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VENTURE CAPITAL AND SEQUENTIAL INVESTMENTS*

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Abstract

We analyze sequential investment decisions in an innovative project that depend on the investor's information about the project failure risk and its potential final value. We consider the feedback effects between learning about the project parameters and the continuous adjustment of the investment strategy. Investors decide sequentially about the speed of investment and the optimal degree of involvement. We develop three types of predictions from our theoretical model and test these predictions in a large sample of venture capital investment in the U.S. for the period of 1987-2002.

First, the investment flow starts cautiously if the failure risk is high and accelerates as the projects mature. Second, the investment flow reacts positively to information that arrives while the project is developed. We find that interim information is more significant for investment decisions than the information prior to the project launch. Third, investors distribute their investments over more funding rounds if the failure risk is larger.

KEYWORDS: Venture Capital, Sequential Investment, Stage Financing, Intertemporal Returns

JEL CLASSIFICATION: D83, D92, G11, G24.

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Non-Technical abstract

This paper aims to understand the relationship between project-related information and the sequentially optimal investment decisions. It develops a theory that analyzes how venture capitalists make optimal dynamic investment decisions as a function of their information about failure risk and potential final value. The theoretical analysis depicts the progress of the project as a continuous process of development and research. Specifically, the model considers a continuous-time model representing the complete investment cycle of an innovative project under uncertainty, characterized by uncertainty about the likelihood of success, the timing of the stopping decision, and about the final value in case of success. At each point in time, information that the project should be abandoned may arrive. Thus, the model incorporates a simple stopping problem. The signal that the project should be abandoned arises with a given probability, derived from the Pareto distribution with parameter λ . Crucially, the analysis distinguishes between information that investors know up-front, and information they learn as the project advances. The model focuses on two essential dimensions of the sequential investment decisions: the speed with which the project is undertaken (optimal capital flow), and the optimal staging decision, reflecting the investors' decision on their optimal degree of their involvement.

The decision about the optimal speed of investment is characterized by the following trade-off: a larger investment flow into the project promises faster success, but is likely to reduce the efficiency of the investment. The investors control the optimal investment flow at every point in time. Moreover, each financing round necessitates a thorough review and valuation exercise that involves several parties and a multilateral negotiation process. With these resource constraints and transaction costs in mind, it is then optimal to review the project only at certain intervals, even if this implies a temporarily suboptimal investment path. The costs of each investment decision are modeled as a loss that is proportional to the current value of the project. The goal is to specifically understand the intertemporal pattern of stage financing and its interaction with the available information. Critically, the determination of the financing rounds, their expected duration and the associated investment flow and the intermediate milestones are endogenous in our model. In this analysis, investors make lumpy investment decisions that are optimized as a function of the expected value and the probability of failure.

The theoretical contributions fall into three groups. First, it is shown that as a project advances and the probability of eventual success increases, investment flows should be optimally increasing. For the same reason, a project with a higher estimated final value or a higher anticipated chance to succeed will also be allocated a larger investment flow throughout. This is because with an increase in the probability to succeed, accelerating becomes a more valuable option, even if it makes investment more costly. The model predicts that if the density of the information arrival leading to abandonment is sufficiently front-loaded then the returns will be decreasing even though the increasing investment flows imply an acceleration of the discovery process. Second, the analysis shows that the optimal staging sequence depends on the value of the real option to abandon. The higher the estimated final value of the project is, and the larger the estimated success probability, the fewer rounds will be used. Also, echoing the earlier result on the optimal investment path for continuous decisions, the investment flow will increase from one round to the next. Third, learning about the expected final value or the failure probability is shown to be incorporated in all subsequent investment decisions. If there is a positive news update then the value of the project will increase as well as the investment flow.

The model predictions are then put to an empirical test, using a comprehensive sample of the US venture capital data for the period 1987-2002, covering the majority of all venture investments in the U.S. over that period. The empirical findings lend support to the main predictions of our model: First, investors proceed cautiously if the failure risk is high, and they accelerate investment, in spite

of the cost of doing so, as projects mature. They also invest faster if they hold favorable information about the project. As a project advances and the probability of eventual success increases, investment flows are increasing. We show that at the same time, the returns of the projects are decreasing over the investment cycle. Second, the investment flow reacts positively to information that arrives during the development of the project. Initially, investors seem to have little screening ability about the eventual probability of success, but they seem to hold some information about the final value of the project in the event of a successful completion. In many cases investors get information that leads to a change in the estimated failure risk or exit value of the project, as inferred from the dynamics of the project valuation. Moreover, such information updates lead investors to adjust the investment path optimally. The subsequent investment flow as well as the size of each round and the number of subsequent rounds react in the way predicted. Consistent with the model, investors receive updates over the course of the investment cycle that allow them to better estimate the final value. These updates again give rise to a change in the investment flow and the number and size of subsequent rounds that is consistent with the pattern predicted by our model. Third, if the failure risk is large then investors tend to adopt a more hands-on approach by adjusting their investment strategy more frequently. The design of financing rounds follows the optimal pattern predicted by our model: the investment size and the investment flow is increasing from one round to the next, and projects with a high initial estimate of the final value or an optimistic appraisal of the probability to succeed will use less rounds than less valuable or more risky projects.

1 Introduction

1.1 Motivation

An innovative project typically has to go through many steps of exploration and development that all require capital outlays before it is completed. Moreover, it carries a substantial failure risk and it is difficult to predict at which point in time evidence might emerge that would lead to its abandonment. The optimal investment policy depends on the information available, but the progress of research uncovers new information about the project and reduces uncertainty that in turn will influence the optimal continuation strategy. The venture capital industry is a powerful example of the importance of these feedback effects in the financing of innovation. But similar issues also arise for innovative projects within large organizations or in publicly funded research.

The purpose of this paper is to understand the relationship between project-related information and the sequentially optimal investment decisions. We develop a theory that analyzes how investors - venture capitalists or other sponsors that provide the financing and help shepherding a project to success - make optimal dynamic investment decisions as a function of their information about failure risk and potential final value. In our theoretical model, we distinguish between information that investors know up-front, and information they learn as the project advances, and we analyze their relative impact on subsequent investment decisions.

We then put the predictions of our model to an empirical test. We use a comprehensive sample of the US venture capital data to examine whether we find support for our predicted relationships. Our empirical findings lend support to the main predictions of our model: First, investors proceed cautiously if the failure risk is high, and they accelerate investment, in spite of the cost of doing so, as projects mature. They also invest faster if they hold favorable information about the project. Second, the investment flow reacts positively to information that arrives during the development of the project, and interim learning seems to be more important for the determination of the investment path and a better predictor of the final outcome than ex ante information. Third, if the failure risk is large then investors tend to adopt a more hands-on approach by adjusting their investment strategy more frequently.

More specifically, we consider a continuous-time model representing the complete investment cycle of an innovative project under uncertainty, characterized by: *(i)* uncertainty about the likelihood of success; *(ii)* uncertainty about the timing of the stopping decision; *(iii)* uncertainty about the final value in case of success; and *(iv)* interim learning about the failure risk and final value after

the project is launched. Our model depicts the progress of the project as a continuous process of development and research. At each point in time, information that the project should be abandoned may arrive. Thus, the model incorporates a simple stopping problem. The signal that the project should be abandoned arises with a given probability, derived from the Pareto distribution with parameter λ . The family of Pareto distributions has the property that the conditional probability of failure is decreasing over time.

In our stylized model, we focus on two essential dimensions of the sequential investment decisions. The first dimension is that investors determine the speed with which the project is undertaken, or the optimal capital flow. Experiments can often be conducted in parallel, but there is a cost to doing so because sequential learning from one experiment to the next is curtailed, or because of the shortages of critical resources, such as research staff and management. Therefore, the decision about the optimal speed of investment is characterized by the following trade-off: a larger investment flow into the project promises faster success, but is likely to reduce the efficiency of the investment. The investors control the optimal investment flow at every point in time. In our analysis, we are particularly interested in the following questions: *(i)* How fast to develop a project? *(ii)* How to change the investment pace with the progress of the project? *(iii)* How to adjust the financing speed if new information arises that changes the expectation of key parameters, notably failure risk and final value in the event of success?

For investors, in addition to the stopping decision and the decision on the optimal investment flow, there is also the question of the optimal degree of their involvement. This is the second investment dimension that we take into account. Venture capitalists normally provide financing in infrequent financing rounds or stages, lasting typically from a few months to over a year or more. They also define milestones that must be met before a certain fraction of the funds is released. In the venture capital industry, the time of fund managers is often considered as one of the most critical resources (see ? and ?). Each financing round necessitates a thorough review and valuation exercise, it typically involves several parties (venture financing is often syndicated among several funds and the managers are involved as well) and a multilateral negotiation process. With these resource constraints and transaction costs in mind, it is then optimal to review the project only at certain intervals, even if this implies a temporarily suboptimal investment path.¹

¹Venture capitalists are also involved via continuous monitoring, e.g. through frequent visits and board representation. We view these monitoring activities as complementary to the financing round decisions. But compared with the information on financing rounds, the information on monitoring is soft and typically unavailable or only through

We add these considerations to our continuous-time model. Our goal is to specifically understand the intertemporal pattern of stage financing and its interaction with the available information. Critically, the determination of the financing rounds, their expected duration and the associated investment flow and the intermediate milestones are endogenous in our model. In this analysis, investors make lumpy investment decisions that are optimized as a function of the expected value and the probability of failure. We model the cost of each investment decision as a loss that is proportional to the current value of the project. With this analysis, we add the following questions to our investigation: *(iv)* how do transaction costs and the need for lumpy investment decisions affect the optimal investment path? *(v)* What is the optimal sequence of stage financing?

The contributions of our theoretical analysis fall into three groups. First, we show that as a project advances and the probability of eventual success increases, investment flows should be optimally increasing. For the same reason, a project with a higher estimated final value or a higher anticipated chance to succeed will also be allocated a larger investment flow throughout. This is because with an increase in the probability to succeed, accelerating becomes a more valuable option, even if it makes investment more costly. In fact, our model shows that investment flow should be increasing over time as a pure information effect, because the risk recedes that the project fails. Our model predicts that if the density of the information arrival leading to abandonment is sufficiently front-loaded then the returns will be decreasing even though the increasing investment flows imply an acceleration of the discovery process. Second, we show that the optimal staging sequence depends on the value of the real option to abandon. The higher the estimated final value of the project is, and the larger the estimated success probability, the fewer rounds will be used. Also, echoing our result on the optimal investment path for continuous decisions, the investment flow will increase from one round to the next. Third, we show that learning about the expected final value or the failure probability will be incorporated in all subsequent investment decisions. If there is a positive news update then the value of the project will increase as well as the investment flow. At the same time, the number of subsequent investment rounds will decrease, and the capital allocation for each of these rounds will increase.

We then take these predictions to a large sample of venture capital investment in the U.S. for the period 1987-2002, covering the majority of all venture investments in the U.S. over that period (?). The venture capital data are attractive for three reasons: first, they allow us to study these effects in a broad sample of projects across different industries. Second, because of the staged nature

questionnaire-based data.

of venture financing, interim valuation data are available that aggregate the state of the beliefs concerning the prospects of any particular project. We can use these values and in particular changes in the valuations to extract information about what investors have learnt since the last capital infusion. Third, the fact that venture-backed projects are independent companies makes it possible to track the timing of stopping decisions or the feedback between information arrival and sequential investment decisions more accurately than, say, for research projects launched within organizations in which there is more discretion for window-dressing and the investment information is more opaque.

The results of our empirical investigation lend support to our theoretical predictions as follows. First, we document empirically that as a project advances and the probability of eventual success increases, investment flows are increasing. We show that at the same time, the returns of the projects are decreasing over the investment cycle. Taken together, these two observations imply that learning about the eventual prospect of a project are sufficiently concentrated at the beginning. In our model specification this means that the parameter λ of the Pareto distribution must be sufficiently high. Second, our evidence shows that initially, investors seem to have little screening ability about the eventual probability of success, but they seem to hold some information about the final value of the project in the event of a successful completion. We show that as the project advances, in many cases investors get information that leads to a change in the estimated failure risk or exit value of the project, as inferred from the dynamics of the project valuation. Moreover, such information updates lead investors to adjust the investment path optimally: the subsequent investment flow as well as the size of each round and the number of subsequent rounds react in the way predicted by our model. Consistent with our model, we find that investors also receive updates over the course of the investment cycle that allow them to better estimate the final value. These updates again give rise to a change in the investment flow and the number and size of subsequent rounds that is consistent with the pattern predicted by our model. Third, we show that the design of financing rounds follows the optimal pattern predicted by our model: the investment size and the investment flow is increasing from one round to the next, and projects with a high initial estimate of the final value or an optimistic appraisal of the probability to succeed will use less rounds than less valuable or more risky projects.

1.2 Related Literature

Our paper is related to three different strands of the literature. First, there is a literature on the role of learning in the financing of innovation. ? analyzes the decision of venture capitalists into which industry they invest in a multi-armed bandit learning model. Using similar data to ours, he finds evidence that both learning as well as forward-looking expectation drive investment decisions. The main difference is that while ? looks at the selection of entire investment projects, we consider the speed and structure of the investment cycle once a project has been selected. ? discuss the learning impact of a venture capital's past investments on the size and the direction of follow-up funds by the same venture capital firm. They find a positive feedback effect between fund performance and the size of follow-on funds. They also explore the speed of learning of limited partners relative to that of general partners, and argue that the evidence supports asymmetric learning. In contrast to our paper, they do not consider the interaction between learning and investment within a single portfolio firm. Several papers discuss non-trivial abandonment decisions that depend on learning. ? consider a project with a given failure risk in which the arrival time of the final discovery, and hence the total cost to deliver it, are uncertain. This implies that the value of the project decreases over time until either success arrives or the project is optimally abandoned. They focus on the information rent of an agent who can divert the continuous investment flow, and show that the project may be financially constrained as these information rents increase in the expected funding horizon. ? present a paper in which the critical constraint is the expertise of the venture capitalists, similar to ?. Because of the opportunity cost linked to their labor input, the venture capitalists abandon projects earlier than would be socially optimal if projects are considered in isolation. In contrast to these papers, in our learning model we focus on the investment flow and the staging sequence.

Second, there is a literature on the optimal dynamic pattern of investments in the presence of a real option to abandon. ? focus on the evolution of the risk profile that are due to changes from a purely technical risk in the early stages to more diverse sources of risk in later stages. Mostly, the theoretical literature has focused on the use of stage financing as a tool to alleviate agency problems. ? consider the real option of abandoning the project in venture capital financing and highlight the role of stage financing in this regard. They consider a contract design problem to alleviate moral hazard and show that the entrepreneur's optimal equity share decreases as uncertainty about the project's ultimate success recedes. In a similar vein, in ? stage financing gives the investor more

flexibility to abandon the project, but may expose the entrepreneur to hold up problems, thus hurting effort incentives. Other papers focus on contracting issues. ? presents a dynamic, incomplete contracting model in which stage financing can reduce the bargaining power of opportunistic entrepreneurs who can repudiate their financial obligations. ? consider a two-period incomplete contracting model and analyze the problem of an entrepreneur who can manipulate short-term results for purposes of “window-dressing”. They show that stage financing is a means to mitigate this problem. Other incomplete contracting models that incorporate stage financing include ?. ? compare traditional round financing with milestone financing, where venture capitalists commit to the financial terms of multiple funding stages conditional on achieving certain benchmarks. All of these papers suggest that a higher frequency of milestones and financing rounds should translate into a more effective use of the abandonment decision, and hence smaller agency costs and better investment performance.

There is also a substantial empirical literature on stage financing, starting with the seminal analysis of ?. Subsequent work analyzes the contingent contract clauses that are either explicit or implied by staging in more detail (see e.g. ?; ?). By and large, these papers confirm many of the theoretical predictions on stage financing. A similar confirmation of the theoretical predictions appears in ? who shows that venture capitalists with better access to information about their venture, as proxied by geographical proximity, use staging less frequently.

Finally, there is a substantial literature on the valuation and the returns in venture capital. In contrast to our paper, this literature looks at venture capital as an asset class and studies the returns of venture capital from a performance-based perspective of a diversified investor. Among these studies our paper is most closely related to two studies that calculate company-level returns for venture-related investments, namely ? and ?. Both studies are interested in understanding the risk-return trade-off, and their focus is on reducing the impact of sample selection bias. ? calculates returns for each financing round separately and limits the sample to final valuations from IPOs and trade sales, whereas we take an integrated approach that solicits as many observations as possible at each round. ? include round valuations just as we do, but for a different objective of creating a performance index. Other papers in the risk-return literature, such as ? and ? look at returns at the fund level. They focus on cash distributions and thus consider only the final value of exited investments. By contrast, our paper focuses on the interaction between information-driven returns and investments, and looks at interim results and sequential investments at the portfolio-company level.

2 Model

We develop a model for a new venture that describes it as a dynamic investment problem under uncertainty in continuous time $t \in [0, \infty)$. The true value of the completed project is initially uncertain. It is known that the final value is either 0 or $Y > 0$, and that the uncertainty about the true value of the project can be resolved over time as follows. The value Y of the venture can only be realized if the project can be successfully developed. We model the development of the venture as a continuous investment process, denoted by i_t , which moves the venture from a starting position, denoted by $k_0 \in (0, K]$, to a final position, denoted by $K > 0$. The position of the project at time t is defined by $k_t \in [0, K]$. The investor is risk-neutral, and the discount rate is given by $r > 0$.

The investment flow i_t at time t controls the rate at which the state of the development k_t is moving forward, through the law of motion:

$$dk_t = \gamma \sqrt{i_t} dt. \quad (1)$$

The current investment i_t increases the speed at which the project is developed in a concave manner - or to put it differently, increasing the speed will increase the total cost of investment in a standard convex manner. The parameter $\gamma > 0$ describes the marginal effect of investment on the speed of development and a larger value of γ represents a project that is easier to develop.

The project is realized successfully with a final value Y if the final position K is reached. In contrast, if the project fails to progress beyond some position k_t , then the project is terminated with a final value 0. The location of the breakdown point is uncertain and given by a prior distribution $F(k)$. For most of the analysis, we shall restrict our attention to the class of Pareto distributions. The Pareto distribution is given by

$$F(k) \triangleq 1 - \left(\frac{\underline{k}}{k} \right)^\lambda,$$

The class of Pareto distributions is parameterized by two variables, \underline{k} and λ . The variable \underline{k} is a given strictly positive lower bound on the starting point of the project and $\lambda > 0$ identifies the skewness of the distribution. The project is successfully concluded if it reaches the terminal position K . The prior probability of success starting at $k_0 = \underline{k}$ is given by:

$$1 - F(K) = \left(\frac{\underline{k}}{K} \right)^\lambda.$$

Conversely, the prior probability that the project will fail during the development phase is given by $F(K)$. The hazard rate of failure, in other words, the conditional probability of failure at k_t is given by

$$h(k_t) \triangleq \frac{f(k_t)}{1 - F(k_t)} = \frac{\lambda}{k_t}.$$

The conditional failure rate is decreasing in the state of the project and a project with a larger λ has a uniformly former higher rate of failure and consequently a lower prior (and posterior) probability at every k_t that it reaches the terminal point K . The instantaneous failure probability, given the current investment i_t , is therefore given by:

$$dk_t \cdot h(k_t) = \left(\gamma \sqrt{i_t} \cdot \frac{f(k_t)}{1 - F(k_t)} \right) dt = \left(\gamma \sqrt{i_t} \cdot \frac{\lambda}{k_t} \right) dt.$$

A venture capital project is now characterized by (Y, K, γ, λ) . The terminal point K describes the length of the development process until the venture can go public or be sold, the parameter λ describes the failure rate of the project and γ identifies the marginal productivity of the monetary funds to develop the project.

The value of the venture depends on the investment policy $\mathbf{i} \triangleq (i_t)_{t=0}^T$. If the project is a success, then the payoff Y will be realized at some time $T \triangleq T(\mathbf{i})$ where the terminal time T naturally depends on the investment flow \mathbf{i} . Along the way, the project requires investments which have to be deducted from the initial net present value. On the other had, if the project is failure, then the investment flow will be stopped at some random position k_t . In this case the project will only incur costs until the moment of failure and not receive any positive returns at all. Conditional on a given investment policy \mathbf{i} , we can then associate to every time t a position $k_t = k_t(\mathbf{i})$ which is reached at time t , provided that we did not observe a failure before k_t . The expected net present value from an investment policy $\mathbf{i} = (i_t)_{t=0}^T$ at time $t = 0$ is then given by:

$$V(k_0) = (1 - F(K)) e^{-rT} Y - \int_0^T (1 - F(k_t)) e^{-rt} i_t dt. \quad (2)$$

3 Sequential Investment

The optimal investment policy can be analyzed as a dynamic programming problem under uncertainty. At every point time, the investment flow carries a cost equal to the investment, $-i_t$, and generates one of two possible outcomes. The project may either fail at the current position or it will pass successfully though the current position. In the event of a failure, the value of the project

drops from the current value, denoted by $V(k_t)$, to 0. In the event of a successful passage the position increases by dk_t and the value increase correspondingly by $V'(k_t)$. The dynamic programming equation for the optimal investment policy is now given by:

$$rV(k_t) = \max_{i_t \in \mathbb{R}} \left\{ -i_t + \gamma\sqrt{i_t} \left(V'(k_t) - \frac{\lambda}{k_t} V(k_t) \right) \right\}. \quad (3)$$

The value of the project depends on the flow of investment i_t in period t through three channels: (i) the direct cost of the investment i_t , (ii) the failure rate $\gamma\sqrt{i_t} \cdot \lambda/k_t$, and (iii) the rate of change $\gamma\sqrt{i_t}$ in the position of the project.

The rate of change $\gamma\sqrt{i_t}$ in the position k_t is a concave function of the current investment i_t . The optimal investment policy, therefore, is the result of an optimal trade-off between the speed of investment and the cost of building up the asset. The optimal investment at point k_t is determined by the first order conditions for the dynamic programming equation (3):

$$\frac{1}{2}\gamma \left(V'(k_t) - \frac{\lambda}{k_t} V(k_t) \right) = \sqrt{i_t},$$

and solving for the investment flow we obtain:

$$i_t^* = \left(\frac{\gamma}{2} \left(V'(k_t) - \frac{\lambda}{k_t} V(k_t) \right) \right)^2. \quad (4)$$

We can insert the optimal investment flow i_t^* into the value function (3) and obtain an ordinary differential equation for the evolution of the value of the venture:

$$rV(k_t) = \frac{\gamma^2}{4} \left(V'(k_t) - \frac{\lambda}{k_t} V(k_t) \right)^2. \quad (5)$$

We observe from (4) and (5) that the optimal investment i_t^* is linear in the flow value of the venture at time t :

$$i_t^* = rV(k_t). \quad (6)$$

We can rewrite the differential equation in its canonical form as:

$$V'(k_t) = \frac{\lambda}{k_t} V(k_t) + \frac{2}{\gamma} \sqrt{rV(k_t)}. \quad (7)$$

With a change of variable given by $W(k_t) \triangleq \sqrt{V(k_t)}$, we can transform the above differential equation into a nonlinear first order differential equation which we can solve explicitly by variation of parameters. We obtain:

Proposition 1 (Investment Policy)

1. The optimal investment policy i^* is increasing and convex in the state k_t .
2. The optimal investment policy i^* is decreasing and concave in the failure rate λ .
3. The optimal investment policy i^* is increasing and convex in Y .

The intuition of Proposition 1 is that the value of the project increases with the gradual resolution of uncertainty about its final success. The more likely it becomes that the project will succeed, the higher is its current value, making it optimal to speed up the discovery in spite of the convex increase in the associated cost of investment. In fact, as equation (4) shows, there is a linear relationship between the optimal investment flow i_t^* and the value of the venture that varies only with the discount rate.

We note that our model is built on a central premise: from the perspective of a risk-neutral investor, the expected return of the investor is constant over time and given by

$$R \triangleq 1 + r.$$

In our model, the failure event is characterized by a fall of the value to zero. In the absence of a failure event, we observe a change in the position k_t given by:

$$dk_t = \gamma \sqrt{i_t} dt.$$

The constant return R is therefore composed of a return in the event of a failure, which is given by 0, and the *surviving return* in the event of a successful continuation, defined by R_t . We therefore have

$$R = \Pr(\text{failure}_t) \cdot 0 + \Pr(\text{survival}_t) \cdot R_t.$$

Given that the instantaneous failure rate in period t is simply

$$\gamma \sqrt{i_t} \frac{\lambda}{k_t},$$

the surviving return R_t in period t is implicitly defined by:

$$R = \left(1 - \gamma \sqrt{i_t} \frac{\lambda}{k_t}\right) \cdot R_t,$$

and explicitly given by:

$$R_t = \frac{R}{1 - \gamma \sqrt{i_t} \frac{\lambda}{k_t}}. \tag{8}$$

We can also express the surviving return R_t using the value function given in (3) and describe the surviving return of the project in terms of the net change in the continuation value $\dot{V}_t - i_t$ relative to the value V_t of the project:

$$R_t = \frac{\dot{V}_t - i_t}{V_t}.$$

We can infer from (8) that the surviving return R_t is controlled by the product of the conditional failure rate λ/k_t and the investment intensity $\gamma\sqrt{i_t}$. As the conditional failure rate is declining in k_t and k_t is increasing over time, it follows that the conditional failure rate is declining over time as well. This may contribute to a decline of the surviving returns over time. On the other hand, the investment intensity $\gamma\sqrt{i_t}$ is increasing over time since the project becomes more valuable as the successful completion of the project appears to be more likely. The intertemporal profile of the surviving return is then determined by the trade-offs between failure rate and optimal responsiveness of the investment to the arrival of new information.

Proposition 2 (Surviving Returns)

1. *For a given terminal value Y , the surviving returns R_t are decreasing over time as long as the failure rate λ and the length of the development phase K are not too high.*
2. *For a given terminal value Y , the surviving returns R_t are increasing over time if the failure rate λ and/or the length of the development phase K are too high.*

4 Staging and Learning

4.1 Staging

We turn now to the second aspect in the investment decisions of the investors, namely the optimal degree of their involvement into the project. This extension captures the reality of investment transaction costs brought about by multi-party bargaining, contracting and the evaluation of the project which leads to a discrete number of financing decisions and rounds. We extend our baseline model to include stage financing that we portray as the decision on the optimal degree of implication into the project. The idea is that continuous involvement is costly for the investors. Therefore, they reevaluate the investment policy only infrequently. We assume that each investor intervention triggers a cost that is sufficiently elevated to force the optimal financing path to occur in discrete lump-sum installments. We express this cost as a constant fraction of the current project value. The

investment decision at the beginning of each round includes a decision about the funding volume as well as a decision about the constant flow of the investment i during this round.

Thus, we depict the staging decision as the result of the trade-off between the transaction costs of a new funding round *versus* the flexibility to adjust the speed of investment. The objective of this analysis is to understand the optimal structure of stage financing based on this trade-off. Importantly, the staging decision is fully endogenous in the sense that both the number of stages as well as their duration are chosen by investors in reaction to their information at the beginning of each round.

The cost associated with each new investment round is introduced as follows. We assume that during the process of putting together and negotiating each new funding round, there is a probability of $1 - p$, that the agreement between the parties involved is not reached and the project is abandoned. With the complementary probability p the relevant funding agreement is reached and the project continues. The probability $1 - p$ by which the agreement fails to be completed can alternatively be viewed as a transaction cost that is proportional to the project value.

We now describe the optimal investment with stage financing. We denote by $i_{l,m}$ the optimal investment at stage l if the entire project is financed in m stages (conditional on surviving until the final state K), with $l \leq m$. Similarly, we denote by $V_{l,m}(k_t)$ the value function of the project in stage l and state k_t if the project is supposed to be funded in m stages until the successful completion of the project.

If the project is funded in a single stage, i.e. it is funded in the initial state k_0 with the objective of maintaining a given investment level i until the positive or negative termination of the object, then the value function is given by the unique solution of the first order differential equation:

$$rV_{1,1}(k_t) = -i_{1,1} + \lambda\sqrt{i_{1,1}} \left(V'_{1,1}(k_t) - \frac{\lambda}{k_t} V_{1,1}(k_t) \right), \quad (9)$$

subject to the boundary condition $V_{1,1}(K) = Y$. The value function can be explicitly solved and we obtain:

$$V_{1,1}(k_t) = Y \left(\frac{k_t}{K} \right)^\lambda e^{\frac{k_t - K}{\sqrt{i_{1,1}}} \frac{r}{\gamma}} - \frac{\sqrt{i_{1,1}}}{(\lambda - 1)\gamma} \left(k_t - K \left(\frac{k_t}{K} \right)^\lambda e^{\frac{k_t - K}{\sqrt{i_{1,1}}} \frac{r}{\gamma}} \right).$$

The optimal investment policy given the initial state k_0 can be obtained implicitly by the first order condition of $V_{1,1}(k_0)$ with respect to $i_{1,1}$. The terms on the rhs of the equation represent the benefit and the cost of pursuing the project at a fixed intensity level i . The first term represent the time discounted probability that the project is successfully realized. The second term represents the expected cost of developing the project.

We now consider the role of stage financing. The value function $V_{1,1}(k_t)$ is determined by the optimal investment funding to complete the venture in a single round starting at k_t . The cost of the stage funding is given by the commitment to a specific investment flow over the round horizon. If the project is developing well, then the investors will react with the infusion of new funds and a new, and presumably higher, investment flow. Given that a renewal of the funding is not certain, but might lead a failure of the project with probability $1 - p$, the question then becomes, at which level of development k_t does it become optimal to complete the development of the venture in multiple rather than in a single stage of funding.

Next, if the project is to be funded in two stages, then the optimal funding policy starting at the initial position k_0 has to make three distinct choices: (i) it has to determine the initial funding level $i_{1,2}$, (ii) the continued funding level $i_{2,2}$ and (iii) the state k_1 in which the funding is supposed to be renewed. Conditional on the optimal funding level $i_{2,2}$ given the renewal stage k_1 , the value function in the initial state k_0 is therefore given as the solution to the following optimization problem:

$$V_{1,2}(k_0) = \max_{i_{1,2}, k_1} \left(\frac{\sqrt{i_{1,2}}}{(\lambda - 1)\gamma} \left(k_0 - k_1 \left(\frac{k_0}{k_1} \right)^\lambda e^{\frac{k_0 - k_1}{\sqrt{i_{1,2}}} \frac{r}{\gamma}} \right) + p \left(\frac{k_0}{k_1} \right)^\lambda e^{\frac{k_0 - k_1}{\sqrt{i_{1,2}}} \frac{r}{\gamma}} V_{2,2}(k_1) \right). \quad (10)$$

We observe that the value function $V_{2,2}(k_1)$ has the same properties as $V_{1,1}(k_0)$ except that the associated investment level $i_{1,1}$ is determined earlier at k_0 rather than at k_1 . But in either case, the optimal investment choice provides the necessary funds in a single round and until the project is completed.

If the investment is provided over several rounds, then the investment decision in each round, and in particular in state k_0 is formally a *joint* optimal control and stopping problem. As we are interested in the interaction between information arrival and investment decisions, we focus our analysis on the following simple question: for a given position k_t , will it be optimal to undertake the remaining investment for the interval $K - k_t$ still to be run in a single round or in two rounds? In other words, we determine the optimal switching decision at which it becomes optimal to comprise the remaining interval $K - k_t$ into one round rather than in two. The comparative statics of this switching point, denoted by k^* gives us all the hypotheses that we need for our empirical investigation.

The control problem then is the determination of the investment flow i and the stopping problem regards the determination of the renewal state k_1 . While we restrict the analysis here to the optimal determination of funding and renewal with two stages, due to the recursive structure of the funding problem, the optimality conditions and the qualitative properties of the optimal funding decision

extend naturally from two to finitely many funding stages.

The first result characterizes the largest development state k^* at which it is still optimal to provide funding in two rounds rather than in one single round. It also shows how the optimality of multiple stage funding is determined by the primitives of the model, namely the returns of the venture Y , the failure rate λ of the Pareto distribution and the probability p of a successful renewal of the funding policy.

Proposition 3 (Optimality of Staging)

1. *The investment levels satisfy: $i_{1,2} < i_{1,1} < i_{2,2}$.*
2. *There is a unique k^* such that it is optimal to add one more stage of funding.*

The relationship between the staging decision and the investment decision is depicted in Figure 1 and Figure 2.

INSERT FIGURE 1 AND FIGURE 2 HERE

If the project is funded in a single stage, then the value function V_1 is continuously increasing until it reaches the terminal value Y . If on the other hand, the project is funded in two stages, then the increase in value is initially smaller as the initial investment is smaller and the project has still to secure the second funding round. If at the stopping point k_1 , the funding for a second round can be secured, then the associated value observes an upward jump from $V_2^-(k_1) = V_{2,2}(k_1)$ to $V_2^+(k_1) = V_{1,2}(k_1)$, where the value before the jump, $V_{2,2}(k_1)$, and the value after the jump, $V_{1,2}(k_1)$ satisfy the following relationship: $V_{2,2}(k_1) = pV_{1,2}(k_1)$. Given the optimality of staging, we now investigate the temporal structure of the staging. In particular, we are interested in the length of each staging round as we come closer to a successful completion of the venture. In particular, we show that as the length of the remaining development increases, $K - k_0$, increases, or alternatively the initial starting point, k_0 , decreases, then it will eventually become optimal to switch from a single stage funding to a multiple stage funding policy. As Figure 2 illustrates, the advantage of the multiple stage funding policy is that it allows the investment flow to be adjusted upwards as the project moves closer to completion.

Proposition 4 (Structure of Staging)

1. *The state k^* is decreasing in Y and increasing in λ .*
2. *The length of the first stage is increasing in the renewal rate p .*
3. *The flow of funding is increasing in Y over all funding stages.*

In particular, an implication of Proposition 3.2 is that a project with a larger return Y will see fewer rounds of funding, as the delay or impasse resulting from a renewal of the funding leads to a higher opportunity cost for a project with a larger possible return Y .

4.2 Learning

So far, we analyzed the dynamic development of the venture with an essentially binary information structure. Either the project progressed and in consequence the future prospect improved, or the venture failed and the funding was terminated. In this final extension we would like to accommodate interim learning while the project is carried out. In particular, we are interested in learning during the project in the sense that the progress of the project uncovers information that may change the expectation about the future probabilities of failure and success. This interim arrival of information is naturally of interest as the current development may also give rise to additional information about the future likelihood of success. Consequently, we shall extend the basic model to accommodate the arrival of new information about the likelihood of success. More specifically, we shall assume that the venture starts with a given probability of failure $\lambda > 0$. At a random time, the current probability of failure λ is replaced by a new failure probability, which can be either lower or higher than the current failure probability λ , with $\lambda_l < \lambda < \lambda_h$. We shall assume that the expected failure probability is equal to the current failure probability, or

$$\lambda = \alpha \lambda_h + (1 - \alpha) \lambda_l.$$

The current failure probability can therefore be interpreted as the current estimate of the true, but currently unknown failure probability which is given by λ_h with probability α and λ_l with probability $1 - \alpha$. We observe that a jump to lower failure probability λ_l represents a positive shock from the point of view of the investors, and conversely an upwards jump to λ_h represents a negative shock as it lowers the expected value of the venture. The new information about the failure probability is assumed to arrive with a constant failure probability given by ρ . The dynamic investment problem can now be represented by the usual dynamic programming equation:

$$rV(k_t) = \max_{i_t} \left\{ -i_t + \sqrt{i_t} \gamma \left(V'(k_t) - \frac{\lambda}{k_t} V(k_t) + \rho ((1 - \alpha) V_l(k_t) + \alpha V_h(k_t) - V(k_t)) \right) \right\}. \quad (11)$$

The investment problem represented by (11) is similar to the earlier model, with the exception of the additional jump terms $V_l(k_t)$ and $V_h(k_t)$. The value function represent the continuation value of the venture conditional on knowing that the true failure probability is either λ_l or λ_h , respectively. While the continuation values, $V_l(k_t)$ and $V_h(k_t)$, have the same form as the value function in the basic model, the initial value function here does not permit an explicit solution as we incorporate the possibility of jump to different failure probabilities. Nonetheless, the implicit solution allows us to obtain a number of important comparative static results.

Proposition 5 (Survival Probability and Investment)

Following a positive shock of λ to λ_l :

- 1. the probability of eventual success increases;*
- 2. the investment flow jumps upwards.*

5 Hypothesis Development

In this Section, we summarize the hypotheses of our theoretical model in order to confront them with venture capital evidence.

Initial Valuation, Time of Information Arrival, and Return Dynamics We can use our model to explore typical patterns of learning in venture-backed investment projects. Prior to launching a project, investors hold beliefs about the prospects (e.g. final value at exit) and the risks of the project, but they may also receive information after the project is launched. We distinguish between three hypotheses regarding the arrival of information: *(i)* in the uninformed investor hypothesis, the investors cannot discriminate between the prospects of individual projects and use the expected values for failure risk and the value of success; *(ii)* in the ex ante information hypothesis, the investors can discriminate between the prospects of individual firms, yet the bulk of the information is available at the project launch; *(iii)* in the interim information hypothesis, the investors obtain valuable information on the project terminal value and failure risk over the course of the investment cycle.

We distinguish between these hypotheses by using the initial valuations and the evolution of the valuations over the venture capital investment cycle. We start from the premise that, at the time of inception, the value of innovative projects consists essentially of the expectation of the future value

of the project in the event of success. They typically have little or no assets. Therefore, variations in the present value of the project are mainly explained by differences in the expected final value at exit if the venture is successful, or the estimated probability of success.

Under the ex ante information hypothesis, ex ante information on the failure risk should be impounded in the initial project valuation. We investigate this hypothesis by analyzing whether the initial project value predicts the ultimate success probability. It could also be the case that investors have ex ante information on the final value. In this case we expect variation in the initial values to be correlated with the final values in the event of success. Alternatively, if the initial value of the firm and the ultimate success are uncorrelated, then this lends support to either the uninformed investor or the interim information hypothesis. Under these two hypotheses, we also expect little correlation between initial and final value.

Moreover, we can discriminate between the uninformed investor and the interim information hypothesis by analyzing the relationship between interim valuations and ultimate success: under the interim information hypothesis, we expect that projects with a large value increase during the investment cycle, i.e. high abnormal returns, are more likely to succeed, whereas we expect no correlation under the uninformed investor hypothesis. We also expect, under the interim information hypothesis, that high abnormal returns over the investment cycle are linked to higher exit values, whereas we expect no relationship under the uninformed investor hypothesis. Thus, by comparing the predictive power of initial and interim valuations for final outcomes, we can draw inferences on the importance of ex ante information and interim information. The same is true for the relationship between investment behavior (e.g. investment flow following the first round) and final outcome (e.g. IPO, M&A, or going down).

Our model also allows us to analyze the dynamics of the failure risk over the investment cycle. Our model is based on the premise of constant expected returns over the lifetime of the project. That is, in a risk-neutral setting the value increase in each round is just an adequate compensation for the failure risk, and hence should decrease as the project is developed to maturity.² For the sake of the argument and in contrast to our assumption, suppose for a moment that the hazard rate of dropping out at any given position $k_t \in (0, K)$ during the investment cycle would be constant.

²If the investors were risk-averse, then the expected or unconditional returns should be decreasing as the project matures, given that the failure risk decreases. The magnitude of this effect may not be large for moderate levels of risk aversion. This appears to be roughly consistent with our finding of decreasing means in the expected returns. Our (weak) result of increasing medians contradicts the hypothesis for risk-averse investors.

With the increase in investment flows, this would imply an increasing speed in the discovery process over time and hence an increase in the returns of the project, since the return is just an adequate compensation for the dynamics of the failure risk. This thought experiment underlines the strong implications contained in the following two elements of our empirical analysis: *(i)* the investment flows are increasing over time and *(ii)* the returns generally decrease from one round to the next. The only way how these two observations can be reconciled is if the conditional hazard rate is declining at a sufficiently high rate (Proposition 2). This condition is satisfied in our model with a sufficiently large value of the parameter λ of the Pareto distribution.

Investment Flow Our model shows that investments optimally react to uncertainty, and that investment flow will increase if there is less uncertainty about the project outcome. In particular, our model explains that as a venture projects matures, it should exhibit larger investments, and higher outlays over any given period of time.

The fundamental prediction of our model is that there should be a positive relationship between project valuation and investment flow. This relationship between the project's valuation and the investment flow should hold throughout the investment cycle. This effect should hold whether the firm's value is high because the final value Y is high or because the failure rate λ is low, or both. The reason is that both a higher final value and a lower failure risk translate into a larger present value of the project, and the model shows that a project's investment flow is closely linked to current project valuation. Therefore, we also expect to find that the investment flow is increasing both in measures of the expected final value and the expected failure risk.

Section 4 extends our model to allow for the interim information hypothesis. Our theoretical analysis explores the possibility that there is interim learning about the failure risk λ of the project. If the firm learns positive news about λ , then this has two consequences. First, a lower λ means an increase in the current value of the project V_t , and hence a positive abnormal return at the time the good news is received. Second, the investment flow should optimally increase. The inverse relationship holds if the firm receives bad news about λ . Thus, the model predicts that subsequent investment flows increase with abnormal returns. The same argument would hold if there were interim learning about the final value of the project Y . Good news about Y translates into an increase in the current value and hence a positive abnormal return, and at the same time leads to an upwards adjustment in the optimal investment flow.

Staging Frequency Section 4 explicitly considered that funding may be provided in lumpy amounts even though investments are made continuously. The renewal of the funding decision enhances the value of the real option to abandon the project. Our analysis shows that shorter financing rounds will occur if the information that investing produces is more valuable for the abandonment decision. In particular, the model explains that the staging frequency should be lower for projects with a higher success probability.

In our model the contracting costs are proportional to the current project value. The analysis shows that the staging frequency should be lower for projects with a high expected exit value. The reason is that the expected loss of adding one round increases in the expected final value, whereas the potential savings if there is early abandonment are constant. Thus, we expect the number of rounds to be a decreasing function of the initial value of the project. This is true whether the variations in the project's value are driven by differences in the expected final value or in the expected failure rate.

Considering interim learning about the project's failure risk, a reduction in the estimated failure probability λ means, first, an increase in the current project value and hence in the abnormal return. At the same time, in reaction to an increase of the value of the firm, the subsequent financing will be undertaken in fewer rounds. Therefore, the model predicts that the number of subsequent rounds until successful completion decreases with the initial abnormal return. The same negative relationship between the initial abnormal return and the number of subsequent rounds holds if there is interim learning about the final value of the project.

Size and Duration of Financing Rounds A separate set of predictions addresses the duration and capital raised in each financing round. The real option of abandonment is most valuable when the uncertainty about ultimate success is highest. As shown in Section 4, as the project advances and investors become more confident about ultimate success, they are willing to travel a longer distance $[k_l, k_{l+1})$ in a single financing round l . Thus, the model leads to the prediction that the size or the volume of the investment rounds is increasing from one round to the next. If the investment flow were constant, then the round duration would also be increasing. However, as Proposition 5 shows, the optimal funding flow/intensity also increases from one round to the next. Therefore, the overall impact on round durations is ambiguous, and they could increase as well as decrease as the project advances.

Moreover, our model predicts that the capital raised in a round is a decreasing function of the

failure rate λ , and an increasing function of the expected final value Y and the cost of adding a contracting stage $1-p$. At the same time, the investment flow increases in Y and decreases in λ . Therefore, the model predicts that the investment size increases in Y and decreases in λ , but the impact on round duration is again ambiguous.

Interim learning about the project’s failure risk implies that capital raised should increase after a positive shock.

Total Project Duration Our model implies that projects with an above-average initial valuation will have a consistently higher investment flow. Therefore, the model predicts that they will be completed faster.

6 Data Description and Empirical Methodology

Our data of venture capital investments are provided by Sand Hill Econometrics (SHE) and contain the majority of US investments in the period from January 1987 to March 2002. SHE combines and extends two databases, Venture Xpert (formerly Venture Economics) and Venture One, which are extensively used in the venture capital literature. According to ? and ?, the Venture Xpert data contain the majority of the investments. SHE has spent substantial time and effort to ensure the accuracy of the data. This includes removing investment rounds that did not actually occur, adding investment rounds that were not in the original data, and consolidating rounds, so that each round corresponds to a single actual investment by one or more venture capitalists ? and ? use different versions of this data set. The data in ? end in June 2000 and the data in ? end in 2005.

The data contains firm level information and venture capital investment round level information. At the firm level, we focus on the following variables: a unique firm ID, industry category (healthcare, IT, retail, or others), and the exit type (IPO, merger & acquisition, out of business, restart or unknown). A firm with unknown exit may be alive at the end of the sample period or exited at a unknown time point before March 2002. The round observations are linked to firms via the unique firm IDs. At the round level, we use the following variables for each round: the date stamp of the round, the round status (seed, first, early, late, mezzanine, restart, IPO, acquisition, busted round, or unknown), the business status of the firm during the current round (start up, in development, beta-testing, in clinical trials, shipping, profitable, restart, or unknown), the amount (million dollars) raised in the current round, post money valuation of the firm and an exit

dummy that equals one if the current round is an exit round. We filter the data by keeping firms that have at least one round before the exiting round, removing firms that exit as a restart or have restart rounds. We further aggregate the business status information by combining in development, beta-testing, and in clinical trials as one status called “in development”, and combining shipping, profitable as “in production”.

Table 1 reports summary statistics for the firms in the data. Panel A reports the break-down of the firms according to industries and exits. It shows the typical composition of venture capital samples, with more than half of the companies in IT-related activities, 15% in Healthcare and 9% in retail. 70% of companies have unknown exits, and 25% exited via either IPO or a trade sale. Panels B and C report average round frequencies and round durations, respectively, for the same breakdown.

While some of the firms with unknown exits might be alive at the end of the sample period, many others might have already been liquidated by then. Failures are incompletely documented in the data because liquidation is less visible than IPOs or trade sales. If firms with unknown exits are more likely to be already liquidated than alive, excluding all firms with unknown exits from our analysis would lead to a biased sample of venture capital backed firms that overrepresents successful ventures. Further, if the firms with a documented failure systematically differ from firms that have been liquidated but do not have a documented failure, excluding all firms with unknown exits would lead to biased results, particularly concerning the analysis of the determinants of exit types and final values of venture capital backed firms. To mitigate the possible sample selection bias, we distinguish “zombies” - firms that were liquidated before March 2002 but have no documented exit in the data set - from firms with unknown exits. Specifically, for each firm with unknown exit, we estimate the length of the period (in months) for which the capital raised in the last recorded round would keep a firm alive. If the duration between the last recorded round and March 2002 is longer than this “survival time”, we assume the firm went down at the end of the “survival time”. Otherwise, we assume that the firm is alive at the end of the sample period. The empirical analysis in this paper uses not only firms with documented exits but also firms with estimated exits.

We use the following procedure to estimate the “survival time” after the last recorded round for firms with unknown exits. First, we estimate the amount of capital consumed per month after the last recorded round for each firm. Second, we divide the raised amount in the last recorded round with the estimated monthly capital consumption, and obtain the “survival time” in months. In our first step, we first run a round level regression of monthly capital consumption (raised capital

divided by the number of months between the current and next rounds, in log) for all rounds except the last recorded rounds, on the amount of capital raised (in log), the post money valuation (in log), industry dummies, business status dummies, and dummies for future exit types. The rationale is that the amount of capital consumed per month by a firm is determined by the amount of capital raised, the size of the firm, the industry and business status of the firm, and the quality of the firm (proxied by the final exit type). The R^2 of the regression is 0.75, which seems to indicate that the monthly capital consumption is explained reasonably well by this regression. The regression results suggest that the amount of capital raised and the post money valuation are positive and significant at the 1% level, which indicates that larger firms consume more capital per month and firms with more capital raised consume more capital per month. In addition, retail firms consume more capital per month than firms in other industries, which is significant also at the 1% level. Firms “in development” and “in production” consume less capital than start up firms, which is significant at the 1% level. Moreover, exit type dummies also have significant coefficients. We run another regression with the post money valuation excluded, and obtain similar results and a R^2 of 0.74. After running these two regressions, we estimate the amount of capital consumed per month after the last recorded rounds for firms with unknown exits, using estimated coefficients from the regressions and the explanatory variables for the last recorded rounds. We use the coefficients from the first regression for the last recorded rounds with post money valuations observed, and use the coefficients from the second regression for rounds with unobserved post money valuations. We let the exit dummies to be 0, assuming that the firms’ exits would follow the same distribution of the exits of the exited firms. In our second step, we divide the raised amount with the estimated capital consumption per month, and obtain the “survival time.” If the survival time is long enough to go beyond the end of the sample period, we assume that the firm is alive. If the survival time ends before the end of the sample period, we add an exit round for the firm, assuming that this firm raised \$0 in the exit round, and went down with \$1 post money valuation.

Starting with Table 2, all the tables and analysis are based on firms with estimated exits whenever the exits are unknown. Table 2 reports the same summary statistics as Table 1, but uses firms with documented *and* estimated exits using the procedure discussed above. Compared with Table 1, the number of liquidated firms increases dramatically - from 938 in Table 1 to 10,857 in Table 2. The percentage of firms that are liquidated increases from 5% in Table 2 to 57% in Table 2. The summary statistics for pre-exit rounds and duration before exits also show differences.

Table 3 reports summary statistics for venture capital investment rounds prior to firms’ exits.

Panel A reports the number of rounds for firms with different exit types in different industries. Panels B, C, and D report the means and standard deviations, which are calculated using rounds with corresponding information available, of pre-financing duration (the number of months between the previous round and the current round), investment amount (million dollars), and the ratio of investment amount to post-money valuation. The table shows that the pre-financing duration, investment amount, and the ratio of investment amount to post-money valuation are similar across industries and different exit types.

7 Empirical Results

Initial Valuation, Time of Information Arrival, and Return Dynamics We start by exploring our alternative hypotheses concerning the typical arrival time of information. Table 4 reports evidence from a comparison of successful projects (firms exiting via IPO or M&A) and unsuccessful ones (firms that are going down). Under the ex ante information hypothesis, valuable information about the prospects of a particular project is mostly known ex ante. If the information were about the success probability, then we would expect projects with higher initial values to succeed more often. As Table 4 shows, this is not the case: successful projects actually have lower initial values compared with unsuccessful ones. Moreover, when we look at investment behavior we also find evidence that is inconsistent with the ex ante information hypothesis. Table 4 shows that there is no significant difference in investment behavior between failed and successful projects. First-round investments and the ratio of first-round investment to initial value are fairly constant across projects, regardless of their ultimate outcome. Further, under the ex ante information hypothesis, the first round investment flow, which reflects investors' ex ante belief of the final exits, should help predict the prospect of the project: a larger investment flow indicates a higher probability of success. Table 4 shows the opposite: IPO/M&A firms have smaller first round investment flows than firms that went down.

Additional evidence is provided by Table 5, which presents results of probit regressions on whether firms exit successfully (IPO or M&A) or exit as failures (including estimated failures). The initial value is not significant or - in one regression - is weakly significant but with the wrong sign according to the ex ante information hypothesis. Thus, the results about initial valuation and initial investment behavior resolutely reject the hypothesis that investors have ex ante discriminatory capabilities about the estimated success probability of a project.

We can also test whether investors hold relevant ex ante information about the expected final value in the case of success. Indeed, initial values and final values are correlated, and Table 6 shows that the initial value has clear predictive power for the exit value (significant at the 1% level) in the event of success. Hence we can conclude that investors have some ex ante information about potential final values, but not about the chances to succeed.

According to the interim information hypothesis, the estimates about the ultimate success probability and/or the exit value of the project evolve with the progress of the project. In many of our tests of the interim information hypothesis, we focus on the information content in the first round for which we have the most observations. From now on, we use the abnormal return from one round to the next to measure the information content of that round. We define the abnormal return as the component of the raw return that is orthogonal to the information already known by investors before the round, including the industry and business status of the project and the status of the round, *and* the common return of the whole venture capital asset class. The common return is assumed to be driven by the financial market instead of the specific prospect of the underlying project. As a result, the abnormal return captures information regarding the specific project that is unknown before the round.

We estimate abnormal returns as:

$$\log\left(\frac{V_{i,t_{k+1}} - I_{i,t_{k+1}}}{V_{i,t_k}}\right) = (t_{k+1} - t_k)\rho' \text{Industry}_i + (t_{k+1} - t_k)\lambda' \text{Business}_i + \sum_{s=t_k+1}^{t_{k+1}} (\log(R_{m,s})) + \varepsilon_{i,k+1}.$$

The subscript i, t_{k+1} denotes the month in which round $k + 1$ is raised for firm i ; $I_{i,t_{k+1}}$ is the amount of capital raised in round $k + 1$; $V_{i,t_{k+1}}$ is the post money value for round $k + 1$; Industry_i is a 3×1 vector of dummies corresponding to healthcare, IT, and retail firms; Business_i is a (3×1) vector of dummies corresponding to start up, in development, and in production; $\log(R_{m,s})$ is the log venture capital market returns, which is assumed to be affected by a vector of unknown market factors that vary over time and factor loadings that are constant for all venture capital investments; $\varepsilon_{i,k}$ is the portion of the log return that is not explained by market factors or information already known by investors, and thus is the "abnormal return". Note that in the regression, $\log(R_{m,s})$ is essentially the coefficient of a dummy variable for month s . The regression is similar to the repeat sales regression for the construction of real estate price indexes.³ We pool all rounds in the data set without missing variables and run the above regression. The monthly abnormal log return

³See ? for the original regression and ?, among others, for an application to real-estate markets.

for firm i from round $k - 1$ to round k , which is denoted by $AR_{i,k}$, is constructed from regression residuals as follows:

$$AR_{i,k} = \frac{\widehat{\varepsilon}_{i,k}}{t_k - t_{k-1}}.$$

As Table 4 shows, one of the most powerful results of our study is that projects that ultimately succeed are likely to receive a positive news update during the initial financing round. The abnormal return following the first round is strongly positive for projects that exit successfully, and significantly negative for all other firms (t -value for the difference: 25.335). The probit analysis in Table 5 confirms this effect when including other explanatory variables with a comparable level of significance. In addition, consistent with the hypothesis that interim information also updates the belief about the final value, Table 6 shows that the abnormal return following the first round is positively related to the exit value. Taken together, these results provide solid support for the interim information hypothesis.

We conclude that investors learn during the first investment round about the failure probability. In fact, information arrival *after* the launch of a project, more precisely during the first round, is a strong predictor of ultimate success, as opposed to ex ante information (initial valuations or first-round investment behavior). In addition, investors also learn about the final value, but parts of their expectation of exit values seems to be ex ante knowledge. The last regression in Table 6 shows that in fact both ex ante information (contained in the initial value) as well as interim information (contained in $AR_{i,2}$) explain the final value. In other words, the final value of a project seems to be partially contained in the ex ante information, and partially to be the result of interim learning as expressed in the abnormal returns over the project's investment cycle.

We turn to the exploration of the dynamics of risk and return. Our basic model assumes that the risk, conditional on survival, of dropping out is decreasing, as captured by the parameter λ of the Pareto distribution. This assumption of decreasing failure risk receives some support in Table 7, which shows that the probability of failing is decreasing over time. Note that while information content in a round is better measured by abnormal returns, raw returns seem a more appropriate measure for the total failure risk, including the expected and not expected (learned) components. Our learning model implies that round returns for surviving companies should show a decreasing trend as the failure risk subsides. This is indeed the case, as Table 7 shows ("surviving return").⁴ The results in Table 7 are in fact even more supportive of our model than these numbers suggest:

⁴We cannot test for the significance of these numbers as they are just the proportions of surviving projects in each round.

the probabilities express the average fraction of surviving projects in each round. Our model, however, predicts the survival probability in terms of units of the $[0, K]$ -investment cycle, which is the survival probability per invested dollar. Table 8 lends support to our premise of constant expected returns. We find that the means of the unconditional returns are increasing over time whereas the median returns are decreasing. Therefore, we conclude that it is difficult to reject the hypothesis of constant expected returns on which our risk-neutral model is based.

Investment Flows We turn to our predictions on investment flow (investment spending per month). Our model leads to the prediction that investment flows are inversely related to a project's estimated failure risk λ . We do not observe estimated or actual failure risk λ directly, but can approximate them in two ways. First, we observe the ultimate project outcome, which is determined by the actual failure risk. Note that the ex ante information hypothesis indicates that investors' estimated failure risk should correlate with the actual failure risk. If the ex ante information hypothesis is correct, we should expect ultimately successful exits (via IPO or trade sale) to be more likely for projects with an above-average investment flow. As the first three regressions in Table 5 show, there is no evidence for this effect. A possible indirect measure of the estimated failure risk λ is the ratio of final value to initial value. Should the ex ante information hypothesis be true, then the higher the ratio, the larger would be the expected cumulative failure probability and hence λ . Again, our regression results are not significant (and thus not reported in our tables). Therefore, consistent with our earlier results, Table 5 provides strong evidence that investors have no ex ante information regarding the chance of success.

The model also predicts that the investment flow increases with the expected final value of the project. Assuming that the investors hold ex ante information regarding the final values, a test is only possible for successfully completed projects, using their actually realized exit values as a proxy for expected final values. Regression 3 in Table 6 presents the results for this test, which seem to support our hypothesis: the coefficient of the first round investment flow is significant at the 1% level. We conclude that investors seem to have some ex ante information regarding the expected exit values.

In our model, the investment flow reacts to uncertainty, and it will optimally increase if there is less uncertainty about the ultimate project outcome. Table 9 provides strong support for this hypothesis (at the 1% level for both means and medians). Consistent with our model, investment flows are increasing over time. Table 9 also shows that venture projects exhibit larger investment

volumes as they advance from one round to the next (at 1% level for both means and medians).

Additional predictions on the investment flow which imply multivariate relationships are tested using OLS regressions; the evidence is provided in Tables 10 and 11. First, the model predicts that the investment flow increases in the valuation of the project, and implies that this positive relationship should hold for the first as well as the later rounds of the investment process. In Table 10, regressions for all firms (unconditional) and IPO and M&A firms substantiate this positive relationship for the first round. Further, since investors have *ex ante* information about the final value of the project, conditional on success (this hypothesis is validated by our results in Table 6 discussed earlier), final values should positively correlate with the initial investment flow. This is indeed the case, as the conditional regressions in Table 10 show. Table 11 provides evidence that the positive correlation between project valuation and investment flow holds throughout the investment process. In Table 11, regressions 2 and 5 show strong evidence for this effect (significant at 1% level). It is useful to note that we use the pre-money company value at the beginning of each round. Regressions 3 and 5 in Table 11 show that the optimal investment flows are autocorrelated and thus persistent throughout the project investment cycle (significant at 1% level). This is consistent with our model which predicts the persistence of the autocorrelation as a mirror image of the evolution of project valuation over time. We observe that the regression 5 controls for the sunk cost (the total investment amount raised before) of investors in firms, which would be significant if investors are not willing to realize losses and tend to continue financing bad projects. While the sunk cost is significant in regression 4, it is not significant in regression 5, which includes the lagged firm value, the lagged investment flow, and the lagged abnormal return. Note that, in Table 11, we use dummies to control for mezzanine rounds because they are likely bridge financing rounds prior to successful exits, and may not reflect the learning phase of the project.

We already presented evidence in favor of the interim information hypothesis. If this hypothesis is true then it has clear implications for investment behavior that can easily be tested: investment flow should be increasing in the most recent abnormal return in each round that we use as a proxy for interim information. If the last observed abnormal return $AR_{i,k}$ is higher, then the project should have received a more positive information update. All regressions in Table 11 support this prediction, and show strong evidence consistent with the interim information hypothesis. The effect seems to have a concave shape as the quadratic term for the abnormal return is negative and highly significant as well.

Staging Frequency With regard to the determinants of the round frequency, our model implies that larger financing rounds will occur if there is less uncertainty resolved for every dollar of investment. It predicts that the staging frequency should be lower for projects with a high success probability.

To test this hypothesis, we need to turn to regressions for completed projects that explain the number of rounds over the entire investment cycle. Since we do not observe the risk variable λ directly, we use the ratio of exit value to initial value for completed projects as a proxy, and assume that investors have ex ante information regarding final values, which is supported by our earlier evidence. The first line in Table 12 shows the results, with the total number of financing rounds as a dependent variable. In all regressions in Table 12, there is a positive and highly significant sign (at 1% level) for our proxy for λ . Moreover, our theoretical results imply a negative sign when we regress the number of rounds on the final value. Regressions 3 and 4 of Table 12 show indeed strong evidence (significant at 1% level) in favor of this hypothesis.

Considering interim learning about the project’s failure risk or final value, we predict that a positive information release that makes the project more valuable or less risky leads to less subsequent financing rounds. As regressions 2 and 4 in Table 12 show, this is indeed the case. The relationship is again nonlinear, as witnessed by the quadratic terms of $AR_{i,2}$.

Size and Duration of Financing Rounds Our model leads to very clear predictions on the investment size (capital raised) of each financing round. The model predicts that it is increasing from one round to the next. This is indeed the case for the means and the medians of the investment volume, as Panel B of Table 9 shows. By contrast, the predictions on round duration (in months) of each financing round are ambiguous, since increasing investment volume (the dollar amount provided in a given round) and investment flow have countervailing effects on the round duration. Interestingly, we do not find any clear patterns for round durations (not reported in tables), consistent with our model’s ambiguous predictions for round duration.

Our model predicts that a higher project valuation, reflecting either a high expected final value or a low expected failure probability, should translate into a larger investment volume in every round. The result of our regression analysis are presented in Table 13. We find strong evidence in favor of this hypothesis as regressions 2 and 5 show. Also, an above-average investment size in round $k - 1$, explained by a high exit value, low failure probability or high contracting cost, should also translate into an above-average investment volume in round k . This is indeed the case, as the

positive and significant signs for variable $\log(\text{investment}_{i,k-1})$ in Table 13 (regressions 3 and 5) shows. Note that we control for the sunk cost and mezzanine rounds in this table as well.

We also explore the implications of the interim information hypothesis for investment volume. The model implies that positive interim information releases should lead to an increase in the capital raised in each subsequent round. This is indeed the case as the highly significant and positive coefficient on the interim learning variable $AR_{i,2}$ shows in Table 13, which again exhibits a nonlinear effect.

Total project duration Our model implies that projects with an above-average initial valuation will be completed faster as they benefit from a persistently higher investment flow. We find clear evidence in support of this prediction in Table 14. Table 14 further substantiates that favorable information updates, which are proxied by abnormal returns, and lower failure risk, which is proxied by the reciprocal of $\log(\text{exitvalue}_i/\text{value}_{i,1})$, increase investment flows and thus help projects to be completed faster.

8 Conclusion

We investigated a stylized model to analyze how investors make optimal dynamic investment decisions in an innovative project as a function of their information about failure risk and potential final value. We consider the complete investment cycle and assume that information leading to the failure of the project may arise at any time, but at a decreasing probability. The investors choose the optimal speed of investment with a convex cost function. They also choose an optimal sequence of financing stages. We model the cost of each investment decision as a loss that is proportional to the current value of the project, so that investment decisions will only occur in discrete intervals.

The results of our theoretical analysis are the following. As the project advances and the probability of eventual success increases, investment flows should be optimally increasing. Therefore, decreasing surviving returns require that the failure risk decreases over time. If the probability of the information arrival is sufficiently front-loaded then the returns will be increasing. Thus despite an increase in the investment flow it follows that the information-sensitive development of the projects proceeds more quickly. The optimal staging sequence depends on the value of the real option to abandon: The higher the estimated final value of the project is, and the larger the estimated success probability, the fewer rounds will be used. Finally, we show that information

updates about the expected final value or the failure probability will be incorporated in all subsequent investment decisions. If the value of the project increases then the subsequent investment flow will increase. At the same time, the number of subsequent investment rounds will decrease, and the capital allocation for each of these rounds will increase.

In our empirical tests of these predictions, we find that investment flows are increasing over time as predicted. Our evidence shows that initially, investors seem to have little ability to predict the eventual probability of success, but have some forecasting ability about the final project value conditional on success. The design of the financing rounds follows the optimal pattern predicted by our model: the investment size and the investment flow is increasing from one round to the next, and projects with a high initial estimate of the final value or an optimistic appraisal are likely to succeed with fewer rounds than less valuable or more risky projects. As the project advances, frequently investors get information that leads them to reappraise the failure risk of the project. We show that such information updates lead them to adjust the investment path optimally: the subsequent investment flow as well as the size of each round and the number of subsequent rounds react in the way predicted by our model.

9 Appendix

The appendix contains the proofs of all propositions in the main body of the text.

Proof of Proposition 1. We consider the dynamic investment problem with uncertainty. The state variable k_t is simply the current position of the venture, but there is a possibility of implosion which is constant per unit of distance. The value function is given by:

$$rV(k_t) = \max_{i_t \in \mathbb{R}} \left\{ -i_t + \gamma \sqrt{i_t} \left(V'(k_t) - \frac{\lambda}{k_t} V(k_t) \right) \right\}, \quad (12)$$

and the resulting optimal investment is

$$i_t^* = \left(\frac{\gamma}{2} \left(V'(k_t) - \frac{\lambda}{k_t} V(k_t) \right) \right)^2. \quad (13)$$

The resulting ordinary differential equation for the evolution of the value of the venture:

$$rV(k_t) = \left(\frac{\gamma}{2} \right)^2 \left(V'(k_t) - \frac{\lambda}{k_t} V(k_t) \right)^2 \quad (14)$$

can be represented in its canonical form:

$$V'(k_t) = \frac{\lambda}{k_t} V(k_t) + \frac{2}{\gamma} \sqrt{rV(k_t)}. \quad (15)$$

With a change of variable given by $W(k_t) \triangleq \sqrt{V(k_t)}$, we can transform the above differential equation into a nonlinear first order differential equation. The differential equation (6) is a nonlinear first order differential equation which we can solve explicitly by variation of parameters (see ?).

We observe that

$$W'(k_t) = \frac{1}{2} \frac{V'(k_t)}{\sqrt{V(k_t)}}, \quad (16)$$

and hence

$$V'(k_t) = 2W(k_t) W'(k_t).$$

Replacing $V(k_t)$ and $V'(k_t)$ by $W(k_t)$ and $W'(k_t)$ in (15) we get:

$$W'(k_t) = \frac{\sqrt{r}}{\gamma} + \frac{\lambda}{2k_t} W(k_t). \quad (17)$$

The solution of the differential equation (17) subject to the boundary condition:

$$W(K) = \sqrt{Y},$$

is now given by:

$$W(k) = \sqrt{Y} \left(\frac{k}{K} \right)^{\frac{1}{2}\lambda} - \frac{2\sqrt{r}}{\gamma(2-\lambda)} \left(K \left(\frac{k}{K} \right)^{\frac{1}{2}\lambda} - k \right) \quad (18)$$

For the failure rate $\lambda = 2$, the value function becomes linear in k and is given by:

$$W(k) = \sqrt{Y} \frac{k}{K} - \frac{2\sqrt{r}}{\gamma}. \quad (19)$$

Consequently, the value function $V(k)$, based on the general solution of $W(k)$ in (18) is given by:

$$V(k) = (W(k))^2.$$

We can now immediately establish the properties (1)- (3) of the optimal investment i_t^* by using the linear relationship (6). We write

$$i^*(k, Y, \lambda) \triangleq r \left(\sqrt{Y} \left(\frac{k}{K} \right)^{\frac{1}{2}\lambda} - \frac{2\sqrt{r}}{\gamma(2-\lambda)} K \left(\frac{k}{K} \right)^{\frac{1}{2}\lambda} - k \right)^2.$$

- (1.) We obtain by elementary calculus that $\partial i^*/\partial k > 0$ and $\partial i^{*2}/\partial^2 k > 0$.
- (2.) We obtain by elementary calculus that $\partial i^*/\partial \lambda < 0$ and $\partial i^{*2}/\partial^2 \lambda < 0$.
- (3.) We obtain by elementary calculus that $\partial i^*/\partial Y > 0$ and $\partial i^{*2}/\partial^2 Y > 0$. ■

Proof of Proposition 2. The surviving returns $R_t \triangleq 1 + r_t$ are given by:

$$r_t = \frac{-i_t + \sqrt{i_t} \gamma V'(k_t)}{V(k_t)}.$$

Using the characterization of the optimal investment given by (6), we get

$$r_t = \frac{-rV(k_t) + \sqrt{rV(k_t)} \gamma V'(k_t)}{V(k_t)} = -r + \frac{\sqrt{r} \gamma V'(k_t)}{\sqrt{V(k_t)}}.$$

Using the same change of variable as in Proposition 1, we find that

$$r_t = -r + 2\sqrt{r} \gamma W'(k_t).$$

Using (17) to replace $W'(k_t)$ we have $r_t \triangleq r(k_t)$:

$$r(k_t) = -r + 2\sqrt{r} \gamma \left(\frac{\lambda}{2k} W(k_t) + \frac{\sqrt{r}}{\gamma} \right),$$

and we now ask whether $r'(k_t)$ is positive or negative:

$$r'(k_t) = 2\sqrt{r} \gamma \left(-\frac{\lambda}{2k^2} W(k_t) + \frac{\lambda}{2k} W'(k_t) \right),$$

and using (17) again to replace $W'(k_t)$ we get

$$\begin{aligned}
r'(k_t) &= 2\sqrt{r}\gamma \left(-\frac{\lambda}{2k^2} W(k_t) + \frac{\lambda}{2k} \left(\frac{\lambda}{2k} W(k_t) + \frac{\sqrt{r}}{\gamma} \right) \right) \\
&= 2\sqrt{r}\gamma \left(-\frac{\lambda}{2k^2} W(k_t) \left(1 - \frac{\lambda}{2} \right) + \frac{\lambda}{2k} \frac{\sqrt{r}}{\gamma} \right) \\
&= \frac{\sqrt{r}\gamma\lambda}{k} \left(-\frac{1}{k} W(k_t) \left(1 - \frac{\lambda}{2} \right) + \frac{\sqrt{r}}{\gamma} \right). \tag{20}
\end{aligned}$$

Since we have a solution for the value function $W(k)$, we can insert it and compute the values. We have

$$\left(-\frac{1}{k} W(k_t) \left(1 - \frac{\lambda}{2} \right) + \frac{\sqrt{r}}{\gamma} \right),$$

and inserting $W(k)$ we get

$$\left(-\frac{1}{k} \left(\sqrt{Y} \left(\frac{k}{kK} \right)^{\frac{1}{2}\lambda} - \frac{2\sqrt{r}}{2\gamma - \lambda\gamma} \left(\frac{K}{k} \left(\frac{k}{K} \right)^{\frac{1}{2}\lambda} - 1 \right) \right) \left(1 - \frac{\lambda}{2} \right) + \frac{\sqrt{r}}{\gamma} \right),$$

or

$$\left(-\left(\sqrt{Y} \left(\frac{k}{kK} \right)^{\frac{1}{2}\lambda} - \frac{2\sqrt{r}}{\gamma(2-\lambda)} \left(\frac{K}{k} \left(\frac{k}{K} \right)^{\frac{1}{2}\lambda} \right) \right) \left(1 - \frac{\lambda}{2} \right) \right).$$

The second term is negative for $\lambda > 2$. To sign the first term, we simplify to:

$$-\left(\sqrt{Y} \frac{1}{k} \left(\frac{k}{K} \right)^{\frac{1}{2}\lambda} - \frac{2\sqrt{r}}{\gamma(2-\lambda)} \left(\frac{k}{K} \right)^{\frac{1}{2}\lambda-1} \right),$$

or

$$-\left(\sqrt{Y} - \frac{2K\sqrt{r}}{\gamma(2-\lambda)} \right) \left(\frac{k}{K} \right)^{\frac{1}{2}\lambda} \frac{1}{k}. \tag{21}$$

Thus if K is not too large and λ is sufficiently below 2, then the surviving returns are declining everywhere. Conversely if K is large or λ above or just below 2, then the surviving returns are increasing in k_t and hence in time everywhere. ■

Proof of Proposition 3. We first describe the optimal investment with stage financing. We denote by $i_{l,m}$ the optimal investment at stage l if the entire project is financed in m stages (conditional on surviving until the final state K), with $l \leq m$. Similarly, we denote by $V_{l,m}(k_t)$ the value function of the project in stage l and state k_t if the project is supposed to be funded in m stages until the successful completion of the project.

If the project is funded in a single stage, i.e. it is funded in the initial state k_0 with the objective of maintaining a given investment level i until the positive or negative termination of the object,

then the value function is given by the unique solution of the first order differential equation:

$$rV_{1,1}(k_t) = -i_{1,1} + \lambda\sqrt{i_{1,1}} \left(V'_{1,1}(k_t) - \frac{\lambda}{k_t} V_{1,1}(k_t) \right), \quad (22)$$

subject to the boundary condition $V_{1,1}(K) = Y$. The explicit solution of the differential equation (22) is given by:

$$\begin{aligned} V_{1,1}(k_t) &= \left(Y + \sqrt{i_{1,1}} \frac{K}{(\lambda-1)\gamma} \right) \left(\frac{k_t}{K} \right)^\lambda e^{\frac{k_t-K}{\sqrt{i_{1,1}}} \frac{r}{\gamma}} - \sqrt{i_{1,1}} \frac{k_t}{(\lambda-1)\gamma} \\ &= Y \left(\frac{k_t}{K} \right)^\lambda e^{\frac{k_t-K}{\sqrt{i_{1,1}}} \frac{r}{\gamma}} - \frac{\sqrt{i_{1,1}}}{(\lambda-1)\gamma} \left(k_t - K \left(\frac{k_t}{K} \right)^\lambda e^{\frac{k_t-K}{\sqrt{i_{1,1}}} \frac{r}{\gamma}} \right). \end{aligned} \quad (23)$$

The optimal investment policy given the initial state k_0 can be obtained implicitly by the first order condition of $V_{1,1}(k_0)$ with respect to $i_{1,1}$. If we define $j \triangleq \sqrt{i_{1,1}}$, then we have

$$\frac{dV_{1,1}(k_0)}{dj} = \frac{K}{(\lambda-1)\gamma} \left(\frac{k_0}{K} \right)^\lambda e^{\frac{k_0-K}{j} \frac{r}{\gamma}} - \frac{k_0}{(\lambda-1)\gamma} - \left(\frac{k_0-K}{j^2} \frac{r}{\gamma} \right) \left(Y + j \frac{K}{(\lambda-1)\gamma} \right) \left(\frac{k_0}{K} \right)^\lambda e^{\frac{k_0-K}{j} \frac{r}{\gamma}} = 0,$$

or

$$K \left(\frac{k_0}{K} \right)^\lambda e^{\frac{k_0-K}{j} \frac{r}{\gamma}} - k_0 - \left(\frac{k_0-K}{j^2} \frac{r}{\gamma} \right) (Y(\lambda-1)\gamma + jK) \left(\frac{k_0}{K} \right)^\lambda e^{\frac{k_0-K}{j} \frac{r}{\gamma}} = 0,$$

or

$$K - k_0 \left(\frac{K}{k_0} \right)^\lambda e^{\frac{K-k_0}{j} \frac{r}{\gamma}} - \left(\frac{k_0-K}{j^2} \frac{r}{\gamma} \right) (Y(\lambda-1)\gamma + jK) = 0,$$

and finally:

$$\left(\frac{K}{k_0} \right)^{1-\lambda} - \left(\left(\frac{k_0}{K} \right)^\lambda - \left(\frac{K}{k_0} \right)^{1-\lambda} \right) \frac{r}{\gamma} \left(\frac{Y(\lambda-1)\gamma}{j^2} + \frac{K}{j} \right) = e^{\frac{K-k_0}{j} \frac{r}{\gamma}}. \quad (24)$$

Based on (24), the solution to the optimal investment policy

$$i_{1,1} \triangleq i^*(Y, k_0, \lambda) \quad (25)$$

for a single stage investment can be shown to be strictly increasing in Y and k_0 and strictly decreasing in λ .

If, in contrast, the project is funded in two stages, then the optimal funding policy starting at the initial position k_0 has to make three distinct choices: it has to determine the initial funding level $i_{1,2}$, the continued funding level $i_{2,2}$ and the funding renewal state k_1 . Conditional on the optimal funding level given the renewal stage k_1 , the value function in the initial state k_0 is therefore given as the solution to the following optimization problem:

$$V_{1,2}(k_0) = \max_{i_{1,2}, k_1} \left(\frac{\sqrt{i_{1,2}}}{(\lambda-1)\gamma} \left(k_0 - k_1 \left(\frac{k_0}{k_1} \right)^\lambda e^{\frac{k_0-k_1}{\sqrt{i_{1,2}}} \frac{r}{\gamma}} \right) + p \left(\frac{k_0}{k_1} \right)^\lambda e^{\frac{k_0-k_1}{\sqrt{i_{1,2}}} \frac{r}{\gamma}} V_{2,2}(k_1) \right). \quad (26)$$

We can insert (23) into (26) to get

$$V_{1,2}(k_0) = \max_{\{i_{1,2}, i_{2,2}, k_1\}} \left\{ \frac{\sqrt{i_{1,2}}}{(\lambda-1)\gamma} \left(k_0 - k_1 \left(\frac{k_0}{k_1} \right)^\lambda e^{\frac{k_0-k_1}{\sqrt{i_{1,2}}} \frac{r}{\gamma}} \right) + p \left(\frac{k_0}{k_1} \right)^\lambda e^{\frac{k_0-k_1}{\sqrt{i_{1,2}}} \frac{r}{\gamma}} \left(\left(Y + \frac{\sqrt{i_{2,2}} K}{(\lambda-1)\gamma} \right) \left(\frac{k_1}{K} \right)^\lambda e^{\frac{k_1-K}{\sqrt{i_{2,2}}} \frac{r}{\gamma}} - \frac{\sqrt{i_{2,2}} k_1}{(\lambda-1)\gamma} \right) \right\},$$

or alternatively:

$$V_{1,2}(k_0) = \max_{\{i_{1,2}, i_{2,2}, k_1\}} \left\{ \frac{\sqrt{i_{1,2}} k_1}{(\lambda-1)\gamma} \left(\frac{k_0}{k_1} - \left(\frac{k_0}{k_1} \right)^\lambda e^{\frac{k_0-k_1}{\sqrt{i_{1,2}}} \frac{r}{\gamma}} \right) + p \left(\frac{k_0}{K} \right)^\lambda Y e^{\left(\frac{k_0-k_1}{\sqrt{i_{1,2}}} + \frac{k_1-K}{\sqrt{i_{2,2}}} \right) \frac{r}{\gamma}} + p e^{\frac{k_0-k_1}{\sqrt{i_{1,2}}} \frac{r}{\gamma}} \frac{\sqrt{i_{2,2}} K}{(\lambda-1)\gamma} \left(\left(\frac{k_0}{K} \right)^\lambda e^{\frac{k_1-K}{\sqrt{i_{2,2}}} \frac{r}{\gamma}} - \frac{k_1}{K} \left(\frac{k_0}{k_1} \right)^\lambda \right) \right\}.$$

We now establish the results of this proposition.

(1.) We first observe that $i_{2,2} > i_{1,1}$ by the comparative static property of the optimal investment policy $i^*(Y, k, \lambda)$ obtained above for the single stage funding policy. After all, the investment funding $i_{2,2}$ of the project conditional on renewing the project is like a single stage funding, but at a higher level of the state k_t . Suppose next that at $k = k^*$, we have $V_{1,1}(k^*) = V_{1,2}(k^*)$. We want to show $i_{1,2} < i_{1,1}$. In the two stage funding environment, the optimal renewal occurs at k_1 and conditional on renewal, we have a value function $V_{2,2}(k_1)$. By construction, we have $V_{2,2}(k_1) > V_{1,1}(k_1)$. We now show that it follows from here that $pV_{2,2}(k_1) \leq V_{1,1}(k_1)$. The proof is by contradiction. If $pV_{2,2}(k_1) > V_{1,1}(k_1)$, then clearly starting at k^* , and having the advantage of determining the investment level to optimally arrive at k_1 , the two stage funding policy could do at least as well as the one stage funding policy which runs through the state k_1 with intensity $i_{1,1}$, only to realize the project at K . But as the initial funding policy in the two stage level seeks to determine the optimal intensity to arrive at a stopping point k_1 with a value $pV_{2,2}(k_1) \leq V_{1,1}(k_1)$, it follows that it will choose a strictly lower investment policy $i_{1,2}$ than $i_{1,1}$ was determined not to reach $V_{1,1}(k_1)$ optimally, but rather the higher value $V_{1,1}(K)$.

(2.) We establish the uniqueness of k^* by a single crossing argument. We first observe that the value functions $V_{1,1}(k_t)$ and $V_{1,2}(k_t)$ are continuous and differentiable in k_t . We then show that if

$$V_{1,1}(k^*) = V_{1,2}(k^*), \quad (27)$$

then

$$V'_{1,1}(k^*) > V'_{1,2}(k^*).$$

The argument will come from the fact that at k^* the value function of each program, $V_{1,1}$ and $V_{1,2}$,

respectively, satisfies:

$$rV_{1,1}(k^*) = \left\{ -i_{1,1} + \lambda\sqrt{i_{1,1}} \left(V'_{1,1}(k^*) - \frac{\lambda}{k^*} V_{1,1}(k^*) \right) \right\}, \quad (28)$$

and

$$rV_{1,2}(k^*) = \left\{ -i_{1,2} + \lambda\sqrt{i_{1,2}} \left(V'_{1,2}(k^*) - \frac{\lambda}{k^*} V_{1,2}(k^*) \right) \right\}. \quad (29)$$

We can now express the value function $V_{l,m}(k^*)$ in terms of the investment level $i_{l,m}$ and the first derivative of the value function $V'_{l,m}(k^*)$ and get

$$V_{l,m} = \frac{\sqrt{i_{l,m}} \lambda V'_{l,m}(k^*) - i_{l,m}}{r + \frac{\sqrt{i_{l,m}}}{k} \lambda^2} \quad (30)$$

We can determine the sign of the difference $V'_{1,1}(k^*) - V'_{1,2}(k^*)$ by analyzing how $V'_{1,1}(k^*)$ and $V'_{1,2}(k^*)$ respectively have to differ in the face of the different investment intensities, $i_{1,2} < i_{1,1}$, established in part 1 of this proposition, yet while maintaining the hypothesis of equal values given by (27). We determine how $V'(k^*)$ changes as we change the investment level i from $i_{1,2}$ to $i_{1,1}$. We omit the subscripts l, m and use the relationship of the value function given by (30):

$$V(k^*, i) = \frac{\sqrt{i} \lambda \frac{\partial V(k^*, i)}{\partial k} - i}{r + \frac{\sqrt{i}}{k} \lambda^2} \quad (31)$$

As we increase i from $i_{1,2}$ to $i_{1,1}$ and consider $k = k^*$, the value $V(k^*, i)$ is supposed to stay constant. We suppress the dependence on k^* and write $V(i) \triangleq V(k^*, i)$ and $W(i) = \partial V(k^*, i) / \partial k$. In addition, as we are only interested in the sign of $W'(i)$, we define $j \triangleq \sqrt{i}$, and hence we obtain from (31):

$$\lambda W(j) = j + \frac{V(j)r}{j} + V(j) \frac{\lambda^2}{k}. \quad (32)$$

We would like to establish the sign of $W'(j)$ and differentiating (32) by j , we find that:

$$\lambda W'(j) = 1 - \frac{V(j)r}{j^2},$$

as by construction $V'(j) = 0$, or alternatively

$$\lambda j^2 W'(j) = j^2 - V(j)r. \quad (33)$$

We complete the argument by establishing that:

$$j^2 > V(j)r. \quad (34)$$

Now we observe that if the investors were allowed to determine the investment flow optimally in every instant, then we would have, as established in Proposition 1:

$$j^2 = \frac{\gamma^2}{4} \left(W - \frac{\lambda}{k} V \right), \quad (35)$$

or

$$j^2 = V(j) r.$$

But as we are now consider the optimal investment decision subject to staging, the optimal investment $i_{1,m}$ has to be determined with respect to some average valuation over the course of the investment round, and thus as the value is increasing in the current position k , we find that at the beginning of the funding round the investment flow $i_{1,m}$ and hence its root j , displays

$$j^2 > \frac{\gamma^2}{4} \left(W - \frac{\lambda}{k} V \right),$$

which establishes (34). ■

Proof of Proposition 4. (1.) The optimal investment decisions to determine $i_{1,1}$ and $i_{1,2}$ represent solutions to a similar problem, except that the terminal value of the investment problem of $i_{1,1}$ is given by Y whereas the terminal value of the investment problem of $i_{1,2}$ is **given** by some fraction of Y , say $q \cdot Y$, with $q \in (0, 1)$. But as the optimal investment problem of $i_{1,2}$ is taking the solution to the optimal stopping problem at k_1 as given, it is the case that the smaller benefit, $q \cdot Y$ is reached at an earlier point, namely, $k = k_1$. It follows that we can represent the optimal investment decision of $i_{1,1}$ and $i_{1,2}$ by

$$i_{1,1} \in \arg \max_{i \in \mathbb{R}_+} \{p_K(i) \cdot Y - c_K(i)\},$$

and

$$i_{1,2} \in \arg \max_{i \in \mathbb{R}_+} \{p_{k_1}(i) \cdot q \cdot Y - c_{k_1}(i)\},$$

respectively. The term $p_k(i)$ represents the discounted probability that a positive terminal value is realized in position k given an investment flow i and the term $c_k(i)$ represents the associated discounted cost to reach the position k with a constant investment flow i . By hypothesis, the value of these problems is equal at k^* , or

$$p_K(i_{1,1}) \cdot Y - c_K(i_{1,1}) = p_{k_1}(i_{1,2}) \cdot q \cdot Y - c_{k_1}(i_{1,2}). \quad (36)$$

Since the cost of reaching K is strictly larger than reaching k_1 , we have

$$c_K(i_{1,1}) > c_{k_1}(i_{1,2}),$$

but this implies by (36) that

$$p_K(i_{1,1}) > p_{k_1}(i_{1,2}) \cdot q, \quad (37)$$

hence it follows by the envelope theorem that a marginal increase in Y is more beneficial to the single round funding regime by (37), which establishes that k^* is decreasing in Y .

The argument for an increase of k^* in response to an increase in the parameter λ of the failure rate λ/k_t is similar to the above argument regarding Y .

(2.) The marginal benefit of extending k_1 is increased by an increase in p and hence it will lead to an increase in k_1 despite the increase in the marginal cost.

(3.) This follows immediately from the Proposition 3.1. ■

Proof of Proposition 5. (1.) The undiscounted probability of success given a constant failure probability λ is given as the solution to the differential equation

$$0 = \left(P' - \frac{\lambda}{k} P \right)$$

with boundary condition $P(K) = 1$. We therefore have the exact solution

$$P(k) = Ck^\lambda$$

and with the boundary condition

$$P(K) = CK^\lambda = 1$$

we have

$$C = K^{-\lambda}$$

and

$$P(k) = \left(\frac{k}{K} \right)^\lambda \quad (38)$$

The undiscounted probability before the resolution of uncertainty is given by

$$0 = P'(k) - \frac{\lambda}{k} P(k) + \rho((1 - \alpha) P_l(k) + \alpha P_h(k) - P(k)), \quad (39)$$

and inserting $P_l(k)$ and $P_h(k)$ from (38) we get:

$$0 = P'(k) - \frac{\lambda}{k} P(k) + \rho \left((1 - \alpha) \left(\frac{k}{K} \right)^{\lambda_l} + \alpha \left(\frac{k}{K} \right)^{\lambda_h} - P(k) \right).$$

By (39), the probability $P(k)$ is an average of $P_l(k)$ and $P_h(k)$ and the result follows from $P_l(k) > P(k) > P_h(k)$.

(2.) The optimal investment policy before the resolution of uncertainty about the failure rate is given as the solution to the dynamic programming equation:

$$rV(k_t) = \max_{i_t} \left\{ -i_t + \sqrt{i_t} \gamma \left(V'(k_t) - \frac{\lambda}{k} V(k_t) + \rho((1-\alpha)V_l(k_t) + \alpha V_h(k_t) - V(k_t)) \right) \right\},$$

with the solution given by:

$$i_t = \left(\frac{\gamma \left(V'(k_t) - \frac{\lambda}{k} V(k_t) + \rho((1-\alpha)V_l(k_t) + \alpha V_h(k_t) - V(k_t)) \right)}{2} \right)^2,$$

and hence the value function is given by

$$rV(k_t) = \frac{1}{4} \left(\left(V'(k_t) - \frac{\lambda}{k} V(k_t) + \rho((1-\alpha)V_l(k_t) + \alpha V_h(k_t) - V(k_t)) \right) \right)^2$$

and so:

$$i_t = 2rV(k_t).$$

But as $V_h(k_t) < V(k_t) < V_l(k_t)$, it follows that $i_{t,h} < i_t < i_{t,l}$, which completes the proof. ■