Liquidity shocks, roll-over risk and debt maturity

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Abstract

We develop an infinite horizon model of an economy in which banks finance long term assets by placing non-tradeable debt among savers. Banks choose the overall principal, interest rate, and maturity of their debt taking into account two opposite forces. On the one hand, savers are exposed to preference shocks that make them inclined towards short maturities. On the other, banks are exposed to systemic liquidity crises in which refinancing maturing debt is specially expensive. Banks' funding structure decisions are affected by their anticipated terms of access to refinancing during crises, which are competitively determined during them. We show that the unregulated equilibrium exhibits inefficiently short debt maturities. The inefficiency is due to pecuniary externalities that result from combining banks' refinancing constraints with the competitive pricing of liquidity during crises.

Keywords: liquidity premium, maturity struture, systemic crises, liquidity regulation, pecuniary externality.

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1 Introduction

The recent financial crisis has evidenced the importance of funding maturity structure for financial stability and has extended the view among regulators and policy-makers that maturity mismatch in the financial system is excessive and is not properly treated by the current regulatory framework that focuses mainly on solvency risk (e.g., Tarullo, 2009). Our current understanding of the unfolding of the financial crisis highlights the huge pre-crisis increase in maturity transformation by the shadow banking sector. Investment banks, hedge funds and a number of other institutions made themselves heavily exposed to refinancing risk in wholesale debt markets. When the first losses on the subprime positions arrived in early 2007, the panic in money markets combined with the important refinancing risk to which the system was exposed was a key lever in spreading the crisis to all the financial system, and finally, to the whole economy (Brunnermeier, 2009).

Taking some distance from the details of the current crisis, this paper aims to improve our understanding of the linkage between maturity mismatch and roll-over risk in banking. We develop a simple infinite horizon model in which banks finance long term assets by placing non-tradeable debt among a continuum of small unsophisticated savers. Banks choose the overall principal, interest rate and maturity of their debt contracts. When the debt matures it has to be refinanced. Most of the time banks are able to replace their maturing debt with new debt offered to savers qualitatively identical to those who acquired the maturing debt in some prior period. But in some special random events, that we call *systemic liquidity crises*, "normal" refinancing strategies fail and banks have to rely on more expensive "bridge financing" funds supplied by more sophisticated investors. Guaranteeing the access to bridge financing during systemic crises imposes a constraint on banks' debt structure decisions during normal times.[1] Our main result is that, because of a pecuniary externality, the equilibrium maturity of the debt issued by banks is, from a social welfare perspective, inefficiently short. This externality arises from the combination of banks' refinancing (or "bridge financing") constraint with the competitive pricing of liquidity during

[1] In the logic of the model, guaranteeing the access to refinancing during the crisis dominates strategies based on getting exposed to bankruptcy during the crisis. Alternative model specifications implying some probability of going bankrupt in a crisis lead to qualitatively similar positive and normative results as under the current specification.

systemic crises.[2]

We model bank dynamics under some simplifying assumptions (on the contracting and funding structure choice set) directed to highlight the main trade-offs behind banks' debt maturity decisions and, specifically, their implications for role-over risk during systemic crises. Under these assumptions, banks' dynamic optimization problem is characterized by a small set of state variables. Banks offer savers debt contracts that promise the payment of an interest rate per period and the repayment of a fixed principal at maturity. The choice of such maturity is done taking into account two opposing forces.

First, savers face uncertainty on their preference for early consumption. When they first buy the debt, they discount the immediate future at a relatively low rate, but they are exposed to experiencing "liquidity needs" that make them more impatient. In those cases, savers suffer disutility from having to wait until their debts mature in order to consume. Short maturity of debt reduces savers' disutility from delaying consumption when they suffer those needs and allows banks to reduce the interest rate that they have to pay to attract savers in the first place.

Second, the economy is subject to recurrent systemic liquidity crises that are characterized by a common shock to the liquidity preferences of all savers, both outstanding debtholders and potentially new ones. As opposed to normal times in which firms have no difficulties to roll-over their debt, during systemic liquidity crises banks have to rely on the expensive funds supplied by more sophisticated investors. Banks have incentives to set a long maturity of debt in order to reduce their expected refinancing needs during crises.

For a given expectation on the equilibrium cost of funds during crises these two opposing forces give rise endogenously to an optimal debt structure decision. On the other hand, the equilibrium cost of funds during crises is determined by supply and demand forces.

The source of inefficiency in our model is that banks do not take into account the general equilibrium effect of their funding structure decisions on the price of funds during crises. By coordinating an increase in the maturity of debt a social planner is able to reduce the aggregate demand of funds during crises. This decreases the equilibrium cost of funds in a crisis, which implies a transfer of wealth from the suppliers of the funds during crises (the

[2] Pecuniary externalities are a common source of inefficiency in models with financial constraints. For a recent example and further discussion see Lorenzoni (2008).

"bridge financiers") to their users (the banks). This redistribution relaxes banks' refinancing constraint and produces an aggregate welfare gain.[3]

The introduction of liquidity shocks to investors preferences as a motivation for short debt maturity is inspired by Diamond and Dybvig (1983). However, our model is more general in the time and refinancing dimensions.[4] For analytical tractability, we take from He and Xiong (2009a) the assumption that debt contracts mature with a constant probability per period. In that paper, however, such probability is taken as given and the authors focus on a dynamic coordination problem among the short-term investors. They show that investors may stop refinancing a currently solvent firm (or bank) in anticipation that investors in the future might do so if the firm's fundamental value deteriorates. We structure the model so as to understand forces—other than coordination problems—which may lead to endogenous funding maturity decisions and discrepancies between individual optimization and social efficiency.

One precedent in the first dimension of our contribution (having endogenous funding maturity decisions) is the analysis in He and Xiong (2009b), who embed a debt-overhang logic in the context of a structural credit risk model a la Leland (1994), where the firm may readjust its capital structure over time by issuing equity. In their model, debt structures combine short and long-term debt. Short-term debt is assumed to be cheaper than long-term debt, but long-term debt produces less occasions in which the firm may end up going bankrupt because of refinancing problems not solved with an equity issue due to the debt-overhang problem. Their model focuses on the "within the firm" conflict of interest between equity and debt holders around debt roll-overs, whereas ours highlights the consequences of the "across firms" externality associated with debt maturity decisions and the working of the refinancing market in systemic crises.

Modeling the preference for short maturities à la Diamond and Dybvig (1983) offers a complementary approach to that taken in most corporate finance analyses of funding matu-

[3] Confirming the intuition about the operation of a pecuniary externality, we show that the allocation resulting from an unregulated equilibrium might improve, in social welfare terms, by introducing a subsidy to banks' use of "bridge financing" during systemic crises. [Discussion to be completed.]

[4] Moving beyond the typical three-period setup in which liquidity risk and refinancing problems have been analyzed by most of the banking literature (see Allen and Gale, 2007, for an excellent overview) may be considered an advantage per se. For instance, it may help in the challenge of incorporating these issues into dynamic macroeconomic models.

rities, where short-term debt is advantageous because of its disciplinary effect on managers (e.g. Flannery, 1994, Leland, 1998, and, in a banking context, Diamond and Rajan, 2001) or because it allows firms with private information to profit from future rating upgrades (e.g. Flannery, 1986, and Diamond, 1991).

Our paper is also related to recent papers on roll-over risk and debt maturity motivated by the theoretical problems raised by the financial crisis. Morris and Shin (2004, 2009) study roll-over risk as the result of a coordination problem between short-term creditors. Acharya, Gale, and Yorulmazer (2009) show that high roll-over frequency can reduce a risky security's collateral value. Finally, Brunnermeier and Oehmke (2009) study the conflict of interests between long-term and short-term creditors during debt crises and show that it leads to an inefficiently short-term maturity structure.

The paper is organized as follows. Section 2 presents the model. In Section 3 we define equilibrium. Section 4 analyzes the determination of savers' required maturity premia, i.e. the interest rate payments that savers will demand for each possible debt maturity that banks may choose. Section 5 considers banks' optimal funding decisions for a given cost of refinancing during systemic liquidity crises. Section 6 characterizes equilibrium. Section 7 examines the social efficiency properties of equilibrium. Section 8 (to be written) discusses the results and possible extensions. Section 9 concludes. The Appendix contains all the proofs.

2 The model

We consider an infinite horizon economy in which time is discrete and indexed by $t = 0, 1, 2, ...$ The economy is populated by two wide classes of long-lived risk-neutral agents: possibly-patient *savers* and impatient *experts* who enter and exit the economy in an overlapping generation fashion described below. As their names suggest, a first difference between savers and experts is their degree of sophistication, which implies that only the experts have the skills needed to extract value from some of the existing investment opportunities. The second difference between them is their intertemporal preferences. Some savers are normally born patient, with a per-period discount rate ρ_P, while experts are always impatient, with a discount rate $\rho_I > \rho_P$. In fact, savers may randomly and irreversibly become impatient, with

a discount rate also equal to ρ_I, according to some process that will be described below.

Differences in sophistication and time-preferences across agents provide a prima facie case for "banking," which will combine features of delegated fund management (experts will manage savers' funds) and maturity transformation (bank liabilities will facilitate to savers the withdrawal and consumption of their savings once they become impatient).

All agents are born with an endowment normalized to an infinitesimal unit of funds and face an early consumption-savings decision. For savers the only alternative to consumption is to invest in the securities issued by *banks*.[5] If they invest in these securities, they may also face a decision on whether to consume or keep saving at some later stages in their lives (e.g. when the securities issued by the banks mature). For experts the early consumption-saving decision is more complicated because, as specified below, their "expertise" implies a wider set of investment opportunities, including becoming *bankers* and undertaking some *alternative private investment projects*.

The banks in this economy are essentially identified by possessing potentially-perpetual illiquid assets that only experts know to manage so as to keep producing positive returns. We consider the situation in which there is a measure-one continuum of banks to start with, each with one unit of identical bank assets. Bank assets are vulnerable in that they would be destroyed (and yield some small liquidation value) if the corresponding bank failed to have an expert that manages it for a period. Under our assumptions these bank-destroying events will not materialize in equilibrium because banks in trouble will be bought by new coalitions of experts precisely to avoid their wasteful liquidation.[6] To keep things simple we will not discuss the possibility that new bank assets (or banks) are formed.

An important feature of the economy is its exposure to *systemic liquidity crises*. A systemic liquidity crisis is a temporary random event in which all existing patient agents become simultaneously impatient—for analytical convenience, we normalize the duration of a crisis to just one period. Formally a systemic crisis is an aggregate shock to preferences,

[5] Equivalently, we can assume that β is determined by the risk-free return of some alternative short-term asset (e.g. government bonds) in which savers can always invest and disinvest without the mediation of an expert. In this case, what we call "consumption" of the patient savers might correspond to investing such asset until they become impatient, point at which they would consume.

[6] Appendix B considers the possibility that a bank deviates to a funding structure which implies its possible liquidation in crisis periods, providing explicit parametric conditions for such a deviation not to be profitable.

but this shock can be interpreted as a reduced-form for other exogenous phenomena that temporarily damage savers' confidence in the viability of the banks.[7] Systemic crises will disturb the banks because of their involvement in maturity transformation. Indeed, in spite of anticipating that systemic crises may occur, banks will finance themselves in *normal times* by rolling over short-maturity debt placed among patient savers. Debt maturity is endogenously short because banks profit from offering savers the option of an early exit (i.e. the withdrawal of their funds or the cancellation of a rolling-over order) once they become impatient.[8] Most of the times savers' turn impatient due to purely idiosyncratic (and, thus, socially diversifiable) reasons, in which case banks face no difficulty in replacing them with other patient savers.

In a systemic crisis, however, all savers turn impatient and banks' normal refinancing strategies fail. As described below, in equilibrium bankers will end up solving their refinancing needs in crisis times by appealing to the more expensive funds that other experts normally invest in other assets. These experts will be compensated with equity from the banks' in trouble (i.e. diluting the stakes of the preexisting bankers) and will become bankers from that point onwards.

The logic of our main results relies on the maintained assumption that banks cannot offer contracts contingent on the realization of the idiosyncratic and aggregate preference shocks, or that give banks the option to postpone debt repayments at will. If they could, banks could be interested in offering contracts that mature when the investor becomes impatient, except

[7] To rationalize this loss of confidence, we might think of an extension in which banks have a small probability of becoming worthless (or having their assets destroyed), perhaps due to managerial mistakes or heterogeneous exposure to (sufficiently bad) shocks to fundamentals. In such a world, if investors learned that this risk has materialized in a (small) fraction of still publicly unindentified banks, their optimal attitude towards refinancing bank liabilities (e.g. demanding higher compensation until the uncertainty disappears) might be equivalent to the modeled shock to their preferences. Of course, these possibilities would have to be taken into account in the original valuation of bank liabilities and in banks' funding structure decisions.

[8] We assume that bank debt is non-tradeable, so that savers that turn impatient do not have the option to sell it in a secondary market. In practice a lot of bank debt, starting with retail deposits and certificates of deposit, but including also interbank deposits, debt acquired in the course of sales with repurchase agreements (repos), and commercial paper issued over the counter, is non-tradeable. Explaining why bank debt is non-tradeable is beyond the scope of the paper. Some explanations in the literature allude to information and incentive problems that might be worsened by secondary trading or make the secondary market too illiquid (REF. to be included). Other point to the fact that secondary trading might damage the insurance role played by bank deposits (REF. to be included). Transaction costs, given the frequency of issuance and the relatively small size of each issue, provide a third class of explanations (REF. to be included).

if this occurs in a crisis (i.e. when all savers turn impatient at the same time). Such contracts would prescribe that maturity is postponed until the crisis is over. If these contracts were not feasible, banks might still benefit from having the option to postpone debt repayments at will.

When verifiability problems and other contractual or institutional frictions make explicit or implicit contingency on individual and aggregate impatience unfeasible, as we assume, banks' funding structures have to trade-off the value of providing savers with protection against idiosyncratic shocks—by offering them short debt maturities—with the cost of increasing the bank's vulnerability to a systemic shock. In general, equilibrium structures will offer only *partial protection* against *both* types of shocks.

Other things equal, the chosen maturities will always be "too long" for the investor that turns impatient (and would like to be able to disinvest immediately) and "too short" for the bank that experiences a systemic crisis (that would have preferred not having a maturing portion of debt during such event). For reasons that will be shown below, banks' individually optimal decisions will not coincide with the decisions that a social planner facing the same underlying frictions and constraints would like them to make. So in our economy the unregulated competitive equilibrium is inefficient and there is a rationale for the regulation of banks' leverage and/or funding maturity decisions.

After this overview, the next subsections provide full details about the various ingredients of the model.

2.1 Aggregate shocks

For the descriptions that follow, it is necessary to differentiate between periods in which the economy is in a normal state, $s_t = N$, and periods in which it is in a systemic liquidity crisis state, $s_t = C$. For analytical convenience, we assume $\Pr[s_{t+1} = C \mid s_t = N] = \varepsilon$ and $\Pr[s_{t+1} = C \mid s_t = C] = 0$, so that crises have a constant probability of following any normal period but never last for more than one period. Thus a period should be empirically interpreted as "the standard duration of a crisis."

2.2 Agents

In each period t a sufficiently large continuum of new risk-neutral savers and experts enter the economy, each endowed with a unit of funds. The measures of each of these classes of new agents are large relative to the size of the refinancing and management needs of the banking sector.

2.2.1 Savers

Except during periods of systemic liquidity crisis ($s_t = C$), a sufficiently large measure of savers are born patient, with a discount rate ρ_P. In normal states ($s_t = N$), patient savers have a purely idiosyncratic (independent) probability $\gamma \in [0, 1]$ of turning irreversibly impatient, with a discount rate $\rho_I > \rho_P$ from thereon. During crises, both entering and existing savers are or become impatient with probability one.

Entering savers decide on whether to invest their endowment in the assets offered by banks (described below) or to consume them. Savers who opt for the first alternative, may face similar (re)investment decisions during their lifetime. Savers who decide to consume their savings become irrelevant for the rest of the economy from thereon.

We assume that savers learn about their own preferences before learning about the aggregate state of the economy and, more importantly, that they make their consumption plans in between both stages.[9] We assume that changing consumption plans after knowing the aggregate state (or postponing the consumption decision to that stage) will entail a cost $\varkappa$ per unit of planned consumption.

2.2.2 Experts

Experts are always impatient, with a constant discount rate ρ_I. When they enter the economy they have the opportunity of undertaking some irreversible private investment project with a cost of one and a net present value (discounted at the rate ρ_I) of z. The parameter $z \in [0, \overline{\phi}]$ is heterogeneously distributed over the population of entering experts according to

[9] In practice, consumption planning may include the search and order of the goods to buy but also making the orders required to access the funds needed to pay for them (e.g. the cancelation of an automatically renewable term deposit).

a differentiable and strictly increasing function $F(\phi)$, with $F(0) = 0$ and $F(\overline{\phi}) = \overline{F}$, which for each ϕ gives the measure of the population of entering experts with $z \leq \phi$.

On occasions, especially in crisis periods, entering experts will have the alternative of becoming part of the population of active bankers, in the terms specified below. Bankers' decision problems will be further discussed when describing the banks.

We assume that experts impatience is always large enough for them not to accumulate any wealth in any form different from their private investments or their bank shares. Finally, we assume that each expert can only devote her expertise to a single venture (private project or bank) at a time. This simplifies the analysis, by helping exclude the possibility that experts who undertook private projects in some prior period devote their dividends to becoming bankers during systemic crises. To do so, they would have to abandon their projects, but we assume that their irreversibility would make this alternative unprofitable.

2.3 The banking sector

The banking sector is initially made up of a measure-one continuum of banks. Each bank has the same fixed amount of assets with residual value L in case of liquidation. Productive bank assets yield a constant cash flow $\mu > 0$ per period. Bank assets only remain productive if continuously managed by an expert or coalition of experts (i.e. one or several *bankers*). The best use for unproductive banks assets is liquidation.

Each bank is initially managed by one or several bankers who hold 100% of its equity.[10] The bankers decide each bank's initial funding structure at some initial normal period (say, $t = 0$). In principle, bankers might have the occasion to reoptimize or revise their banks' funding structure in subsequent periods (and, at least marginally, when part of the contracts involved in such structure mature). However, in order to keep tractability in an infinite horizon setup, we will adopt assumptions regarding the set of funding arrangements among which bankers can choose so as to make their dynamic optimization problem have a simple

[10] For simplicity, we assume away both permanent and temporary ownership of bank equity by ordinary savers. The impossibility of permanent equity ownership might be justified in reference to some (unmodeled) governance problem that would make outside equityholders to be recurrently expropriated by the managers. The impossibility of temporary equity ownership by savers—as a solution to banks' refinancing problems during a systemic crisis—might be justified as a result of typical conflicts of interest and free riding problems around a bankruptcy process (or by simply saying that the transaction costs associated with the conversion of debt into equity are prohibitively high).

Markovian structure, with two relevant states that coincide with the two aggregate states of the economy $s_t = N, C$.

2.3.1 Normal times funding

To analyze decisions concerning the maturity of funding, we assume that each bank's initial funding structure consists of a continuum of ex ante equal infinitesimal-size non-tradeable debt contracts which can be collectively described as a triple (D, δ, r), where D is the overall principal (and par value of the contracts at the issuing period), δ is the constant probability with which each infinitesimal contract matures in each period, and r is the constant per-period interest rate paid on the non-matured contracts. Moreover, we assume that each of these contracts matures independently both within banks and across banks.[11]

With this specification, each contract's maturity is random and has the property that the expected time to maturity, if the contract has not yet matured, is constant (and equal to $1/\delta$). Though arguably unrealistic at the individual contract level, this assumption makes the problems of the banks offering the contracts and the savers buying them very tractable. At the level of the bank, these independently-arrived maturities produce essentially the same effect as having the same overall debt D made up of uniform perfectly-staggered fixed-maturity contracts which are rolled-over (or replaced by identical contracts) as they mature.[12]

Obviously the debt issued by banks will always be more attractive to patient savers than to any impatient saver or expert. Hence all the initial holders of the debt involved in (D, δ, r), issued in a normal period, will be patient savers (whose number has been assumed to be sufficiently large). Under (D, δ, r), the bank will be obliged to pay interest equal to rD in each period and will have refinancing needs of δD resulting from the fraction of contracts that mature. In normal periods, δD can be covered with the revenue raised by just replacing the maturing contracts with identical contracts placed among savers who are

[11] Having some degree of correlation in contract maturities within a bank will make it potentially more vulnerable to systemic liquidity crises. The case of perfectly correlated maturities within a bank (and independent across banks) is as tractable as our benchmark case. It can be shown that in that case all results are qualitatively identical to the ones obtained in the benchmark case, but banks produce less value to their shareholders and less social welfare. So the assumed independent maturity contracts are superior from both an individual bank perspective and a social perspective.

[12] Indeed, under this interpretation, the corresponding fixed maturity would be exactly $1/\delta$, if this were an integer. See Leland and Toft (1996) for a model with fixed-maturity contracts in continuous time.

or remain patient in that period. Thus, after each normal period, the bank will have a free cash flow of $\mu - rD$ that can be paid to the bankers as a dividend.

2.3.2 Refinancing during crises

In a systemic crisis, the bank's refinancing problem is more complicated, because the absence of patient savers will make the "funding" of the repayments δD unaffordable by just selling debt equivalent to a fraction δ of the initial debt (D, δ, r). If in this context banks issued some new debt with arbitrary maturity and arbitrary interest payments, then the bank might exit the crisis with a different financial structure than it entered it. Describing the state variable of the bank's dynamic problem might require keeping track of a complex combination of heterogenous liabilities, which would easily become analytically untractable.

The following assumptions regarding the course of events in crisis periods help us preserve the simplicity of the dynamic problem:

1. *Dividends.* Bankers learn about the state of the economy after having received and consumed dividends of $\mu - rD$.[13]

2. *Savers' consumption plans vs. expert refinancing.* Savers' cost of reverting their consumption plans, $\varkappa$, is larger than the opportunity cost of funds $z = \phi$ of the relevant marginal entering expert in a crisis period. Moreover, the frequency of systemic crises is low enough for savers to plan to consume their entire savings as soon as they learn to be impatient.

3. *Experts' bridge financing.* Experts can be offered to refinance δD in exchange for an equity stake in the bank. If this arrangement is feasible, the bank operates with lower debt, $(1 - \delta)D$, during the crisis period and in the period after the crisis restores its originally optimal debt structure (D, δ, r) by re-issuing an amount δD of such debt.

4. *Bankruptcy.* If the bank is unable to refinance its maturing debt δD, creditors file for bankruptcy. Bankruptcy entails the liquidation of the bank and the division of its

[13] For sufficiently impatient bankers and a sufficiently small likelihood of suffering a systemic crisis, paying out (and consuming) these dividends in normal periods is optimal for bankers. This is the case even if bankers had the possibility of holding precautionary savings with which to refinance their own banks when a crisis comes (e.g. by investing in an outside risk-free asset at the rate ρ_L until a crisis occurs).

liquidation value L among creditors.

The logic and motivation for these assumptions is quite self-explanatory. The first simplifies the algebra and could be removed without material qualitative or quantitative effect on the results.[14] The second captures a realistic feature of systemic crisis—the failure of banks' standard financing channels—and is instrumental to pushing banks into the bridge financing provided by experts. Such experts (that in reality might correspond to hedge funds, distant sovereign funds, and other sophisticate investors that in normal times are not important for banks' funding), apart from being impatient, are assumed to have alternative profitable investment opportunities to which they have to renounce to finance the bank. This will make banks' refinancing during crises specially costly, and, given the underlying heterogeneity in z, will make the relevant cost increasing in the aggregate refinancing needs of the banking industry.

The third assumption establishes a formula for the bank to satisfy its refinancing needs during a crisis and be able (and willing) to restore its pre-crisis debt structure (D, δ, r) immediately afterwards. After the crisis, the bank will be governed by shareholders who include the bridge financiers and any residual pre-crisis owner. All of them are impatient and, collectively, will have the same objective function and face the same market conditions, as when the bank chose its original debt structure.

Finally, the forth assumption sets a (sufficiently bad) outside option for the bankers who attempt to refinance their bank during the crisis. In fact, a sufficiently low L (relative to the cost of bridge financing) will not only push them into trying to obtain experts' bridge financing for δD, but it will also lead the bankers to choose an initial debt structure (D, δ, r) that makes such bridge financing feasible, ruling out bankruptcy as an equilibrium phenomenon. To simplify the presentation, we will directly assume that avoiding bankruptcy is optimal, relegating to Appendix B the analysis of the conditions under which this optimality holds.

[14] In the numerical examples below, the dividends $\mu - rD$ end up being very small relative to the refinancing needs δD, so their omission would only reduce very marginally the (excess) refinancing costs suffered in a crisis.

3 Equilibrium with bridge financing

Before formally defining an equilibrium for our economy, notice that we will focus on the case in which banks choose debt structures (D, δ, r) compatible with obtaining bridge financing during crises. This implies that in crises some entering experts become shareholders of the banks. The competitive market for bridge financing in each crisis period will determine the share of equity α that the bridge financiers receive. In equilibrium, this share α will have to be enough to compensate some marginal entering expert for the opportunity cost of her funds, which we will denote ϕ.

The heterogeneity in the value of the private investment opportunities of the entering experts and the fact that banks' aggregate refinancing needs in the crisis are δD implies that clearing the market for bridge financing is equivalent to having $F(\phi) = \delta D$ (which in turn requires $\delta D \leq \overline{F}$). Since $F(\cdot)$ is strictly increasing, we can equivalently write this condition as $\phi = F^{-1}(\delta D) \equiv \Phi(\delta D)$,where $\Phi(\cdot)$ is strictly increasing and differentiable, with $\Phi(0) = 0$ and $\Phi(\overline{F}) = \overline{\phi}$. We will refer to ϕ as the *excess cost of liquidity* during a crisis and to $\Phi(\cdot)$ as the *inverse supply of liquidity* during a crisis.

We are now ready to define an equilibrium with bridge financing:

Definition 1 *Given the exogenous parameters of the model ε, ρ_P, ρ_I, γ, μ, and the function $\Phi(\cdot)$, an equilibrium with bridge financing is a tuple $((D^e, r^e, \delta^e), \phi^e)$ describing a debt structure for banks (D^e, r^e, δ^e) and an excess cost of liquidity during a crisis ϕ^e such that:*

1. *Patient savers accept the debt contracts involved in (D^e, r^e, δ^e).*

2. *Among the class of debt structures that allow banks to be refinanced during crises, (D^e, r^e, δ^e) maximizes the normal-period value of each bank to its initial owners.*

3. *The market for liquidity during crises clears in a way compatible with the refinancing of all banks, i.e. $\phi^e = \Phi(\delta^e D^e)$.*

In the next sections we undertake the steps necessary to characterize the existence and uniqueness of this equilibrium, as well as its positive and normative properties. We will start looking at the conditions upon which the contracts involved in some debt structure (D, r, δ)

are acceptable to patient savers in normal periods. This will determine a participation constraint relevant for banks' debt structure optimization. Then, for any given cost of liquidity in a crisis, ϕ, we will write down the equation that describes the value of each bank to its shareholders in a normal period, the condition for the feasibility of bridge financing during a crisis (the *bridge financing constraint*), and finally the optimization problem that, conditional on ϕ, determines bankers' optimal choice of (D, r, δ) subject to the bridge financing constraint. Finally, we will establish the properties of equilibrium.

4 Savers' required maturity premium

In this section we analyze the conditions upon which the debt contracts associated with a debt structure (D, r, δ) are acceptable to savers during normal times. In that case, the debt structure will be feasible when first put in place and the refinancing of its per-period maturing fraction δD will be feasible, again in normal periods, under exactly the same conditions as in the replaced contracts.

We have assumed that banks issue their debt at par, so for the purposes of this section we can abstract from D and focus on the valuation of a debt contract with a principal of one. From a saver's perspective, given that the bank will fully pay back its maturing debt even in crisis periods, the valuation of such contract does not depend on the aggregate state of the economy per se but on whether he is patient ($i = P$) or impatient ($i = I$).

Let U_i be the value function of a saver at each of these states $i = P, I$, just after the interest rate r of the current period is paid. These value functions must satisfy the following system of equations:

$$\begin{aligned} U_P &= \frac{1}{1+\rho_P}\{r + \delta + (1-\delta)[(1-\varepsilon)(1-\gamma)V_P + ((1-\varepsilon)\gamma + \varepsilon)V_I]\}, \\ U_I &= \frac{1}{1+\rho_I}\left[r + \delta + (1-\delta)V_I\right]. \end{aligned} \quad (1)$$

To explain them, notice that the different discount factors multiply the payoffs and continuation values relevant under each individual state $i = P, I$. The contract pays r with probability one in each next period. Additionally it matures with probability δ, in which case it pays also back its principal of one. With probability $1 - \delta$, it does not mature and

then its continuation value depends on the investor's individual state in the next period. To understand the terms multiplying U_P and U_I in the right hand side (RHS) of the equations in (1), notice that impatience is an absorbing state that any patient saver can reach in each following period either idiosyncratically, with probability γ, if such period is normal (which happens with probability $1-\varepsilon$) or, with probability one, if a systemic crisis arrives (which happens with probability ε).

Denoting by $\pi \equiv (1-\varepsilon)\gamma + \varepsilon$ the unconditional probability that a patient saver becomes impatient in the next period, the solution to the system (1) yields the following closed-form expressions for U_P and U_I:

$$U_P(r,\delta) = \frac{r+\delta}{\rho_I+\delta} \frac{\rho_I+\delta+(1-\delta)\pi}{\rho_P+\delta+(1-\delta)\pi}, \tag{2}$$

$$U_I(r,\delta) = \frac{r+\delta}{\rho_I+\delta},$$

where the second multiplicative factor affecting $U_P(r,\delta)$ makes it clear that $\rho_P < \rho_I$ implies, of course, $U_P(r,\delta) > U_I(r,\delta)$. Naturally, both valuations are increasing in r and, for $r < \rho_I$, increasing in δ.

Banks issue their debt in normal periods, when patient savers are abundant, so the relevant condition for the acceptability of some terms (r,δ) is having $U_P(r,\delta) \geq 1$. It is obvious that, for any given δ, a bank maximizing its initial owners' value will offer contracts with the minimal interest rate r that satisfies $U_P(r,\delta) = 1$. Denoting such interest rate by $r(\delta)$ and using (2), we obtain:

$$r(\delta) = \frac{\rho_I\rho_P + \delta\rho_P + (1-\delta)\pi\rho_I}{\rho_I+\delta+(1-\delta)\pi}. \tag{3}$$

From these derivations, we can state the following result:

Proposition 1 *The minimal interest rate acceptable to patient savers under the debt contracts described above are given by a function $r(\delta)$ which is strictly decreasing and convex, with $r(0) = \rho_I \frac{\rho_P+\pi}{\rho_I+\pi} \in (\rho_P, \rho_I)$ and $r(1) = \rho_P$.*

This result highlights the value of offering short debt maturities to the savers in our model. The intuition is quite straightforward. A contract maturing after just one period

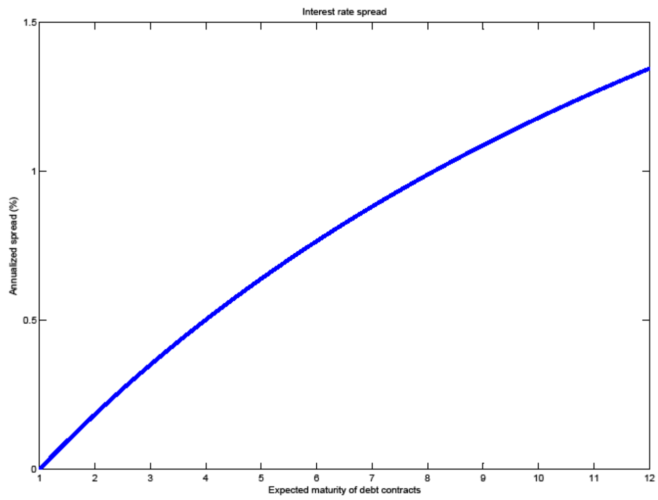

Figure 1: Interest rate spread vs. $1/\delta$

$(\delta = 1)$ would allow all patient savers to ensure that they can consume their savings as soon as they turn impatient, which implies $r(1) = \rho_P$. For any lower δ, the expected maturity of the contract, $1/\delta$, gets lengthened, which means that the saver bears the risk of turning impatient and having to postpone his consumption until his contract matures. Compensating the cost of waiting via a larger interest rate generates a maturity premium, $r(\delta) - \rho_P > 0$, increasing in the expected time to maturity $1/\delta$. Figure **??** illustrates the behavior of $r(\delta)$ under some specific values of the parameters.[15]

5 Banks' optimal funding structures

In this section, treating the cost of liquidity in a crisis, ϕ, as a given constant, we write down the equation that describes the value of each bank to its shareholders in a normal period, the condition for the feasibility of bridge financing during a crisis (the *bridge financing constraint*), and finally the optimization problem that, given ϕ, determines bankers' optimal

[15] This and all other figures rely on a baseline parameterization in which one period is one month, $\Phi(x) = 0.3x$, and the remaining parameters take the following values: agents' annualized discount rates are $\rho_P = 2\%, \rho_I = 6\%$; the annualized yield on bank assets is $\mu = 4\%$; the expected time until the arrival of an idiosyncratic preference shock is 2 years (implying $\gamma = 1/24$); and the expected time between systemic crises is 10 years (implying $\varepsilon = 1/120$).

choice of (D, r, δ) subject to the corresponding bridge financing constraint.

For the purposes of this section, we will take savers' participation constraint into account by assuming that the debt structures (D, r, δ) considered by the banks always set $r = r(\delta)$. This reduces the dimensionality of banks' problem and allows us to refer their debt structures as (D, δ). In the equations we will keep writing r rather than $r(\delta)$ except when presentationally convenient.

5.1 Value of a bank in normal times

Let $E(D, \delta; \phi)$ (or, in brief form, E) be the value of a bank to its shareholders at a normal period, immediately after having paid the dividends due to cash flows generated in the prior period (if applicable). And let $V(D, \delta; \phi) = D + E(D, \delta; \phi)$ be the total market value of the bank at the same stage of a normal period. Notice that when the bank first adopts the structure (D, δ), the initial bankers appropriate D out of what savers pay for the corresponding debt. Hence optimal debt structures will maximize $V(D, \delta; \phi)$.

A bank's equity value in normal times $E(D, \delta; \phi)$ satisfies the following Bellman equation:

$$\begin{aligned} E(D, \delta; \phi) &= \frac{1}{1+\rho_I}\{(\mu - rD) + (1-\varepsilon)E(D, \delta; \phi) + \\ &\quad + \varepsilon(1-\alpha)\frac{1}{1+\rho_I}[\mu - (1-\delta)rD + \delta D + E(D, \delta; \phi)]\}. \end{aligned} \tag{4}$$

To explain the equation, recall that ρ_I is bankers' discount rate and that after each normal period bankers have been assumed to obtain (and immediately consume) the dividend $\mu - Dr$. If the next period is a normal period (i.e. with probability $1-\varepsilon$), bankers additionally obtain the normal-period continuation value $E(D, \delta; \phi)$.[16] If, instead, a systemic crisis arrives (with probability ε), refinancing the bank involves accessing bridge financing, which in turns implies relinquishing a fraction α of the equity to the bridge financiers.

To explain the factor $\frac{1}{1+\rho_I}[\mu - (1-\delta)rD + \delta D + E]$ in the expression above, notice that this accounts for the total value of the equity of banks after being refinanced in the crisis period. Such value is expressed in terms of the payoffs and continuation values received one period ahead, once the crisis period is over. The term $\mu - (1-\delta)rD$, within the square

[16] Notice that the terms associated with a normal period do not reflect the negative cash flows due to the maturing debt δD since they are exactly cancelled out with the issuance of an identical amount of replacing debt.

brackets, corresponds to the dividends paid in that period, which are inflated by the fact that, during the crisis, the bank's outstanding debt was temporarily reduced to $(1-\delta)D$. The term δD precisely accounts for the fact that, once back to normal times, the bank reissues the debt that was "bridge financed" by the new shareholders (and uses the proceeds to pay a special dividend). After completing that transaction the bank's normal-times original debt structure is fully restored and shareholders' continuation value becomes $E(D,\delta;\phi)$ again.

Before continuing, let us discuss how α is determined. Bridge financiers are called to supply their funds for the (temporary) funding of the maturing debt δD. Compensating the *marginal* bridge financier implies paying in present value terms $(1+\phi)\delta D$ for the obtained funds. Using the expression for the continuation value of the bank's equity in crisis periods explained above, the condition for the participation of the bridge financiers becomes:

$$\alpha[\mu-(1-\delta)rD+\delta D+E(D,\delta;\phi)]\geq(1+\rho_I)(1+\phi)\delta D. \tag{5}$$

Obviously bankers will obtain bridge financing in exchange for the minimal α which satisfies (5), which will then hold with equality. Since we must have $\alpha\leq 1$, it follows from (5) that the feasibility of bridge financing eventually requires

$$\mu+E(D,\delta;\phi)\geq[(1+\rho_I)(1+\phi)\delta+(1-\delta)r-\delta]D, \tag{6}$$

which we will call the *bridge financing constraint* (BF).

Now, since (5) holds with equality, we can rewrite the Bellman equation (4) in the following terms:

$$\begin{aligned} E(D,\delta;\phi) \;&=\; \frac{1}{1+\rho_I}\{(\mu-rD)+(1-\varepsilon)E(D,\delta;\phi)+ \\ &\quad +\varepsilon\{\frac{1}{1+\rho_I}[\mu-r(1-\delta)D+\delta D+E(D,\delta;\phi)]-(1+\phi)\delta D\}\}, \end{aligned} \tag{7}$$

where the term multiplied by the probability of a crisis, ε, show that the excess cost of bridge financing is indeed internalized by the initial bankers.

We can solve for $E(D,\delta;\phi)$ in (7), finding:

$$E(D,\delta;\phi)=\frac{1}{\rho_I}\left[\mu-r(\delta)D-\frac{\varepsilon}{1+\rho_I+\varepsilon}\{[(1+\rho_I)\phi+\rho_I]-r(\delta)\}\delta D\right], \tag{8}$$

which has a very intuitive interpretation:

1. $\frac{1}{\rho_I}$ is the present value of a perpetual unit cash flow discounted at bankers' discount rate.

2. μ is the unlevered cash flow of the bank; the remaining terms are proportional to the amount of debt D.

3. $r(\delta)$ is the interest rate paid on debt in normal periods.

4. $\frac{\varepsilon}{1+\rho_I+\varepsilon}\{[(1+\rho_I)\phi+\rho_I]-r(\delta)\}$ reflects the differential cost of refinancing the amount of maturing debt δD every time a crisis arrives. It can be decomposed in two factors:

   (a) $\frac{\varepsilon(1+\rho_I)}{1+\rho_I+\varepsilon}$, which is the net present value multiplier for crisis-period cash flows;

   (b) $\frac{1}{1+\rho_I}\{[(1+\rho_I)\phi+\rho_I]-r(\delta)\}$ reflects that the debt that matures in a crisis is "bridge financed". This means that at the end of the crisis period (so the discounting) such debt "costs" $[(1+\rho_I)\phi+\rho_I]$ rather than $r(\delta)$.

Using (8), the total market value of the bank can then be written as:

$$V(D,\delta;\phi)=D+E(D,\delta;\phi)=\frac{\mu}{\rho_I}+\frac{\rho_I-r(\delta)}{\rho_I}D-\frac{\varepsilon\{[(1+\rho_I)\phi+\rho_I]-r(\delta)\}}{1+\rho_I+\varepsilon}\delta D, \qquad (9)$$

where the first term is the value of the unlevered bank, the second term (which is positive since $r(\delta)<\rho_I$, by Proposition 1) reflects the value of financing the bank with debt claims held by savers potentially more patient than the bankers, and the third term reflects costs due to facing refinancing problems during systemic crises. Quite intuitively, the last term can be made zero by choosing $\delta=0$, i.e. financing the bank with perpetual debt. In fact, Proposition 1 implies $r(0)<\rho_I$, so that debt financing (or financing with initially patient savers) is a source of value for the banks in this economy even with $\delta=0$.

In fact, unless ϕ is excessively large, banks can generate even more value in this economy by getting involved in maturity transformation, i.e. choosing funding structures with $\delta>0$. But showing this requires looking at bank's optimization problem in full detail, which is what we do next.

5.2 Optimal debt structure problem

Formally, the maximization problem of the bank can be written as:

$$
\begin{array}{lll}
\max\limits_{D\geq 0,\ \delta\in[0,1]} & V(D,\delta;\phi)=D+E(D,\delta;\phi) & \\
\text{s.t.} & E(D,\delta;\phi)\geq 0 & \text{(LL)} \\
 & \mu+E(D,\delta;\phi)-[(1+\rho_I)(1+\phi)\delta+(1-\delta)r-\delta]D\geq 0 & \text{(BF)}
\end{array}
\qquad (10)
$$

The first constraint imposes the non-negativity of the bank's equity value in normal periods, and we will refer to it as bankers' limited liability constraint (LL). It is easy to realize from equation (8) that satisfying (LL) implies in particular $\mu - r(\delta)D \geq 0$, which is the condition on the non-negativity of bankers dividends.

The second constraint is the already discussed bridge financing constraint (6). In fact, it is natural to think of it as parallel to (LL), since it comes from having required $\alpha \leq 1$ or, equivalently, initial bankers' limited liability in crisis times. It can be shown that (6) is generally tighter than (8) (and both impose the same constraint on D for $\delta = 0$ since in that case the bank is immune to the systemic crises).[17] So we can safely ignore (LL) in our discussion.

The following technical assumptions help us prove the existence and uniqueness of the solution to the bank's optimization problem:[18]

Assumption 1 The function Φ is upper bounded by $2\frac{1+\rho_P}{1+\rho_I}-1$.

Assumption 2 $\pi < \frac{1-\rho_I}{2}$.

Proposition 2 *For any given excess cost of liquidity during a crisis $\phi \leq 2\frac{1+\rho_P}{1+\rho_I}-1$, the bank's maximization problem has a unique solution (D^*,δ^*). In the solution:*

1. *The bridge financing constraint is binding, i.e. in each crisis bridge financiers take 100% of the bank's equity.*

2. *Optimal debt maturity $1/\delta^*$ is increasing in ϕ and the optimal amount of maturing debt per period $\delta^* D^*$ is decreasing in ϕ. (In fact, if $\delta^* \in (0,1)$, both δ^* and $\delta^* D^*$ are strictly decreasing in ϕ.)*

[17] For a formal argument that uses (4), see the proof of Proposition 2 in Appendix A.

[18] We have checked numerically that the results in Proposition 2 below are also true when these assumptions do not hold. In any case, these sufficient conditions do not impose tight restrictions on the parameters of the model.

These results are very intuitive:

- Even if the bank does not get involved in maturity transformation ($\delta = 0$), its value is increasing in D, making it interested in choosing the maximum D compatible with the constraints. If maturity transformation generates value, this tendency remains, so (BF) must necessarily be binding at the optimum.[19]

- As the excess cost of liquidity in a crisis ϕ increases, the value of maturity transformation diminishes which implies the choice of a lower δ^* (i.e. a longer expected maturity). For given δ, (BF) becomes tighter, leading (or forcing) banks to reduce the amount of funding $\delta^* D^*$ demanded to bridge financiers during crises. Although, we have no formal proof regarding D^*, in all our numerical examples total debt D^* is also decreasing in ϕ.

Although not included in the formal result above, we can also prove that δ^* is independent from μ, while D^* is increasing in μ.

6 The competitive equilibrium

We have just discussed the solution to banks' optimization problem for any given excess cost of liquidity in a crisis ϕ. Such problem embedded savers' participation constraint. The only remaining condition for equilibrium is finding the value of ϕ for which banks' funding structures are compatible with the clearing of the market for bridge financing in crisis periods. The following result regarding the existence and uniqueness of an equilibrium is based on standard arguments that rely on the continuity and monotonicity of the excess demand function in the market for liquidity during a crisis.

Proposition 3 *The equilibrium of the economy* $((D^e, r^e, \delta^e), \phi^e)$ *exists and is unique.*

The next result shows the effects of shifts in the supply of crisis liquidity:

[19] The full dilution of the original equity stakes of the bank in each crisis is an arguably unrealistic implication of having crisis which are all of the same severity. If we introduce heterogeneity in this dimension, for example, by introducing random shifts in the inverse supply of liquidity curve $\Phi(x)$, the bridge financing constraint might only be binding (or even not satisfied, inducing bankruptcy) in the most severe crises.

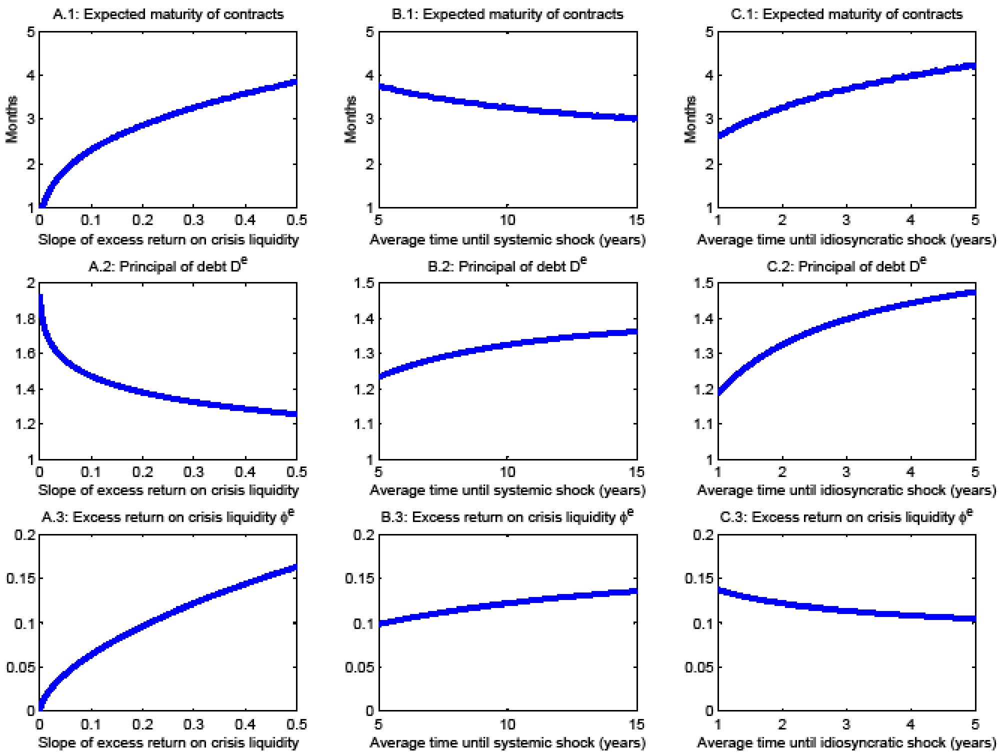

Figure 2: Effect of changes in the parameters on the competitive equilibrium.

Proposition 4 *If the inverse supply of liquidity curve* $\Phi(x)$ *shifts upwards (implying strictly larger excess cost of liquidity* ϕ *for each demand* $x > 0$*), the equilibrium tuple* $((D^e, r^e, \delta^e), \phi^e)$ *changes as follows:* δ^e *and* $\delta^e D^e$ *fall,* r^e *and* ϕ^e *increase. If initially* $\delta^e \in (0, 1)$*, all these variations are strict.*

The results in Proposition 4 are illustrated in the first column of graphs in Figure **??** where we plot the competitive equilibrium of the economy as a function of the slope a of the excess return on crisis liquidity curve $\Phi_a(x) = ax$. As funds during crises become more expensive banks set a longer maturity of debt to reduce their refinancing needs during crises (Panel A.1). At the same time, each bank generates less value per unit of debt, which through the bridge financing constraint happens to force the bank to reduce its debt (Panel

A.2). Finally, banks' reaction to the change in a partially offsets the direct effect of the increase in a in the excess return on crisis liquidity (producing the concave curve depicted in Panel A.3).[20]

The second and third columns of graphs in Figure **??** show the effects of increasing the time to the arrival of systemic and idiosyncratic shocks, respectively (i.e. the effects of reducing the frequency of each of these shocks). As systemic liquidity shocks become less frequent banks become less worried about the "costly" refinancing suffered during crises and thus shorten the maturity of their debt (Panel B.1). Maturity transformation produces more value and the bridge financing constraint is relaxed, so leverage increases (Panel B.2). As a consequence the equilibrium excess cost of liquidity in a crisis increases (Panel B.3). The first-round response of a bank to a reduction in the frequency of systemic crises is partially offset as all other banks also shorten their debt maturities and increase their leverage, which induces the increase in ϕ^e and gives banks second-round incentives to take stabilizing decisions in the opposite direction.

On the other hand, when idiosyncratic liquidity shocks become less frequent, savers disutility due to delaying consumption is reduced. In this situation, the bank might react with a reduction in the interest rate. But it can do better by increasing the maturity of its debt (at the cost of a smaller reduction in the interest rate) and thus eliminating part of its costly refinancing needs during systemic liquidity crises (Panel C.1). At the same time the bank is able to issue more debt as it is able to generate more value due to the fall in its funding costs (Panel C.2). Finally, notice that the effect due to the lengthening of maturity dominates the effect of increasing leverage, producing on the net a fall in the equilibrium excess cost of crisis liquidity (Panel C.3).

7 Efficiency properties of the competitive equilibrium

In this section we want to study the social efficiency properties of the competitive equilibrium. We structure the section in two parts. In the first we solve the welfare maximization problem of a social planner who had the ability to directly control or regulate banks' funding structure

[20] This result is similar to the prediction in He and Xiong (2009b) that banks will shift to long-term funding if bond markets become more illiquid.

decisions subject to the same type of constraints that banks face when solving their private value maximization problems; we compare the solution of this problem with the unregulated competitive equilibrium characterized in previous sections and we find that the latter features inefficiently short debt maturities. Inspired by the fact that debt maturities are precisely too short because they make banks' refinancing in a crisis excessively difficult, the second part of this section examines the welfare implications of introducing a subsidy on banks' refinancing costs during systemic crises. We find that for reasonably low opportunity costs of public funds, some positive degree of subsidization is indeed welfare improving.

7.1 Inefficiency of the unregulated equilibrium

Let us suppose that a social planner can regulate both the amount of debt D issued by banks and the maturity of contracts offered to creditors, i.e. the parameter δ. In our economy only bank shareholders (both the initial bankers and those who become bankers when providing bridge financing during systemic crises) obtain a surplus. Thus the natural objective function for the social planner in this economy is the present value of the net payoffs that banks generate for current and future bankers.

Because of the heterogeneity of their alternative investment opportunities, all entering experts who become bridge financiers in crisis periods obtain the difference between the competitive excess cost of liquidity during a crisis ϕ and the net present value of their alternative project z. From the condition for the clearing of the market for crisis liquidity, each particular choice of (D, δ) will imply some $\phi = \Phi(\delta D)$ Hence, the surplus obtained by bridge financiers in a crisis period can be computed as:

$$u(D,\delta) = \int_0^{D\delta} (\Phi(\delta D) - \Phi(x))\, dx = \delta D \Phi(\delta D) - \int_0^{D\delta} \Phi(x) dx.$$

And the present value (evaluated at a normal period) of the surplus bridge financiers obtain along all future crises can be written as

$$U(D,\delta) = \frac{1}{\rho_I} \frac{(1+\rho_I)\varepsilon}{1+\rho_I+\varepsilon} u(D,\delta).$$

Using the expression for $U(D, \delta)$ and our prior expression (9) for the total market value of the bank to its initial owners in a normal period, $V(D, \delta; \Phi(D\delta))$, the objective function

of the social planner can be expressed as:

$$\begin{aligned}W(D,\delta) &= V(D,\delta;\Phi(D\delta))+U(D,\delta)\\ &= \frac{\mu}{\rho_I}+\frac{\rho_I-r(\delta)}{\rho_I}D-\frac{\varepsilon\{[(1+\rho_I)\phi+\rho_I]-r(\delta)\}}{1+\rho_I+\varepsilon}\delta D-\\ &\quad -\frac{1}{\rho_I}\frac{(1+\rho_I)\varepsilon}{1+\rho_I+\varepsilon}\int_0^{D\delta}\Phi(x)dx.\end{aligned}$$

And its maximization problem can be written as:[21]

$$\begin{array}{lll}\max\limits_{D\geq 0,\ \delta\in[0,1]} & V(D,\delta;\Phi(D\delta))+U(D,\delta) & \\ \text{s.t.} & \mu+E(D,\delta;\Phi(D\delta))-[(1+\rho_I)(1+\Phi(D\delta))\delta+(1-\delta)r-\delta]D\geq 0 & \text{(BF')}\end{array} \tag{11}$$

This problem is different from the bank's optimization problem (equation (10)) in two dimensions:

1. The social planner takes into account the surplus that bridge financiers obtain.
2. The social planner internalizes the effect of banks' funding structure decisions on the (equilibrium) excess cost of liquidity during crises (which explains why we write $\Phi(D\delta)$ in the place occupied by ϕ in individual banks' problems).

A first interesting result that may help us understand the sources of inefficiency in this economy is the following:

Proposition 5 *If the total amount of debt D issued by banks is exogenously fixed, the competitive equilibrium of the model is socially efficient.*

This "efficiency" result refers to a hypothetical situation in which the social planner were able to regulate δ without changing some given (perhaps independently regulated) D. The result shows that moving δ away from the equilibrium value $\overline{\delta}^e$ that would arise in the fixed-D situation would not produce any net welfare gain. The reason for this is that changing δ in that situation would amount to a pure redistribution of value between bridge financiers

[21] Recall that the constraint called (LL) in (10) can be ignored because it is implied by the bridge financing constraint.

and the initial bankers (e.g. a lower δ would reduce the induced excess cost of crisis liquidity ϕ but the increase in bankers value V would be exactly offset, in the margin, by the decline in bridge financiers' value U).[22]

Now let $D^s(\delta)$ give, for every δ, be the unique principal of debt such that the bridge financing constraint, as expressed in (11), is binding, i.e.

$$\mu + E(D, \delta; \Phi(D\delta)) - [(1 + \rho_I)(1 + \Phi(D\delta))\delta + (1 - \delta)r - \delta]D = 0$$

It is possible to prove that the solution of the social planner problem lies on this curve.

The following proposition states the main efficiency result of the paper:

Proposition 6 *Let $((D^e, \delta^e), \phi^e)$ be the competitive equilibrium of the economy. If $\delta^e \in (0, 1)$ then a social planner can increase social welfare by choosing some $\delta^s < \delta^e$, i.e. a longer expected debt maturity than in the competitive equilibrium allocation. More precisely, we have*

$$\left.\frac{dW(D^s(\delta), \delta)}{d\delta}\right|_{\delta=\delta^e} < 0, \quad \left.\frac{dD^s(\delta)}{d\delta}\right|_{\delta=\delta^e} < 0, \text{ and } \left.\frac{d(D^s(\delta)\delta)}{d\delta}\right|_{\delta=\delta^e} > 0.$$

The reason why a social planner can improve the competitive equilibrium allocation is that it internalizes that a reduction in the maturity of contracts reduces the excess cost of liquidity during systemic crises and this, in turn, relaxes banks' bridge financing constraints. In fact, it allows banks to increase the overall principal of their debt in a way which turns out to be welfare improving.

This suggests that a social planner would resolve the trade-offs relevant for the exploitation of the maturity-transformation opportunities that exist in the economy differently from the manner the unregulated competitive equilibrium would do. In particular, it would make less use of the intensive margin (the debt maturity margin δ) and greater use of the extensive margin (the leverage margin D).[23]

Interestingly, $D^s\delta^s < D^e\delta^e$ implies moving from equilibrium to the socially optimal allocation would reduce the net present value of the surplus appropriated by future bridge

[22] If for whatever reasons the social planner gave more weight in the social welfare function to the initial bankers than to the potential bridge financiers, then, even for fixed D, there would be social gains from imposing some regulated $\overline{\delta} < \overline{\delta}^e$.

[23] This finding offers a new perspective for the joint assessment of some of the re-regulation proposals emerged from the experience accumulated in the recent financial crisis, which taken altogether seem to imply the need for reducing *both* the leverage of the financial system *and* its reliance on short-term funding.

financiers, which suggests that the great beneficiaries of regulating debt maturity in this economy would be the already existing bankers.

Figure 3 illustrates the comparison between the equilibrium and the socially-efficient bank funding structures in some specific parameterizations of the model

[Further discussion of this result: To be completed. Some additional intuition is currently provided in the Introduction.]

7.2 Effects of subsidizing bridge financing

The fact that debt maturities in the unregulated equilibrium are too short precisely because they make banks' refinancing in a crisis excessively difficult (which tightens banks bridge financing constraints in an inefficient manner) suggest that, in the absence of a direct regulation of debt maturity, subsidizing banks access to bridge financing during systemic crises might be welfare improving. [To be completed]

8 Discussion and extensions

[To be written]

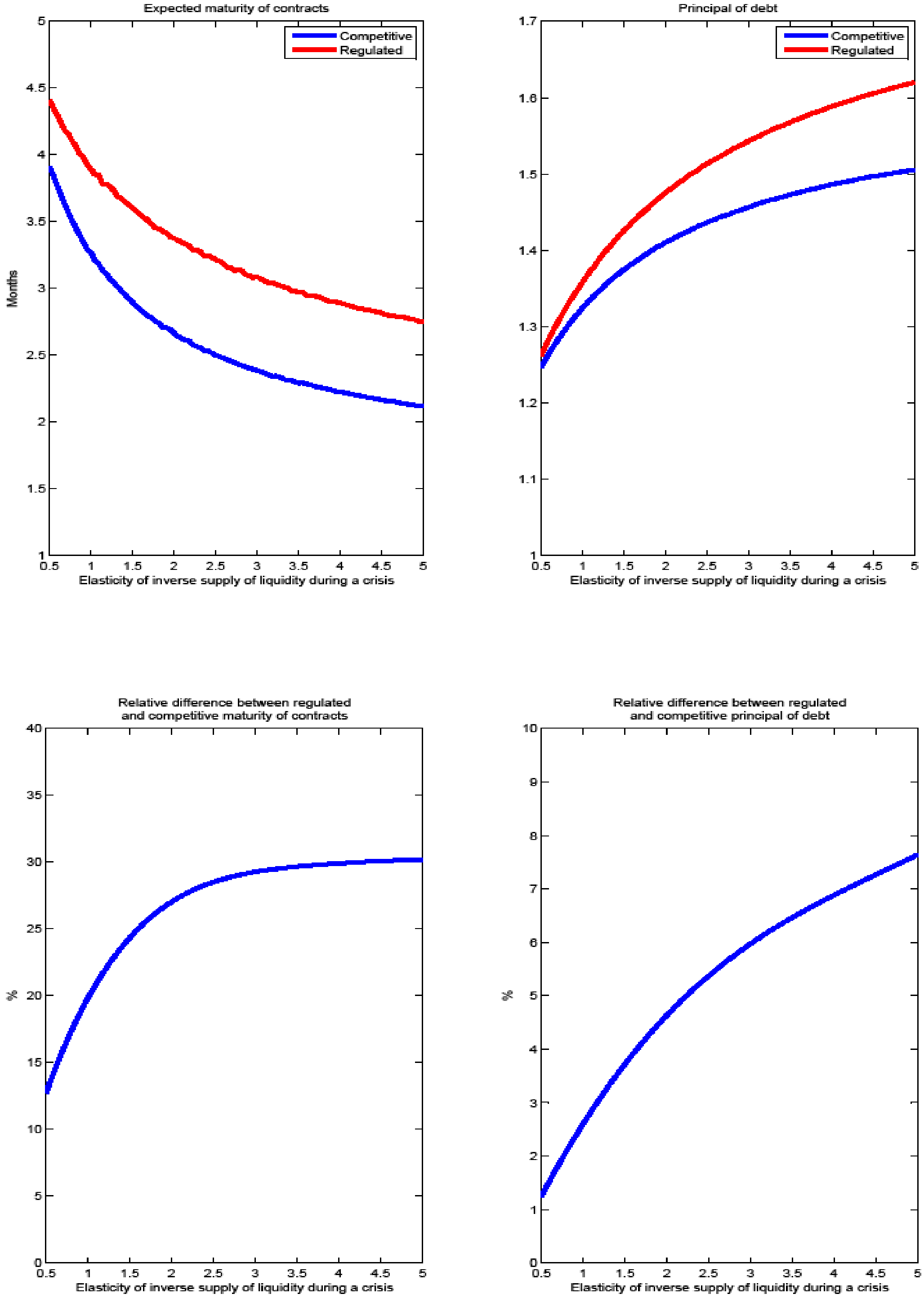

Figure 3: Equilibrium vs. socially-efficient funding structures

9 Conclusion

We have developed an infinite horizon equilibrium model in which banks that invest in long-lived assets decide the overall principal, interest rate payments, and maturity of their debt. The model contains a microfoundation for savers' preference for short maturities in line with the traditional Diamond and Dybvig (1983) formulation, which is simplified and adapted to the needs of a recursive dynamic formulation. Banks' incentive not to set debt maturities as short as savers might ceteris paribus prefer, comes from the fact that there are events (called systemic liquidity crises) in which their normal financing channels fail and they have to turn to more expensive sources of funds.

We identify a pecuniary externality that renders the unregulated competitive equilibrium of the model socially inefficient. It turns out that if a social planner induces banks to choose some longer debt maturity than the one they would uncoordinatedly decide, social welfare increases. This is because longer maturities reduce banks' aggregate refinancing needs during crises and, consequently, the equilibrium cost of funding in those times.

The pecuniary externality arises from the combination of the ex-post competitive pricing of funds during crises and the constraints that banks have to satisfy ex-ante in order to guarantee their refinancing capacity during crises. Albeit different in many details, including the infinite horizon formulation and its expression in terms of banks' funding maturity decisions, this basic result of our model is similar in nature to the result concerning the inefficiency of the private provision of liquidity in the presence of aggregate shocks obtained in the model of Holmström and Tirole (1998) and many other finite-horizon models in the banking literature (see, Allen and Gale, 2007).

Appendix

A Proofs

This appendix contains the proofs of the propositions included in the body of the paper.

Proof of Proposition 1 Using (3) it is a matter of simple algebra to obtain that:

$$r'(\delta) = \frac{-\pi(1+\rho_I)(\rho_I-\rho_P)}{(\rho_I+\delta+(1-\delta)\pi)^2} < 0,$$

$$r''(\delta) = \frac{2\pi(1-\pi)(1+\rho_I)(\rho_I-\rho_P)}{(\rho_I+\delta+(1-\delta)\pi)^3} > 0.$$

The other properties stated in the proposition are immediate.■

Proof of Proposition 2 The proof is organized in a sequence of steps.

1. If (BF) is satisfied then (LL) is also satisfied Using equation (8) we have that (LL) can be written as:

$$0 \leq E(D,\delta;\phi) = \frac{1}{\rho_I}(\mu - rD) - \frac{1}{\rho_I}\frac{(1+\rho_I)\varepsilon}{1+\rho_I+\varepsilon}\left(1+\phi-\frac{1+r}{1+\rho_I}\right)\delta D,$$

while (BF) can be written, using (6), as

$$\begin{aligned} 0 \leq & \frac{1}{1+\rho_I}(\mu - r(1-\delta)D + \delta D + E(D,\delta;\phi)) - (1+\phi)\delta D = \\ = & \frac{1}{1+\rho_I}(\mu - rD + E(D,\delta;\phi)) - \left(1+\phi-\frac{1+r}{1+\rho_I}\right)\delta D = \\ = & \frac{1}{1+\rho_I}(\mu - Dr) - \left(1+\frac{1}{\rho_I}\frac{\varepsilon}{1+\rho_I+\varepsilon}\right)\left(1+\phi-\frac{1+r}{1+\rho_I}\right)\delta D. \end{aligned}$$

Now, since $1+\frac{1}{\rho_I}\frac{\varepsilon}{1+\rho_I+\varepsilon} = \frac{(1+\rho_I)(\rho_I+\varepsilon)}{\rho_I(1+\rho_I+\varepsilon)} > \frac{(1+\rho_I)\varepsilon}{\rho_I(1+\rho_I+\varepsilon)}$ we conclude that whenever (BF) is satisfied, (LL) is also satisfied.

2. Notation and useful bounds Using equation (8) we can write:

$$V(D,\delta;\phi) = D + E(D,\delta;\phi) = \frac{1}{\rho_I}\mu + D\Pi(\delta;\phi),$$

where

$$\Pi(\delta,\phi) = 1 - \frac{1}{\rho_I}\left[\left(1-\frac{\varepsilon}{1+\rho_I+\varepsilon}\delta\right)r + \frac{(1+\rho_I)\varepsilon}{1+\rho_I+\varepsilon}\delta\left(\phi+\frac{\rho_I}{1+\rho_I}\right)\right]$$

can be interpreted as the value the bank generates to its shareholders per unit of debt. Using Proposition 1 we can see that the function $\Pi(\delta, \phi)$ is concave in δ. If we define $C(\delta, \phi) = (1+\rho_I)(1+\phi)\delta + (1+r)(1-\delta)$, (BF) can be written as:

$$\frac{1+\rho_I}{\rho_I}\mu + (\Pi(\delta,\phi) - C(\delta,\phi))D \geq 0. \tag{12}$$

Using Proposition 1 we can see that the function $C(\delta, \phi)$ is convex in δ.

Now, let us compute the partial derivatives of $C(\delta, \phi)$ with respect to (w.r.t.) δ for later use:

$$\begin{aligned} \frac{\partial C}{\partial \delta} &= (1+\rho_I)(1+\phi) - (1+r) + \frac{dr}{d\delta}(1-\delta)), \\ \frac{\partial^2 C}{\partial \delta^2} &= -2\frac{dr}{d\delta} + \frac{d^2 r}{d\delta^2}(1-\delta). \end{aligned} \tag{13}$$

The assumption $\phi \leq 2\frac{1+\rho_P}{1+\rho_I} - 1$ implies $1 + \phi \leq 2\frac{1+r(1)}{1+\rho_I} \leq 2\frac{1+r(\delta)}{1+\rho_I}$ for all δ. This gives the following bounds that are independent from ϕ:

$$\begin{aligned} C(\delta,\phi) &\geq 1 + r(\delta). \\ \frac{\partial C(\delta,\phi)}{\partial \delta} &\leq 2(1+r(\delta)) - (1+r(\delta)) = 1 + r(\delta). \end{aligned} \tag{14}$$

And we can check that the following relationship between $\Pi(\delta, \phi)$ and $C(\delta, \phi)$ is satisfied:

$$\Pi(\delta,\phi) = 1 - \frac{1}{\rho_I}\frac{(1+\rho_I)}{1+\rho_I+\varepsilon}\left[r(\delta) + \frac{\varepsilon}{1+\rho_I}(C(\delta,\phi)-1)\right]. \tag{15}$$

Finally, we have that, for all δ:

$$\begin{aligned} \frac{d^2 r}{d\delta^2} + \frac{dr}{d\delta} &= \frac{2\pi(1-\pi)(1+\rho_I)(\rho_I-\rho_P)}{(\rho_I+\delta+(1-\delta)\pi)^3} - \frac{\pi(1+\rho_I)(\rho_I-\rho_P)}{(\rho_I+\delta+(1-\delta)\pi)^2} \\ &= \frac{\pi(\rho_I-\rho_P)(1+\rho_I)}{(\rho_I+\delta+(1-\delta)\pi)^3}\left(2(1-\pi) - (\rho_I+\delta+(1-\delta)\pi)\right) \geq \\ &\geq \frac{\pi(\rho_I-\rho_P)(1+\rho_I)}{(\rho_I+\delta+(1-\delta)\pi)^3}\left(1-2\pi-\rho_I\right) \geq 0, \end{aligned} \tag{16}$$

where in the last inequality we have used the assumption $\pi < \frac{1-\rho_I}{2}$. To save on notation, we will drop from now on the arguments of these functions when it does not lead to ambiguity.

3. $D^* = 0$ is not optimal It suffices to realize that $\frac{\partial V(D,0;\phi)}{\partial D} = \Pi(0,\phi) = 1 - \frac{r(0)}{\rho_I} > 0$.

4. The solution (D^*, δ^*) of the maximization problem in equation (10) exists, is unique, and satisfies (BF) with equality, i.e. $\frac{1+\rho_I}{\rho_I}\mu + (\Pi(\delta^*,\phi) - C(\delta^*,\phi))D^* = 0$

We are going to prove existence and uniqueness in the particular case that there exist $\delta_\Pi, \delta_C \in [0,1]$ such that $\frac{\partial \Pi(\delta_\Pi,\phi)}{\partial \delta} = \frac{\partial C(\delta_C,\phi)}{\partial \delta} = 0$. This will ensure that the solution of the maximization problem is interior in δ. The other cases are treated in an analogous way but might give rise to corner solutions in δ.[24]

First, since $\Pi(\delta,\phi)$ is concave in δ we have that $\frac{\partial \Pi(\delta,\phi)}{\partial \delta} \geq 0$ iff $\delta \leq \delta_\Pi$. Since $C(\delta,\phi)$ is convex in δ we have that $\frac{\partial C(\delta,\phi)}{\partial \delta} \geq 0$ iff $\delta \geq \delta_C$. Differentiating equation (15) w.r.t. δ we obtain:

$$\begin{aligned} \frac{\partial \Pi}{\partial \delta} &= -\frac{1}{\rho_I}\frac{1+\rho_I}{1+\rho_I+\varepsilon}\left[\frac{dr}{d\delta} + \frac{\varepsilon}{1+\rho_I}\frac{\partial C}{\partial \delta}\right], \\ \frac{\partial^2 \Pi}{\partial \delta^2} &= -\frac{1}{\rho_I}\frac{1+\rho_I}{1+\rho_I+\varepsilon}\left[\frac{d^2 r}{d\delta^2} + \frac{\varepsilon}{1+\rho_I}\frac{\partial^2 C}{\partial \delta^2}\right], \end{aligned} \tag{17}$$

which implies $\delta_C < \delta_\Pi$.

Now, let (D^*, δ^*) be a solution to the maximization problem. The first order conditions (FOC) that characterize an interior solution (D^*, δ^*) are:

$$\begin{aligned} (1+\theta)\Pi - \theta C &= 0, \\ (1+\theta)\frac{\partial \Pi}{\partial \delta} - \theta\frac{\partial C}{\partial \delta} &= 0, \\ \theta\left[\frac{1+\rho_I}{\rho_I}\mu + (\Pi - C)D^*\right] &\geq 0, \\ \theta &\geq 0, \end{aligned} \tag{18}$$

where θ is the Lagrange multiplier associated with (BF) and we have used that $D^* > 0$ in order to eliminate it from the second equation.

If $\theta = 0$ then the second equation implies $\delta^* = \delta_\Pi$ and thus $\Pi(\delta^*,\phi) \geq \Pi(0,\phi) > 0$ and the first equation is not satisfied. Therefore we must have $\theta > 0$ so that (BF) is binding at the optimum. Now we can eliminate θ from the previous system of equations, which gets reduced to:

$$\frac{\partial \Pi(\delta^*,\phi)}{\partial \delta} C(\delta^*,\phi) = \frac{\partial C(\delta^*,\phi)}{\partial \delta} \Pi(\delta^*,\phi), \tag{19}$$

$$\frac{1+\rho_I}{\rho_I}\mu = \left[C(\delta^*,\phi) - \Pi(\delta^*,\phi)\right] D^*. \tag{20}$$

We are going to show that equation (19) has a unique solution in δ. For $\delta \leq \delta_C < \delta_\Pi$, we have $\frac{\partial C}{\partial \delta} \leq 0 < \frac{\partial \Pi}{\partial \delta}$ and thus the left hand side (LHS) of (19) is strictly bigger than the RHS. For $\delta \geq \delta_\Pi > \delta_C$, we have $\frac{\partial \Pi}{\partial \delta} \leq 0 < \frac{\partial C}{\partial \delta}$ and thus RHS of (19) is strictly bigger.

[24] More precisely, if for all $\delta \in [0,1]$ $\frac{\partial C(\delta,\phi)}{\partial \delta} > 0$ we might have $\delta^* = 0$ and if for all $\delta \in [0,1]$, $\frac{\partial \Pi(\delta,\phi)}{\partial \delta} > 0$ we might have $\delta^* = 1$.

Now, the function $\frac{\partial C(\delta,\phi)}{\partial\delta}\Pi(\delta,\phi)$ is strictly increasing in the interval (δ_C, δ_Π) since both terms are positive and increasing. Thus, it suffices to prove that for $\delta \in (\delta_C, \delta_\Pi)$ the function $\frac{\partial\Pi(\delta,\phi)}{\partial\delta}C(\delta,\phi)$ is decreasing.[25] Using the system of equations (17), we have:

$$\frac{\partial}{\partial\delta}\left(\frac{\partial\Pi}{\partial\delta}C\right) = \frac{\partial^2\Pi}{\partial\delta^2}C + \frac{\partial\Pi}{\partial\delta}\frac{\partial C}{\partial\delta} =$$
$$= -\frac{1}{\rho_I}\frac{(1+\rho_I)}{1+\rho_I+\varepsilon}\left[\left[\frac{d^2r}{d\delta^2} + \frac{\varepsilon}{1+\rho_I}\frac{\partial^2 C}{\partial\delta^2}\right]C + \left[\frac{dr}{d\delta} + \frac{\varepsilon}{1+\rho_I}\frac{\partial C}{\partial\delta}\right]\frac{\partial C}{\partial\delta}\right],$$

while, using the system of equation (13) and the bounds in (14), we have:

$$\frac{\partial}{\partial\delta}\left(\frac{\partial\Pi}{\partial\delta}C\right) \leq -\kappa(1+r)\left[(1+\frac{\varepsilon}{1+\rho_I}(1-\delta))\left(\frac{d^2r}{d\delta^2}+\frac{dr}{d\delta}\right) + \right.$$
$$\left. +\varepsilon\left(1+\phi-\frac{1+r}{1+\rho_I}\right)\right] + 2\kappa(1+r)\frac{\varepsilon}{1+\rho_I}\frac{dr}{d\delta} \leq 0,$$

where $\kappa = \frac{1}{\rho_I}\frac{(1+\rho_I)}{1+\rho_I+\varepsilon}$ and, in order to establish the last inequality, we have used $\frac{d^2r}{d\delta^2} + \frac{dr}{d\delta} \geq 0$, $\frac{dr}{d\delta} < 0$ and $1+\phi-\frac{1+r}{1+\rho_I} > 0$. This concludes the proof on the existence and uniqueness of a δ^* that satisfies the necessary FOC in (19).

Now, for given δ^*, the other necessary FOC (20) determines D^* uniquely.[26]

4. δ^* is independent from μ and D^* is increasing in μ Equation (19) determines δ^* and is independent from μ. Then equation (20) shows that D^* is increasing in μ.

5. δ^* is decreasing in ϕ and, if $\delta^* \in (0,1)$, it is strictly decreasing Let $\delta(\phi)$ be the solution of the maximization problem of the bank for given ϕ. Let us assume that $\delta(\phi)$ satisfies the FOC (19). The case of corner solutions is analyzed in an analogous way. We have the following partial derivatives w.r.t. ϕ that will be used later:

$$\begin{aligned}
\frac{\partial C}{\partial\phi} &= (1+\rho_I)\delta, \qquad (21)\\
\frac{\partial^2 C}{\partial\phi\partial\delta} &= 1+\rho_I,\\
\frac{\partial\Pi}{\partial\phi} &= -\frac{1}{\rho_I}\frac{\varepsilon}{1+\rho_I+\varepsilon}\frac{\partial C}{\partial\phi} = -\frac{1}{\rho_I}\frac{(1+\rho_I)\varepsilon}{1+\rho_I+\varepsilon}\delta,\\
\frac{\partial^2\Pi}{\partial\phi\partial\delta} &= -\frac{1}{\rho_I}\frac{(1+\rho_I)\varepsilon}{1+\rho_I+\varepsilon},
\end{aligned}$$

[25] This is not trivial since $C(\delta,\phi)$ is increasing.

[26] Let us observe that for all δ $C(\delta,\phi) \geq 1 > \Pi(\delta,\phi)$.

and the following useful relationships:

$$
\begin{aligned}
C-\frac{\partial C}{\partial \delta}\delta &= (1+r)-\frac{dr}{d\delta}(1-\delta)\delta, \\
\Pi-\frac{\partial \Pi}{\partial \delta}\delta &= 1-\frac{1}{\rho_I}\frac{1+\rho_I}{1+\rho_I+\varepsilon}\left[r-\frac{dr}{d\delta}\delta+\frac{\varepsilon}{1+\rho_I}\left(C-\frac{\partial C}{\partial \delta}\delta-1\right)\right].
\end{aligned} \tag{22}
$$

We have proved in Step 3 above that the function $\frac{\partial \Pi}{\partial \delta}C-\frac{\partial C}{\partial \delta}\Pi$ is decreasing in δ around $\delta(\phi)$. In order to show that $\delta(\phi)$ is decreasing, it suffices to show that the derivative of this function w.r.t. ϕ is negative. Using the equations (17),(21),(22) we obtain:

$$
\begin{aligned}
\frac{\partial}{\partial \phi}\left[\frac{\partial \Pi}{\partial \delta}C-\frac{\partial C}{\partial \delta}\Pi\right] &= \frac{\partial^2 \Pi}{\partial \delta \partial \phi}C+\frac{\partial \Pi}{\partial \delta}\frac{\partial C}{\partial \phi}-\frac{\partial^2 C}{\partial \delta \partial \phi}\Pi-\frac{\partial C}{\partial \delta}\frac{\partial \Pi}{\partial \phi}= \\
&= -\frac{1}{\rho_I}\frac{(1+\rho_I)\varepsilon}{1+\rho_I+\varepsilon}\left(C-\frac{\partial C}{\partial \delta}\delta\right)-(1+\rho_I)\left(\Pi-\frac{\partial \Pi}{\partial \delta}\delta\right)= \\
= -\frac{1}{\rho_I}\frac{1+\rho_I}{1+\rho_I+\varepsilon}&\left(\varepsilon(1+r)-\varepsilon\frac{dr}{d\delta}(1-\delta)\delta\right)-(1+\rho_I) \\
&+\frac{1}{\rho_I}\frac{1+\rho_I}{1+\rho_I+\varepsilon}\left((1+\rho_I)\left(r-\frac{dr}{d\delta}\delta\right)+\varepsilon(1+r)-\varepsilon\frac{dr}{d\delta}(1-\delta)\delta-\varepsilon\right) \\
&= -(1+\rho_I)-\frac{1}{\rho_I}\frac{1+\rho_I}{1+\rho_I+\varepsilon}\left[(1+\rho_I)\left(\frac{dr}{d\delta}\delta-r\right)+\varepsilon\right].
\end{aligned}
$$

Now we have $\frac{d}{d\delta}\left(\frac{dr}{d\delta}\delta-r\right)=\frac{d^2r}{d\delta^2}\delta \geq 0$ and thus $\frac{dr}{d\delta}\delta-r \geq \frac{dr}{d\delta}\delta-r\big|_{\delta=0}=-r(0)$, and finally:

$$
\begin{aligned}
\frac{\partial}{\partial \phi}\left[\frac{\partial \Pi}{\partial \delta}C-\frac{\partial C}{\partial \delta}\Pi\right] &\leq -(1+\rho_I)-\frac{1}{\rho_I}\frac{1+\rho_I}{1+\rho_I+\varepsilon}\left[-(1+\rho_I)r(0)+\varepsilon\right] \\
&< -(1+\rho_I)+\frac{1}{\rho_I}(1+\rho_I)r(0)=-(1+\rho_I)\left(1-\frac{r(0)}{\rho_I}\right)<0.
\end{aligned}
$$

This concludes the proof that $\frac{d\delta}{d\phi}<0$.[27]

6. $D^*\delta^*$ is decreasing with ϕ. If $\delta^*>0$ it is strictly decreasing Let $\delta(\phi), D(\phi)$ be the solution of the maximization problem of the bank for given ϕ. We have:

$$
\frac{1+\rho_I}{\rho_I}\mu=\left[C(\delta(\phi),\phi)-\Pi(\delta(\phi),\phi)\right]D(\phi).
$$

Let $\phi_1<\phi_2$. In Step 5 we showed that $\delta(\phi_1)\geq\delta(\phi_2)$. If $\delta(\phi_2)=0$ then trivially $D(\phi_1)\delta(\phi_1)\geq D(\phi_2)\delta(\phi_2)=0$. Let us suppose that $\delta(\phi_2)>0$. Since trivially $\Pi(\delta(\phi_1),\phi_1)D(\phi_1)\geq$

[27] In the case of corner solution $\delta^*(\phi)=1$, we might have $\frac{d\delta^*}{d\phi}=0$ and obviously for $\delta^*(\phi)=0$, $\frac{d\delta^*}{d\phi}=0$.

$\Pi(\delta(\phi_2);\phi_2)D(\phi_2)$, we must have $C(\delta(\phi_1),\phi_1)D(\phi_1) \geq C(\delta(\phi_2),\phi_2)D(\phi_2)$. Now, suppose that $D(\phi_1)\delta(\phi_1) \leq D(\phi_2)\delta(\phi_2)$, then we have the following two inequalities:

$$\begin{aligned} (1+\rho_I)(1+\phi_1)D(\phi_1)\delta(\phi_1) &< (1+\rho_I)(1+\phi_2)D(\phi_2)\delta(\phi_2), \\ (1+r(\delta(\phi_1)))(1-\delta(\phi_1)) &\leq (1+r(\delta(\phi_2)))(1-\delta(\phi_2)), \end{aligned}$$

that imply $C(\delta(\phi_1),\phi_1)D(\phi_1) < C(\delta(\phi_2),\phi_2)D(\phi_2)$, but this contradicts our assumption. Thus, $D(\phi_1)\delta(\phi_1) > D(\phi_2)\delta(\phi_2)$.■

Proof of Proposition 3 Let us denote $(D(\phi),\delta(\phi))$ the solution of the bank's optimization problem for every excess return on crisis liquidity $\phi \geq 0$. Proposition 2 states that $D(\phi)\delta(\phi)$ is decreasing in ϕ. For $\phi \in [0,\overline{\phi}]$ let us define $\Sigma(\phi) = \Phi(D(\phi)\delta(\phi)) - \phi$. This function represents the difference between the excess cost of liquidity during a crisis by banks' decisions and banks' expectation on such variable. Since Φ is an increasing function on the aggregate demand of funds during a crisis the function $\Sigma(\phi)$ is strictly decreasing. Because of the uniqueness of the solution to the problem that defines $(D(\phi),\delta(\phi))$, the function is also continuous. Moreover, we trivially have $\Sigma(0) \geq 0$ and $\lim_{\phi\to\infty}\Sigma(\phi) = -\infty$. Therefore there exists a unique $\phi^e \in \mathbb{R}^+$ such that $\Sigma(\phi^e) = 0$. By construction $D(\phi^e),\delta(\phi^e),\phi^e$is the unique equilibrium of the economy. Conditions for having an interior equilibrium in $\delta^e \in (0,1)$ will be provided in a next version of the paper.■

Proof of Proposition 4 We are going to follow the notation used in the proof of Proposition 3. Let Φ_1, Φ_2 be two curves describing the inverse supply of liquidity during a crisis and assume they satisfy $\Phi_1(x) > \Phi_2(x)$ for all x. Let us denote $\Sigma_i(\phi) = \Phi_i(D(\phi)\delta(\phi)) - \phi$ for $i = 1,2$. By construction we have $\Sigma_1(\phi_1^e) = 0$. Let us suppose that $\phi_1^e < \phi_2^e$. Then we would have:

$$\Sigma_2(\phi_2^e) = \Phi_2(D(\phi_2^e)\delta(\phi_2^e)) - \phi_2^e \leq \Phi_1(D(\phi_2^e)\delta(\phi_2^e)) - \phi_2^e < \Phi_1(D(\phi_1^e)\delta(\phi_1^e)) - \phi_1^e = \Sigma_1(\phi_1^e) = 0, \tag{23}$$

where in the first inequality we use the assumption $\Phi_2(x) \leq \Phi_1(x)$ for $x \geq 0$, and in the second inequality we use that if $\phi_1^e < \phi_2^e$ then $D(\phi_2^e)\delta(\phi_2^e) \leq D(\phi_1^e)\delta(\phi_1^e)$ (Proposition 2), and that $\Phi_1(\cdot)$ is increasing.

Notice that the sequence of inequalities in (23) implies $\Sigma_2(\phi_2^e) < 0$, which contradicts the definition of ϕ_2^e. We must therefore have $\phi_1^e \geq \phi_2^e$. Now Proposition 2 implies that $\delta_1^e \leq \delta_2^e, D_1^e\delta_1^e \leq D_2^e\delta_2^e, r_1^e \geq r_2^e$. Let us suppose that $\delta_2^e \in (0,1)$ then the first inequality in (23) is strict, since $D_2^e\delta_2^e > 0$, and we can straightforwardly check that the previous argument

implies $\phi_1^e > \phi_2^e$. Now, since $\delta_2^e \in (0,1)$, Proposition 2 implies that $\delta_1^e < \delta_2^e$, $D_1^e\delta_1^e < D_2^e\delta_2^e$, and $r_1^e > r_2^e$.■

Proof of Proposition 5 We are going to follow the notation in the proof of Proposition 2. Using the definition of $W(D,\delta)$ we have

$$\begin{aligned}
\frac{\partial W(D,\delta)}{\partial \delta} &= \frac{\partial V(D,\delta;\Phi(D\delta))}{\partial \delta} + \frac{\partial V(D,\delta;\Phi(D\delta))}{\partial \phi} D\Phi'(D\delta) + \frac{\partial U(D,\delta)}{\partial \delta} \\
&= \frac{\partial V(D,\delta;\Phi(D\delta))}{\partial \delta} - \frac{1}{\rho_I}\frac{(1+\rho_I)\varepsilon}{1+\rho_I+\varepsilon}\delta D^2\Phi'(D\delta) + \\
&\quad + \frac{1}{\rho_I}\frac{(1+\rho_I)\varepsilon}{1+\rho_I+\varepsilon}\delta D^2\Phi'(D\delta) \\
&= \frac{\partial V(D,\delta;\Phi(D\delta))}{\partial \delta} = D\frac{\partial \Pi(\delta,\Phi(D\delta))}{\partial \delta} \qquad (24)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^2 W(D,\delta)}{\partial \delta^2} &= D\frac{\partial^2 \Pi(D,\delta;\Phi(D\delta))}{\partial \delta^2} + \frac{\partial^2 \Pi(D,\delta;\Phi(D\delta))}{\partial \delta \partial \phi} D^2\Phi'(D\delta) \\
&= \frac{\partial^2 \Pi(D,\delta;\Phi(D\delta))}{\partial \delta^2} - \frac{1}{\rho_I}\frac{(1+\rho_I)\varepsilon}{1+\rho_I+\varepsilon} D^2\Phi'(D\delta) < 0,
\end{aligned}$$

where in the last inequality we have used that $\Pi(D,\delta;\phi)$ is concave in δ and that $\Phi'(\cdot) > 0$. Notice that $W(D,\delta)$ is concave in δ.

Denote the exogenous amount of debt referred in the proposition as $\overline{D} > 0$. Let (δ^e,ϕ^e) be the equilibrium of the economy in which banks do not decide $\overline{D}$. Let us suppose that $\delta^e \in (0,1)$; the argument if $\delta^e = 1$ is analogous and will be omitted for brevity.[28] By analogy with the system of equations in (18), the competitive equilibrium is characterized by

$$\begin{aligned}
(1+\theta)\frac{\partial \Pi(\delta^e,\phi^e)}{\partial \delta} - \theta\frac{\partial C(\delta^e,\phi^e)}{\partial \delta} &= 0, \qquad (25) \\
\theta\left[\frac{1+\rho_I}{\rho_I}\mu + (\Pi(\delta^e,\phi^e) - C(\delta^e,\phi^e))\overline{D}\right] &\geq 0, \\
\theta &\geq 0, \\
\phi^e &= \Phi(\overline{D}\delta^e).
\end{aligned}$$

Now, let δ^s be the solution to the social planner problem. We can distinguish two cases: i) $\theta = 0$. In this case the system of equations (25) implies $\frac{\partial \Pi(\delta^e,\phi^e)}{\partial \delta} = 0$. Now, if we use equation (24) we have

$$\frac{\partial W(\overline{D},\delta^e)}{\partial \delta} = \overline{D}\frac{\partial \Pi(\delta^e,\Phi(\overline{D}\delta^e))}{\partial \delta} = \overline{D}\frac{\partial \Pi(\delta^e,\phi^e)}{\partial \delta} = 0$$

[28] As in the competitive equilibrium with an endogenous overall principal of debt we must have $\delta^e > 0$.

and, therefore, δ^e maximizes the (concave) function $W(\overline{D},\delta)$. Thus, in this case $\delta^s=\delta^e$.
ii) $\theta>0$. In this case, from equation (17) we conclude that $\frac{\partial\Pi(\delta,\phi)}{\partial\delta}<0$ implies $\frac{\partial C(\delta,\phi)}{\partial\delta}>0$. Now, the first equation in the system (25) implies $\frac{\partial\Pi(\delta^e,\phi^e)}{\partial\delta}>0$, $\frac{\partial C(\delta^e,\phi^e)}{\partial\delta}-\frac{\partial\Pi(\delta^e,\phi^e)}{\partial\delta}>0$. So we have

$$\frac{\partial W(\overline{D},\delta^e)}{\partial\delta}=\overline{D}\frac{\partial\Pi(\delta^e,\phi^e)}{\partial\delta}>0,$$

and, since $W(\overline{D},\delta)$ is concave, we have $W(\overline{D},\delta)<W(\overline{D},\delta^e)$ for all $\delta<\delta^e$. Now, given that δ^e satisfies (BF) with equality, in order to prove that $\delta^s=\delta^e$ it suffices to show that for $\delta>\delta^e$ (BF) is not satisfied. In fact, we have

$$\begin{aligned}\frac{\partial}{\partial\delta}\left[C(\delta,\Phi(\overline{D}\delta))-\Pi(\delta,\Phi(\overline{D}\delta))\right] &= \frac{\partial C}{\partial\delta}-\frac{\partial\Pi}{\partial\delta}+\left(\frac{\partial C}{\partial\phi}-\frac{\partial\Pi}{\partial\phi}\right)\overline{D}\Phi'(\overline{D}\delta),\\ \frac{\partial^2}{\partial\delta^2}\left[C(\delta,\Phi(\overline{D}\delta))-\Pi(\delta,\Phi(\overline{D}\delta))\right] &= \frac{\partial^2 C}{\partial\delta^2}-\frac{\partial^2\Pi}{\partial\delta^2}+2\left(\frac{\partial^2 C}{\partial\phi\partial\delta}-\frac{\partial^2\Pi}{\partial\phi\partial\delta}\right)\overline{D}\Phi'(\overline{D}\delta)\\ &\quad+\left(\frac{\partial C}{\partial\phi}-\frac{\partial\Pi}{\partial\phi}\right)\overline{D}^2\Phi''(\overline{D}\delta).\end{aligned}$$

From this expression it is easy to check that $\frac{\partial^2}{\partial\delta^2}\left[C(\delta,\Phi(\overline{D}\delta))-\Pi(\delta,\Phi(\overline{D}\delta))\right]>0$ and, using that $\frac{\partial C(\delta^e,\phi^e)}{\partial\delta}-\frac{\partial\Pi(\delta^e,\phi^e)}{\partial\delta}>0$, we obtain

$$\frac{\partial}{\partial\delta}\left[C(\delta,\Phi(\overline{D}\delta))-\Pi(\delta,\Phi(\overline{D}\delta))\right]\Big|_{\delta=\delta^e}>0.$$

Combining both results we conclude that, for all $\delta>\delta^e$,

$$\frac{1+\rho_I}{\rho_I}\mu=\left(C(\delta^e,\Phi(\overline{D}\delta^e))-\Pi(\delta^e,\Phi(\overline{D}\delta^e))\right)\overline{D}<\left(C(\delta,\Phi(\overline{D}\delta))-\Pi(\delta,\Phi(\overline{D}\delta))\right)\overline{D}$$

and thus (BF) is not satisfied.■

Proof of Proposition 6 We are going to follow the notation used in the proof of Proposition 2. The proof is organized in five steps:

1. Preliminaries We have seen in the proof of Proposition 5 that:

$$\frac{\partial W(D,\delta)}{\partial\delta}=\frac{\partial V(D,\delta;\Phi(D\delta))}{\partial\delta}=D\frac{\partial\Pi(\delta,\Phi(D\delta))}{\partial\delta}. \tag{26}$$

Similarly we have

$$\frac{\partial W(D,\delta)}{\partial D}=\frac{\partial V(D,\delta;\Phi(D\delta))}{\partial D}=\Pi(\delta,\Phi(D\delta)). \tag{27}$$

2. (BF) is binding at the socially optimal debt structure This is a statement that has been done in the main text just before Proposition 6. The proof is analogous to the one for the maximization problem of the bank that we did in Step 4 of the proof of Proposition 2. The only difference is that ϕ is not taken as given but as the function $\Phi(D\delta)$ in D and δ.

3. Definition of function $D^c(\delta)$ and its properties Let $(D^e, \delta^e), \phi^e$ be the competitive equilibrium. Let us assume that $\delta^e < 1$. By definition of equilibrium we have $\phi^e = \Phi(D^e\delta^e)$. For every δ let $D^c(\delta)$ be the unique principal of debt such that (BF) is binding, i.e.:

$$\frac{1+\rho_I}{\rho_I}\mu = \left[C(\delta,\phi^e) - \Pi(\delta,\phi^e)\right] D^c(\delta). \tag{28}$$

Differentiating w.r.t. δ:

$$\left[\frac{\partial C(\delta,\phi^e)}{\partial \delta} - \frac{\partial \Pi(\delta,\phi^e)}{\partial \delta}\right] D^c(\delta) + \left[C(\delta,\phi^e) - \Pi(\delta,\phi^e)\right] \frac{dD^c(\delta)}{d\delta} = 0. \tag{29}$$

Using the characterization of δ^e in equation (19), the inequalities $C(\delta,\phi^e) \geq 1 > \Pi(\delta,\phi^e)$ imply $\frac{\partial C(\delta^e,\phi^e)}{\partial \delta} - \frac{\partial \Pi(\delta^e,\phi^e)}{\partial \delta} > 0$ and, then, we can deduce from the equation above that $\frac{dD^c(\delta^e)}{d\delta} < 0$. Since (BF) is binding at the optimal debt structure we can think of the bank problem as maximizing the univariate function $V(D^c(\delta),\delta;\phi^e)$. Hence δ^e must satisfy the necessary FOC for an interior solution to the maximization of $V(D^c(\delta),\delta;\phi^e)$:

$$\frac{dV(D^c(\delta),\delta^e;\phi^e)}{d\delta} = 0 \Leftrightarrow D^c(\delta^e)\frac{\partial \Pi(\delta^e,\phi^e)}{\partial \delta} + \Pi(\delta^e,\phi^e)\frac{dD^c(\delta^e)}{d\delta} = 0, \tag{30}$$

which multiplying by δ^e can be written as

$$D^c(\delta^e)\frac{\partial \Pi(\delta^e,\phi^e)}{\partial \delta^e}\delta^e = \Pi(\delta^e,\phi^e)\left(-\frac{dD^c(\delta^e)}{d\delta}\delta^e\right).$$

Since $\frac{\partial\left(\Pi - \frac{\partial \Pi}{\partial \delta}\delta\right)}{\partial \delta} = -\frac{\partial^2 \Pi}{\partial \delta^2}\delta \geq 0$ and $\Pi(0,\phi) - \frac{\partial \Pi(0,\phi)}{\partial \delta}0 > 0$, we have $\Pi(\delta,\phi) > \frac{\partial \Pi(\delta,\phi)}{\partial \delta}\delta$ for all $\delta \in [0,1]$ and the previous equation implies

$$D^c(\delta^e) > -\frac{dD^c(\delta^e)}{d\delta}\delta^e \Leftrightarrow \left.\frac{d\left(D^c(\delta)\delta\right)}{d\delta}\right|_{\delta=\delta^e} > 0.$$

4. Evaluation of $\left.\frac{d(D^s(\delta))}{d\delta}\right|_{\delta=\delta^e}$ and $\left.\frac{d(D^s(\delta)\delta)}{d\delta}\right|_{\delta=\delta^e}$ For every δ, let $D^s(\delta)$ be the unique principal of debt such that (BF) is binding, i.e.

$$\frac{1+\rho_I}{\rho_I}\mu = \left[C(\delta,\Phi(D^s(\delta)\delta)) - \Pi(\delta,\Phi(D^s(\delta)\delta))\right] D^s(\delta).$$

Differentiating w.r.t. δ, we obtain

$$\left[\frac{\partial C(\delta,\Phi)}{\partial\delta}-\frac{\partial\Pi(\delta,\Phi)}{\partial\delta}\right]D^s(\delta)+\left[C(\delta,\Phi(D^s(\delta)\delta))-\Pi(\delta,\Phi(D^s(\delta)\delta))\right]\frac{dD^s(\delta)}{d\delta}+ \tag{31}$$
$$+\left[\frac{\partial C(\delta,\Phi)}{\partial\phi}-\frac{\partial\Pi(\delta,\Phi)}{\partial\phi}\right]\Phi'(D^s(\delta)\delta))\frac{d\left(D^s(\delta)\delta\right)}{d\delta}=0.$$

By construction, $D^s(\delta^e)=D^c(\delta^e)=D^e$. Now, subtracting equation (29) from equation (31) at the point $\delta=\delta^e$ we obtain

$$[C(\delta^e,\phi^e)-\Pi(\delta^e,\phi^e)]\left(\frac{dD^s(\delta^e)}{d\delta}-\frac{dD^c(\delta^e)}{d\delta}\right)+\left[\frac{\partial C(\delta^e,\phi^e)}{\partial\phi}\right. \tag{32}$$
$$\left.-\frac{\partial\Pi(\delta^e,\phi^e)}{\partial\phi}\right]\Phi'(D^e\delta^e)\left.\frac{d\left(D^s(\delta)\delta\right)}{d\delta}\right|_{\delta=\delta^e}=0.$$

Suppose that $\left.\frac{d(D^s(\delta)\delta)}{d\delta}\right|_{\delta=\delta^e}\leq 0$, then we would have $\frac{dD^s(\delta^e)}{d\delta}\geq\frac{dD^c(\delta^e)}{d\delta}$, since trivially $\frac{\partial C(\delta^e,\phi^e)}{\partial\phi}-\frac{\partial\Pi(\delta,\phi^e)}{\partial\phi}>0$. But then

$$\left.\frac{d\left(D^s(\delta)\delta\right)}{d\delta}\right|_{\delta=\delta^e}=D^s(\delta^e)+\frac{dD^s(\delta^e)}{d\delta}\delta^e>D^c(\delta^e)+\frac{dD^c(\delta^e)}{d\delta}\delta^e=\left.\frac{d\left(D^c(\delta)\delta\right)}{d\delta}\right|_{\delta=\delta^e}>0,$$

which contradicts the hypothesis. We must thus have $\left.\frac{d(D^s(\delta)\delta)}{d\delta}\right|_{\delta=\delta^e}>0$, in which case equation (32) implies $\frac{dD^s(\delta^e)}{d\delta}<\frac{dD^c(\delta^e)}{d\delta}<0$.

5. Evaluation of $\left.\frac{dW(D^s(\delta),\delta)}{d\delta}\right|_{\delta=\delta^e}$ Using equations (26) and (27), we have:

$$\begin{aligned}\frac{dW(D^s(\delta),\delta)}{d\delta} &= \frac{\partial W(D^s(\delta),\delta)}{\partial\delta}+\frac{\partial W(D^s(\delta),\delta)}{\partial D}\frac{dD^s(\delta)}{d\delta}\\ &= D^s(\delta)\frac{\partial\Pi(\delta,\Phi(D^s(\delta)\delta))}{\partial\delta}+\Pi(\delta,\Phi(D^s(\delta)\delta))\frac{dD^s(\delta)}{d\delta}.\end{aligned}$$

And, using $\frac{dD^s(\delta^e)}{d\delta}<\frac{dD^c(\delta^e)}{d\delta}$ and (30), we obtain:

$$\left.\frac{dW(D^s(\delta),\delta)}{d\delta}\right|_{\delta=\delta^e}<D^s(\delta^e)\frac{\partial\Pi(\delta^e,\phi^e)}{\partial\delta}+\Pi(\delta^e,\phi^e)\frac{dD^c(\delta^e)}{d\delta}=0.\blacksquare$$

B Equilibrium with liquidation?

In this appendix—to be completed—we will examine the possibility that banks decide to expose themselves to the risk of defaulting on their debt obligations and being liquidated during systemic crises. Starting from the candidate equilibrium with bridge financing, we will consider the value for a single bank of deviating to a debt structure which violates the bridge financing constraint that hence implies that, at the arrival of the first systemic liquidity crisis, the bank is unable to finance the repayment of its maturity debt and, then, goes bankrupt, and gets liquidated. Obviously the debt of this deviant bank will be valued by savers in correct anticipation of this course of events. The analysis will yield the specific conditions under which this type of deviation is not profitable and the equilibrium with bridge financing gets fully confirmed as such.

References

Acharya, Viral, Douglas Gale, and Tanju Yorulmazer (2009), "Rollover Risk and Market Freezes," mimeo, New York University.

Allen, Frankling, and Douglas Gale (2007), *Understanding Financial Crises*, Clarendon Lectures in Finance, Oxford University Press.

Blanchard, Olivier, Giovanni Dell'Ariccia, and Paolo Mauro (2010), "Rethinking Macroeconomic Policy", IMF Staff Position Note, SPN/10/03.

Brunnermeier, Markus (2009), "Deciphering the Liquidity and Credit Crunch 2007-08", *Journal of Economic Perspectives*, 23, 77-100.

Brunnermeier, Markus, and Martin Oehmke (2009), "The Maturity Rat Race," mimeo, Princeton University.

Diamond, Douglas (1991), "Debt Maturity Structure and Liquidity Risk," *Quarterly Journal of Economics,* 106, 709-737.

Diamond, Douglas, and Philip H. Dybvig (1983), "Bank Runs, Deposit Insurance, and Liquidity," *Journal of Political Economy,* 91(3), 401-419.

Diamond, Douglas, and Raghuram G. Rajan (2001), "Liquidity Risk, Liquidity Creation, and Financial Fragility: A Theory of Banking," *Journal of Political Economy*, 109, 287-327.

Flannery, Mark (1986), "Asymmetric Information and Risky Debt Maturity Choice," *Journal of Finance,* 41, 19-37.

Flannery, Mark (1994), "Debt Maturity and the Deadweight Cost of Leverage: Optimally Financing Banking Firms," *American Economic Review,* 84, 320-331.

He, Zhiguo, and Wei Xiong (2009a), "Dynamic Debt Runs," mimeo, Princeton University.

He, Zhiguo, and Wei Xiong (2009b), "Liquidity and Short-term Debt Crises," mimeo, Princeton University.

Holmström, Bengt, and Jean Tirole (1998), "Private and Public Supply of Liquidity," *Journal of Political Economy*, 106, 1-40.

Leland, Hayne (1994), "Corporate Debt Value, Bond Covenants, and Optimal Capital Structure," *Journal of Finance*, 49, 1213-1252.

Leland, Hayne (1998), "Agency Costs, Risk Management, and Capital Structure," *Journal of Finance,* 53, 1213-1243.

Leland, Hayne, and Klaus Bjerre Toft (1996), "Optimal Capital Structure, Endogenous Bankruptcy, and the Term Structure of Credit Spreads," *Journal of Finance*, 51, 987-1019.

Lorenzoni, Guido (2008), "Inefficient Credit Booms," *Review of Economic Studies*, 75, 809-833.

Morris, Stephen, and Hyun Song Shin (2004), "Liquidity Black Holes," *Review of Finance,* 8, 1-18.

Morris, Stephen, and Hyun Song Shin (2009), "Illiquidity Component of Credit Risk," mimeo, Princeton University.

Tarullo, Daniel (2009), "Regulatory Reform," Testimony before the Committee on Financial Services, U.S. House of Representatives, Washington, D.C., October 29, 2009.