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A Copula Approach to Cross-Market Diversification[*]

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Abstract

Equity markets are vulnerable to rare and extreme events such as crashes and booms. Extremely positive or negative returns in the US stock market significantly affect the rest of developed markets. We use the copula method to show that stocks traded in European emerging markets typically show very little sensitivity to extreme events in the US market, despite a high degree of asymmetry between the co-movements in the lower and upper tail of return distributions. This finding makes them a good candidate for cross-market diversification.

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Non-technical Summary

When diversifying their portfolios investors typically seek for cost-effective ways to increase the number of positions they hold. Apart from usual cross-industry diversification, further benefits can be gained by investing in foreign securities because they tend to be less correlated with domestic investments. To perform a proper cross-market diversification, the knowledge of the dynamics and interdependency of international financial markets is essential. It is also crucial for subsequent risk management of such geographically diverse portfolios.

The conventional approach to modeling uncertainty is to specify a joint distribution of the random variables as a product of marginal and conditional distributions. A disadvantage with the typical marginal-and-conditional approach is that the required number of probability assessments can grow exponentially with the number of variables. Analysts respond by searching diligently for conditional independence among variables to reduce the assessment burden.

In this paper we discuss an alternative in which a joint distribution is constructed using a copula, requiring only marginal distributions and measures of dependence among the random variables. A copula is a function that joins or "couples" a multivariate distribution function to its one-dimensional marginal distribution functions. Using the copula that underlies the multivariate normal distribution, a complete copula-based joint distribution can be constructed using assessed rank-order correlations and marginal distributions, thereby reducing the number of required assessments and relaxing the need to search for conditional independence.

We study the possibility of a cross-market diversification using daily data for nine EEM value-weighted indices and the corresponding series for the S&P 500 index. The data was obtained from Thomson Datastream and covers the period from October 20, 1996 to September 22, 2006, a series of 1545 observations. All the indices are denominated in US Dollars. They display a marked departure from normality, since the values of Jarque-Bera statistics are significantly above the critical values for the chi-square distribution. Based on the coefficients of linear correlation only, the indices could be separated into three well-distinguished groups: (1) markets with small negative linear correlation with S&P 500 (Slovenia, Croatia, and Bulgaria); (2) markets with relatively small positive linear correlation with S&P 500 (Romania and Slovakia); and (3) markets whose linear correlation coefficient with S&P 500 is above 0.10 (Hungary, Czech Republic, Turkey, and Poland).

Apart from the linear correlation coefficient, we use two other measures of concordance (conditional correlation) between the indices: Kendall's τ and Spearman's ρ_S. We can thus identify groups of indices with high probability of having simultaneous incidence of extremely high values (like, for example SBI (Slovenia), SOFIX (Bulgaria), and CROBEX (Croatia)) or extremely low values (BET (Romania), PX (Czech Republic), ISE (Turkey), and WIG (Poland)). We find that all concordance measures are positive across the emerging market indices, which is indicative of a co-moving tendency in extreme events.

The only negative values are obtained for concordance of SBI, SOFIX, and CROBEX indices with S&P 500, which is in qualitative agreement with negative linear correlations between these markets. Perhaps more importantly, the obtained values for τ and ρ_S point to a very low degree of correlation between booms and crashes in the US and similar extreme events in EEMs. This conclusion is even more important when one takes into account that distributions of stock price returns throughout EEMs typically have more fat tails than in the US.

Additionally, we document the differences between the lower and upper tail dependence in order to assess the tail dependence asymmetry. To introduce a proper measure of tail dependence, we consider the likelihood that one event with probability lower than v occurs in one variable, given that an event with probability lower than v occurs in the other. If we compute this dependence measure far in the lower tail, we obtain the so-called *lower tail index*. If we choose to condition everything with respect to the S&P 500 index, then the lower tail index can be interpreted as the probability of a crash in one of the EEMs simultaneously with a crash in the US market. Similarly, we can define the *upper tail index* and interpret it as the probability that a positive jump in an EEM stock market occurs at the same time as in the US. Unlike τ and ρ_S, whose sample analogs do not depend on the choice of copula, the tail indices will differ depending on the choice of the copula model. Copula functions could be distinguished according to the balance between the weight put on the upper and lower tails. The ability of copulas to capture tail dependence is a crucial one for our study. Therefore, we will focus on several copula models that have positive lower or upper tail dependences, such as Clayton, rotated Clayton, Gumbel, Student and Joe-Clayton.

To determine which copula fits our data the best, we rank each of the pairs X and Y according to their canonical maximum likelihood, Akaike Information Criterion and Bayesian Information Criterion, where X is an EEM index while Y is S&P 500. It turns out that Student copula is the best fit for all market indices, except BUX (Hungary) where Joe-Clayton is the best choice. The problem with the Student copula is its symmetric tail dependence, which may not be realistic for some markets that display sharp asymmetry in the likelihood of extreme events. For example, there is a pronounced left coupling displayed among Turkish and US market, which is captured by rotated Gumbel copula. This is implying that the assumption of joint normality is violated in a dangerous direction in this case. Thus, when most needed, cross-market diversification may sometimes be of limited use as a mean of reducing portfolio risk. However, most of the EEMs display very little tendency of simultaneous positive or negative jumps, which makes them a good candidate for cross-market diversification.

Extension of this study to time-varying copulas with non-zero tail dependence is a subject of our ongoing research. It will be crucial for further analysis of extreme behavior of stock markets and verification of the effectiveness of cross-market diversification, especially in light of the ongoing global financial crisis.

1 Introduction

When diversifying their portfolios investors typically seek for cost-effective ways to increase the number of positions they hold. Apart from usual cross-industry diversification, further benefits can be gained by investing in foreign securities because they tend to be less correlated with domestic investments. To perform a proper cross-market diversification, the knowledge of the dynamics and interdependency of international financial markets is essential. It is also crucial for subsequent risk management of such geographically diverse portfolios.

One of the key issues in risk management is the aggregation of individual risk into overall portfolio risk. This is especially important when dealing with securities that trade in genuinely different markets. The technical problem here is that, at some point in the process, one has to make assumptions about the dependence structure between the factors that drive individual risk. The standard practice in assessing the overall risk of a portfolio is to assume that asset prices are driven by jointly normal random variables. The assumption of joint normality (or ellipticity) is often implicitly made through the use of linear correlation as the measure of dependence. Perhaps the best known example is JP Morgan's CreditMetrics (1997), where credit ratings are driven by unobserved jointly normal distributions. However, different joint distributions with the same correlation matrix can give rise to different Value-at-Risk (VaR) forecasts.

A straightforward approach in dealing with the dependency problem is to consider time series for the return on the entire portfolio rather than a set of univariate return series. In this case it is possible to empirically assess the distribution of the portfolio return and calculate its VaR directly, without worrying about dependence or correlation. For example, Engle & Manganelli (1999) propose a modified GARCH model to describe the evolution of VaR directly, while McNeil & Frey (2000) suggest using a GARCH model to estimate the conditional mean and variance of the portfolio return first, and then modeling the distribution of the residuals by employing extreme value theory and historical simulation, to provide estimates for the VaR.

However, when considering problems such as the choice of optimal portfolio, one inevitably has to understand the dependence structure between individual assets. A possible approach is to use a time-dependent correlation. Correlation is very sensitive to the patterns of market volatility. More specifically, empirical evidence of Longin & Solnik (1995) documents that correlation is higher in periods of larger volatilities. An alternative is to note that any multivariate distribution function F can be split into two parts. The first part is the set of univariate distribution functions (the marginals) of each of the random time series, F_i, while the second part is the function (the copula) that outlines the dependence structure between the series, C. The relation between F and F_i through C can then be written in the form

$$F\left(x_1,\ldots,x_n\right)=C\left(F_1(x_1),\ldots,F_n(x_n)\right).$$

The copula approach in risk management seems natural if one wants to capture different sources of risk – the individual and the portfolio risk. Another key advantage when using copulas is that they are able to capture the co-movement of different time series under extreme conditions. Extreme events, such as crashes and booms, are very rare and difficult to assess probabilistically. Yet, their impact can have a long-lasting effect not only on individual investors' portfolios, but also on the economy as a whole. It is therefore of uttermost importance to have a technical tool for managing the extreme-event risk.

From the face value, it seems that cross-market diversification alone may not be enough to hedge the risk of extreme equity returns. For example, having a "well-diversified" portfolio that consisted only of US and EU stocks would not be of much use on October 17, 1987 or immediately after September 11, 2001. The reason is that, although the crashes that happened on these dates occurred in the US market, they had a very big impact with similar consequences throughout all major exchanges in Europe.

Such examples reveal that there is a tendency for developed and highly integrated markets to react in the similar fashion when one of them is facing a crash or a boom. Longin &

Solnik (2001) show, however, that the correlation between stock return series tends to be higher in market downturns than in market upturns, a fact for which standard symmetric models of multivariate stock returns cannot account. Indeed, the authors reject joint normality for the negative tail of the multivariate distribution, but not for the positive tail. In other words, there seems to be significant dependence in the lower tail of the joint distribution, which cannot be explained by assuming joint normality with its implied zero tail-dependence.

To achieve a good diversification, and yet avoid any co-movement in extreme events, a good alternative might be to invest in the markets where such co-movement is negligible, especially in case of downturns. In this paper we investigate whether European emerging markets (EEM) are a good choice in that respect. To this end, we compare the return series for the US market with nine market indices in emerging Europe, taking the Standard & Poor 500 Composite Index as a proxy for the US market portfolio. We use copula method to assess whether EEMs are correlated with the US stock market when the latter is facing extremely high or low returns. Our approach is somewhat similar to that of Patton (2001a), Fortin & Kuzmics (2002) and De La Peña *et al.* (2004).

By calculating several standard measures of concordance, we show that there is a little to none tendency of EEMs to have unusually high or low returns during times when US market is in the upturn or downturn. To discern between asymmetric impacts of extremely positive and extremely negative returns in the S&P 500, we fit appropriate copulas to each of the individual EEM return series and compute the left and right tail indices. Overall, EEMs seem to be inert to extreme movements of the US market. This makes them a good candidate for cross-market hedging of extreme event risk.

The remainder of the paper is organized as follows. In Section 2 we present the basic properties of copulas and several families of copulas useful for further analysis. For further survey see Hutchinson & Lai (1990), Dall'Aglio (1991), Joe (1997) and Nelsen (2006) and conference proceedings edited by Ruschendorf *et al.* (1996), Benes & Stepan (1997) and Cuadras *et al.* (2002). Section 3 describes our estimation procedure for copula dependence,

while Section 4 examines the dependence structure between nine EEMs and the US market. To deal with fat tails and asymmetry in the return distributions, we test several copula models and find the best fitting one for each of the indices. We conclude in Section 5.

2 The copula theory

The essence of the copula approach is that a joint distribution of random variables can be expressed as a function of the marginal distributions. To make this notion precise, we review few essential mathematical results.

The standard "operational" definition of a copula is a multivariate distribution function defined on the unit cube $[0,1]^n$, with uniformly distributed marginals. This definition is very natural if one considers how a copula is derived from a continuous multivariate distribution function. Indeed, in this case the copula is simply the original multivariate distribution function with transformed univariate margins. This definition however obscures some of the problems we face when trying to construct copulas using other techniques, i.e. it does not say what is meant by a multivariate distribution function. For that matter, we start with a slightly more abstract definition, returning to the "operational" one later. The overview of copula theory presented below follows the lines of Patton (2001a, 2001b). A very readable and thorough introduction to the theory of copulas may be found in Joe (1997) and Nelsen (2006).

2.1 The copula and transformations of random variables

Let $U_t \equiv F_t(X_t)$ and $V_t \equiv G_t(Y_t)$. We say that U_t and V_t are the probability integral transforms of X_t and Y_t.

Theorem 1 *(Fischer, 1932)* If F_t and G_t are continuous distribution functions, then $U_t \equiv F_t(X_t) \sim Unif(0,1)$ and $V_t \equiv G_t(Y_t) \sim Unif(0,1)$.

This theorem enables us to formally introduce the copula. Let $U_t \equiv F_t(X_t)$ and $V_t \equiv G_t(Y_t)$, as above. Next, we find the joint density of U and V. We denote the joint density of U and V as c, which turns out to be the "copula density". Since F and G are strictly increasing and continuous, we have that $X = F^{-1}(U)$ and $Y = G^{-1}(V)$, while

$$\frac{\partial X}{\partial U} = \left(\frac{\partial U}{\partial X}\right)^{-1} = \left(\frac{\partial F(X)}{\partial X}\right)^{-1} = f(X)^{-1}$$

and

$$\frac{\partial Y}{\partial V} = \left(\frac{\partial V}{\partial Y}\right)^{-1} = \left(\frac{\partial G(Y)}{\partial Y}\right)^{-1} = g(Y)^{-1}.$$

Note that $\frac{\partial X}{\partial V} = \frac{\partial Y}{\partial U} = 0$. Then,

$$c(u,v) = \frac{h\left(F^{-1}(u), G^{-1}(v)\right)}{f\left(F^{-1}(u)\right) \cdot g\left(G^{-1}(v)\right)} \tag{1}$$

Equation (1) shows that the copula density of X and Y is equal to the ratio of the joint density, h to the product of the marginal densities, f and g. From this expression we can obtain a first result on the properties of copulas: if X and Y are independent, then the copula density takes the value 1 everywhere, since in that case the joint density is equal to the product of the marginal densities. Since we know that the marginal densities of U and V are uniform, by Theorem 1 above, we thus have that if X and Y are independent the joint distribution of U and V is the bivariate $Uniform(0,1)$ distribution.

We can also use equation (1) to derive an expression for h as a function of x and y instead:

$$h(x,y) = f(x) \cdot g(y) \cdot c\left(F(x), G(y)\right) \tag{2}$$

Equation (2) is the "density version" of Sklar's (1959) theorem: the joint density, h, can be decomposed into product of the marginal densities, f and g, and the copula density, c. Sklar's theorem holds under more general conditions than the ones we imposed for this illustration. The connection between copulas and general bivariate distributions is given by the Sklar's theorem, perhaps the most important result regarding copulas.

2.2 The theory of the conditional copula

For an introduction to the general theory of copulas the reader is referred to Nelsen (2006) or Chapter 6 of Schweizer & Sklar (1983). We start with a few very basic, but very important, definitions based on those in Nelsen (2006). The second condition below refers to the volume of a rectangle $[x_1,x_2]\times[y_1,y_2]\subset\mathbb{R}^2$, denoted by V_{H_t}. This is simply the probability of observing a point in the region $[x_1,x_2]\times[y_1,y_2]$. It is expressed in the following way as it generalizes more easily to the multivariate case.

Definition 1 *A conditional bivariate distribution function is a right continuous function* $H_t:\mathbf{R}^2\to[0,1]$ *with the properties:*

1. $H_t(x,-\infty\,|\,\mathcal{F}_{t-1})=H_t(-\infty,y\,|\,\mathcal{F}_{t-1})=0$ and $H_t(-\infty,\infty\,|\,\mathcal{F}_{t-1})=1$
2. $V_{H_t}([x_1,x_2]\times[y_1,y_2]) = H_t(x_2,y_2\,|\,\mathcal{F}_{t-1}) - H_t(x_2,y_1\,|\,\mathcal{F}_{t-1}) - H_t(x_1,y_2\,|\,\mathcal{F}_{t-1}) + H_t(x_1,y_1\,|\,\mathcal{F}_{t-1}) \ge 0$

   $\forall x_1,x_2,y_1,y_2\in\bar{\mathbb{R}} \wedge x_1\le x_2 \wedge y_1\le y_2$ where $\mathcal{F}_{t-1}$ is some conditioning set.

The first condition provides the upper and lower bounds on the distribution function. The second condition ensures that the probability of observing a point in the region $[x_1,x_2]\times[y_1,y_2]$ is non-negative.

Let us now define the conditional copula.

Definition 2 *A two-dimensional conditional copula is a function* $C_t:[0,1]\times[0,1]\to[0,1]$ *with the following properties:*

1. $C_t(u,0\,|\,\mathcal{F}_{t-1})=C_t(0,v\,|\,\mathcal{F}_{t-1})=0 \wedge C_t(u,1\,|\,\mathcal{F}_{t-1})=u \wedge C_t(1,v\,|\,\mathcal{F}_{t-1})=v\,\forall u,v\in[0,1]$
2. $V_{C_t}([u_1,u_2]\times[v_1,v_2]\,|\,\mathcal{F}_{t-1}) = C_t(u_2,v_2\,|\,\mathcal{F}_{t-1}) - C_t(u_2,v_1\,|\,\mathcal{F}_{t-1}) - C_t(u_1,v_2\,|\,\mathcal{F}_{t-1}) + C_t(u_1,v_1\,|\,\mathcal{F}_{t-1}) \ge 0$

   $\forall u_1,u_2,v_1,v_2\in[0,1] \wedge u_1\le u_2 \wedge v_1\le v_2$, where $\mathcal{F}_{t-1}$ is some conditioning set.

The first condition of Definition 2 provides the lower bound on the distribution function, and ensures that the marginal distributions, $C_t\left(u,1 \mid \mathrm{F}_{t-1}\right)$ and $C_t\left(1,v \mid \mathrm{F}_{t-1}\right)$, are uniform. The condition that V_{C_t} is non-negative has the same interpretation as the second condition of Definition 1 it simply ensures that the probability of observing a point in the region $[u_1,u_2]$¥$[v_1,v_2]$ is non-negative. By drawing on the above conditions for the conditional copula, and extending its domain to $\mathbf{R}^2$, we may alternatively define a conditional copula as the conditional bivariate distribution of a pair of random variables (U_t,V_t) having margins that are $\mathit{Uniform}(0,1)$. The extension of the domain to $\mathbf{R}^2$ is accomplished as follows:

$$\text{Let } C_t^*(u,2 \mid \Phi_{t-1}) = \begin{cases} 0 & \text{for } u < 0 \text{ or } v < 0, \\ C_t(u,v \mid \Phi_{t-1}) & \text{for } (u,v) \in [0,1] \times [0,1], \\ u & \text{for } u \in [0,1], v > 1, \\ v & \text{for } u > 1, v \in [0,1], \\ 1 & \text{for } u > 1, v > 1, \end{cases} \tag{3}$$

Now it becomes clear: the copula is the joint distribution function of the probability integral transforms of each of the variables X_t and Y_t with respect to their marginal distributions, F_t and G_t. We now move on to an extension of the key result in the theory of copulas: Sklar's (1959) theorem for conditional distributions:

Theorem 2 *Let H_t be a conditional bivariate distribution function with continuous margins F_t and G_t, and let F_{t-1} be some conditioning set. Then there exists a unique conditional copula $C_t : [0,1]$¥$[0,1]$Æ$[0,1]$ such that*

$$H_t(x,y \mid \Phi_{t-1}) = C_t\left(F_t(x \mid \Phi_{t-1}), G_t(x \mid \Phi_{t-1}) \mid \Phi_{t-1}\right) \ \forall x,y \in \Upsilon^2 \tag{4}$$

Conversely, if C_t is a conditional copula and F_t and G_t are the conditional distribution functions of two random variables X_t and Y_t, then the function H_t defined by equation (4) is a bivariate conditional distribution function with margins F_t and G_t.

From Sklar’s Theorem we see that for continuous multivariate distribution functions the univariate margins and the multivariate dependence structure can be separated, and the dependence structure can be represented by a copula. The density-function equivalent to equation (4) is useful for maximum likelihood analysis, and is obtained quite easily, provided that F_t and G_t are differentiable, and H_t and C_t are twice differentiable:

$$h_t(x,y \mid \Phi_{t-1}) = f_t(x \mid \Phi_{t-1}) \cdot g_t(y \mid \Phi_{t-1}) \cdot c_t(u,v \mid \Phi_{t-1}), \forall (x,y) \in \Upsilon^2, \tag{5}$$

where $u \equiv F_t(x \mid \Phi_{t-1})$ and $v \equiv G_t(y \mid \Phi_{t-1})$. The expression in equation (5) is precisely the same as that in equation (2), which we obtained using the theory on the distribution of transformations of random variables. Taking the logarithm of both sides we obtain:

$$L_{XY} = L_X + L_Y + L_C. \tag{6}$$

In other words, the joint log-likelihood is equal to the sum of the marginal log-likelihoods and the copula log-likelihood.

We can also obtain a corollary to Theorem 2 analogous to that of Nelson's (1999) corollary to Sklar's Theorem, which enables us to extract the conditional copula from any conditional bivariate distribution function, but first we need the definition of the “quasi-inverse” of a function.

Definition 3 *The quasi-inverse, F^{-1} of a distribution function F is defined as:*

$$F^{-1}(u) = \inf\{x : F(x) \geq u\}, \forall u \in [0,1]. \tag{7}$$

If F is strictly increasing then the above definition returns the usual functional inverse of F, but more importantly it allows us to consider inverses of non-strictly increasing functions.

Corollary 1 *Let H_t be any conditional bivariate distribution with continuous marginal distributions, margins F_t and G_t, and let F_t^{-1} and G_t^{-1} denote the (quasi-) inverses of the marginal distributions. Finally, let Φ_{t-1} be some conditioning set. Then there exists a unique conditional copula $C_t : [0,1] \times [0,1] \rightarrow [0,1]$ such that*

$$C_t(u, v \mid \Phi_{t-1}) = H_t\left(F_t^{-1}(u \mid \Phi_{t-1}), G_t^{-1}(v \mid \Phi_{t-1}) \mid \Phi_{t-1}\right) \quad \forall u, v \in [0,1] \tag{8}$$

This corollary completes the idea that a bivariate distribution function may be decomposed into three parts. Given any two marginal distributions and any copula we have a joint distribution, and from any given joint distribution we can extract the implied marginal distributions and copula.

2.3 Families of Copulas

If one has a collection of copulas, then using Sklar's theorem, one can construct bivariate distributions with arbitrary margins. Thus, for the purposes of statistical modeling, it is desirable to have a collection of copulas at one's disposal. A great many examples of copulas can be found in the literature, most are members of families with one or more real parameters. (Members of such families are often denoted by C_θ, $C_{\alpha,\beta}$, etc.) We now present a very brief overview of some parametric families of copulas. Extensive surveys of families of copulas can be found in Hutchinson & Lai (1990), Joe (1997) and Nelsen (2006).

2.3.1 The Farlie-Gumbel-Morgenstern family

$$C_\theta(u, v) = uv + \theta uv(1 - u)(1 - v), \; \theta \in [-1, 1] \tag{9}$$

These are the only copulas whose functional form is a polynomial quadratic in u and in v. They are commonly denoted FGM copulas. Members of the FGM family are symmetric, i.e., $C_{\theta}(u,v) = C_{\theta}(v,u)$, $\forall (u,v) \in [-1,1]^2$. A pair (X,Y) of random variables is said to be *exchangeable* if the vectors (X,Y) and (Y,X) are identically distributed. For identically distributed continuous random variables, exchangeability is equivalent to the symmetry of the copula.

2.3.2 Copulas cubic in u and v

$$C(u,v) = uv + uv(1-u)(1-v)\left[\alpha uv + \beta u(1-v) + \gamma v(1-u) + \delta(1-u)(1-v)\right], \quad (10)$$

where α, β, γ and δ are real constants chosen so that the points (α,β), (α,γ), (δ,β) and (δ,γ) all lie in the set

$$[-1,2] \times [-2,1] \cap \left\{(x,y) \mid x^2 - xy + y^2 - 3x + 3y \le 0\right\}.$$

When $\alpha = \beta = \gamma = \delta = \theta$, C is quadratic rather than cubic in u and in v, and the copulas are members of the FGM family. Unlike FGM copulas, copulas cubic in u and in v may be asymmetric. For further details, see Nelsen (2006).

2.3.3 Gaussian copulas

Let $N_{\rho}(x,y)$ denote the standard bivariate normal joint distribution function with correlation coefficient ρ. Then C_{ρ}, the copula corresponding to N_{ρ}, is given by $C_{\rho}(u,v) = N_{\rho}\left(\Phi^{-1}(u), \Phi^{-1}(v)\right)$, where Φ denotes the standard normal (Gaussian) distribution function. Since there is no closed form expression for Φ^{-1}, there is neither a closed form expression for N_{ρ}. However, N_{ρ} can be evaluated approximately in order to

construct bivariate distribution functions with the same dependence structure as the standard bivariate normal distribution function but with non-normal marginals.

Definition 4 *For a pair* X, Y *of random variables with marginal distribution functions* F, G *respectively, and joint distribution function* H*, the marginal survival functions* F, G *and joint survival function* H *are given by* $F(x) = P[X > x]$, $G(y) P[Y > y]$ *and* $H(x,y) = P[X > x,\ Y > y]$ *respectively. The function* C*, which couples the joint survival function to its marginal survival functions, is called the* survival copula*:* $X(x,y) = C(F(x), G(y))$.

It is easy to show that C is a copula, and is related to the (ordinary) copula of X and Y via the equation $C(u,v) u + v - 1 +$ copula$(1 - u, 1 - v)$. See Nelsen (2006) for details.

2.3.4 Archimedean copulas

Let ϕ be a continuous, strictly decreasing function from $[-1,1]$ to $[0,\infty]$, such that $\phi(1) = 0$, and let ϕ^{-1} denote the "pseudo-inverse" of f for $t \in [0, \phi(0)]$, with $\phi^{-1}(t) = 0$ for $t \geq \phi(0)$. The copulas of the form

$$C(u,v) = \phi^{-1}(\phi(u) + \phi(v)) \tag{11}$$

are called Archimedean. When $\phi(0) = \infty$, we say that C is strict, and when $\phi(0) < \infty$, we say that C is non-strict. When C is strict, $C(u,v) > 0$, $\forall (u,v) \in (0,1]^2$.

The following is a short list of some generators and the names associated with the copula in the literature. The parameter interval gives the values of θ for which the generator ϕ is convex (limits may be required at some values in the interval). See Nelsen (2006) for further examples.

Generator	$\theta \in$	Copula
$\phi(t) = \left(t^{-\theta} - 1\right) / \theta$	$[-1, \infty)$	Clayton
$\phi(t) = \ln\left[\left(1 - \theta(1 - t)\right) / t\right]$	$[-1, 1)$	Ali-Mikhail-Haq
$\phi(t) = (-\ln t)^{\theta}$	$[1, \infty)$	Gumbel-Hougaard
$\phi(t) = -\ln\left[\left(e^{-\theta t} - 1\right) / \left(e^{-\theta} - 1\right)\right]$	$(-\infty, \infty]$	Frank

Archimedean copulas are widely used in applications (especially in finance, insurance, etc.) due to their simple form and nice properties. For example, most but not all extend to higher dimensions via the associative property. Procedures exist for choosing a particular member of a given family of Archimedean copulas to fit a data set (Genest & Rivest, 1993; Wang & Wells, 2000). However, there does not seem to be a natural statistical property for random variables with an associative copula.

2.3.5 Mixture copulas

Mixture copulas are formed by two copulas, called here the components of the mixture. If C_1 and C_2 are copulas and $w_1, w_w \geq 0, w_1 + w_2 = 1$ are weights, the following equation

$$C(u,v) = w_1 C_1(u,v) + w_2 C_2(u,v) \tag{12}$$

defines a copula, called a mixture copula. The density of the mixture copula is obtained in the same way. Mixture copulas can be used to combine the properties of the component copulas, for instance when C_1 has lower tail dependence but not upper tail dependence, and C_2 the opposite way. Extension to more than two components is straightforward, but less

useful, at least for bivariate copulas. Continuous mixture, defined by integrals, can also be considered, see for example Nelson (1998).

Mixture copulas have already been used in financial research. Fortin & Kuzmics (2002) have recognized the necessity of improving the fit achieved with the common one-parameter copula models by means of mixtures. They fitted mixtures of an elliptical model (a Gaussian or a *t* copula) and a non-elliptical model (a Clayton, a rotated Gumbel or a rotated Joe copula) to a set of European stocks, testing the weight of the non-elliptical component by means of a likelihood ratio test. De la Peña, Chollete & Lu (2004) have also argued in favor of mixture copulas.

With the aim of gaining flexibility when modeling the dependence structure, Hu (2006) proposed a mixture copula approach to measure cross-market dependence, using four stock market indices from developed markets. Her model is a mixture of a Gaussian copula, a Gumbel copula, and a Gumbel survival copula. In this analysis we will test several copula models to see which one makes the best fit to our data.

3 Estimation of copulas

Two main approaches for fitting copula models to bivariate data have been already considered in the literature. The first one, which is appropriate for one-parameter copulas, uses the relationship between a correlation measure and the parameter of the copula. Unfortunately, this approach cannot be applied for copulas with more than one parameter, such as mixture copulas. However, for Archimedean copulas the closed-form solutions can be found for an important concordance measure (Kendall's τ, cf. Section 4), making this method popular in the literature.

The second approach searches parameter estimates through the optimization of a measure of goodness-of-fit. One such measure is the likelihood function. The maximum likelihood

(ML) estimation approach is widely known and much favored in academic research. Other measures derived from the likelihood, such as the Akaike information criterion (AIC) and the Schwarz, or Bayesian, information criterion (BIC) are currently used in model selection in a great deal of statistical analysis, specially in these fields when the analyst must choose among several models the one that fits better to data analyzed. The extension of the ML approach to non-Archimedean copulas is usually done by means of the so-called expectation-maximization (EM) algorithm.

Other measures of goodness-of-fit are distance measures, of L^2 or L^∞ type, between the empirical (i.e. data-based) copula and the theoretical copula. Here, the parameter estimates are those for which the distance used takes the minimum value. Although there is no standard way to extend the methods based on these measures to mixture copulas, they still remain valid for assessing the fit and, moreover, they are much simpler to grasp than the likelihood-based measures. Graphical methods for identifying the copula model, like the chi-plots, introduced in Fisher & Switzer (2001) and used by Abberger (2003), or the Kendall plots, introduced by Genest & Boies (2003), have not been used here, since we have favored an approach based on the examination of the tail dependence, followed by a comparison among plausible models based on numerical measures. This Section contains an introduction and a brief discussion for each of these estimation methods. It partly follows Pedreira (2005).

Let us suppose that the true copula belongs to a parametric family $C = C_\theta,\ \theta \in \Theta$, with reasonable mathematical properties. Then, consistent and asymptotically normally distributed estimates of the association parameter θ can be obtained through the optimization of the likelihood function. There are three basic ML estimation procedures for copulas. The first one is called the *exact maximum likelihood* (EML) method. The EML estimation could be computationally intensive, because it requires the joint estimation of the specific parameters of the marginal distributions as well as the common parameters of the dependence structure. The EML estimates depend strongly on the correct specification of the marginals, this being an important aspect that mines the use of the EML. Also, the misspecification of the marginals may lead to severe biases in the estimation of the

dependence parameters, and vice versa. The second approach consists in a two-step procedure. In the first step, the parameters for the univariate marginal distributions are estimated. Once the distribution parameters are computed, in the second step the structure parameters are estimated, given the marginals. This two-step procedure is known as the *method of inference functions for the margins*, or IFM procedure. Estimating through IFM, as in the EML method, requires one to be very careful when specifying the model, because errors, due to a misspecification of the margins, may also lead to severe bias in the estimation of the association parameters in the second step. The third method is called the *canonical maximum likelihood* (CML). The CML method estimates the association parameters θ of the copula without assuming any parametric form of the marginal distributions. Fortin & Kuzmics (2002) used the ML method to fit various mixture models to the disturbances of GARCH (1,1) models, for which the marginal distributions were assumed to be t distributions. Fermanian & Scaillet (2002) and Fantazzini (2005) present evidence that, when using EML or IFM, the efficiency loss in the parameter estimation is severe, although this is not the case for CML method. Furthermore, they state that “this suggests that if one has doubt about the correct modeling of the margins, there is probably little to loose but lots to gain from shifting towards a semiparametric approach”. Hu (2006) also argues in favor of the use of CML. Due to the fact that we do not know which is the correct specification for each of the distribution margins, we use the semiparametric CML method. In this way we have the advantage that we do not need to specify the marginal distribution and, therefore, the selected approach is robust and free of the distribution margin misspecification error.

Let X be a variable and $(x_1, x_2, \ldots.., x_N)$ a set of observations of X. The empirical cumulative density function (cdf) of X is an approximation to the true cdf of X, which can be computed by the formula

$$F_X(x) = \frac{\mathrm{No}(x_t \le x)}{N}, \tag{13}$$

where $\mathrm{No}(x_t \le x)$ denotes the number of observations $(x_1, x_2, \ldots, x_N)$ not greater than x. A set of joint observations (x_t, y_t) of X and Y can be transformed into a set of points (u_t, v_t) in the unit square by defining $u_t = F_X(x_t)$ and $v_t = F_Y(y_t)$. The CML method consists in maximizing the likelihood of the data set formed by these points. If $c(u, v; \theta)$ is the density of the copula model, the likelihood of this data set is

$$
\begin{aligned}
L(\theta) &= \prod_{t=1}^{N} c(u_t, v_t; \theta) \\
\log L(\theta) &= \sum_{t=1}^{N} \log c(u_t, v_t; \theta) \qquad (14) \\
\tilde{\theta} &= \arg\max_{\theta} \log L(\theta)
\end{aligned}
$$

4 Results

4.1 Data

We use daily data for nine EEM value-weighted indices and the corresponding series for the S&P 500 index. The data was obtained from Thomson Datastream and covers the period from October 20, 1996 to September 22, 2006, a series of 1545 observations. All the indices are denominated in US Dollars. The description and summary statistics for daily returns of these indices are shown in Table 1. The summary statistics include annualized mean returns, standard deviations, skewness, kurtosis, Jarque-Bera statistics, and coefficients of linear correlation with S&P 500. Evidently, the values of Jarque-Bera statistics show that all the series display a marked departure from normality: the critical value of the chi-square distribution with two degrees of freedom is only 0.21 for a confidence level as high as 0.10.

Table 1. Summary statistics for daily returns on nine emerging-market indices and S&P 500, between October 20, 1996 and September 22, 2006.

Index	**SBI**	**SOFIX**	**CROBEX**	**BET**	**BUX**	**SAX**	**PX**	**ISE**	**WIG**	**S&P 500**
Country	Slovenia	Bulgaria	Croatia	Romania	Hungary	Slovakia	Czech Rep.	Turkey	Poland	US
Mean (p.a.)	25.9	48.5	31.4	45.3	24.8	34.9	30.7	16.5	19.5	0.5
St. dev. (p.a.)	16.1	31.5	23.3	25.9	24.2	22.3	22.6	52.8	22.7	17.3
Skewness	0.8402	0.5044	0.3750	0.5867	–0.1940	0.1063	–0.0211	0.2256	–0.0061	0.2402
Kurtosis	31.8233	33.2198	21.7537	14.3063	4.6909	5.3938	5.6504	8.9809	3.9703	5.8092
Jarque-Bera	53510.36	58687.87	22609.09	8290.79	192.41	369.64	449.84	2306.91	59.98	520.15
Linear correlation	–0.0483	–0.0323	–0.0336	0.0469	0.1582	0.0745	0.1559	0.1217	0.1790	1.0000

4.2 Correlation, concordance and departure from normality

Based on the coefficients of linear correlation only, the indices could be separated into three well distinguished groups:

- Markets with small negative linear correlation with S&P500 (Slovenia, Croatia, and Bulgaria).
- Markets with relatively small positive linear correlation with S&P500 (Romania and Slovakia).
- Markets whose linear correlation coefficient with S&P500 is above 0.10 (Hungary, Czech Republic, Turkey, and Poland).

Apart from the linear correlation coefficient, we use two other measures of concordance between the indices: Kendall's tau (τ) and Spearman's rho (ρ_S). They are defined, respectively, by:

$$\tau = 4\iint_{I^2} C(v,z)\,\mathrm{d}C(v,z) - 1$$

and

$$\rho_S = 12\iint_{I^2} C(v,z)\,\mathrm{d}v\,\mathrm{d}z - 3,$$

for any pair of random variables with a copula *C*. Using the sample analogs, we compute these two measures for all pairs of indices in our dataset. The results are given in Tables 2 and 3.

In order to interpret these results, we use the property that Kendall's τ can be written equivalently as

$$\tau = 4\mathrm{E}\left[C(U_1,U_2)\right] - 1,$$

where both U_1 and U_2 are standard uniform and have joint distribution C (see, for example, Cherubini *et al.*, 2004). In other words, Kendall's τ can be regarded as a normalized expected value. On the other hand, Spearman's ρ_S can be cast into the following form:

$$\rho_S = 12\mathrm{E}\left[U_1 U_2\right] - 3$$

or, equivalently,

$$\rho_S = \frac{\mathrm{cov}\big(F_1(X), F_2(Y)\big)}{\sqrt{\mathrm{var}\big(F_1(X)\big)\mathrm{var}\big(F_2(Y)\big)}},$$

where $U_1 = F_1(X)$ and $U_2 = F_2(Y)$, while *X* and *Y* are two pairs of indices. Spearman's ρ_S is therefore the rank correlation (in the sense of correlation of integral transforms) of *X* and *Y*.

Thus, from Tables 2 and 3 we can identify groups of indices with high probability of having simultaneous incidence of extremely high or extremely low values. These are, for example:

- SBI (Slovenia), SOFIX (Bulgaria), and CROBEX (Croatia).
- BET (Romania), PX (Czech Republic), ISE (Turkey), and WIG (Poland).

It is worth noting that for all emerging markets in the sample the concordance measures are positive, indicating joint co-moving tendencies in extreme events. The negative values are obtained for concordance of SBI, SOFIX, and CROBEX indices with S&P 500, which is in

Table 2. The values of Kendall's τ for the daily returns on nine emerging-market indices and S&P 500.

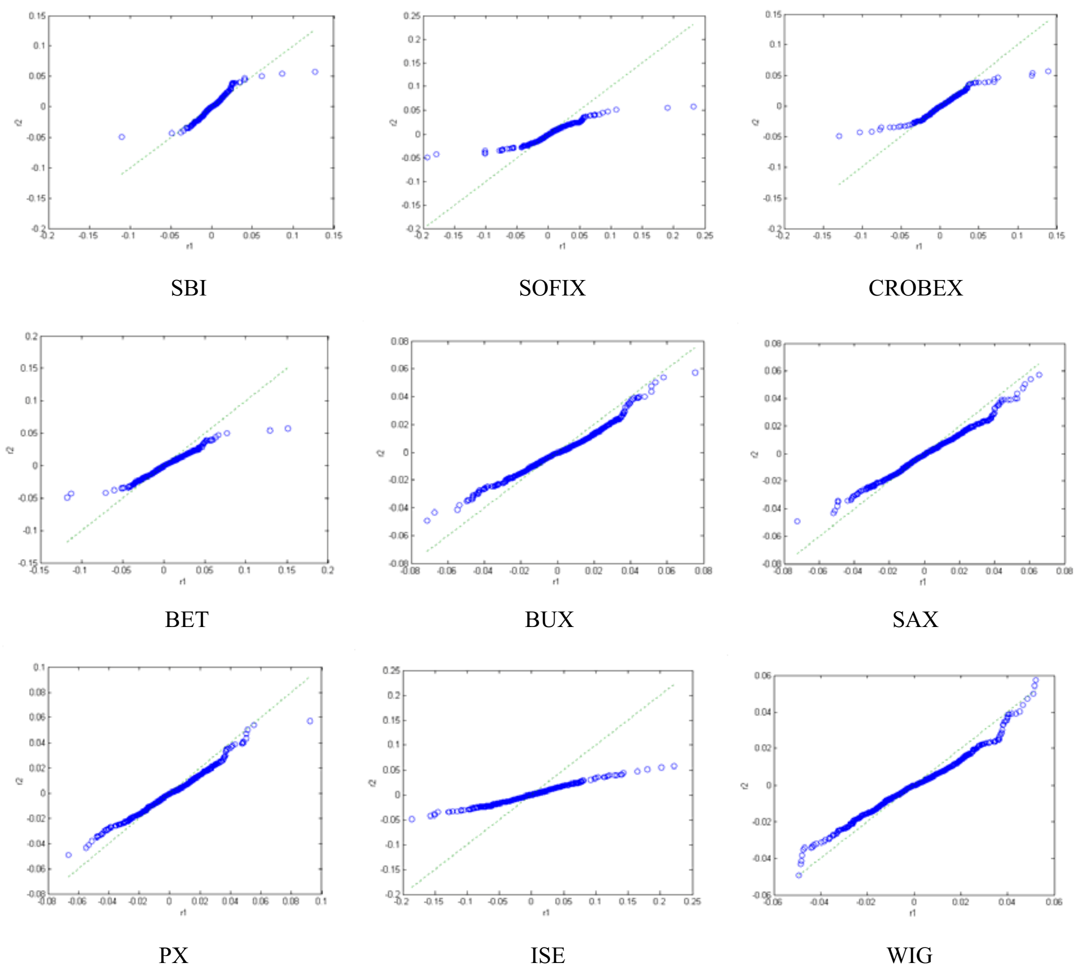
Index	SBI	SOFIX	CROBEX	BET	BUX	SAX	PX	ISE	WIG	S&P500
SBI	1.00									
SOFIX	0.24	1.00								
CROBEX	0.24	0.17	1.00							
BET	0.07	0.03	0.05	1.00						
BUX	0.14	0.09	0.13	0.06	1.00					
SAX	0.12	0.08	0.09	0.07	0.06	1.00				
PX	0.08	0.06	0.09	0.11	0.26	0.14	1.00			
ISE	0.05	0.05	0.07	0.04	0.18	0.06	0.18	1.00		
WIG	0.10	0.07	0.11	0.09	0.35	0.07	0.28	0.21	1.00	
S&P500	–0.03	–0.03	–0.02	0.02	0.09	0.05	0.09	0.07	0.13	1.00

Table 3. The values of Spearman's ρ_S for the daily returns on nine emerging-market indices and S&P 500.

Index	SBI	SOFIX	CROBEX	BET	BUX	SAX	PX	ISE	WIG	S&P500
SBI	1.00									
SOFIX	0.34	1.00								
CROBEX	0.34	0.25	1.00							
BET	0.10	0.05	0.08	1.00						
BUX	0.21	0.14	0.19	0.10	1.00					
SAX	0.18	0.12	0.13	0.10	0.10	1.00				
PX	0.12	0.08	0.13	0.16	0.38	0.20	1.00			
ISE	0.07	0.07	0.11	0.06	0.27	0.08	0.27	1.00		
WIG	0.14	0.10	0.17	0.13	0.50	0.10	0.40	0.30	1.00	
S&P500	–0.05	–0.05	–0.03	0.03	0.14	0.07	0.14	0.10	0.19	1.00

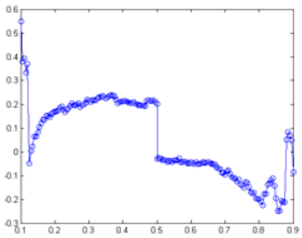
a qualitative agreement with the coefficients of linear correlation between these markets (Table 1). Perhaps more importantly, the obtained values for τ and ρ_S point to a very low degree of correlation between booms and crashes in the US and similar extreme events in EEMs. This conclusion is even more important when one takes into account that distributions of stock price returns in EEMs typically have more fat tails than in the US. The values of skewness and kurtosis in Table 1 indicate that the extreme events are indeed more likely to occur in these markets. This is particularly pronounced for Slovenian, Bulgarian, Croatian and Turkish stock markets. The S-shape of the quantile-quantile plots in Figure 1 is a good and intuitive illustration of this finding.

Figure 1. Quantile-quantile plots for the nine emerging market indices vs. S&P 500.

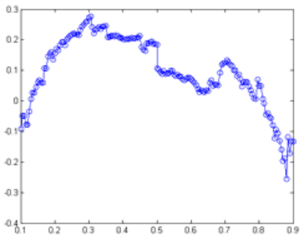


Clearly, all return distributions in our sample exhibit a substantial departure from normality. If returns were normally distributed, their exceedance correlations would have the tent-shaped distributions, see Ang & Chen (2001). Figure 2 shows the exceedance correlations for the EEM indices. It provides a clear graphical representation of the asymmetric movements between the EEMs and the US stock market. We can observe that the exceedance correlations are quite far from symmetric. Typically, there is a sharp break evident at the midpoint, where the conditioning changes from corr(X, Y | X>0, Y>0), using the positive quadrant, to corr(X, Y | X<0, Y<0), using the negative quadrant, where X is a return on a EEM index, and Y is return on S&P 500. This is a good indicator of large asymmetric co-movements during booms and crashes. For example, Slovenian SBI index is positively correlated with S&P 500 when returns are negative, but negatively correlated when S&P 500 performs well. Similarly, Turkish stock market tends to be very sensitive to declines of US stocks, having a high probability of decline itself; however, its ISE index displays a mild positive correlation with the S&P 500 when it rises. On the other hand, CROBEX, BET, BUX, and PX have unusually high level of exceedance correlation of returns with S&P 500 in the positive quantiles.

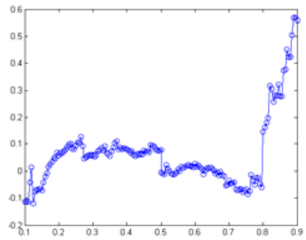
Figure 2. Exceedence correlations for the nine emerging market indices with respect to S&P 500.



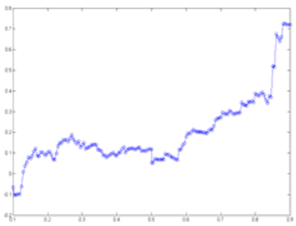
SBI



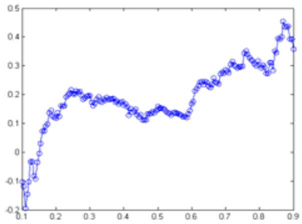
SOFIX



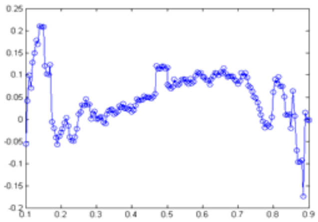
CROBEX



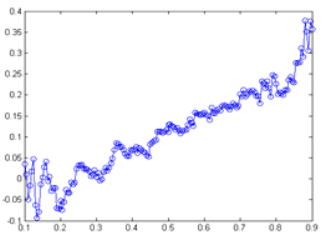
BET



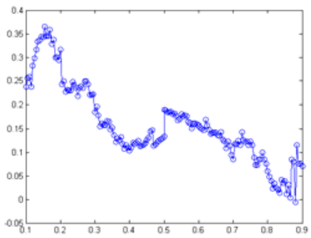
BUX



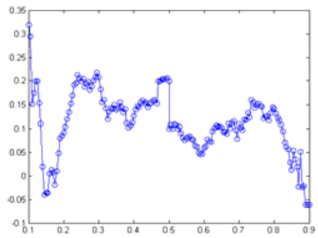
SAX



PX



ISE



WIG

Additionally, in order to investigate tail dependence asymmetry, we document the differences between the coefficients of lower and upper tail dependence. To introduce a proper measure of tail dependence, we consider the likelihood that one event with probability lower than v occurs in one variable, given that an event with probability lower than v occurs in the other. If we compute this dependence measure far in the lower tail, we obtain the so-called *lower tail index* (cf. Cherubini *et al.*, 2004):

$$\lambda_L = \lim_{v \to 0^+} \frac{C(v,v)}{v}.$$

If we choose to condition everything with respect to the S&P 500 index, then λ_L can be interpreted as the probability of a crash in one of the EEM simultaneously with a crash in the US stock market. Similarly, we can define the *upper tail index* as

$$\lambda_U \equiv \lim_{v \to 1} \frac{1 - 2v + C(v,v)}{1 - v},$$

and interpret it as the probability that EEM price boom occurs at the same time as in the US. Unlike τ and ρ_S, whose sample analogs do not depend on the choice of C,[1] the tail indices λ_L and λ_U will differ depending on the choice of the copula model. Copula functions could be distinguished according to the balance between the weight put on the upper and lower tails. Some copula functions stress more on the upper tail (Gumbel copula), while others stress more on the lower tail (Clayton copula). Some copulas have zero tail indices by definition. These are, for example, Gaussian, Plack, and Frank copulas. On the other hand, Clayton and rotated Gumbel by definition have no upper tail dependence ($\lambda_U = 0$); similarly, rotated Clayton and Gumbel copulas have no lower tail dependence ($\lambda_L = 0$). Finally, Student-t copula has symmetric tail dependence ($\lambda_L = \lambda_U$), while the mixture Joe-Clayton model is an example of copula with $\lambda_L \neq \lambda_U$. The ability of copulas to capture tail dependence is the crucial one for our study. Therefore, we will focus on several copula models that have positive lower or upper tail dependence. These are Clayton, rotated Clayton, Gumbel, Student-t, and Joe-Clayton.

[1] The sample analogs are also consistent estimators of τ and ρ_S.

Table 4. Best-fitting and second best-fitting models for EEM indices with the corresponding values of upper and lower tail index.

Index	Best-fitting model	l_L	l_U	Second best-fitting model	l_L	l_U
SBI	Student	0.0162	0.0162	Clayton	0.0000	–
SOFIX	Student	0.0305	0.0305	Clayton	0.0000	–
CROBEX	Student	0.0012	0.0012	Clayton	0.0000	–
BET	Student	0.0001	0.0001	Joe-Clayton	0.0000	0.0015
BUX	Joe-Clayton	0.0136	0.0538	Student	0.0046	0.0046
SAX	Student	0.0012	0.0012	Joe-Clayton	0.0000	0.0073
PX	Student	0.0027	0.0027	Joe-Clayton	0.0312	0.0403
ISE	Student	0.0349	0.0349	Rotated Gumbel	0.1221	–
WIG	Student	0.0027	0.0027	Joe-Clayton	0.0218	0.0757

To determine which copula fits our data best, we rank each of the pairs X and Y according to their CML, AIC and BIC scores. Here, X is the return on an EEM index, while Y is the return on S&P 500. The results are summarized in Table 4. Since all the three criteria give the same rankings, for each EEM index we report a unique best-fitting and second best-fitting model. The corresponding values for l_L and l_U are also given. It turns out that Student-t copula is the best fit for all market indices except BUX (Hungary), where Joe-Clayton is the best choice. The problem with the Student-t copula is its symmetric tail dependence, which, as we discussed above, may not be realistic for some markets that display sharp asymmetry in the likelihood of extreme events. This is why we also report tail indices for the second best-fitting models. For example, there is a pronounced left coupling displayed among Turkish and US market, which is nicely captured by rotated Gumbel copula. The corresponding value of l_L indicates that, if the joint distribution is indeed dictated by this model, there is a 12.21% probability of simultaneous market crash in Turkey and in the US. This implies that the assumption of joint normality is violated in a dangerous direction. Thus, when most needed, cross-market diversification may sometimes be of a limited use as a mean of reducing portfolio risk.

Based on tail dependencies, other EEMs display very little tendency of simultaneous booms and crashes with the US stock market. Exceptions are, to some extent, Slovenia ($l_L = l_U = 0.0162$) and Bulgaria ($l_L = l_U = 0.0305$), given that the joint distribution of SBI and SOFIX with the S&P 500 is modeled by the Student-t copula, and Hungary ($l_L = 0.0136$ and $l_U = 0.0538$), given that the joint distribution of BUX and S&P 500 is modeled by the asymmetric Joe-Clayton copula. The pronounced asymmetric behavior indicates that BUX and S&P 500 are substantially more correlated when they both increase than when they both decrease.

5 Conclusion

In this paper we used the copula method to investigate the extreme co-movements of US stocks with those from nine European emerging markets. Using the data for stock market indices, we have shown that the stocks traded in European emerging markets show very little sensitivity to extreme events in the US market. At the same time, we have found a high degree of asymmetry between the co-movements in the lower and upper tail of distributions of returns on indices. This indicates that European emerging markets on average tend to react differently to increases and decreases in the US market. The asymmetries do not seem to follow any particular rule. The Gaussian copula model, often adopted in practice, is not able to model such asymmetries. One part of the problem is the presence of fat tails, and the other is a strong asymmetry displayed by the tails. Since the Gaussian copula has a null asymptotic coefficient for both lower and upper tail dependence, it will incorrectly model the extreme dependence. On the other hand, Student copula can capture fat tails, but not the tail dependence asymmetry. The Clayton, Joe-Clayton or rotated Gumbel copulas are able to capture both effects, but may sometimes fail in describing the interior of the joint distribution. This tradeoff between different copula models has to be considered carefully in practice.

Extension of this study to time-varying copulas with non-zero tail dependence (cf. Patton 2004, 2006a, and 2006b) would be a reasonable next step. It will be crucial for further analysis of extreme behavior of the stock markets and verification of the effectiveness of cross-market diversification, especially in the light of the ongoing global financial crisis.

References

Abberger, K. (2003). "A simple graphical method to explore tail-dependence in stock-return pairs." Working paper, University of Konstanz.

Ang, A. and J. Chen. (2001). "Asymmetric correlations of equity portfolios." Working paper.

Benes, V. and J. Stephan. (1997). *Distributions with Given Marginals and Moment Problems.* Kluwer, Dordrecht, Netherlands.

Cherubini, U., E. Luciano, and W. Vecchiato. (2004). *Copula methods in finance.* Wiley.

Cuadras, C. M., J. Fortiana, and J. A. Rodriguez Lallena. (2002). *Distributions with Given Marginals and Statistical Modelling.* Kluwer, Dordrecht, Netherlands.

Dall'Aglio, G. (1991). "Frechet classes: the beginnings." In G. Dall'Aglio, S. Kotz & G. Salinetti (Eds.), *Advances in Probability Distributions with Given Marginals.* Kluwer, Dordrecht, Netherlands.

De la Peña, V., C. Loran, and C-C. Lu. (2004). "Co-movement of international financial markets." Working paper, Columbia University.

Engle, R. F. and S. Manganelli (1999). "CAViaR: Conditional autoregressive value at risk by regression quantiles." NBER working paper, 7341.

Fantazzini, D. (2005). "The econometric modeling of copulas: a review with extensions." Working paper, University of Pavia.

Fermanian, J-D. and O. Scaillet. (2002). "Nonparametric estimation of copulas for time series." *Journal of Risk*, 5, 25–54.

Fischer, R. A. (1932). *Statistical Methods for Research Workers.* Liver and Loyd, Edinburgh, UK.

Fisher, N. I. (1997). *Encyclopedia of Statistical Sciences*, volume 1. Wiley.

Fisher, N. I. and P. Switzer. (2001). "Graphical assessment of dependence – is a picture worth 100 tests?" *The American Statistician*, 55, 233–239.

Fortin, I. and C. Kuzmics. (2002). "Tail dependence in stock-return pairs." *Reihe Oekonomie*, 126, 1–35.

Genest, C. and J-C. Boies. (2003). "Detecting dependence with Kendall plots." *The American Statistician*, 57, 1–10.

Genest, C. and L-P. Rivest. (1993). "Statistical inference procedures for bivariate Archimedean copulas." *Journal of the American Statistical Association*, 88, 1034–1043.

Hu, L. (2006). "Dependence patterns across financial markets: a mixed copula approach." *Applied Financial Economics*, 16(10), 717–729.

Hutchinson, T. P. and C. D. Lai. (1990). *Continuous Bivariate Distributions, Emphasizing Applications.* Rumsby Scientific Publishing.

Joe, H. (1997). "Multivariate Models and Dependence Concepts," volume *Monographs* in *Statistics and Probability 73*. Chapman and Hall, London.

JP Morgan (1997). *Introduction to CreditMetrics.* Technical Document. J. P. Morgan: New York.

Ling, H. (2003). "Dependence patterns across financial markets: a mixed copula approach." Working Paper, Ohio State University.

Longin, F. and B. Solnik (1995). "Is the correlation in international equity returns constant: 1960–1990?" *Journal of International Money and Finance*, 14, 3–26.

Longin, F. and B. Solnik (2001). "Extreme correlations of international equity markets" *Journal of Finance*, 56, 649–676.

McNeal, A. J. and R. Frey (2000). "Estimation of tail-related risk measures for heteroskedastic time series: An extreme value approach." *Journal of Empirical Finance*, 7, 271–300.

Nelsen, R. B. (2006). *An Introduction to Copulas*. Springer, New York.

Patton, A. J. (2001a). "Modelling time-varying exchange rate dependence using the conditional copula." Working Paper, University of California, San Diego.

Patton, A. J. (2001b). "On the importance of skewness and asymmetric dependence in stock returns for asset allocation." Working Paper, University of California, San Diego.

Patton, A. J. (2004). "On the out-of-sample importance of skewness and asymmetric dependence for asset allocation." *Journal of Financial Econometrics*, 2(1), 130–168.

Patton, A. J. (2006a). "Estimation of multivariate models for time series of possibly different lengths." *Journal of Applied Econometrics*, 21, 147–173.

Patton, A. J. (2006b). "Modelling asymmetric exchange rate dependence." *International Economic Review*, 47(2), 527–556.

Pedreira Collazo, E. (2005). "Portfolio Selection and Dependence Structure in Emerging Markets." PhD thesis, IESE Business School.

Ruschendorf L., B. Schweizer, and M. D. Taylor. (1996). *Distributions with Fixed Marginals and Related Topics.* Institute of Mathematical Statistics, Hayward, CA.

Schweizer B., and A. Sklar. (1983). *Probabilistic Metric Spaces.* North-Holland, New York.

Sklar, A. (1959). "Fonctions de repartition *n* dimensions et leurs marges." Publications de l'Intitut de Statistique de l'Universit de Paris, 8, 229–231.

Wang W., and M. T. Wells. (2000). "Model selection and semiparametric inference for bivariate failure-time data." *Journal of the American Statististical Association*, 95, 62–76.