution to the equation system

$$\Psi_1(\xi_\sigma, \mu, \pi^\sigma; \overline{\pi}^\sigma) := \begin{pmatrix} F_\sigma(\xi_\sigma; \overline{\pi}^\sigma) \\ \sum_{s\in S} \pi_{s\sigma} \Delta_{s\sigma}^{h\hat{h}}(y_\sigma, \mu)\, \hat{r}_s \end{pmatrix} = 0,$$

[9] If s is an insurable state, $\bar{\Delta}_{s\sigma}^{h\hat{h}}(\overline{\mu}_\sigma^{h\hat{h}}(\overline{\pi}^\sigma)) = 0$, since equation (8) holds for $\pi = \overline{\pi}$.

for a given pair of agents h and $\hat{h}$. Then, by Lemma 4.3, a Pareto improving reallocation exists conditional on σ. For any choice of $\pi^\sigma \in \Pi^\sigma_{\overline{\pi}}$, we can always choose $\{\pi^{\sigma'}\}_{\sigma' \neq \sigma}$, so that $\pi \in \Pi_{\overline{\pi}}$, and find a weakly Pareto improving reallocation for all $\sigma' \neq \sigma$. Moreover, if π^σ is in a generic subset of $\Pi^\sigma_{\overline{\pi}}$, the corresponding π is in a generic subset of $\Pi_{\overline{\pi}}$.

The Jacobian of Ψ_1, evaluated at a zero $(\overline{\xi}_\sigma, \mu, \pi^\sigma)$ of Ψ_1, is

$$D_{\xi_\sigma,\mu,\pi^\sigma}\Psi_1 = \left(\begin{array}{cc|c} D_{\xi_\sigma}F_\sigma(\overline{\xi}_\sigma;\overline{\pi}^\sigma) & 0 & 0 \\ \hline \multicolumn{2}{c|}{*} & \ldots\bar{\Delta}^{h\hat{h}}_{s\sigma}(\mu)\,\hat{r}_s\ldots_{s\in S} \end{array}\right). \tag{10}$$

Since this is a block-triangular matrix, its rank is equal to the sum of the ranks of the diagonal blocks. The upper left block has full row rank, and $\{\bar{\Delta}^{h\hat{h}}_{s\sigma}(\mu)\}_{s\in S}$ is nonzero by Lemma 4.4. Indeed, since $\hat{r}_s$ and $\hat{r}_{s'}$ are not collinear, for any pair of states s, s', and $\sum_{s\in S}\pi_{s\sigma}\bar{\Delta}^{h\hat{h}}_{s\sigma}(\mu)\hat{r}_s = 0$ at any zero of Ψ_1, we must have $\bar{\Delta}^{h\hat{h}}_{s\sigma}(\mu) \neq 0$ for at least three states in S. This implies that the lower right block of (10) has full row rank.

Therefore, the Jacobian $D_{\xi_\sigma,\mu,\pi^\sigma}\Psi_1$ has full row rank, at every zero of Ψ_1. Thus Ψ_1 is transverse to zero, and $\Psi^{-1}_{1\pi^\sigma}(0) = \varnothing$, for every π^σ in a generic subset of $\Pi^\sigma_{\overline{\pi}}$. □

Theorem 4.1 holds for π in a generic subset of $\Pi_{\overline{\pi}}$. It does not specify this subset, however. A π^σ for which the π-efficiency condition (8) is violated for a pair of agents h and $\hat{h}$ will depend on $\{\bar{\Delta}^{h\hat{h}}_{s\sigma}\}_{s\in S}$, and hence we cannot say whether $\overline{\pi}$-equilibria are π-inefficient for a given choice of π, without reference to a particular equilibrium. On the other hand, it is clear from (8) that a change in information that affects only the relative likelihood of insurable events (i.e. $\{\pi_{s\sigma}\}_{s\in S_k}$ proportional to $\{\overline{\pi}_{s\sigma}\}_{s\in S_k}$, for all $k \in K$, $\sigma \in \Sigma$) does not admit an ex-post Pareto improvement. Agents' marginal utilities for assets remain collinear after such a change in information, which we can therefore deem to be payoff-irrelevant. This leads us to the following definition:[10]

Definition 4.2 *Given $\overline{\pi} \in \Pi$, an information structure $\pi \in \Pi_{\overline{\pi}}$ is* <u>*payoff-relevant*</u> *if $\{\pi_{s\sigma}\}_{s\in S_k}$ is not proportional to $\{\overline{\pi}_{s\sigma}\}_{s\in S_k}$, for some $\sigma \in \Sigma$, and for some insurable event $S_k \in \mathcal{S}(R)$.*

Of course, if $\{\pi_{s\sigma}\}_{s\in S_k}$ is not proportional to $\{\overline{\pi}_{s\sigma}\}_{s\in S_k}$, then S_k must be nontrivial. If markets are incomplete (so that a nontrivial insurable event exists), the set of payoff-relevant information structures is a generic subset of $\Pi_{\overline{\pi}}$.[11] Lemma 4.4 suggests that a payoff-relevant information structure π may admit an ex-post Pareto improvement. Indeed, this is generically the case:

Theorem 4.2 *Suppose markets are incomplete, and there exists a portfolio of the first $J-1$ assets with a nonnegative nonzero payoff. Then, for any $\overline{\pi}$, and a*

[10] The payoff-relevance condition applies to a change in information from $\overline{\pi}$ to π. However, it is easier to speak of π as being payoff-relevant, taking $\overline{\pi}$ as given.

[11] As we have noted before, if R is in general position, there is only one insurable event, namely the whole set S; in this case, any π for which π^σ is not proportional to $\overline{\pi}^\sigma$, for some σ, is payoff-relevant.

payoff-relevant π, there is a generic subset of Ω such that every $\overline{\pi}$-equilibrium is π-inefficient.

Proof:

Consider a $\overline{\pi} \in \Pi$, and a payoff-relevant $\pi \in \Pi_{\overline{\pi}}$. Fix a σ for which the payoff-relevance condition applies. Given the assumption in the theorem that there exists a portfolio of the first $J-1$ assets with a nonnegative nonzero payoff, we can assume, without loss of generality, that an asset $j' \neq J$ has this property.

We will use the following result, which can be deduced from Lemma 5 of Geanakoplos and Mas-Colell (1989):

FACT 1 *Consider nonzero scalars θ_s, θ'_s, $s \in S$, such that $\{\theta_s\}_{s \in S_k}$ is not proportional to $\{\theta'_s\}_{s \in S_k}$, for some insurable event $S_k \in \mathcal{S}(R)$. Then, $\operatorname{diag}_s[\theta_s]R$ and $\operatorname{diag}_s[\theta'_s]R$ do not have the same column span.*

Let
$$R^* := \Big(\operatorname{diag}_s[\pi_{s\sigma}]R \quad \operatorname{diag}_s[\overline{\pi}_{s\sigma}]R\Big).$$

Since π is payoff-relevant, Fact 1 implies that $\operatorname{rank}(R^*) \geq J+1$. Let $r^j \in \mathbb{R}^S$ denote the payoff of asset j. We pick an asset j_1 such that $\operatorname{diag}_s[\pi_{s\sigma}]\, r^{j_1}$ lies outside the column span of $\operatorname{diag}_s[\overline{\pi}_{s\sigma}]R$. We pick a second asset j_2 as follows. If $\operatorname{rank}(R^*) > J+1$, we choose j_2 so that $\operatorname{diag}_s[\pi_{s\sigma}]\, r^{j_2}$ lies outside the column span of $\operatorname{diag}_s[\overline{\pi}_{s\sigma}]R$. If, on the other hand, $\operatorname{rank}(R^*) = J+1$, we choose j_2 to be either j' or J (if j_1 happens to be the same as j' or J, we pick j_2 to be the other asset). Thus the matrix

$$\hat{R}^* := \Big(\operatorname{diag}_s[\pi_{s\sigma}](r^{j_1} \;\; r^{j_2}) \quad \operatorname{diag}_s[\overline{\pi}_{s\sigma}]R\Big) \tag{11}$$

either has rank $J+2$, or has rank $J+1$ and $\operatorname{diag}_s[\pi_{s\sigma}]\, r^{j_2}$ lies in the column span of $\operatorname{diag}_s[\overline{\pi}_{s\sigma}]R$.

Let $\hat{r}_s \in \mathbb{R}^2$ be the payoff in state s of assets j_1 and j_2. We will show that, for a generic subset of Ω, there is no solution to the equation system

$$\Psi_2(\xi_\sigma, \mu, \omega; \overline{\pi}^\sigma, \pi^\sigma) := \begin{pmatrix} F_\sigma(\xi_\sigma, \omega; \overline{\pi}^\sigma) \\ \sum_{s \in S} \pi_{s\sigma} \Delta^{h_1 h_2}_{s\sigma}(y_\sigma, \mu, \omega)\, \hat{r}_s \end{pmatrix} = 0.$$

This is the same system of equations as in the proof of Theorem 4.1, for a (possibly) different choice of $\hat{r}_s$, and of the pair of agents. Here we will be perturbing ω instead of π^σ.

The Jacobian of Ψ_2, evaluated at a zero $(\overline{\xi}_\sigma, \mu, \omega)$ of Ψ_2, is

$$D_{\xi_\sigma,\mu,\omega}\Psi_2 = \left(\begin{array}{c|c|c} * & 0 & \operatorname{diag}_h\big[\ldots \overline{\pi}_{s\sigma}\, \overline{u}^{h\prime\prime}_{s\sigma} r_s \ldots_s\big] \\ \hline D_{\xi_\sigma} g_\sigma & 0 & 0 \\ \hline * & -\sum_s \pi_{s\sigma}\, \overline{u}^{h_2\prime}_{s\sigma} \hat{r}_s & \big[\ldots \pi_{s\sigma}\, \overline{u}^{h_1\prime\prime}_{s\sigma} \hat{r}_s \ldots_s\big] \qquad * \end{array}\right),$$

which is row-equivalent to

$$\left(\begin{array}{c|c|c|c} * & -\sum_s \pi_{s\sigma}\,\overline{u}^{h_2\prime}_{s\sigma}\hat{r}_s & \ldots \pi_{s\sigma}\,\overline{u}^{h_1\prime\prime}_{s\sigma}\hat{r}_s \ldots_s & * \\ \hline * & 0 & \ldots \overline{\pi}_{s\sigma}\,\overline{u}^{h_1\prime\prime}_{s\sigma} r_s \ldots_s & 0 \\ \hline * & 0 & 0 & \mathrm{diag}_{h\neq h_1}\big[\ldots \overline{\pi}_{s\sigma}\,\overline{u}^{h\prime\prime}_{s\sigma} r_s \ldots_s\big] \\ \hline D_{\xi_\sigma} g_\sigma & 0 & 0 & 0 \end{array}\right).$$

We wish to show that this matrix has full row rank. Since $D_{\xi_\sigma} g_\sigma$ has full row rank, as does the matrix $\mathrm{diag}_{h\neq h_1}\big[\ldots \overline{\pi}_{s\sigma}\,\overline{u}^{h\prime\prime}_{s\sigma} r_s \ldots_s\big]$, it suffices to show that

$$\left(\begin{array}{c|c} -\sum_s \pi_{s\sigma}\,\overline{u}^{h_2\prime}_{s\sigma}\hat{r}_s & \ldots \pi_{s\sigma}\,\overline{u}^{h_1\prime\prime}_{s\sigma}\hat{r}_s \ldots_s \\ \hline 0 & \ldots \overline{\pi}_{s\sigma}\,\overline{u}^{h_1\prime\prime}_{s\sigma} r_s \ldots_s \end{array}\right)$$

has full row rank, $J+2$. This matrix is column-equivalent to

$$\left(\begin{array}{c|c} \sum_s \pi_{s\sigma}\,\overline{u}^{h_2\prime}_{s\sigma}\hat{r}_s & \ldots \pi_{s\sigma}\,\hat{r}_s \ldots_s \\ \hline 0 & \ldots \overline{\pi}_{s\sigma}\, r_s \ldots_s \end{array}\right),$$

the transpose of which can be written as follows:

$$A_2 := \left(\begin{array}{c|c} \big[\ldots \pi_{s\sigma}\,\overline{u}^{h_2\prime}_{s\sigma} \ldots_s\big](r^{j_1} \quad r^{j_2}) & 0 \\ \hline \mathrm{diag}_s[\pi_{s\sigma}](r^{j_1} \quad r^{j_2}) & \mathrm{diag}_s[\overline{\pi}_{s\sigma}]R \end{array}\right).$$

We now exploit the properties of the matrix $\hat{R}^*$, given by (11). If $\mathrm{rank}(\hat{R}^*) = J+2$, A_2 has full column rank and we are done. If not, we have

$$A_2 \sim_C \left(\begin{array}{c|c} \big[\ldots \pi_{s\sigma}\,\overline{u}^{h_2\prime}_{s\sigma} \ldots_s\big]\, r^{j_2} & * \\ \hline 0 & \mathrm{diag}_s[\pi_{s\sigma}]\, r^{j_1} \qquad \mathrm{diag}_s[\overline{\pi}_{s\sigma}]R \end{array}\right).$$

Since asset j_2 has a nonnegative nonzero payoff, the upper left block of this matrix is a nonzero scalar, while the rank of the lower right block is $J+1$. Therefore, A_2 has full column rank for this case as well.

We have shown that the Jacobian $D_{\xi_\sigma,\mu,\omega}\Psi_2$ has full row rank, at every zero of Ψ_2. Thus Ψ_2 is transverse to zero, and $\Psi^{-1}_{2\omega}(0) = \varnothing$, for every ω in a generic subset of Ω. $\square$

Theorem 4.2 shows that any payoff-relevant π violates (8), with an appropriate perturbation of $\{\bar{\Delta}^{h\hat{h}}_{s\sigma}\}_{s\in S}$, i.e. for a generic subset of Ω. This generic subset will clearly depend on π (unlike the generic subset $\hat{\Omega}$ in Theorem 4.1, which does not depend on the choice of π in $\hat{\Pi}_{\overline{\pi}}$). Thus, while Theorem 4.2 improves upon Theorem 4.1 by identifying a generic subset of π's for which equilibrium allocations are π-inefficient, it does not specify a generic subset of Ω for which a Pareto improving

reallocation exists for all such π's. We now show that, under some restrictions on asset payoffs, there is a generic subset of Ω for which a Pareto improvement can be attained for a large, albeit not necessarily generic, subset of π's that can be explicitly identified.

In order to develop this result in a notationally convenient manner, we will assume that R takes the block-diagonal form $\mathrm{diag}_k[R_k]$ and, if there is a trivial insurable event, then S_K is one such event. Due to Lemma 4.2, this assumption is without loss of generality. In particular, it implies that asset J pays off only in event S_K.[12]

Suppose markets are incomplete, there are at least two insurable events, and the initial information structure $\overline{\pi}$ is uninformative. We consider information structures π that affect the relative probabilities of states in a nontrivial insurable event S_k, $k \neq K$, but not in S_K (if S_K is trivial, then the latter condition is irrelevant). As we will verify in the proof of Theorem 4.3 below, this implies that $\overline{\mu}_\sigma^{h\hat{h}}(\pi^\sigma) = \overline{\mu}_\sigma^{h\hat{h}}(\overline{\pi}^\sigma)$, and $\bar{\Delta}_{s\sigma}^{h\hat{h}}(\overline{\mu}_\sigma^{h\hat{h}}(\overline{\pi}^\sigma))$ is σ-invariant for all $s \in S_k$. In Lemma A.1 in the Appendix we show that $\bar{\Delta}_{s\sigma}^{h\hat{h}}(\overline{\mu}_\sigma^{h\hat{h}}(\overline{\pi}^\sigma))$ is nonzero, for all $s \in S_k$, for a generic subset of Ω (thus strengthening the non-collinearity condition (9) for a particular value of μ, namely $\mu = \overline{\mu}_\sigma^{h\hat{h}}(\overline{\pi}^\sigma)$). The π-efficiency condition (8) is then violated if the change in information is independent across sufficiently many different values of σ, i.e. if the matrix

$$\mathbf{\Pi}_{S_k} := \begin{pmatrix} \vdots \\ \ldots \pi_{s\sigma} \ldots_{s \in S_k} \\ \vdots_\sigma \end{pmatrix}$$

has sufficiently high rank. Formally, we have:

Theorem 4.3 *Suppose markets are incomplete, there are at least two insurable events, and $\overline{\pi}$ is uninformative. Let S_k be a nontrivial insurable event, $k \neq K$. Then, for a generic subset of Ω, every $\overline{\pi}$-equilibrium is π-inefficient, for all $\pi \in \Pi_{\overline{\pi}}$ satisfying*

(a) $\mathrm{rank}(\mathbf{\Pi}_{S_k}) > S_k - J_k$; *and*

(b) $\{\pi_{s\sigma}\}_{s \in S_K}$ *is collinear with* $\{\overline{\pi}_{s\sigma}\}_{s \in S_K}$, *for all* $\sigma \in \Sigma$.

Proof:

Consider a $\overline{\pi}$-equilibrium allocation $\overline{y}$, a nontrivial insurable event S_k, and a $\pi \in \Pi_{\overline{\pi}}$ satisfying the conditions of the theorem. Suppose $\overline{y}$ is π-efficient. Then, by Lemma 4.3, for an arbitrary pair of agents h and $\hat{h}$,

$$\sum_{s \in S_k} \pi_{s\sigma}\, \bar{\Delta}_{s\sigma}^{h\hat{h}}\big(\overline{\mu}_\sigma^{h\hat{h}}(\pi^\sigma)\big)\, r_s = 0, \qquad \forall \sigma \in \Sigma. \tag{12}$$

[12] If S_K is trivial, then asset J pays off only in the last state. Of course, S_K may be nontrivial, and may even be the whole state space S (if there is only one insurable event).

Exploiting the fact that asset J pays off only in the event S_K, and using condition (b) in the statement of the theorem, we see that $\overline{\mu}^{h\hat{h}}_{\sigma}(\pi^{\sigma}) = \overline{\mu}^{h\hat{h}}_{\sigma}(\overline{\pi}^{\sigma})$, for all $\sigma \in \Sigma$. Therefore, condition (12) reduces to

$$\sum_{s \in S_k} \pi_{s\sigma}\, \bar{\Delta}^{h\hat{h}}_{s\sigma}\big(\overline{\mu}^{h\hat{h}}_{\sigma}(\overline{\pi}^{\sigma})\big)\, r_s = 0, \qquad \forall \sigma \in \Sigma. \tag{13}$$

Due to the assumption that $\overline{\pi}$ is uninformative, the marginal utilities $\overline{u}^{h\prime}_{s\sigma}$ and $\overline{u}^{\hat{h}\prime}_{s\sigma}$ are σ-invariant, and so is $\overline{\mu}^{h\hat{h}}_{\sigma}(\overline{\pi}^{\sigma})$. It follows that $\bar{\Delta}^{h\hat{h}}_{s\sigma}(\overline{\mu}^{h\hat{h}}_{\sigma}(\overline{\pi}^{\sigma}))$ is σ-invariant. Condition (13) can then be rewritten as follows:

$$\mathbf{\Pi}_{S_k} \mathrm{diag}_{s \in S_k}\big[\bar{\Delta}^{h\hat{h}}_{s\sigma}\big(\overline{\mu}^{h\hat{h}}_{\sigma}(\overline{\pi}^{\sigma})\big)\big] R_k = 0.$$

Since $\mathrm{rank}(\mathbf{\Pi}_{S_k}) > S_k - J_k$, the rank of $\mathrm{diag}_{s \in S_k}[\bar{\Delta}^{h\hat{h}}_{s\sigma}(\overline{\mu}^{h\hat{h}}_{\sigma}(\overline{\pi}^{\sigma}))] R_k$ must be strictly less than J_k. Therefore, $\bar{\Delta}^{h\hat{h}}_{s\sigma}(\overline{\mu}^{h\hat{h}}_{\sigma}(\overline{\pi}^{\sigma})) = 0$ for some $s \in S_k$. By Lemma A.1, this condition is violated for a generic subset of Ω. $\square$

Notice that condition (b) is automatically satisfied if S_K is trivial. The theorem generalizes Theorem 5 of Blume et al. (2006). Their result corresponds to the case where S_K is trivial (they assume that one of the assets is an Arrow security); they also impose a stronger full rank condition (in our notation, their assumption is that $\mathrm{rank}(\mathbf{\Pi}_{S_1} \ldots \mathbf{\Pi}_{S_K}) = S$). While in Theorems 4.1 and 4.2 it sufficed to consider a change in the information structure for only two values of σ, for example an appropriate choice of π^{σ_1} that admits a Pareto improvement conditional on σ_1, and a corresponding choice of π^{σ_2} to satisfy the adding-up condition $\pi \in \Pi_{\overline{\pi}}$, Theorem 4.3 requires an independent change in information across at least $S_k - J_k$ values of σ.[13] The corresponding π must therefore be payoff-relevant for these values. In Theorems 4.1 and 4.2, the set of π's for which $\overline{\pi}$-equilibria are π-inefficient is a generic subset of $\Pi_{\overline{\pi}}$, but this is not the case in Theorem 4.3 unless S_K is trivial.

5 Equilibrium Changes in Welfare

We have shown that, for any $\overline{\pi}$-equilibrium, there is a rich set of information structures that allow a Pareto improvement. For every information structure π in this set, there is an ex-post Pareto improving reallocation of the given assets. A natural question to ask is whether there exists a π such that an associated π-equilibrium Pareto dominates the $\overline{\pi}$-equilibrium under consideration. In this section we provide an affirmative answer to this question, for a generic economy. It is indeed typically possible to find an information structure π, and a corresponding π-equilibrium, such that all agents are better off ex-ante.[14] But it is also possible to find a π such that

[13] Condition (a) allows for the possibility that the relative probabilities of the states in S_k are the same under π and $\overline{\pi}$ for one value of σ.

[14] Whether this is possible ex-post remains an open question.

all agents are worse off. Indeed, the welfare effects in equilibrium of a change in information can go in any direction. The proof of this result exploits the general framework laid out in Citanna et al. (1998), and exposited more fully in Villanacci et al. (2002), Chapter 15.

Given a $\overline{\pi}$-equilibrium allocation $\overline{y}$, and a π-equilibrium allocation y, let

$$U(\pi, y; \overline{y}) := \left\{ \sum_{s,\sigma} \pi_{s\sigma} \Big(u^h \left[\omega_s^h + r_s \cdot y_\sigma^h \right] - u^h \left[\omega_s^h + r_s \cdot \overline{y}_\sigma^h \right] \Big) \right\}_{h \in H}$$

be the vector of differences in the ex-ante expected utilities of agents between the two allocations. We consider local changes in (π, y) in a neighborhood of $(\overline{\pi}, \overline{y})$. With no change in information, there is no change in welfare: $U(\overline{\pi}, \overline{y}; \overline{y}) = 0$. An ex-ante Pareto improvement is attained in equilibrium if $U(\pi, y; \overline{y})$ is nonnegative and nonzero. We show, in fact, that it is possible to generate a local change in U in any direction, for a generic economy. Our result requires that markets are sufficiently incomplete, and that the heterogeneity of agents is not too large.

Theorem 5.1 *Suppose $S \geq 2JH$, $J \geq H$, and R is in general position. Then, for a generic subset of $\mathcal{E}$, for any $\overline{\pi}$-equilibrium allocation $\overline{y}$, there exists a local change in information $d\pi$, and a corresponding local change in the equilibrium allocation dy, such that $U(\overline{\pi} + d\pi, \overline{y} + dy; \overline{y})$ is any desired vector in $\mathbb{R}^H$.*

Proof:

We fix an economy $\overline{e} \in \mathcal{E}$, with information structure $\overline{\pi}$, and consider a $\overline{\pi}$-equilibrium given by $\overline{\xi} = (\overline{y}, \overline{\hat{p}}, \overline{\lambda})$. With a slight abuse of notation, we can write the equilibrium system (7) as $F(\pi, \xi) = 0$. Thus (π, ξ) is a zero of F if and only if ξ is a π-equilibrium. Of course, $F(\overline{\pi}, \overline{\xi}) = 0$. Let

$$\Phi(\pi, \xi; \mathring{y}) := \begin{pmatrix} U(\pi, y; \mathring{y}) \\ F(\pi, \xi) \\ \left\{ \sum_\sigma \pi_{s\sigma} - \overline{\pi}_s \right\}_{s \in S} \end{pmatrix},$$

where y is the allocation corresponding to ξ, and $\mathring{y}$ is any (exogenously given) allocation. Notice that $\Phi(\overline{\pi}, \overline{\xi}; \overline{y}) = 0$. The theorem is established by showing that the Jacobian of $\Phi(\pi, \xi; \mathring{y})$ with respect to (π, ξ) has full row rank at $(\overline{\pi}, \overline{\xi}; \overline{y})$, i.e. $D_{\pi,\xi}\Phi(\overline{\pi}, \overline{\xi}; \overline{y})$ has full row rank. This full rank property implies that there is a local change $(d\pi, d\xi)$ such that $d\Phi(\overline{\pi}, \overline{\xi}; \overline{y})$ is any desired vector. In particular, $(d\pi, d\xi)$ can be chosen so that $dU(\overline{\pi}, \overline{y}; \overline{y}) := U(\overline{\pi} + d\pi, \overline{y} + dy; \overline{y})$ is any desired vector in $\mathbb{R}^H$, $dF(\overline{\pi}, \overline{\xi}) := F(\overline{\pi} + d\pi, \overline{\xi} + d\xi) = 0$, and $\sum_\sigma d\pi_{s\sigma} = 0$ for all $s \in S$. The latter two conditions ensure that $(\overline{\pi} + d\pi) \in \Pi_{\overline{\pi}}$, and $(\overline{\xi} + d\xi)$ is a $(\overline{\pi} + d\pi)$-equilibrium.

The matrix $D_{\pi,\xi}\Phi(\overline{\pi},\overline{\xi};\overline{y})$ is given by

$$\left(\begin{array}{c|c|c} 0 & D_{y,\hat{p}}U(\overline{\pi},\overline{y};\overline{y}) & 0 \\ \hline \mathrm{diag}_\sigma \begin{pmatrix} & \vdots & \\ \dots & \left(\overline{u}^{h'}_{s\sigma} r_s - \overline{\lambda}^h_\sigma \overline{p}_\sigma\right) & \dots_s \\ & \vdots_h & \end{pmatrix} & * & \mathrm{diag}_{h,\sigma}[\overline{\pi}_\sigma \overline{p}_\sigma] \\ \hline 0 & D_{y,\hat{p}}\, g(\overline{\xi}) & 0 \\ \hline \dots I_S \dots_\sigma & 0 & 0 \end{array}\right). \tag{14}$$

Notice that we can eliminate the term $\overline{\lambda}^h_\sigma \overline{p}_\sigma$ in the (2,1) block with column operations involving the last column block. Rearranging rows, (14) is thus row/column-equivalent to

$$\left(\begin{array}{c|c} 0 & \begin{pmatrix} D_{y,\hat{p}}g(\overline{\xi}) \\ D_{y,\hat{p}}U(\overline{\pi},\overline{y};\overline{y}) \end{pmatrix} \quad 0 \\ \hline Q(\overline{y}) & * \end{array}\right), \tag{15}$$

where

$$Q(y) := \left(\begin{array}{c} \mathrm{diag}_\sigma \begin{pmatrix} & \vdots & \\ \dots & u^{h'}_{s\sigma} r_s & \dots_s \\ & \vdots_h & \end{pmatrix} \\ \hline \dots \; I_S \; \dots_\sigma \end{array}\right). \tag{16}$$

In the Appendix we show that, for every competitive equilibrium of a generic economy, under the dimensionality restrictions in the statement of the theorem, the portfolios and marginal utilities of agents satisfy the following conditions:[15]

C1. The vectors $\{\hat{y}^h_\sigma\}_{h\in H,\, h\neq h_1}$ are linearly independent, for all $\sigma \in \Sigma$.

C2. The vectors $(\dots \lambda^{h_1}_\sigma \dots_\sigma)$ and $(\dots \lambda^{h_2}_\sigma \dots_\sigma)$ are not collinear.

C3. $Q(y)$ has full row rank.

Each of these conditions holds for a different generic subset of $\mathcal{E}$ (see Lemmas A.2, A.4 and A.6). In the remainder of the proof, we assume that the economy $\overline{e}$ considered in the foregoing analysis is in the intersection of these three generic subsets. Thus $\overline{e}$ is a generic economy for which conditions **C1**, **C2** and **C3** hold. We will prove that, for such an economy, (15) has full row rank, and hence so does the Jacobian $D_{\pi,\xi}\Phi(\overline{\pi},\overline{\xi};\overline{y})$.

[15] The choice of agent h_1 in condition **C1**, and agents h_1 and h_2 in condition **C2**, is just a matter of convenience.

Since (15) is a block-triangular matrix, it suffices to show full row rank of the two blocks along the north-east diagonal. The matrix $Q(\overline{y})$ has full row rank due to **C3**. The nonzero term in the other diagonal block is

$$\begin{pmatrix} D_{y,\hat{p}}g(\overline{\xi}) \\ D_{y,\hat{p}}U(\overline{\pi},\overline{y};\overline{y}) \end{pmatrix} \sim_R \left(\begin{array}{c|c} \mathrm{diag}_{h,\sigma}[\overline{p}_\sigma^\top] & \mathrm{diag}_\sigma[\ldots \hat{\overline{y}}_\sigma^h \ldots_h]^\top \\ \hline \mathrm{diag}_\sigma[\ldots \hat{I} \ldots_h] & 0 \\ \hline \ldots \mathrm{diag}_h[\overline{\pi}_\sigma \overline{\lambda}_\sigma^h \overline{p}_\sigma^\top] \ldots_\sigma & 0 \end{array}\right), \tag{17}$$

where we have used (3) to evaluate D_yU, the (3,1) block of (17). Consider the top two submatrices of (17). Adding the rows corresponding to agents $h \neq h_1$ to the row for agent h_1, for every σ, we see that (17) is row-equivalent to

$$\left(\begin{array}{c|c} * & \mathrm{diag}_\sigma[\ldots \hat{\overline{y}}_\sigma^h \ldots_{h\neq h_1}]^\top \\ \hline \mathrm{diag}_\sigma[\ldots \overline{p}_\sigma^\top \ldots_h] & \\ \mathrm{diag}_\sigma[\ldots \hat{I} \ldots_h] & 0 \\ \ldots \mathrm{diag}_h[\overline{\pi}_\sigma \overline{\lambda}_\sigma^h \overline{p}_\sigma^\top] \ldots_\sigma & \end{array}\right).$$

We need to verify that this matrix has full row rank. It is block-triangular, and the upper right block has full row rank by **C1**. It remains to show that the lower left block also has full row rank. Rearranging rows and columns, this block is row/column-equivalent to

$$\left(\begin{array}{c} \ldots \mathrm{diag}_\sigma \begin{pmatrix} \overline{p}_\sigma^\top \\ \hat{I} \end{pmatrix} \ldots_h \\ \hline \mathrm{diag}_h[\ldots \overline{\pi}_\sigma \overline{\lambda}_\sigma^h \overline{p}_\sigma^\top \ldots_\sigma] \end{array}\right)$$

$$\sim_R \left(\begin{array}{c|c|c} \mathrm{diag}_\sigma \begin{pmatrix} \overline{p}_\sigma^\top \\ \hat{I} \end{pmatrix} & \mathrm{diag}_\sigma \begin{pmatrix} \overline{p}_\sigma^\top \\ \hat{I} \end{pmatrix} & \ldots \mathrm{diag}_\sigma \begin{pmatrix} \overline{p}_\sigma^\top \\ \hat{I} \end{pmatrix} \ldots_{h\neq h_1,h_2} \\ \hline 0 & \ldots \begin{pmatrix} \overline{\pi}_\sigma \overline{\lambda}_\sigma^{h_1} \overline{p}_\sigma^\top \\ \overline{\pi}_\sigma \overline{\lambda}_\sigma^{h_2} \overline{p}_\sigma^\top \end{pmatrix} \ldots_\sigma & * \\ \hline 0 & 0 & \mathrm{diag}_{h\neq h_1,h_2}[\ldots \overline{\pi}_\sigma \overline{\lambda}_\sigma^h \overline{p}_\sigma^\top \ldots_\sigma] \end{array}\right).$$

This is again a block-triangular matrix. The first diagonal block has full row rank, since $\hat{I} = (I_{J-1}\ \ 0)$ and $(p_\sigma)_J = 1$. The middle diagonal block has full row rank due to **C2**. The third diagonal block clearly has full row rank as well. Hence the whole matrix has full row rank as desired. □

A Appendix

We first provide proofs of Lemmas 4.2 and 4.4 stated in Section 4, and of Lemma A.1 that was invoked in the proof of Theorem 4.3. We then establish conditions **C1**, **C2** and **C3**, that were used in the proof of Theorem 5.1, for a generic subset of $\mathcal{E}$.

Proof of Lemma 4.2:

The matrices R and R' are column-equivalent if and only if $R' = RX$, for some $J \times J$ nonsingular matrix X. Let $\mathcal{S}(R) = \{S_1, \ldots, S_K\}$, and let $\bar{R}_k$ be the $S_k \times J$ submatrix of R consisting of the rows of R corresponding to the states in S_k. Similarly, let $\bar{R}'_k$ be the $S_k \times J$ submatrix of R' corresponding to S_k. Consider a vector $a \in \mathbb{R}^S$, and let $a_k \in \mathbb{R}^{S_k}$ be the elements of a corresponding to S_k. We have $a^\top R' = a^\top RX$ and $a_k^\top \bar{R}'_k = a_k^\top \bar{R}_k X$.

Now suppose $a^\top R' = 0$. Then $a^\top R = \sum_{k \in K} a_k^\top \bar{R}_k = 0$. Since the subspaces $\{L_k\}$ are linearly independent, we must have $a_k^\top \bar{R}_k = 0$, for all k. It follows that $a_k^\top \bar{R}'_k = 0$, for all k, and hence the subspaces $\{L'_k\}$ are linearly independent. Moreover, since $\{L_k\}$ is a maximal set of linearly independent subspaces, so is $\{L'_k\}$. This establishes that $\mathcal{S}(R) = \mathcal{S}(R')$.

We now show that there exists a $J \times J$ nonsingular matrix X such that RX has the block-diagonal structure in the statement of the theorem. Let M_k be the J_k-dimensional subspace of $\mathbb{R}^J$ that is the orthogonal complement of the subspace generated by $\{L_{\hat{k}}\}_{\hat{k} \neq k}$. We claim that the subspaces $\{M_k\}$ are linearly independent. Indeed, consider $m_k \in M_k$ such at $\sum_k m_k = 0$. Then, $\ell_k \cdot \sum_k m_k = 0$, for all $\ell_k \in L_k$. But $\ell_k \cdot m_{\hat{k}} = 0$, for all $\hat{k} \neq k$. Therefore, $\ell_k \cdot \sum_k m_k = \ell_k \cdot m_k = 0$, for all $\ell_k \in L_k$, i.e. m_k is orthogonal to L_k. By the definition of M_k, m_k is orthogonal to $L_{\hat{k}}$, for all $\hat{k} \neq k$. Consequently, m_k is orthogonal to $\mathbb{R}^J$, implying that it is zero. The same argument applies for all values of k.

Let X_k be a $J \times J_k$ matrix whose columns are a basis of M_k. Thus every column of X_k is orthogonal to every row of R that does not correspond to the states in S_k. Therefore, $\bar{R}_{\hat{k}} X_k = 0$, for all $\hat{k} \neq k$. Let $X := (X_1 \ldots X_K)$. Then $RX = \mathrm{diag}_k[R_k]$, where $R_k := \bar{R}_k X_k$, an $S_k \times J_k$ matrix. Since the subspaces $\{M_k\}$ are linearly independent, X is nonsingular. This proves that $R \sim_C \mathrm{diag}_k[R_k]$. Moreover, $\mathrm{rank}(R_k) = \mathrm{rank}(\bar{R}_k) = J_k$. $\square$

Proof of Lemma 4.4:

Let $\hat{S}$ be a subset of S satisfying the conditions of the lemma. We choose a pair of states in $\hat{S}$, in a manner that we specify later in the proof. It is convenient to reorder the states in S so that these are the first two states, labeled s_1 and s_2. It suffices to establish the result for the first two agents, h_1 and h_2. We will show that generically there is no solution to the equation system

$$\Psi_3(\xi_\sigma, \mu, \omega; \overline{\pi}^\sigma) := \begin{pmatrix} F_\sigma(\xi_\sigma, \omega; \overline{\pi}^\sigma) \\ \left\{\Delta_{s\sigma}^{h_1 h_2}(y_\sigma, \mu, \omega)\right\}_{s \in \{s_1, s_2\}} \end{pmatrix} = 0.$$

The system Ψ_3 is obtained from the equilibrium system F_σ by appending one additional variable, μ, and two additional equations.

The Jacobian $D_{\xi_\sigma,\mu,\omega}\Psi_3$, evaluated at a zero $(\bar{\xi}_\sigma, \mu, \omega)$ of Ψ_3, is

$$\left(\begin{array}{c|c|ccc} * & 0 & & \operatorname{diag}_h\left[\ldots \overline{\pi}_{s\sigma}\, \overline{u}^{h\prime\prime}_{s\sigma} r_s \ldots_s\right] & \\ \hline D_{\xi_\sigma} g_\sigma & 0 & & 0 & \\ \hline * & \begin{array}{c} -\overline{u}^{h_2\prime}_{s_1\sigma} \\ -\overline{u}^{h_2\prime}_{s_2\sigma} \end{array} & \operatorname{diag}_{s\in\{s_1,s_2\}}\left[\overline{u}^{h_1\prime\prime}_{s\sigma}\right] & 0_{2\times(S-2)} & * \end{array}\right),$$

which is row-equivalent to

$$\left(\begin{array}{c|c|cc|c} * & \begin{array}{c} -\overline{u}^{h_2\prime}_{s_1\sigma} \\ -\overline{u}^{h_2\prime}_{s_2\sigma} \end{array} & \operatorname{diag}_{s\in\{s_1,s_2\}}\left[\overline{u}^{h_1\prime\prime}_{s\sigma}\right] & 0 & * \\ \hline * & 0 & \ldots \overline{\pi}_{s\sigma}\, \overline{u}^{h_1\prime\prime}_{s\sigma} r_s \ldots_s & & 0 \\ \hline * & 0 & 0 & & \operatorname{diag}_{h\neq h_1}\left[\ldots \overline{\pi}_{s\sigma}\, \overline{u}^{h\prime\prime}_{s\sigma} r_s \ldots_s\right] \\ \hline D_{\xi_\sigma} g_\sigma & 0 & 0 & & 0 \end{array}\right).$$

We wish to show that this matrix has full row rank. Since $D_{\xi_\sigma} g_\sigma$ has full row rank, as does the matrix $\operatorname{diag}_{h\neq h_1}\left[\ldots \overline{\pi}_{s\sigma}\, \overline{u}^{h\prime\prime}_{s\sigma} r_s \ldots_s\right]$, it suffices to show that

$$A_3 := \left(\begin{array}{c|cc} \begin{array}{c} -\overline{u}^{h_2\prime}_{s_1\sigma} \\ -\overline{u}^{h_2\prime}_{s_2\sigma} \end{array} & \operatorname{diag}_{s\in\{s_1,s_2\}}\left[\overline{u}^{h_1\prime\prime}_{s\sigma}\right] & 0 \\ \hline 0 & \ldots \overline{\pi}_{s\sigma}\, \overline{u}^{h_1\prime\prime}_{s\sigma} r_s \ldots_s & \end{array}\right)$$

has full row rank, $J+2$. We have

$$A_3 \sim_C \left(\begin{array}{cc|c|c} & I_2 & \begin{array}{c} -\overline{u}^{h_2\prime}_{s_1\sigma} \\ -\overline{u}^{h_2\prime}_{s_2\sigma} \end{array} & 0 \\ \hline \overline{\pi}_{s_1\sigma}\, r_{s_1} & \overline{\pi}_{s_2\sigma}\, r_{s_2} & 0 & \ldots r_s \ldots_{s\notin\{s_1,s_2\}} \end{array}\right)$$

$$\sim_R \left(\begin{array}{c|c|c} I_2 & \begin{array}{c} -\overline{u}^{h_2\prime}_{s_1\sigma} \\ -\overline{u}^{h_2\prime}_{s_2\sigma} \end{array} & 0 \\ \hline 0 & \overline{\pi}_{s_1\sigma}\, \overline{u}^{h_2\prime}_{s_1\sigma} r_{s_1} + \overline{\pi}_{s_2\sigma}\, \overline{u}^{h_2\prime}_{s_2\sigma} r_{s_2} & \ldots r_s \ldots_{s\notin\{s_1,s_2\}} \end{array}\right).$$

Thus, in order to establish that $\operatorname{rank}(A_3) = J+2$, it suffices to show that the matrix

$$B := \left(\begin{array}{c|c} \overline{\pi}_{s_1\sigma}\, \overline{u}^{h_2\prime}_{s_1\sigma} r_{s_1} + \overline{\pi}_{s_2\sigma}\, \overline{u}^{h_2\prime}_{s_2\sigma} r_{s_2} & \ldots r_s \ldots_{s\notin\{s_1,s_2\}} \end{array}\right)$$

has rank J. We now argue that this is indeed the case, for an appropriate choice of the states s_1 and s_2.

By assumption, $\hat{S}$ contains a nontrivial insurable event, say S_k, so that $J_k < S_k$. If $J_k \leq S_k - 2$, we can choose s_1 and s_2 in S_k such that both r_{s_1} and r_{s_2} are spanned by the vectors $\{r_s\}_{s\in S_k, s\notin\{s_1,s_2\}}$. Then the right submatrix of B has rank J, and hence so does B. Suppose then that $J_k = S_k - 1$. Let s_1 be any state in S_k. The vector r_{s_1} must be in the span of $\{r_s\}_{s\in S_k, s\neq s_1}$ (if this is not the case, then s_1 is an insurable state, contradicting the fact that s_1 belongs to the nontrivial insurable event S_k). If S_k is a strict subset of $\hat{S}$, let s_2 be a state in $\hat{S}$ that is not in S_k. Then r_{s_1} is in the span of $\{r_s\}_{s\in S, s\notin\{s_1,s_2\}}$, so that

$$B \sim_C \left(\begin{array}{c|c} r_{s_2} & \ldots r_s \ldots_{s\notin\{s_1,s_2\}} \end{array} \right),$$

which has rank J.

The only case that remains to be considered, therefore, is the one where $\hat{S} = S_k$, with $J_k = S_k - 1$. As before, let s_1 be any state in S_k. We claim that there is a state $s_2 \in S_k$, such that $\operatorname{rank}(B) = J$. Suppose not. Then, there exist coefficients $\{\alpha_{\hat{s},s}\}_{\hat{s}\in S_k, s\in S}$ such that

$$\overline{\pi}_{s_1\sigma}\, \overline{u}^{h_2\prime}_{s_1\sigma} r_{s_1} + \overline{\pi}_{\hat{s}\sigma}\, \overline{u}^{h_2\prime}_{\hat{s}\sigma} r_{\hat{s}} = \sum_{s\in S, s\notin\{s_1,\hat{s}\}} \alpha_{\hat{s},s}\, r_s, \qquad \forall \hat{s} \in S_k.$$

Since r_{s_1} can be written as a unique linear combination of $\{r_s\}_{s\in S_k, s\neq s_1}$, we must have

$$\overline{\pi}_{s_1\sigma}\, \overline{u}^{h_2\prime}_{s_1\sigma} r_{s_1} = -\sum_{s\in S_k, s\neq s_1} \overline{\pi}_{s\sigma}\, \overline{u}^{h_2\prime}_{s\sigma} r_s,$$

i.e.

$$\sum_{s\in S_k} \overline{\pi}_{s\sigma}\, \overline{u}^{h_2\prime}_{s\sigma} r_s = 0. \tag{18}$$

We now invoke the assumption that there exists a portfolio whose payoff is nonnegative and nonzero in $\hat{S}$.[16] In the case under consideration, $\hat{S} = S_k$. Hence, there exists $x^* \in \mathbb{R}^J$ such that $r_s \cdot x^* \geq 0$ for all $s \in S_k$, and $r_s \cdot x^* > 0$ for some $s \in S_k$. Then we must have $\sum_{s\in S_k} \overline{\pi}_{s\sigma}\, \overline{u}^{h_2\prime}_{s\sigma} r_s \cdot x^* > 0$, which contradicts (18).

We have shown that $D_{\xi_\sigma,\mu,\omega}\Psi_3$ has full row rank, at every zero of Ψ_3. Thus Ψ_3 is transverse to zero, and $\Psi^{-1}_{3\omega}(0) = \varnothing$ for all ω in a generic subset of Ω. $\square$

[16] This is the only place in the proof where we use this assumption. Thus the assumption is needed only for the case where $\hat{S}$ is a nontrivial insurable event S_k, and $J_k = S_k - 1$.

Lemma A.1 *Suppose markets are incomplete, and there are at least two insurable events. Let S_k be a nontrivial insurable event, $k \neq K$. Then, for a generic subset of Ω, at any $\overline{\pi}$-equilibrium allocation $\overline{y}$,*

$$\bar{\Delta}^{h\hat{h}}_{s\sigma}\big(\overline{\mu}^{h\hat{h}}_{\sigma}(\overline{\pi}^{\sigma})\big) \neq 0, \qquad \forall s \in S_k;\, h, \hat{h} \in H;\, \sigma \in \Sigma.$$

Proof :
Let S_k be a nontrivial insurable event, $k \neq K$. It suffices to establish the result for the pair of agents h_1 and h_2, and for one state in S_k, which we take to be the first state s_1, for convenience in writing the matrix computations below. We show that generically there is no solution to the equation system:

$$\Psi_4(\xi_\sigma, \mu, \omega; \overline{\pi}^{\sigma}) := \begin{pmatrix} F_\sigma(\xi_\sigma, \omega; \overline{\pi}^{\sigma}) \\ \Delta^{h_1h_2}_{s_1\sigma}(y_\sigma, \mu, \omega) \\ \mu^{h_1h_2}_{\sigma}(y_\sigma, \omega; \overline{\pi}^{\sigma}) - \mu \end{pmatrix} = 0.$$

The Jacobian $D_{\xi_\sigma,\mu,\omega}\Psi_4$, evaluated at a zero $(\overline{\xi}_\sigma, \overline{\mu}^{h_1h_2}_{\sigma}, \omega)$ of Ψ_4, is

$$\left(\begin{array}{c|c|c}
* & -1 & 0_{1\times 1} \quad \left[\ldots \frac{\overline{\pi}_{s\sigma}\,\overline{u}^{h_1\,\prime\prime}_{s\sigma}\,r^J_s}{\sum_{s\neq s_1}\overline{\pi}_{s\sigma}\,\overline{u}^{h_2\,\prime}_{s\sigma}\,r^J_s}\ldots_{s\neq s_1}\right] \quad * \\
\hline
* & 0 & \operatorname{diag}_h\big[\ldots\overline{\pi}_{s\sigma}\,\overline{u}^{h\,\prime\prime}_{s\sigma}r_s\ldots_s\big] \\
\hline
D_{\xi_\sigma}g_\sigma & 0 & 0 \\
\hline
* & -\overline{u}^{h_2\,\prime}_{s_1\sigma} & \overline{u}^{h_1\,\prime\prime}_{s_1\sigma} \quad 0_{1\times(S-1)} \quad *
\end{array}\right),$$

where we have used the fact that the payoff of asset J in s_1 is zero (since asset J pays off only in S_K). The Jacobian is row-equivalent to

$$\left(\begin{array}{c|c|c|c|c}
* & -1 & 0 & \ldots \frac{\overline{\pi}_{s\sigma}\,\overline{u}^{h_1\,\prime\prime}_{s\sigma}\,r^J_s}{\sum_{s\neq s_1}\overline{\pi}_{s\sigma}\,\overline{u}^{h_2\,\prime}_{s\sigma}\,r^J_s}\ldots_{s\neq s_1} & * \\
\hline
* & -\overline{u}^{h_2\,\prime}_{s_1\sigma} & \overline{u}^{h_1\,\prime\prime}_{s_1\sigma} & 0 & * \\
\hline
* & 0 & \overline{\pi}_{s_1\sigma}\,\overline{u}^{h_1\,\prime\prime}_{s_1\sigma}r_{s_1} & \ldots\overline{\pi}_{s\sigma}\,\overline{u}^{h_1\,\prime\prime}_{s\sigma}r_s\ldots_{s\neq s_1} & 0 \\
\hline
* & 0 & 0 & 0 & \operatorname{diag}_{h\neq h_1}\big[\ldots\overline{\pi}_{s\sigma}\,\overline{u}^{h\,\prime\prime}_{s\sigma}r_s\ldots_s\big] \\
\hline
D_{\xi_\sigma}g_\sigma & 0 & 0 & 0 & 0
\end{array}\right).$$

Since $D_{\xi_\sigma}g_\sigma$ has full row rank, as does the matrix $\operatorname{diag}_{h\neq h_1}\big[\ldots\overline{\pi}_{s\sigma}\,\overline{u}^{h\,\prime\prime}_{s\sigma}r_s\ldots_s\big]$, it suffices to show that

$$A_4 := \left(\begin{array}{c|c|c}
-1 & 0 & \ldots \frac{\overline{\pi}_{s\sigma}\,\overline{u}^{h_1\,\prime\prime}_{s\sigma}\,r^J_s}{\sum_{s\neq s_1}\overline{\pi}_{s\sigma}\,\overline{u}^{h_2\,\prime}_{s\sigma}\,r^J_s}\ldots_{s\neq s_1} \\
\hline
-\overline{u}^{h_2\,\prime}_{s_1\sigma} & \overline{u}^{h_1\,\prime\prime}_{s_1\sigma} & 0 \\
\hline
0 & \overline{\pi}_{s_1\sigma}\,\overline{u}^{h_1\,\prime\prime}_{s_1\sigma}r_{s_1} & \ldots\overline{\pi}_{s\sigma}\,\overline{u}^{h_1\,\prime\prime}_{s\sigma}r_s\ldots_{s\neq s_1}
\end{array}\right)$$

has full row rank, $J+2$. Exploiting once again the fact that $r_{s_1}^J = 0$, we can do a row operation using the last row of A_4 to get

$$A_4 \sim_R \left(\begin{array}{c|c|c} -1 & 0 & 0 \\ \hline -\overline{u}_{s_1\sigma}^{h_2\prime} & \overline{u}_{s_1\sigma}^{h_1\prime\prime} & 0 \\ \hline 0 & \overline{\pi}_{s_1\sigma}\,\overline{u}_{s_1\sigma}^{h_1\prime\prime} r_{s_1} & \ldots \overline{\pi}_{s\sigma}\,\overline{u}_{s\sigma}^{h_1\prime\prime} r_s \ldots_{s\neq s_1} \end{array}\right).$$

Since S_k is a nontrivial insurable event, r_{s_1} is in the span of $\{r_s\}_{s\in S_k, s\neq s_1}$. It follows that the bottom right block has rank J. Hence $\mathrm{rank}(A_4) = J+2$ as desired.

We have established that Ψ_4 is transverse to zero, so that $\Psi_{4\omega}^{-1}(0) = \varnothing$ for all ω in a generic subset of Ω. $\square$

Lemma A.2 *Suppose $J \geq H$. Then, for a generic subset Ω_Y of Ω, condition* **C1** *is satisfied at any equilibrium.*

Proof :
Consider the equation system

$$\Psi_5(\xi_\sigma, \psi_5, \omega) := \begin{pmatrix} F_\sigma(\xi_\sigma, \omega) \\ \psi_5^\top Y_\sigma \\ \psi_5^\top \psi_5 - 1 \end{pmatrix} = 0,$$

where $\psi_5 \in \mathbb{R}^{H-1}$, and Y_σ is the square matrix obtained from $[\ldots \hat{y}_\sigma^h \ldots_h]$ by deleting the first column and the last $J-H$ rows. We will show that, for a generic subset of endowments, this system has no solution, and hence Y_σ has full rank at any equilibrium.

The Jacobian of Ψ_5 is

$$D_{\xi_\sigma,\psi_5,\omega}\Psi_5 = \left(\begin{array}{cc|c} D_{\xi_\sigma} f_\sigma & 0 & D_\omega f_\sigma \\ \hline D_{\xi_\sigma} g_\sigma & 0 & 0 \\ D_{\xi_\sigma}(\psi_5^\top Y_\sigma) & Y_\sigma^\top & 0 \\ 0 & 2\psi_5^\top & 0 \end{array}\right). \tag{19}$$

Note that the matrix

$$D_{y_\sigma}\begin{pmatrix} g_\sigma \\ \psi_5^\top Y_\sigma \end{pmatrix} = \begin{pmatrix} \mathrm{diag}_h[p_\sigma^\top] \\ [\ldots \hat{I} \ldots_h] \\ \begin{pmatrix} 0 & \mathrm{diag}_h[\psi_5^\top\ 0] \end{pmatrix} \end{pmatrix}$$

is row-equivalent to

$$\begin{pmatrix} p_\sigma^\top & 0 & \ldots & 0 \\ \hat{I} & \hat{I} & \ldots & \hat{I} \\ 0 & p_\sigma^\top & \ldots & 0 \\ 0 & [\psi_5^\top\ 0] & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & p_\sigma^\top \\ 0 & 0 & \ldots & [\psi_5^\top\ 0] \end{pmatrix}. \tag{20}$$

Since $(p_\sigma)_J = 1$ and, at any zero of Ψ_5, $\psi_5 \neq 0$, (20) has full row rank, and hence so does the lower left block of (19). Furthermore, $D_\omega f_\sigma$ has full row rank. It follows that the Jacobian $D_{\xi_\sigma,\psi_5,\omega}\Psi_5$ has full row rank, at every zero of Ψ_5. Thus Ψ_5 is transverse to zero, and $\Psi_{5\omega}^{-1}(0) = \varnothing$ for all $\omega \in \Omega_{Y\sigma}$, a generic subset of Ω. The set Ω_Y in the statement of the lemma is given by $\Omega_Y := \cap_{\sigma\in\Sigma}\Omega_{Y\sigma}$. $\square$

In order to establish conditions **C2** and **C3**, we will need to independently perturb the marginal utilities of agents, for different signal realizations σ. This cannot be achieved via endowment perturbations, since endowments do not vary with respect to σ. Instead, we perturb the (initial) information structure π. As a first step we show that, for a generic subset of endowments, marginal utilities vary sufficiently with respect to π, across both s and σ. Let

$$W(\xi_{\sigma_1}, \xi_{\sigma_2}, \pi^1, \pi^2, \omega) := \begin{pmatrix} f_{\sigma_1}(\xi_{\sigma_1}, \pi^1, \omega) \\ \sum_s \pi_{s\sigma_1} - \overline{\pi}_{\sigma_1} \\ f_{\sigma_2}(\xi_{\sigma_2}, \pi^2, \omega) \\ \sum_s \pi_{s\sigma_2} - \overline{\pi}_{\sigma_2} \end{pmatrix},$$

where $\pi^i := \{\pi_{s\sigma_i}\}_{s\in S}$, and $\overline{\pi}_{\sigma_i}$ is an arbitrary constant in $(0,1)$, $i = 1,2$. We show in Lemma A.3 below that, for a generic subset of endowments, $D_{\pi^1,\pi^2}W$ has full row rank at any equilibrium. Note that we are considering perturbations in π^1 and π^2 that leave π_{σ_1} and π_{σ_2} unchanged. In particular this ensures that the perturbations lie in the set Π. Lemma A.3 is invoked in all the lemmas that follow (the assumptions in the statement of Lemma A.3 are implied by those in the statements of Lemmas A.4, A.5 and A.6).

Lemma A.3 *Suppose $S \geq J(H+1)+1$, and R is in general position. Let $\pi \in \Pi$ be an information structure with $\sum_s \pi_{s\sigma} = \overline{\pi}_\sigma$. Then, for a generic subset Ω_W of Ω, at any π-equilibrium, $D_{\pi^1,\pi^2}W$ has full row rank.*

Proof :
We have

$$f_\sigma = \sum_s \pi_{s\sigma}\, u^{h\prime}\left[\omega_s^h + r_s \cdot y_\sigma^h\right] r_s - \overline{\pi}_\sigma \lambda_\sigma^h\, p_\sigma, \qquad \sigma = \sigma_1, \sigma_2.$$

Therefore,

$$D_{\pi^1,\pi^2}W = \mathrm{diag}_{\sigma\in\{\sigma_1,\sigma_2\}} \left(\begin{array}{c} \vdots \\ \ldots\ u_{s\sigma}^{h\prime} r_s\ \ldots_s \\ \vdots_h \\ \hline \ldots\ 1\ \ldots_s \end{array} \right).$$

Let

$$\Gamma_\sigma := \begin{pmatrix} \vdots \\ \ldots\ u_{s\sigma}^{h\prime} r_s\ \ldots_{s\leq JH+1} \\ \vdots_h \end{pmatrix}$$

and $\mathbf{1}^\top = (1\,1\ldots1)_{1\times(JH+1)}$. We claim that, for a generic subset of Ω, at a π-equilibrium, the (square) matrix

$$\breve{\Gamma}_\sigma := \left(\begin{array}{c}\Gamma_\sigma\\ \hline \mathbf{1}^\top\end{array}\right)$$

has full rank, i.e. for $\psi_6 \in \mathbb{R}^{JH+1}$, there is no solution to

$$\Psi_6(\xi_\sigma, \psi_6, \omega) := \begin{pmatrix} F_\sigma(\xi_\sigma,\omega)\\ \Gamma_\sigma(\xi_\sigma,\omega)\,\psi_6\\ \mathbf{1}^\top\psi_6\\ \psi_6^\top\psi_6 - 1\end{pmatrix} = 0.$$

The Jacobian, $D_{\xi_\sigma,\psi_6,\omega}\Psi_6$, is row-equivalent to

$$\left(\begin{array}{c|c|c} * & * & D_\omega\begin{pmatrix} f_\sigma\\ \Gamma_\sigma\,\psi_6\end{pmatrix}\\ \hline 0 & \begin{array}{c}\mathbf{1}^\top\\ \psi_6^\top\end{array} & 0\\ \hline D_{\xi_\sigma} g_\sigma & 0 & 0\end{array}\right). \tag{21}$$

We wish to show that this matrix has full row rank at any zero of Ψ_6. As we have seen already, $D_{\xi_\sigma} g_\sigma$ has full row rank. Also, ψ_6 is orthogonal to $\mathbf{1}$ and nonzero (since $\psi_6^\top\psi_6 = 1$). Hence, due to the block-triangular structure of (21), it suffices to show that the upper right block, given by

$$D_\omega\begin{pmatrix} f_\sigma\\ \Gamma_\sigma\,\psi_6\end{pmatrix} = \left(\begin{array}{c}\operatorname{diag}_h\left[\ldots\ \pi_{s\sigma}\,u^{h''}_{s\sigma}\,r_s\ \ldots_s\right]\\ \hline \operatorname{diag}_h\left[\left(\ldots\psi_{6s}u^{h''}_{s\sigma}\,r_s\ldots_{s\le JH+1}\right)\quad(0\ldots0)\right]\end{array}\right),$$

has full row rank. This matrix is row-equivalent to a block-diagonal matrix, with blocks indexed by h. The h'th block is:

$$\left(\begin{array}{c|c}\ldots\pi_{s\sigma}\,u^{h''}_{s\sigma}\,r_s\ldots_{s\le JH+1} & \ldots\pi_{s\sigma}\,u^{h''}_{s\sigma}\,r_s\ldots_{s>JH+1}\\ \hline \ldots\psi_{6s}u^{h''}_{s\sigma}\,r_s\ldots_{s\le JH+1} & 0\end{array}\right). \tag{22}$$

This matrix is block-triangular as well, and its upper right block has full row rank since it has at least J columns and R is in general position. It remains to show that the lower left block of (22) has full row rank. Let $\bar{S}$ be the subset of states for which $\psi_{6s} \neq 0$. This is a nonempty subset at any zero of Ψ_6.[17] Then we have $\sum_{s\in\bar{S}}\psi_{6s}u^{h'}_{s\sigma}r_s = 0$. Since R is in general position, and $u^{h'}_{s\sigma}$ is nonzero for all s, we must have $\#\bar{S} \ge J+1$. Full row rank of the lower left block of (22) now follows from the general position of R.

[17] $\bar{S}$ may depend on the zero of Ψ_6 we are considering, but this does not affect our argument.

We have shown that the Jacobian $D_{\xi_\sigma,\psi_6,\omega}\Psi_6$ has full row rank, at every zero of Ψ_6. Thus Ψ_6 is transverse to zero, and $\Psi_{6\omega}^{-1}(0)=\varnothing$ for all $\omega\in\Omega_{W\sigma}$, a generic subset of Ω. This establishes that $\breve{\Gamma}_\sigma$ has full row rank. It follows that $D_{\pi^1,\pi^2}W$ has full row rank for the generic subset of endowments $\Omega_W := \Omega_{W\sigma_1}\cap\Omega_{W\sigma_2}$. $\square$

Lemma A.4 *Suppose $S\geq J(H+1)+1$, and R is in general position. Then, for a generic subset of $\Omega\times\Pi$, condition* **C2** *is satisfied at any equilibrium.*

Proof :
We restrict endowments to lie in the generic subset Ω_W of Ω for which $D_{\pi^1,\pi^2}W$ has full row rank (see Lemma A.3), and show that, for a generic subset of Π, there is no solution to

$$\Psi_7(\xi_{\sigma_1},\xi_{\sigma_2},\pi^1,\pi^2) := \begin{pmatrix} F_{\sigma_1}(\xi_{\sigma_1},\pi^1) \\ F_{\sigma_2}(\xi_{\sigma_2},\pi^2) \\ \sum_s \pi_{s\sigma_1}-\overline{\pi}_{\sigma_1} \\ \sum_s \pi_{s\sigma_2}-\overline{\pi}_{\sigma_2} \\ \lambda_{\sigma_1}^{h_1}\lambda_{\sigma_2}^{h_2}-\lambda_{\sigma_2}^{h_1}\lambda_{\sigma_1}^{h_2} \end{pmatrix} = 0.$$

The Jacobian, $D_{\xi_{\sigma_1},\xi_{\sigma_2},\pi^1,\pi^2}\Psi_7$, is row/column-equivalent to

$$\left(\begin{array}{ccc|c} & * & & D_{\pi^1,\pi^2}W \\ \hline D_{y_{\sigma_1}}g_{\sigma_1} & 0 & * & \\ 0 & D_{y_{\sigma_2}}g_{\sigma_2} & * & 0 \\ 0 & 0 & D_{p_{\sigma_1},p_{\sigma_2},\lambda_{\sigma_1},\lambda_{\sigma_2}}\left[\lambda_{\sigma_1}^{h_1}\lambda_{\sigma_2}^{h_2}-\lambda_{\sigma_2}^{h_1}\lambda_{\sigma_1}^{h_2}\right] & \end{array}\right).$$

Since $D_{y_\sigma}g_\sigma$ has full row rank, and $\lambda_\sigma^h\neq 0$ for all σ, it follows that the lower left block of the above matrix has full row rank. Since $D_{\pi^1,\pi^2}W$ also has full row rank, the Jacobian $D_{\xi_{\sigma_1},\xi_{\sigma_2},\pi^1,\pi^2}\Psi_7$ has full row rank. Thus Ψ_7 is transverse to zero, and $\Psi_{7\pi}^{-1}(0)=\varnothing$ for all π in a generic subset of Π. $\square$

In order to establish condition **C3**, we will employ finite-dimensional perturbations of the agents' utility functions. This procedure requires the following result.

Lemma A.5 *Suppose $S\geq J(H+1)+1$, $J\geq H$, and R is in general position. Then, for a generic subset $\Omega_C\times\Pi_C$ of $\Omega\times\Pi$, at any equilibrium, $c_{\hat{s}\hat{\sigma}}^h\neq c_{\check{s}\check{\sigma}}^h$ for all $(\hat{s},\hat{\sigma})\neq(\check{s},\check{\sigma})$, and for all $h\in H$.*

Proof :
Consider first the case where $\hat{s}\neq\check{s}$ and $\hat{\sigma}=\check{\sigma}$. Without loss of generality, we can take $\hat{s}=s_1$ and $\check{s}=s_2$, and prove the result for the first agent, h_1, for a given σ. We will show that, for a generic subset of Ω, there is no solution to

$$\Psi_8(\xi_\sigma,\omega) := \begin{pmatrix} F_\sigma(\xi_\sigma,\omega) \\ c^{h_1}_{s_1\sigma} - c^{h_1}_{s_2\sigma} \end{pmatrix} = 0.$$

Here we use $c^h_{s\sigma}$ as shorthand notation for $\omega^h_s + r_s \cdot y^h_\sigma$, not as an additional variable. The Jacobian, $D_{\xi_\sigma,\omega}\Psi_8$, is row-equivalent to

$$\left(\begin{array}{c|c} * & D_\omega \begin{pmatrix} f_\sigma \\ c^{h_1}_{s_1\sigma} - c^{h_1}_{s_2\sigma} \end{pmatrix} \\ \hline D_{\xi_\sigma} g_\sigma & 0 \end{array}\right),$$

and

$$D_\omega \begin{pmatrix} f_\sigma \\ c^{h_1}_{s_1\sigma} - c^{h_1}_{s_2\sigma} \end{pmatrix} = \left(\begin{array}{cccc} \multicolumn{4}{c}{\mathrm{diag}_h[\ldots\ \pi_{s\sigma}\, u^{h\prime\prime}_{s\sigma}\, r_s\ \ \ldots_s]} \\ \hline 1 & -1 & 0_{1\times(S-2)} & \multicolumn{1}{|c}{0} \end{array}\right). \tag{23}$$

Note that the zero block in the lower right of (23) corresponds to all agents other than h_1. The matrix (23) is row-equivalent to

$$\left(\begin{array}{ccc|c} \pi_{s_1\sigma}\, u^{h_1\prime\prime}_{s_1\sigma}\, r_{s_1} & \pi_{s_2\sigma}\, u^{h_1\prime\prime}_{s_2\sigma}\, r_{s_2} & [\ldots\pi_{s\sigma}\, u^{h_1\prime\prime}_{s\sigma} r_s \ldots_{s>2}] & \\ & & & 0 \\ 1 & -1 & 0 & \\ \hline \multicolumn{3}{c|}{0} & \mathrm{diag}_{h\neq h_1}[\ldots\pi_{s\sigma}\, u^{h\prime\prime}_{s\sigma}\, r_s\ldots_s] \end{array}\right) \tag{24}$$

The lower right submatrix of (24) has full row rank due to the general position of R. The $(1,3)$ block of the upper left submatrix of (24) also has full row rank for the same reason (note that the assumed dimensionality condition implies that $S \geq J+2$). Consequently, the whole matrix (24) has full row rank, and hence so does (23). Furthermore, due to the full row rank of $D_{\xi_\sigma} g_\sigma$, the Jacobian $D_{\xi_\sigma,\omega}\Psi_8$ has full row rank as well. Thus Ψ_8 is transverse to 0, and $\Psi^{-1}_{8\omega}(0) = \varnothing$ for all $\omega \in \Omega_{1\sigma}$, a generic subset of Ω. Let $\Omega_1 := \cap_{\sigma\in\Sigma}\Omega_{1\sigma}$.

Now consider the case where $\hat\sigma \neq \check\sigma$, while $\hat s$ may or may not be equal to $\check s$. Without loss of generality, we can take $\hat\sigma = \sigma_1$, $\check\sigma = \sigma_2$, and $\hat s = s_1$, and prove the result for the first agent, h_1. We restrict the set of endowments to the generic subset $\Omega_Y \cap \Omega_W$, for which condition **C1** holds and $D_{\pi^1,\pi^2}W$ has full row rank (see Lemmas A.2 and A.3), and show that, for a generic subset of Π, there is no solution to

$$\Psi_9(\xi_{\sigma_1},\xi_{\sigma_2},\pi^1,\pi^2) := \begin{pmatrix} F_{\sigma_1}(\xi_{\sigma_1},\pi^1) \\ F_{\sigma_2}(\xi_{\sigma_2},\pi^2) \\ \sum_s \pi_{s\sigma_1} - \overline{\pi}_{\sigma_1} \\ \sum_s \pi_{s\sigma_2} - \overline{\pi}_{\sigma_2} \\ c^{h_1}_{s_1\sigma_1} - c^{h_1}_{\check s\sigma_2} \end{pmatrix} = 0.$$

We have

$$D_{\xi_{\sigma_1},\xi_{\sigma_2},\pi_1,\pi_2}\Psi_9 \sim_R \left(\begin{array}{c|c} * & D_{\pi^1,\pi^2}W \\ \hline \left(\begin{array}{c|c} D_{\xi_{\sigma_1}}\begin{pmatrix} c^{h_1}_{s_1\sigma_1} \\ g_{\sigma_1}\end{pmatrix} & * \\ \hline 0 & D_{\xi_{\sigma_2}} g_{\sigma_2}\end{array}\right) & 0 \end{array}\right).$$

Since $D_{\pi^1,\pi^2}W$ has full row rank, it suffices to show that the lower left block of the above matrix has full row rank. This block is itself block-triangular, and $D_{\xi_{\sigma_2}} g_{\sigma_2}$ has full row rank. The other diagonal term is:

$$D_{\xi_{\sigma_1}}\begin{pmatrix} c^{h_1}_{s_1\sigma_1} \\ g_{\sigma_1}\end{pmatrix} = \left(\begin{array}{c|c|c} r_{s_1}^\top \quad 0_{1\times J(H-1)} & 0 & 0 \\ \hline \mathrm{diag}_h[p_{\sigma_1}^\top] & [\ldots \hat{y}^h_{\sigma_1} \ldots_h]^\top & 0 \\ \hline \ldots \hat{I} \ldots_h & 0 & 0 \end{array}\right).$$

Consider the middle row block of this matrix. Adding the rows corresponding to all agents other than h_1 to the first row of this block, and using the market-clearing condition, we get:

$$D_{\xi_{\sigma_1}}\begin{pmatrix} c^{h_1}_{s_1\sigma_1} \\ g_{\sigma_1}\end{pmatrix} \sim_R \left(\begin{array}{c|c|c|c} r_{s_1}^\top & 0 & 0 & 0 \\ \hline p_{\sigma_1}^\top & \ldots p_{\sigma_1}^\top \ldots_{h\neq h_1} & 0 & 0 \\ \hline 0 & \mathrm{diag}_{h\neq h_1}[p_{\sigma_1}^\top] & [\ldots \hat{y}^h_{\sigma_1} \ldots_{h\neq h_1}]^\top & 0 \\ \hline \hat{I} & \ldots \hat{I} \ldots_{h\neq h_1} & 0 & 0 \end{array}\right)$$

$$\sim_R \left(\begin{array}{c|c|c} 0 & \mathrm{diag}_{h\neq h_1}[p_{\sigma_1}^\top] & [\ldots \hat{y}^h_{\sigma_1} \ldots_{h\neq h_1}]^\top \quad 0 \\ \hline \begin{array}{c} p_{\sigma_1}^\top \\ \hat{I} \end{array} & \ldots \begin{pmatrix} p_{\sigma_1}^\top \\ \hat{I} \end{pmatrix} \ldots_{h\neq h_1} & 0 \\ \hline r_{s_1}^\top & 0 & 0 \end{array}\right).$$

This matrix has a block-triangular structure. Since condition **C1** holds, the top right diagonal block has full row rank. The middle diagonal block has full row rank since $\hat{I} = (I_{J-1} \;\; 0)$ and the Jth element of p_{σ_1} is nonzero. The bottom left block is nonzero by the general position of R.

It follows that the Jacobian $D_{\xi_{\sigma_1},\xi_{\sigma_2},\pi^1,\pi^2}\Psi_9$ has full row rank, at every zero of Ψ_9. Thus Ψ_9 is transverse to zero, and $\Psi_{9\pi}^{-1}(0) = \varnothing$ for all $\pi \in \Pi_C$, a generic subset of Π. The set Ω_C in the statement of the lemma is given by $\Omega_C := \Omega_1 \cap \Omega_Y \cap \Omega_W$. $\square$

The space of utility functions $\mathcal{U}$ is infinite-dimensional. For the genericity arguments that we use in order to establish condition **C3**, it suffices to consider a finite-dimensional submanifold of $\mathcal{U}$. This submanifold consists of linear perturbations of the von Neumann-Morgenstern utility index of each agent in the neighbor-

hood of consumption in each state (s, σ), for a given equilibrium. This is a standard construction (see, for example, Citanna et al. (1998)).[18]

Consider an economy $(\breve{\omega}, \breve{u}, \breve{\pi}) \in \Omega_C \times \mathcal{U}^H \times \Pi_C$, and a corresponding equilibrium with consumption allocation $\{\breve{c}^h_{s\sigma}\}_{h\in H, s\in S, \sigma\in\Sigma}$. By Lemma A.5, the consumption level of agent h, $\breve{c}^h_{s\sigma}$, is distinct across (s, σ), for all h. Therefore, we can find open intervals $\breve{B}^h_{s\sigma}, B^h_{s\sigma}$ such that $\breve{c}^h_{s\sigma} \in \breve{B}^h_{s\sigma} \subsetneq B^h_{s\sigma} \subset \mathbb{R}_{++}$, where the intervals $B^h_{s\sigma}$ are disjoint across (s, σ). Define $\mathcal{C}^2$ functions $\rho^h_{s\sigma} : \mathbb{R}_{++} \to [0, 1]$ such that $\rho^h_{s\sigma} = 1$ on $\breve{B}^h_{s\sigma}$ and $\rho^h_{s\sigma} = 0$ on the complement of $B^h_{s\sigma}$.[19] Now consider the class of utility functions u^h parametrized by $\nu^h \in \mathbb{R}^{S\Sigma}$:

$$u^h(c, \nu^h) := \breve{u}^h(c) + \sum_{s,\sigma} \rho^h_{s\sigma}(c)\, \nu^h_{s\sigma}(c - \breve{c}^h_{s\sigma}).$$

It can be verified that, for ν^h sufficiently small in norm, $u^h \in \mathcal{U}$. We have

$$u^{h\prime}(c, \nu^h) = \breve{u}^{h\prime}(c) + \sum_{s,\sigma} \rho^{h\prime}_{s\sigma}(c)\, \nu^h_{s\sigma}(c - \breve{c}^h_{s\sigma}) + \sum_{s,\sigma} \rho^h_{s\sigma}(c)\, \nu^h_{s\sigma},$$

so that

$$D_{\nu^h_{s\sigma}} u^{h\prime}(\breve{c}^h_{s\sigma}, \nu^h) = 1. \tag{25}$$

Let $\nu_\sigma := \{\nu^h_{s\sigma}\}_{h\in H, s\in S}$, and $\nu := \{\nu_\sigma\}_{\sigma\in\Sigma}$. In order to show that condition **C3** holds, we will perturb utility functions via perturbations of ν.

Lemma A.6 *Suppose $S \geq 2JH$, $J \geq H$, and R is in general position. Then, for a generic subset of $\mathcal{E}$, condition* **C3** *is satisfied at any equilibrium.*

Proof :
Let Λ_σ be the $JH \times S$ matrix defined by

$$\Lambda_\sigma := \left(\begin{array}{c} \vdots \\ \ldots\ u^{h\prime}_{s\sigma} r_s\ \ldots_s \\ \vdots_h \end{array} \right).$$

We first establish that

$$\Lambda := \left(\begin{array}{c} \Lambda_{\breve{\sigma}} \\ \Lambda_{\hat{\sigma}} \end{array} \right)$$

has full row rank, for all $\breve{\sigma} \neq \hat{\sigma}$. Without loss of generality we can take $\breve{\sigma} = \sigma_1$ and $\hat{\sigma} = \sigma_2$. Let $\hat{\Lambda}$ be the (square) submatrix of Λ consisting of the first $2JH$ columns of Λ. Let $\psi_{10} \in \mathbb{R}^{2JH}$. We will show that, for a generic subset of $\mathcal{U}^H \times \Pi$, there is no solution to

[18] Unlike Citanna et al. (1998), we perturb the gradient of the utility functions instead of their Hessian. Also, we have state-independent separable utility so additional care has to be exercised in perturbing utilities in different states.

[19] The existence of such a "bump" function is well-known. See Guillemin and Pollack (1974), chapter 1.

$$\Psi_{10}(\xi_{\sigma_1},\xi_{\sigma_2},\psi_{10},\nu_{\sigma_1},\nu_{\sigma_2},\pi^1,\pi^2) := \begin{pmatrix} F_{\sigma_1}(\xi_{\sigma_1},\nu_{\sigma_1},\pi^1) \\ F_{\sigma_2}(\xi_{\sigma_2},\nu_{\sigma_2},\pi^2) \\ \sum_s \pi_{s\sigma_1} - \overline{\pi}_{\sigma_1} \\ \sum_s \pi_{s\sigma_2} - \overline{\pi}_{\sigma_2} \\ \hat{\Lambda}(\xi_{\sigma_1},\xi_{\sigma_2},\nu_{\sigma_1},\nu_{\sigma_2})\psi_{10} \\ \psi_{10}^\top\psi_{10} - 1 \end{pmatrix} = 0.$$

We restrict ourselves to the generic subset of endowments and initial signals $(\Omega_R \cap \Omega_C) \times \Pi_C$ for which Lemma A.5 applies, so that we can parametrize utility functions by the vector ν, and for which the number of equilibria is finite. Recall that Ω_C is a subset of Ω_W, for which $D_{\pi^1,\pi^2}W$ has full row rank (see Lemma A.3). The Jacobian, $D_{\xi_{\sigma_1},\xi_{\sigma_2},\psi_{10},\nu_{\sigma_1},\nu_{\sigma_2},\pi^1,\pi^2}\Psi_{10}$, is row-equivalent to

$$\left(\begin{array}{ccc|c|c} & * & & * & D_{\pi^1,\pi^2}W \\ \hline & * & & D_{\nu_{\sigma_1},\nu_{\sigma_2}}(\hat{\Lambda}\psi_{10}) & 0 \\ \hline 0 & 0 & 2\psi_{10}^\top & & \\ D_{\xi_{\sigma_1}}g_{\sigma_1} & 0 & 0 & 0 & 0 \\ 0 & D_{\xi_{\sigma_2}}g_{\sigma_2} & 0 & & \end{array}\right). \tag{26}$$

We wish to show that this matrix has full row rank. Since $D_{\pi^1,\pi^2}W$, $D_{\xi_{\sigma_1}}g_{\sigma_1}$ and $D_{\xi_{\sigma_2}}g_{\sigma_2}$ have full row rank, and $\psi_{10} \neq 0$, it suffices to show that the middle block of (26) has full row rank. Using (25), we see that this block is block-diagonal with respect to $h \in H$ and $\sigma \in \{\sigma_1, \sigma_2\}$, with typical diagonal term given by $[\ldots \psi_{10s} r_s \ldots_{s \leq 2JH}]$. This diagonal term has full row rank by the same argument that we used for the bottom left block of (22). It follows that the Jacobian $D_{\xi_{\sigma_1},\xi_{\sigma_2},\psi_{10},\nu_{\sigma_1},\nu_{\sigma_2},\pi^1,\pi^2}\Psi_{10}$ has full row rank, at every zero of Ψ_{10}. Thus Ψ_{10} is transverse to zero, and $\Psi_{10u\pi}^{-1}(0) = \varnothing$ for all $(u,\pi) \in \mathcal{U}^H \times \Pi_C$.

Now that we have established that Λ has full row rank, consider the matrix Q, defined in (16). We have

$$Q = \left(\begin{array}{c} \operatorname{diag}_\sigma[\Lambda_\sigma] \\ \hline \ldots\ I_S\ \ldots_\sigma \end{array}\right).$$

The upper submatrix of Q has full row rank due to the full row rank of Λ, and clearly the lower submatrix of Q has full row rank as well. If Q does not have full row rank, there exist vectors $a_\sigma \in \mathbb{R}^{JH}$, for all $\sigma \in \Sigma$, and $b \in \mathbb{R}^S$, not all of which are zero, such that $a_\sigma^\top \Lambda_\sigma + b = 0$, for all σ. Moreover, since Λ_σ has full row rank, $b \neq 0$. It follows that the row spaces of $\{\Lambda_\sigma\}_{\sigma\in\Sigma}$ have a nontrivial intersection. But this contradicts the full row rank property of Λ.

The utility perturbations in this proof apply only to the particular equilibrium under consideration. However, we can repeat the same construction for each equilibrium, and take the intersection of the generic subsets for which the Jacobian of Ψ_{10} has full row rank. This intersection is itself a generic subset since the number of equilibria is finite (recall that endowments are restricted to Ω_R). $\square$

References

Blackwell, D. (1951). Comparison of experiments. In *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, pages 93–102. Berkeley: UC Press.

Blume, L., Coury, T., and Easley, D. (2006). Information, trade and incomplete markets. *Economic Theory*, 29:379–394.

Cass, D. and Citanna, A. (1998). Pareto improving financial innovation in incomplete markets. *Economic Theory*, 11:467–494.

Citanna, A., Kajii, A., and Villanacci, A. (1998). Constrained suboptimality in incomplete markets: A general approach and two applications. *Economic Theory*, 11:495–521.

Citanna, A. and Villanacci, A. (2000). Incomplete markets, allocative efficiency and the information revealed by prices. *Journal of Economic Theory*, 90:222–253.

Diamond, P. A. (1967). The role of a stock market in a general equilibrium model with technological uncertainty. *American Economic Review*, 57(4):759–776.

Elul, R. (1995). Welfare effects of financial innovation in incomplete markets economies with several consumption goods. *Journal of Economic Theory*, 65:43–78.

Geanakoplos, J. and Mas-Colell, A. (1989). Real indeterminacy with financial assets. *Journal of Economic Theory*, 47:22–38.

Geanakoplos, J. and Polemarchakis, H. (1986). Existence, regularity, and constrained suboptimality of competitive allocations when the asset market is incomplete. In Walter Heller, R. S. and Starrett, D., editors, *Uncertainty, Information and Communication. Essays in Honor of Kenneth J. Arrow, Volume III.* Cambridge University Press.

Gottardi, P. and Rahi, R. (2001). Efficiency properties of rational expectations equilibria with asymmetric information. Mimeo, London School of Economics.

Green, J. (1981). Value of information with sequential futures markets. *Econometrica*, 49:335–358.

Greenwald, B. and Stiglitz, J. E. (1986). Externalities in economies with imperfect information and incomplete markets. *Quarterly Journal of Economics*, 101:229–264.

Guillemin, V. and Pollack, A. (1974). *Differential Topology.* Prentice Hall.

Hakansson, N. H., Kunkel, J. G., and Ohlson, J. A. (1982). Sufficient and necessary conditions for information to have social value in pure exchange. *Journal of Finance*, 37:1169–1181.

Hirshleifer, J. (1971). The private and social value of information and the reward to inventive activity. *American Economic Review*, 61:561–574.

Magill, M. and Quinzii, M. (1996). *Theory of Incomplete Markets, Volume 1.* MIT Press.

Milne, F. and Shefrin, H. M. (1987). Information and securities: A note on Pareto dominance and the second best. *Journal of Economic Theory*, 43:314–328.

Schlee, E. E. (2001). The value of information in efficient risk sharing arrangements. *American Economic Review*, 91(3):509–524.

Villanacci, A., Carosi, L., Benevieri, P., and Battinelli, A. (2002). *Differential Topology and General Equilibrium with Complete and Incomplete Markets.* Kluwer Academic Publishers.