Liquidity Hoarding[1]

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1 Introduction

One of the most interesting phenomena marking the recent financial crisis was the "freezing" of the interbank market. As early as the fall of 2007, following the collapse of the market for asset backed commercial paper in Europe, banks reported an inability to borrow on the interbank market and central bank borrowing facilities became an essential source of liquidity. Since that time, problems obtaining liquidity in interbank markets have been observed in many countries. Two main explanations have been offered for this phenomenon. The first is counter-party risk. Because of the widespread exposure to sub-prime asset backed securities banks became wary of lending to any bank that might be affected by this or any other source of credit risk. The second explanation was that banks were *hoarding* liquidity, because of fears that their own future access to liquidity might be impaired. The second explanation is not unrelated to the first: in a world of asymmetric information, where rumors of distress are enough to cause a "run" by counter-parties, every bank has to be concerned that it might be perceived as a source of future counter-party risk and lose access to markets for short-term borrowing. Whatever the motivation for hoarding, once banks start to hoard there is the possibility for coordination failure that leads to a break down of markets. Thus, fears of future lack of access, for whatever reason, can lead to hoarding, which restricts access for other banks and provides the motivation for more hoarding.

In this paper, we present a simplified model that allows us to examine liquidity management in general equilibrium. We divide time into four periods or dates. At the first date, there is a large number of agents whom we think of as "banks." Banks choose whether to be liquid, in the sense that they hold both liquid and illiquid assets, or illiquid, in the sense that they hold only illiquid assets. At the second and third dates, some banks receive a random liquidity shock, which we interpret as a demand for payment of senior debt that can only be discharged by a payment of the liquid asset. A bank that receives a liquidity shock and does not have any liquid assets has to sell the illiquid asset in order to discharge its debt. Liquidity is supplied by liquid banks that have not received a liquidity shock.

When deciding whether to supply liquidity in the second period, a liquid bank has several things to consider. One is that it may itself receive a liquidity shock in the next period. If a bank gives up its liquidity today and receives a shock tomorrow, it will be able to obtain "cash" by selling the illiquid asset but the price may be very high if the demand for liquidity is high. This may lead the bank to hoard its liquidity in the second period, rather than supplying it to the market. The precautionary motive is only one reason for hoarding. There is also a speculative motive: in the event that the bank does not receive a liquidity shock at the third period, it may profit from a fire sale of illiquid assets if demand for liquidity is high. Clearly, these motives cannot be separated; liquidity serves both motives simultaneously.

We begin our analysis by studying the constrained-efficient allocation chosen by a central planner who accumulates a stock of liquid assets and distributes them to the banks that report a need for liquidity. Then we analyze a laisser-faire economy in which banks make their own decisions about liquidity accumulation and liquidity provision. Finally, we investigate the constrained (in)efficiency of the laisser-faire economy, and show that there are several

simple interventions that can improve on the laisser-faire allocation.

1.1 Related literature

Some recent papers provide empirical evidence for and discuss liquidity hoarding in interbank markets. Acharya and Merrouche (2009) document that the U.K. banks' liquidity buffers experienced an almost permanent upward shift of 30% in August 2007 (relative to their pre-August levels) and the result was a rise in borrowing costs between banks and an almost complete drying up of liquidity in interbank markets beyond the very short maturities. Heider et al. (2008) provide evidence of liquidity hoarding in the unsecured euro interbank market. They document that until August 9, 2007, the unsecured euro interbank market is characterized by a very low spread and infinitesimal amounts of excess reserves with the European Central Bank (ECB) since, in normal times, banks prefer to lend out excess cash as the interest rate on excess reserves is punitive relative to rates available in interbank markets. They document that the period between August 9, 2007 and the last weekend of September 2008 is characterized by a significantly higher spread, yet excess reserves remain virtually nil. As of September 28, 2008, the spread increases even further to a maximum of 186 basis points. More importantly, we observe a dramatic increase in excess reserves, where the average daily volume in the overnight unsecured interbank market halved. Ashcraft, McAndrews and Skeie (2008) use data on intraday account balances held by banks at the Federal Reserve and Fedwire interbank transactions for a sample of approximately 700 banks that ever lend or borrow during the period September 2007 through August 2008 to estimate all overnight fed funds trades. They present empirical evidence on banks' precautionary hoarding of reserves, their reluctance to lend, and extreme fed funds rate volatility. Afonso, Kovner and Schoar (2010) examine the response of the US Fed Funds market to the bankruptcy of Lehman Brothers and documents that while rates spiked and loan terms became more sensitive to borrower risk, mean borrowing amounts remained stable on aggregate. They argue that it is likely that the market did not expand to meet additional demand for funds, which is consistent with our result on rationing in the interbank market when demand for liquidity is high.

At a general level, our paper is related to Shleifer and Vishny (1992) and Allen and Gale (1994, 1998) that show that when potential buyers of assets are themselves financially constrained, the price of the assets may fall below their fundamental value and be determined by the available liquidity in market, that is, we observe cash-in-the-market prices.[1]

Our paper is related to the literature on portfolio choice of banks and how the level of liquidity is determined endogenously (e.g. Allen and Gale (2004a,b), Gorton and Huang (2004), Diamond and Rajan (2005), and Acharya, Shin and Yorulmazer (2009)). Allen and Gale (2004b), for example, build a model where runs by depositors result in fire-sale liquidation of banking assets. Banks endogenously choose the level of the liquid asset, which they use to purchase banking assets. Since on average the liquid asset has a lower return than

[1] Also, see Allen and Gale (2005) for a review of the literature that explores the relation between asset-price volatility and financial fragility when markets and contracts are incomplete.

the risky asset, banks have to be compensated for holding liquid assets, which is possible in equilibrium if they can purchase the risky asset at a discount in some states of the world, leading to cash-in-the-market pricing. Acharya, Shin and Yorulmazer (2009) analyze banks' portfolio choice problem and show that when the pledgeability of assets is high (low) banks hold less (more) than the socially optimal level of liquidity. The recent work by Diamond and Rajan (2009) build a model, where banks in anticipation of future fire-sales have high expected returns from holding cash. Our paper differs from these papers in various aspects. First, in our paper agents hold liquidity for protecting themselves against future liquidity shocks (precautionary motive) as well as taking advantage of potential sales (strategic motive). Second, in our paper, agents make a portfolio choice initially as well as a choice to lend to needy agents or hoard liquidity for future periods. This adds richness to our model and allow us to analyze the interaction between agents' two choices. Furthermore, this allows us to analyze a rich set of policies such as ex ante liquidity requirements and various ex post lending facilities.

Our paper is related to the literature on interbank markets (e.g. Rochet and Tirole (1996) and Allen and Gale (2000)), and the failure of such markets to transfer liquidity efficiently that justifies regulatory intervention.[2] Goodfriend and King (1988) argue that with efficient interbank markets, central banks should not lend to individual banks, but instead provide liquidity via open market operations, which the interbank market would then allocate among banks. Others, however, argue that interbank markets may fail to allocate liquidity efficiently due to frictions such as asymmetric information about banks' assets (Flannery (1996), Freixas and Jorge (2007)), banks' free-riding on each other's liquidity (Bhattacharya and Gale (1987)), or on the central bank's liquidity (Repullo (2005)), and market power and strategic behavior (Acharya, Gromb and Yorulmazer (2007)).

Our paper, in general, is also related to the papers on runs in wholesale markets (Huang and Ratnovski (2008), Gorton and Metrick (2009), and He and Xiong (2009)), shortening of maturities during stress periods (Brunnermeier and Oehmke (2009)), and drying up of liquidity and market freezes (Acharya, Gale and Yorulmazer (2009)).

2 Constrained efficiency

2.1 Primitives

Time: Time is divided into four dates, indexed by $t = 0, 1, 2, 3$. At the first date, agents choose the amount of liquidity they hold as part of their portfolio. At the second and third dates, agents receive liquidity shocks and trade assets in order to obtain the liquidity they need. At the final date, asset returns are realized.

Assets: There are two assets, a liquid asset that we refer to as 'cash,' and an illiquid asset that we will refer to simply as 'the asset.' Cash can be used to discharge debts and can be stored from period to period. One unit of cash has a return of one unit of consumption at

[2] Also, see Freixas et al. (1999) for an excellent survey on interbank markets.

date 3. The asset cannot be used to discharge debts (unless it is first exchanged for cash). The asset can be stored from period to period. One unit of the asset has a return of $R > 1$ units of consumption at date 3.

Agents: There is a continuum of identical, risk neutral agents, indexed by $i \in [0, 1]$. Each agent has an initial endowment of one unit of the asset at date 0. At a (utility) cost of $c > 0$, an agent can obtain one unit of cash at date 0, in addition to his endowment of the asset.

Liquidity shocks: Agents receive a liquidity shock with some probability at date 1 or date 2. We can think of the liquidity shock as a debt coming due early or a failure to roll over short-term funding. If an agent is not hit by one of these shocks, he pays one unit of cash for this debt at $t = 3$ after the return from the asset is realized. An agent who receives one of these shocks must immediately deliver one unit of cash to discharge the existing debt; otherwise he will be forced to default. If the agent becomes bankrupt, we assume that all his assets are immediately liquidated and, for simplicity, we assume that the liquidation costs consume the entire value of the assets. This extreme assumption can be relaxed, but it greatly simplifies the analysis and does not appear to affect the qualitative results too much. In order to obtain cash, an agent can sell some or all of his holdings of the asset. Agents who receive a liquidity shock at date 1 will not receive a liquidity shock at date 2.

Distributions: At date 1, a fraction θ_1 of the agents require one unit of cash in order to discharge an existing debt; otherwise, they will be forced to default. The random variable θ_1 has a density function $f_1(\theta_1)$ and the c.d.f. is denoted by $F_1(\theta_1)$. At date 2, a fraction θ_2 of the agents who did not receive a liquidity shock at date 1 will receive a liquidity shock. The random variable θ_2 has a density function $f_2(\theta_2)$ and the c.d.f. is denoted by $F_2(\theta_2)$. We assume that both random variables have support $[0, 1]$.

2.2 The planner's problem

Since agents are ex ante identical, we choose total expected surplus as our objective function. In addition to the usual feasibility constraints, the planner operates subject to the constraint that he cannot transfer assets between agents. If the planner were able to transfer assets, he would assign all assets at date 1 to agents who had already received a liquidity shock, thus rendering the liquidity shocks at date 2 irrelevant. To avoid this trivial solution, we restrict the planner's actions to accumulating cash at date 0, distributing cash at dates 1 and 2, and redistributing the consumption good at date 3.

To simplify the exposition, we initially assume that the planner knows an agent's type, that is, he knows which agents have received a liquidity shock at each date. Later, we show that one can achieve the same level of welfare subject to incentive compatibility constraints.

Let $m_0 \geq 0$ denote the quantity of cash held at the end of date 0, let $m_1(\theta_1) \geq 0$ denote the amount of cash held at the end of date 1 in state θ_1, and let $m_2(\theta_1, \theta_2) \geq 0$ denote the amount of cash held at the end of date 2 in state (θ_1, θ_2). Feasibility requires

$$m_0 \geq m_1(\theta_1) \geq m_2(\theta_1, \theta_2), \tag{1}$$

for every value of (θ_1, θ_2). The amount of cash distributed at date 1 in state θ_1 is denoted by $x_1(\theta_1)$ and defined by putting

$$x_1(\theta_1) = m_0 - m_1(\theta_1) \geq 0,$$

for every value of θ_1. The amount distributed at date 2 in state (θ_1, θ_2) is denoted by $x_2(\theta_1, \theta_2)$ and defined by putting

$$x_2(\theta_1, \theta_1) = m_1(\theta_1) - m_2(\theta_1, \theta_2) \geq 0,$$

for every value of (θ_1, θ_2). Efficiency requires that

$$x_1(\theta_1) \leq \theta_1,$$

for every value of θ_1 and that

$$x_2(\theta_1, \theta_2) = \min\{(1 - \theta_1)\theta_2, m_1(\theta_1)\},$$

for every value of (θ_1, θ_2). The planner's policy is effectively determined by the choice of m_0 and $m_1(\theta_1)$, since efficiency requires that

$$\begin{aligned} m_2(\theta_1, \theta_2) &= m_1(\theta_1) - x_2(\theta_1, \theta_2) \\ &= m_1(\theta_1) - \min\{(1 - \theta_1)\theta_2, m_1(\theta_1)\} \\ &= \max\{m_1(\theta_1) - (1 - \theta_1)\theta_2, 0\}, \end{aligned} \tag{2}$$

for every value of (θ_1, θ_2) and efficiency implies that m_0 and $m_1(\theta_1)$ satisfy

$$m_1(\theta_1) \geq m_0 - \theta_1, \tag{3}$$

for every value of θ_1. In what follows we assume that any policy $\mathbf{x} = (x_0, x_1(\cdot))$ satisfies (1), (2) and (3).

The expected output from the planner's policy in state (θ_1, θ_2) is

$$R\{x_1(\theta_1) + x_2(\theta_1, \theta_2) + (1 - \theta_1)(1 - \theta_2)\} + x_1(\theta_1) + x_2(\theta_1, \theta_2) + m_2(\theta_1, \theta_2) \tag{4}$$

The total amount of the asset at date 3 will be equal to the amount of cash distributed to agents who receive a liquidity shock at dates 1 and 2, that is, $x_1(\theta) + x_2(\theta_1, \theta_2)$, plus the number of agents who do not receive a liquidity shock at either date, that is, $(1 - \theta_1)(1 - \theta_2)$. The total amount of cash at date 3 is equal to the amount held by the planner $m_2(\theta_1, \theta_2)$ plus the amount distributed to creditors, $x_1(\theta_1) + x_2(\theta_1, \theta_2)$. Multiplying the amounts of cash and the asset by their respective returns and summing them gives the expression in (4). The total surplus is equal to the expected output minus the cost of obtaining liquidity, that is,

$$\begin{aligned} &R\{x_1(\theta_1) + x_2(\theta_1, \theta_2) + (1 - \theta_1)(1 - \theta_2)\} + x_1(\theta_1) + x_2(\theta_1, \theta_2) + m_2(\theta_1, \theta_2) - cm_0 \\ &= R\{m_0 - m_2(\theta_1, \theta_2) + (1 - \theta_1)(1 - \theta_2)\} + (1 - c)m_0, \end{aligned} \tag{5}$$

where we eliminate the constant term for simplicity. The planner chooses $(x_0, x_1(\cdot))$ to maximize the expected value of (5) subject to the constraints in (1-3).

2.2.1 First-order conditions

Suppose that the planner has m_1 units of cash at the beginning of date 2 and the state is (θ_1, θ_2). There are $(1-\theta_1)\theta_2$ agents in need of cash and the optimal distribution strategy is to supply

$$x_2(\theta_1, \theta_2) = \min\{(1-\theta_1)\theta_2, m_1\}.$$

Thus, the value of m_1 units of cash in state (θ_1, θ_2) is

$$\begin{aligned} V_2(m_1, \theta_1, \theta_2) &= R\min\{(1-\theta_1)\theta_2, m_1\} + m_1 - \min\{(1-\theta_1)\theta_2, m_1\} + \min\{(1-\theta_1)\theta_2, m_1\} \\ &= R\min\{(1-\theta_1)\theta_2, m_1\} + m_1. \end{aligned}$$

For a fixed value of θ_1, the value of m_1 units of cash at the end of date 1 (before θ_2 has been realized) is

$$\begin{aligned} V_2(m_1, \theta_1) &= E[V_2(m_1, \theta_1, \theta_2)|\theta_1] \\ &= R\int_0^{\frac{m_1}{1-\theta_1}} (1-\theta_1)\theta_2 f_2(\theta_2)\, d\theta_2 + m_1 R\left(1 - F_2\left(\frac{m_1}{1-\theta_1}\right)\right) + m_1. \end{aligned}$$

The derivative of V_2 with respect to m_1 is calculated to be

$$\begin{aligned} V_2'(m_1, \theta_1) &= R(1-\theta_1)\frac{m_1}{1-\theta_1} f_2\left(\frac{m_1}{1-\theta_1}\right)\frac{1}{1-\theta_1} - Rm_1 f_2\left(\frac{m_1}{1-\theta_1}\right)\frac{1}{1-\theta_1} + \\ &\quad R\left(1 - F_2\left(\frac{m_1}{1-\theta_1}\right)\right) + 1 \\ &= R\left(1 - F_2\left(\frac{m_1}{1-\theta_1}\right)\right) + 1. \end{aligned}$$

Now consider the planner's problem at date 1. He has m_0 units of cash in state θ_1 and maximizes

$$(R+1)x_1 + V_2(m_0 - x_1, \theta_1)$$

subject to

$$0 \le x_1 \le \min\{m_0, \theta_1\}.$$

If the constraint is non-binding, we must have

$$\begin{aligned} R + 1 &= V_2'(m_0 - x_1, \theta_1) \\ &= R\left(1 - F_2\left(\frac{m_1}{1-\theta_1}\right)\right) + 1, \end{aligned}$$

which is not possible unless $m_0 - x_1 = 0$, a contradiction. Thus, the constraint should always be binding and this implies that the optimal policy is $x_1 = \min\{m_0, \theta_1\}$ or

$$m_1(\theta_1) = \max\{m_0 - \theta_1, 0\}.$$

Thus, the value of m_0 units of cash at the end of date 0, before θ_1 is realized, is equal to

$$\int_0^{m_0} \left[(R+1)\,\theta_1 + V_2\,(m_0 - \theta_1, \theta_1) \right] f_1\,(\theta_1)\, d\theta_1 + (R+1)\, m_0\, (1 - F_1\,(m_0))\,.$$

The derivative is easily calculated to be

$$\begin{aligned}
& \left[(R+1)\, m_0 + V_2\,(0, m_0) \right] f_1\,(m_0) - (R+1)\, m_0 f_1\,(m_0) + \int_0^{m_0} V_2'\,(m_0 - \theta_1, \theta_1)\, f_1\,(\theta_1)\, d\theta_1 + \\
& (R+1)\,(1 - F_1\,(m_0)) \\
= \; & \int_0^{m_0} \left[R\left(1 - F_2\left(\frac{m_0 - \theta_1}{1 - \theta_1}\right)\right) + 1 \right] f_1\,(\theta_1)\, d\theta_1 + (R+1)\,(1 - F_1\,(m_0)) \\
= \; & (R+1)\, F_1\,(m_0) - \int_0^{m_0} R F_2\left(\frac{m_0 - \theta_1}{1 - \theta_1}\right) f_1\,(\theta_1)\, d\theta_1 + (R+1)\,(1 - F_1\,(m_0)) \\
= \; & R + 1 - R \int_0^{m_0} F_2\left(\frac{m_0 - \theta_1}{1 - \theta_1}\right) f_1\,(\theta_1)\, d\theta_1.
\end{aligned}$$

At date 0, the choice of how much liquidity to hold is determined by the cost c and the value of cash at date 1, that is, m_0 will be chosen to satisfy the first-order condition

$$R + 1 - R \int_0^{m_0} F_2\left(\frac{m_0 - \theta_1}{1 - \theta_1}\right) f_1\,(\theta_1)\, d\theta_1 = c.$$

2.3 Incentive compatibility

Let $(m_0^*, m_1^*\,(\theta_1))$ denote the solution to the planner's problem above. The description of the planner's problem assumes that the planner observes the agents who receive receive a liquidity shock at each date. Now suppose that the liquidity shock is private information. In that case, the solution can still be implemented by a direct mechanism via the Revelation Principle if the allocation satisfies the appropriate incentive compatibility conditions.

A direct mechanism is defined by an array $(\mu_1\,(\theta_1)\,, p_1\,(\theta_1)\,, \mu_2\,(\theta_1, \theta_2)\,, p_2\,(\theta_1, \theta_2))$, where $\mu_1\,(\theta_1)$ is the probability that an agent who reports a liquidity shock at date 1 in state θ_1 receives one unit of cash and $p_1\,(\theta_1)$ is the price he pays for it and $\mu_2\,(\theta_1, \theta_2)$ is the probability that an agent who reports a liquidity shock at date 2 in state (θ_1, θ_2) receives a unit of cash and $p_2\,(\theta_1, \theta_2)$ is the price he pays for it. An agent who reports no liquidity shock is assumed without loss of generality to receive no cash and make no payment.

The probabilities of trade are defined by putting

$$\begin{aligned}
\mu_1\,(\theta_1) &= \frac{x_1\,(\theta_1)}{\theta_1} \\
&= \frac{m_0 - m_1\,(\theta_1)}{\theta_1},
\end{aligned}$$

for every value of θ_1 and

$$\begin{aligned}
\mu_2(\theta_1,\theta_2) &= \frac{x_2(\theta_1,\theta_2)}{(1-\theta_1)\,\theta_1} \\
&= \frac{m_1(\theta_2)-m_2(\theta_1,\theta_2)}{(1-\theta_1)\,\theta_1},
\end{aligned}$$

for every value of (θ_1,θ_2).

The price functions are chosen to satisfy the incentive compatibility conditions. The price of cash is expressed in units of the asset, although what is actually traded is the good at date 3. Consider first incentive compatibility at date 2. We assume that the planner remembers which agents reported a shock at date 1 and will not allow those agents to participate at date 2. An agent who has received a shock at date 2 can report no shock, in which case he receives no cash and must default, or he can report a shock in which case, with probability $\mu_2(\theta_1,\theta_2)$, he receives a unit of cash in exchange for $p_2(\theta_1,\theta_2)$ units of the asset. If he defaults, his payoff is zero and if he receives cash his payoff is $R(1-p_2(\theta_1,\theta_2))$, so it is incentive compatible to report truthfully if

$$\mu_2(\theta_1,\theta_2)\,R\,(1-p_2(\theta_1,\theta_2)) \geq 0. \tag{6}$$

Now consider an agent who does not receive a shock and did not report a shock at date 1. If he reports a shock then, with probability $\mu_2(\theta_1,\theta_2)$, he receives a unit of cash in exchange for $p_2(\theta_1,\theta_2)$ units of the asset and, with probability $1-\mu_2(\theta_1,\theta_2)$, he receives nothing. His payoff in the first case is $R(1-p_2(\theta_1,\theta_2))$ and in the second case it is $R-1$. Then it is incentive compatible for him to report truthfully if

$$\mu_2(\theta_1,\theta_2)\,R\,(1-p_2(\theta_1,\theta_2)) + (1-\mu_2(\theta_1,\theta_2))\,(R-1) \leq (R-1). \tag{7}$$

Incentive compatibility at date 1 is a little bit more complicated than at date 2 because one needs to consider what an agent will do at date 2. An agent who has received a liquidity shock can report the shock and receive a unit of cash in exchange for $p_1(\theta_1)$ units of the asset with probability $\mu_1(\theta_1)$, or he can report no shock and default. If he defaults his payoff is zero and if he receives cash his payoff is $R(1-p_1(\theta_1))$, so it is incentive compatible to report truthfully if

$$\mu_1(\theta_1)\,R\,(1-p_1(\theta_1)) \geq 0. \tag{8}$$

Now consider an agent who does not receive a shock. If he reports a shock then, with probability $\mu_1(\theta_1)$, he will receive one unit of cash in exchange for $p_1(\theta_1)$ units of the asset. At date 2, he will receive a liquidity shock with probability θ_2. If he does not receive cash at date 1 and receives a shock at date 2, he will not be allowed to participate in the mechanism at date 2 and must default. Thus, the payoff from reporting a shock is

$$\begin{aligned}
\mu_1(\theta_1)\,E[1-\theta_2]\,R\,(1-p_1(\theta_1)) + \mu_1(\theta_1)\,E[\theta_2]\,R\,(1-p_1(\theta_1)) + (1-\mu_1(\theta_1))\,E[1-\theta_2]\,(R-1) \\
= \mu_1(\theta_1)\,R\,(1-p_1(\theta_1)) + (1-\mu_1(\theta_1))\,E[1-\theta_2]\,(R-1),
\end{aligned}$$

because, with probability $\mu_1(\theta_1) E[1-\theta_2]$, he ends up with $1-p_1(\theta)$ units of the asset and one unit of cash out of which he pays one unit of cash to creditors at $t=3$, with probability $\mu_1(\theta_1) E[\theta_2]$, he ends up with $1-p_1(\theta_1)$ units of the asset, with probability $(1-\mu_1(\theta_1)) E[1-\theta_2]$, he ends up with one unit of the asset and pays one unit of cash to creditors at $t=3$, and, with probability $(1-\mu_1(\theta_1)) E[\theta_2]$, he is forced to default. If the agent does not report a shock, then his continuation payoff is

$$E[\theta_2\mu_2(\theta_1,\theta_2) R(1-p_2(\theta_1,\theta_2)) + (1-\theta_2)(R-1)],$$

because, in state (θ_1,θ_2), with probability $\theta_2\mu_2(\theta_1,\theta_2)$, he receives a liquidity shock at date 2 and receives a unit of cash in exchange for $p_2(\theta_1,\theta_2)$ units of the good, with probability $(1-\theta_2)\mu_2(\theta_1,\theta_2)$, he receives a liquidity shock and is forced to default and, with probability $1-\theta_2$, he does not receive a liquidity shock and ends up with one unit of the asset. Thus, it is incentive compatible for him to respond truthfully at date 1 if

$$\begin{aligned} &\mu_1(\theta_1) R(1-p_1(\theta_1)) + (1-\mu_1(\theta_1)) E[1-\theta_2](R-1) \\ &\leq E[\theta_2\mu_2(\theta_1,\theta_2) R(1-p_2(\theta_1,\theta_2)) + (1-\theta_2)(R-1)]. \end{aligned} \tag{9}$$

Proposition 1 *The incentive compatibility conditions (6), (7), (8) and (9) are satisfied if*

$$\begin{gathered} p_2(\theta_1,\theta_2) \leq 1, \textit{ for every value of } (\theta_1,\theta_2); \\ R(1-p_2(\theta_1,\theta_2)) + 1 \leq R, \textit{ for every value of } (\theta_1,\theta_2); \\ p_1(\theta_1) \leq 1, \textit{ for every value of } \theta_1; \end{gathered}$$

and

$$\mu_1(\theta_1)[(1-Rp_1(\theta_1)) + E[\theta_2](R-1)] \leq E[\theta_2\mu_2(\theta_1,\theta_2)(1-p_2(\theta_1,\theta_2))] R, \textit{ for every value of } (\theta_1,\theta_2).$$

Proof. The first three inequalities are clearly equivalent to (6), (7) and (8). The final condition is equivalent to (9) if we add $E[(1-\theta_2)] R$ to both sides. ■

3 Laisser-faire

3.1 Time line

—Figure 1 about here—

Date 0 At date 0, all agents are endowed with one unit of the asset. They also have the option of acquiring one unit of cash for an additional cost of $c > 0$ units of utility. Let α denote the fraction of agents who choose not to obtain cash. We call the agents who obtain one unit of cash *liquid* agents and those who do not obtain cash are called *illiquid* agents. Then there are α illiquid agents and $1-\alpha$ liquid agents at the end of date 0.

Date 1 At the beginning of date 1, everyone observes θ_1, the fraction of agents who receive the liquidity shock. The liquid agents who receive a shock must use their cash to discharge their debt. Otherwise, they will default and lose their entire wealth. The illiquid agents who receive a liquidity shock can sell some or all of their holdings of the asset in exchange for cash to discharge their debt. We denote the price of one unit of liquidity by $0 \leq p_1 \leq 1$. If some of these agents cannot obtain cash to discharge their debt, they must be indifferent between obtaining cash and default. This will be the case if $p_1 = 1$.

The illiquid agents who do not receive a shock are assumed to remain passive during the period. We will see later that this is the optimal strategy for them. The liquid agents who do not receive a shock have the option of selling their unit of cash for p_1 units of the asset. We refer to the liquid agents who buy the asset as *buyers* and those who do not as *hoarders*.

Date 2 At the beginning of date 2, everyone observes θ_2, the fraction of agents who did not receive a liquidity shock at date 1 and did receive a liquidity shock at date 2. Agents who received a liquidity shock at date 1 cannot receive another liquidity shock and are assumed to have no cash, so there is nothing for them to do at date 2. Without loss of generality we assume they remain inactive.

Next consider the illiquid agents. The illiquid agents who receive a shock can purchase one unit of cash for a price $p_2 \geq 0$. It will be optimal for them to do so as long as $p_2 \leq 1$, but since the buyers have $1 + p_1$ units of the asset, the price may rise above one unit of the asset. The illiquid agents who do not receive a shock have no gains from trade and are assumed not to trade.

Next, consider the buyers. The buyers who receive a liquidity shock can purchase one unit of cash for a price $p_2 \geq 0$. It will be optimal for them to do so as long as $p_2 \leq 1 + p_1$. The remaining buyers who do not receive a shock have no gains from trade and are assumed not to trade.

Finally, consider the hoarders. The hoarders who receive a liquidity shock must use their unit of cash to discharge their debt. Otherwise they will be forced to default and lose all their wealth. The hoarders who do not receive a liquidity shock can supply cash to the illiquid agents and buyers who need cash to discharge their debts. It will be optimal to supply cash as long as $p_2 \geq R^{-1}$.

Date 3 Agents receive the payoffs from their holdings of cash and the asset. Agents who are not hit by the liquidity shock at $t = 1$ and $t = 2$ pay creditors one unit of cash.

3.2 Allocations and payoffs

The allocations of the agents in the first two dates are illustrated in Figure 2.

—Figure 2 about here—

Date 0 At date 0, a fraction $1-\alpha$ of the agents decide to obtain liquidity and α choose not to obtain liquidity, so there are $1-\alpha$ liquid agents with a portfolio $(1,1)$, consisting of one unit of the asset and one unit of cash and α illiquid agents with a portfolio $(1,0)$, consisting of one unit of the asset and no cash.

Date 1 A fraction θ_1 of all agents receive a liquidity shock. The illiquid agents with a shock must sell p_1 units of the asset to obtain cash, in which case their portfolio at the end of the period is $(1-p_1,0)$. Note that when not everyone can obtain cash and some default, the equilibrium price must make the agents indifferent between trading and not trading, that is, $p_1=1$ and the final portfolio is $(0,0)$ for all illiquid agents who receive the shock.

The liquid agents use their unit of cash to discharge their debts and end the period with a portfolio $(1,0)$.

A fraction $1-\theta_1$ of the agents do not receive a liquidity shock. The illiquid agents who do not receive a shock are assumed to remain passive during the period and retain the initial portfolio $(1,0)$. The liquid agents who do not receive a shock have the option of selling their cash for p_1 units of the asset. We let λ denote the fraction of these agents who are buyers. The buyers end the period with the portfolio $(1+p_1,0)$ and the hoarders end the period with the portfolio $(1,1)$.

Date 2 At date 2, all the agents who received a liquidity shock at date 1 are inactive and retain their initial endowments. These comprise the illiquid agents who now hold a portfolio $(1-p_1,0)$ and the liquid agents with a portfolio $(1,0)$.

The active agents at date 2 include illiquid agents who did not receive a shock at date 1. Of these, a fraction $1-\theta_2$ do not receive a shock at date 2. They have no gains from trade and simply hold their initial portfolio $(1,0)$ until date 3 when they receive their payoff. The fraction θ_2 of illiquid agents who do receive a shock, must either sell some or all of the asset for cash or default. It is strictly optimal to sell the asset if $p_2<1$ and impossible to avoid default if $p_2>1$, so these agents will end up with a portfolio $(\max\{0,1-p_2\},0)$ consisting of $\max\{0,1-p_2\}$ units of the asset and no cash.

Now consider the buyers. Of these, the fraction $1-\theta_2$ do not receive a shock at date 2. They have no gains from trade and simply hold their portfolio from date 1 $(1+p_1,0)$ until date 3 when they will receive their payoff. The fraction θ_2 of buyers who receive a shock must either sell p_2 units of the asset for cash or default. It is strictly optimal to sell the asset if $p_2<1+p_1$ and the agents are indifferent between selling and default if $p_2=1+p_1$, so these agents will end up with a portfolio $(1+p_1-p_2,0)$, whether they sell or default.

The hoarders who receive a shock must use their cash to discharge their debts, after which there is no possibility of trade. Thus, a fraction θ_2 of the hoarders end up with a portfolio $(1,0)$ which they hold until date 3. The remaining fraction $1-\theta_2$ of hoarders do not receive a shock and they can use their cash to buy p_2 units of the asset. They will strictly prefer this if $p_2>R^{-1}$ and be indifferent if $p_2=R^{-1}$. Hence, these agents end up either with a portfolio $(1+p_2,0)$ or a portfolio $(1,1)$.

The allocation at date 2 illustrated in Figure 3 below.

—Figure 3 about here—

The terminal payoffs, which are easily calculated from the terminal allocation, are illustrated in Figure 4 below.

—Figure 4 about here—

3.3 Market clearing

In this section, we solve for the market clearing prices p_1 and p_2, beginning at date 2 and working back to date 1. The price at date 1 will be a function of the state θ_1 at date 1 and the price at date 2 will be a function of the state (θ_1, θ_2) at date 2, but for the most part this notation will be suppressed as we take the state as given.

3.3.1 Market clearing at date 2

Suppose that the state of the economy at date 2 is (θ_1, θ_2). We can ignore the agents who received a shock at date 1 and are inactive at date 2. We can also ignore the hoarders who receive a shock at date 2; they will use their own cash to discharge their debts and will have no gains from trade.[3] And we can ignore the buyers and the illiquid agents who do not receive a shock. Since they have assets but no cash and no need for cash, they will have no incentive to trade either.

Thus, there are three groups of agents who might engage in trade at date 2. First, there are the hoarders who do not receive a shock. These are the potential suppliers of liquidity. Then there are the buyers and the illiquid agents who receive a shock. They are the potential demanders of liquidity.

The available supply of cash at date 2 is equal to the number of agents who obtained liquidity at date 0 $(1-\alpha)$, who did not receive a liquidity shock at date 1 (a fraction $1-\theta_1$), who decided to hoard (a fraction $1-\lambda$), and did not receive a liquidity shock at date 2 (a fraction $1-\theta_2$). Thus, the available supply is

$$(1-\alpha)(1-\theta_1)(1-\lambda)(1-\theta_2).$$

It is optimal to supply no cash if $p_2 < R^{-1}$, optimal to supply some cash if $p_2 = R^{-1}$ and optimal to supply all the cash if $p_2 > R^{-1}$. The supply of cash is illustrated in Figure 5A.

—Figure 5 about here—

[3] We are assuming that the agents in this class must discharge their own debt or default and lose the value of any assets they hold. This implies that they cannot trade cash for assets with agents who hold a large number of assets but need cash.

We can construct the demand curve similarly. The number of buyers who receives a liquidity shock at date 2 is equal to the number of agents who obtained liquidity $(1-\alpha)$, who did not receive a liquidity shock at date 1 (a fraction $1-\theta_1$), who chose to buy at date 1 (a fraction λ), and who received a liquidity shock at date 2 (a fraction θ_2). Thus, the maximum demand for cash from buyers is

$$(1-\alpha)(1-\theta_1)\lambda\theta_2.$$

Each of the buyers has $1+p_1$ units of the asset. It is optimal for them to sell all of these assets for cash if $p_2 < 1+p_1$ and to sell some of these assets for cash if $p_2 = 1+p_1$.

The number of illiquid agents demanding cash is equal to the number of illiquid agents at date 0 (α), who did not receive a liquidity shock at date 1 (a fraction $1-\theta_1$), and who received a liquidity shock at date 2 (a fraction θ_2). Thus, the maximum demand for cash from illiquid agents is

$$\alpha(1-\theta_1)\theta_2.$$

Each if these agents has one unit of the asset. It is optimal for them to sell all of their assets for cash if $p_2 < 1$ and optimal for them to sell some of their assets if $p_2 = 1$. The demand function is illustrated in Figure 5B.

In Panel C of Figure 5 we illustrate the different configurations of the demand and supply curves that may arise for different values of the liquidity shock θ_2. It is clear from Panel C that, except for a set of states of probability zero, the intersection of the supply and demand curves will correspond to one of three regimes. The regime in Panel C(i) occurs when the supply of cash is greater than the maximum demand for cash from illiquid agents and buyers. In this regime, some hoarders will not be able to exchange cash for the asset, so they must be indifferent between holding and selling cash. This will occur only if the market clearing price is $p_2 = R^{-1}$. The regime in Panel C(ii) occurs when the supply of cash is sufficient to meet the needs of the buyers and some, but not all, illiquid agents. Then the market will clear if and only if the price is $p_2 = 1$. Finally, the regime in Panel C(iii) occurs when the supply of cash is insufficient to meet even the needs of all the buyers. The market will clear if and only if the price is $p_2 = 1+p_1$.

We can characterize the three different regimes at date 2 in terms of the critical values of θ_2 that divide them. Consider first the regime in Panel C(iii), which occurs if and only if

$$(1-\alpha)(1-\theta_1)(1-\lambda)(1-\theta_2) < (1-\alpha)(1-\theta_1)\lambda\theta_2$$

or

$$(1-\lambda)(1-\theta_2) < \lambda\theta_2$$

This inequality is equivalent to $\theta_2 > \theta_2^{**}$, where θ_2^{**} is implicitly defined by the condition that

$$(1-\lambda)(1-\theta_2^{**}) = \lambda\theta_2^{**}$$

or $\theta_2^{**} = 1-\lambda$.

Next consider the regime in Panel C(ii), which corresponds to

$$(1-\alpha)(1-\theta_1)\lambda\theta_2 < (1-\alpha)(1-\theta_1)(1-\lambda)(1-\theta_2) < (1-\alpha)(1-\theta_1)\lambda\theta_2 + \alpha(1-\theta_1)\theta_2,$$

or

$$(1-\alpha)\lambda\theta_2 < (1-\alpha)(1-\lambda)(1-\theta_2) < (1-\alpha)\lambda\theta_2 + \alpha\theta_2.$$

These inequalities are equivalent to $\theta_2^* < \theta_2 < \theta_2^{**}$, where θ_2^* is defined by

$$(1-\alpha)(1-\lambda)(1-\theta_2^*) = (1-\alpha)\lambda\theta_2^* + \alpha\theta_2^*$$

or $\theta_2^* = (1-\alpha)(1-\lambda)$.

Then it is easy to see that the regime in Panel C(i) occurs if and only if $\theta_2 < \theta_2^*$.

We summarize the preceding discussion in the following proposition.

Proposition 2 *The market-clearing price at date* 2 *is denoted by* $p_2(\theta_1, \theta_2)$ *and defined by*

$$p_2(\theta_1,\theta_2) = \begin{cases} R^{-1} & \text{for } 0 \le \theta_2 < \theta_2^*; \\ 1 & \text{for } \theta_2^* < \theta_2 < \theta_2^{**}; \\ 1 + p_1(\theta_1) & \text{for } \theta_2^{**} < \theta_2 \le 1; \end{cases}$$

where $\theta_2^* = (1-\alpha)(1-\lambda)$ *and* $\theta_2^{**} = 1-\lambda$.

3.3.2 Market clearing at date 1

The analysis of market clearing at date 1 is a bit more complicated, because agents' decisions depend on expectations about date 2. The first step is to show that, in equilibrium, there will always be some agents who buy assets and some who hoard cash at date 1. This requires that the agents with spare cash are indifferent between buying and hoarding. We can show that it is optimal to hoard if and only if $p_1 \le E[p_2]$ and, conversely, it is optimal to buy if and only if $p_1 \ge E[p_2]$. Thus, indifference is equivalent to $p_1 = E[p_2]$. Now consider what will happen if $\lambda = 0$. The excess demand for cash at date 1 implies that $p_1 = 1$, but at date 2 the price p_2 must be less than or equal to one (since there are no buyers) and will sometimes be less than one (when θ_2 is sufficiently small). Then $E[p_2] < 1 = p_1$ contradicting the optimality of hoarding. Conversely, if $\lambda = 1$, the price at date 2 must satisfy $p_2 = 1 + p_1$ because there will be excess demand for cash with probability one, but this violates the optimality condition for buying. Hence, we get the following proposition.

Proposition 3 *For every value of* θ_1,

$$0 < \lambda(\theta_1) < 1$$

in equilibrium at date 1. *Thus, agents holding unneeded cash at date* 1 *are indifferent between hoarding cash and buying the asset in equilibrium, which holds if and only if*

$$p_1(\theta_1) = E[p_2(\theta_1,\theta_2)|\theta_1].$$

Proof. See Section 6 ■

From Proposition 3, we know what $p_1 = E[p_2]$ and from Proposition 2 we know the distribution of p_2 as a function of λ, which allows us to calculate the value of $E[p_2]$ as a function of λ. Let $\tilde{p}(\lambda)$ denote this value for each value of λ. There is a unique value of λ, call it $\bar{\lambda} \in (0,1)$, such that $\tilde{p}\left(\bar{\lambda}\right) = 1$ and $\tilde{p}(\lambda) < 1$ if and only if $\lambda < \bar{\lambda}$. If $p_1 < 1$, then the market-clearing condition tells us that

$$(1-\alpha)(1-\theta_1)\lambda = \alpha\theta_1$$

or

$$\lambda = \frac{\alpha\theta_1}{(1-\alpha)(1-\theta_1)}.$$

On the other hand, $\tilde{p}(\lambda) = 1$ implies that $\lambda = \bar{\lambda}$. Putting these facts together, we can characterize the equilibrium values of p_1 and λ in the following result.

Proposition 4 *The market clears at date* 1 *if and only if the equilibrium values of* λ *and* p_1 *are given by*

$$\lambda(\theta_1) = \min\left\{\frac{\alpha\theta_1}{(1-\alpha)(1-\theta_1)}, \bar{\lambda}\right\}$$

and

$$p_1(\theta_1) = \min\left\{\tilde{p}\left(\frac{\alpha\theta_1}{(1-\alpha)(1-\theta_1)}\right), 1\right\},$$

for every value of $0 \le \theta_1 \le 1$, *where*

$$\tilde{p}(\lambda) = \frac{1 - F_2((1-\alpha)(1-\lambda))(1-R^{-1})}{F_2(1-\lambda)}$$

for every value of $0 \le \lambda \le 1$ *and* $\bar{\lambda}$ *is the unique value of* $\lambda \in (0,1)$ *satisfying* $\tilde{p}(\lambda) = 1$.

3.3.3 Market clearing at date 0

Just as we showed that buyers and the hoarders have the same expected return at date 1, we can show that $0 < \alpha < 1$ in equilibrium at date 0 and that agents must therefore be indifferent between acquiring liquidity and not acquiring it. The calculation of the equilibrium payoffs from each course of action is complicated, but the equilibrium can be simplified considerably as the following result shows.

Proposition 5 *In equilibrium,* $0 < \alpha < 1$, *which implies that agents will be indifferent at date* 0 *between acquiring liquidity and not acquiring it. Agents are indifferent if and only if*

$$\int_0^1 p_1 \left\{1 + (1-\theta_1)(1-F_2(\theta_2^{**}))E\left[\theta_2 \left|\theta_2 > \theta_2^{**}\right.\right]\right\} \; f_1(\theta_1)d\theta_1 = \frac{c}{R}.$$

Proof. See Section 6. ■

3.4 Equilibrium

An equilibrium is described by the endogenous variables α, $\lambda(\theta_1)$, $p_1(\theta_1)$, and $p_2(\theta_1, \theta_2)$ satisfying the following conditions. Define $\tilde{p}(\lambda)$ by putting

$$\tilde{p}(\lambda) = \frac{1 - F_2((1-\alpha)(1-\lambda))(1-R^{-1})}{F_2(1-\lambda)}$$

for every $0 \leq \lambda \leq 1$ and let $\bar{\lambda}$ be the unique value of $0 < \lambda < 0$ satisfying $\tilde{p}(\lambda) = 1$. Then the equilibrium functions $p_1(\theta_1)$ and $\lambda(\theta_1)$ satisfy

$$\lambda(\theta_1) = \min\left\{\frac{\alpha\theta_1}{(1-\alpha)(1-\theta_1)}, \bar{\lambda}\right\}$$

and

$$p_1(\theta_1) = \min\left\{\tilde{p}\left(\frac{\alpha\theta_1}{(1-\alpha)(1-\theta_1)}\right), 1\right\},$$

for every value of $0 \leq \theta_1 \leq 1$.

The equilibrium price function $p_2(\theta_2)$ must satisfy

$$p_2(\theta_1, \theta_2) = \begin{cases} R^{-1} & \text{for } 0 \leq \theta_2 < \theta_2^*(\theta_1), \\ 1 & \text{for } \theta_2^*(\theta_1) < \theta_2 < \theta_2^{**}, \\ 1 + p_1(\theta_1) & \text{for } \theta_2^{**} < \theta_2 \leq 1, \end{cases}$$

where

$$\theta_2^*(\theta_1) = (1-\alpha)(1-\lambda(\theta_1)) \text{ and } \theta_2^{**} = 1 - \lambda(\theta_1).$$

Finally, at date 0, market-clearing requires indifference between acquiring and not acquiring liquidity:

$$\int_0^1 p_1\left\{1 - (1-\theta_1)(1-F_2(\theta_2^{**}))E\left[\theta_2 \,|\theta_2 > \theta_2^{**}\right]\right\} f_1(\theta_1) d\theta_1 = \frac{c}{R}.$$

3.5 Markets for liquidity insurance

We consider a market formed at date 0 in which some agents enter into a contract to acquire liquidity and supply it under certain conditions and other agents simultaneously enter into a contract to supply the asset under certain conditions. The suppliers of liquidity are required to report their type, that is, whether or not they have received a liquidity shock at date 1 and date 2. In the event that they have not reported a shock, they may be required to supply one unit of liquidity, if they have not already done so, in exchange for a specified amount of the asset. The demanders of liquidity similarly are required to report their type, that is, whether or not they have received a liquidity shock at date 1 and date 2. In the event that they have reported a shock, they may be supplied with one unit of cash, if they have not already received it, in exchange for a specified amount of the asset. We let $\hat{p}_1(\theta_1)$ denote

the price of cash at date 1 in state θ_1 and let $\hat{p}_2(\theta_1, \theta_2)$ denote the price of cash at date 2 in state (θ_1, θ_2). Suppose that there exists an equilibrium $\{\alpha, \lambda(\theta_1), p_1(\theta_1), p_2(\theta_1, \theta_2)\}$ and consider the effect of opening a market for liquidity at date 0. The market must satisfy an incentive compatibility constraint to ensure that agents report their types truthfully. At date 1 in state θ_1, one unit of cash can be traded for $p_1(\theta_1)$ units of cash on the spot market. If $p_1(\theta_1) > \hat{p}_1(\theta_1)$, an agent with cash who has not received a liquidity shock is better off reporting a liquidity shock since he could always sell his unit of cash on the spot market for the higher price. Likewise, if $p_1(\theta_1) < \hat{p}_1(\theta_1)$, an agent without cash who has received a liquidity shock would be better off reporting no liquidity shock since he can always buy cash at the lower price. Thus, incentive compatibility at date 1 requires

$$\hat{p}_1(\theta_1) = p_1(\theta_1),$$

for every value of θ_1. A similar argument implies that

$$\hat{p}_2(\theta_1, \theta_2) = p_2(\theta_1, \theta_2),$$

for every value of (θ_1, θ_2). Since the prices are the same, it is clear that the market mechanism cannot improve on the allocation provided by the spot markets.

4 Constrained inefficiency of equilibrium

In this section, we analyze whether agents' private choices of lending at $t = 1$ and acquiring liquidity at $t = 0$ differ from the socially optimal levels of lending at $t = 1$ and liquidity acquisition at $t = 0$, respectively.

First, we try to find the socially optimal level of lending at $t = 1$, denoted by λ^{soc}, that maximizes the expected output at $t = 1$ generated using the assets and cash assuming that the market for asset sales at $t = 2$ will function as in Section 3.3.1 where we characterize the equilibrium.

At $t = 1$, the liquidity shock θ_1 is realized and we can find the expected output for each realization of θ_1. Then we can find λ^{soc} and compare it with the privately optimal level of lending given in Proposition (4).

In calculating the expected output at $t = 1$, we need to consider three different regions for θ_2:

(i) For $\theta_2 < \theta_2^*$, there is enough liquidity at $t = 2$ for all agents that got hit by the liquidity shock at $t = 2$. Hence, no asset needs to be liquidated at $t = 2$.

ii) For $\theta_2^* < \theta_2 < \theta_2^{**}$, there is enough liquidity for all buyers that got hit by the liquidity shock at $t = 2$ but not enough for all illiquid agents that got hit at $t = 2$. Hence, some of the assets held by illiquid agents that got hit at $t = 2$ need to be liquidated prematurely.

iii) For $\theta_2 > \theta_2^{**}$, there is not enough liquidity even for all buyers that got hit by the liquidity shock at $t = 2$. Hence, some of the assets held by buyers that got hit at $t = 2$ and all the assets held by illiquid agents that got hit at $t = 2$ need to be liquidated prematurely.

The following proposition characterizes the constrained efficient level of lending at $t = 1$ and compares it with the equilibrium level of lending at $t = 1$ characterized in Proposition (4).

Proposition 6 *We can characterize the socially optimal level of lending λ^{soc} as follows:*

$$\lambda^{soc}(\theta_1) = \min\left\{\frac{\alpha\theta_1}{(1-\alpha)(1-\theta_1)}, \widetilde{\lambda}\right\},$$

where $\widetilde{\lambda}$ is determined implicitly by the condition

$$F_2\left((1-\alpha)\left(1-\widetilde{\lambda}\right)\right) + F_2\left(1-\widetilde{\lambda}\right) = 1.$$

Furthermore, we obtain $\widetilde{\lambda} > \overline{\lambda}$.

The constrained efficient level of lending has the same structure as the equilibrium level of lending. In particular, as in the equilibrium, the constrained efficient lending requires that the liquidity need of all illiquid agents that got hit by the shock at $t = 1$ be satisfied up to a threshold, denoted by $\widetilde{\theta}_1$, which is higher than the threshold proportion in equilibrium, denoted by $\overline{\theta}_1$, but beyond that point, only a fraction $\widetilde{\lambda}$ of the liquid agents that did not get hit by the shock at $t = 1$ lend to illiquid agents. Hence, in equilibrium there is inefficiently low level of lending at $t = 1$ for $\theta_1 > \overline{\theta}_1$.

Next, we try to show that the private choice of agents to acquire liquidity at $t = 0$ does not correspond to the level of liquidity that maximizes the expected output generated using the assets and cash. Hence, the equilibrium level of liquidity at $t = 0$ is not constrained efficient. To achieve that we calculate the expected output generated by the asset and cash as we did in the analysis of constrained efficiency at $t = 1$. Then we show that at the equilibrium level, the expected output is decreasing in α so that, at the equilibrium, by decreasing the proportion of illiquid agents, we can increase expected output. This gives us the following formal proposition.

Proposition 7 *The equilibrium level of α characterized in Proposition (5) is constrained inefficient.*

Below, we provide a numerical example that illustrates the wedge between the equilibrium and constrained efficient levels of λ and α.

Example 8 *In the following numerical example, we use the parameter values $R = 3$, $c = 2$ and assume that θ_1 and θ_2 are iid and $U[0, 1]$. We find that in equilibrium a fraction $\alpha = 0.14$ of agents choose to become illiquid at $t = 0$, whereas the constrained efficient level of α is 0.067. We also find that in the equilibrium $\overline{\lambda} = 0.364$ and $\overline{\theta}_1 = 0.691$, whereas constrained efficiency requires $\widetilde{\lambda} = 0.462$ and $\widetilde{\theta}_1 = 0.740$.*

We also provide simulation results for the equilibrium and constrained efficient levels of λ as a function of θ_1 (Figure 6a) and α as a function of R (Figure 6b) and c (Figure 6c). In all the simulations, we use $R = 3$, $c = 2$ and the uniform distribution over $[0, 1]$ for the liquidity shocks θ_1 and θ_2 unless we vary the parameter.

—Figure 6 about here—

4.1 Comparative statics

In this section, we provide some comparative statics analysis for lending at $t = 1$. First, we focus on the case when higher liquidity shocks are more likely at $t = 2$. To capture the likelihood of liquidity shocks at $t = 2$, we use two different probability distributions, f_2 and g_2, for θ_2, where g_2 first-order stochastically dominates f_2. Hence, higher proportions of the liquidity shock at $t = 2$ are more likely under the probability distribution g_2.

From the equilibrium condition we have

$$F_2\left(1-\overline{\lambda}_f\right)+F_2\left(\left(1-\alpha\right)\left(1-\overline{\lambda}_f\right)\right)\left(1-\frac{1}{R}\right)=1.$$

Since g_2 first-order stochastically dominates f_2, we obtain

$$G_2\left(1-\overline{\lambda}_f\right)+G_2\left(\left(1-\alpha\right)\left(1-\overline{\lambda}_f\right)\right)\left(1-\frac{1}{R}\right)<1.$$

We know that the LHS of the inequality is decreasing in λ since

$$-g_2\left(1-\lambda\right)-\left(1-\alpha\right)\ g_2\left(\left(1-\alpha\right)\left(1-\lambda\right)\right)\left(1-\frac{1}{R}\right)<0.$$

Hence, we obtain $\overline{\lambda}_f > \overline{\lambda}_g$. We can use a similar argument to show that $\widetilde{\lambda}_f > \widetilde{\lambda}_g$.

We have the following formal proposition.

Proposition 9 *Let f_2 and g_2 be two probability distributions over θ_2, where g_2 first-order stochastically dominates f_2. Let $\overline{\lambda}_f, \widetilde{\lambda}_f$ and $\overline{\lambda}_g, \widetilde{\lambda}_g$ be characterized as in Propositions (4) and (6) under probability distributions f_2 and g_2, respectively. We obtain $\overline{\lambda}_f > \overline{\lambda}_g$ and $\widetilde{\lambda}_f > \widetilde{\lambda}_g$.*

Example 10 *Let θ_2 be distributed uniformly according to the probability distribution $f_2^b = \frac{1}{b-a}$ over the interval $[a, b]$, with $0 \leqslant a < b \leqslant 1$. Note that for a fixed a, for $b' > b$, $f_2^{b'}$ first-order stochastically dominates f_2^b. From the analysis of constrained efficiency, we have*

$$\left(\left(1-\widetilde{\lambda}\right)-a\right)\frac{1}{b-a}+\left(\left(1-\alpha\right)\left(1-\widetilde{\lambda}\right)-a\right)\frac{1}{b-a}=1,$$

so that

$$\widetilde{\lambda}=1-\frac{b-a}{2-\alpha}.$$

Note that $\widetilde{\lambda}_f$ is decreasing in b. From the equilibrium condition, we obtain

$$\left(1-\overline{\lambda}\right)-a+\left((1-\alpha)\left(1-\overline{\lambda}\right)-a\right)\left(1-\frac{1}{R}\right)=b-a,$$

so that

$$\overline{\lambda}=1-\frac{bR+a\,(R-1)}{R+(1-\alpha)\,(R-1)},$$

Note that $\overline{\lambda}$ is decreasing in b. Furthermore, we have

$$\frac{d\left(\widetilde{\lambda}-\overline{\lambda}\right)}{db}=\frac{R}{R+(1-\alpha)\,(R-1)}-\frac{1}{2-\alpha}=\frac{1-\alpha}{(2-\alpha)\,(R+(1-\alpha)\,(R-1))}>0.$$

Hence, as higher shocks become more likely, in the first-order stochastic sense, the wedge between the constrained efficient lending and its equilibrium level increases.

Let f_2 be a symmetric probability distribution over θ_2 with the support $[a,b]$, with $0 \leqslant a < b \leqslant 1$. We can show that the constrained efficiency requires that $\widetilde{\lambda}=1-\frac{b+a}{2-\alpha}$. We know that

$$1-F_2\left(1-\widetilde{\lambda}\right)=F_2\left((1-\alpha)\left(1-\widetilde{\lambda}\right)\right).$$

Since, f_2 is a symmetric probability distribution, the condition above requires that

$$b-\left(1-\widetilde{\lambda}_f\right)=(1-\alpha)\left(1-\widetilde{\lambda}_f\right)-a,$$

so that $\widetilde{\lambda}=1-\frac{b+a}{2-\alpha}$.

Example 11 *Let θ_2 be distributed uniformly according to the probability distribution $f_2=\frac{1}{b-a}$ over the interval $[a,b]$. Furthermore let $a+b=1$ so that the distribution is symmetric around $\frac{1}{2}$. Note that for $b'>b$, $f_2^{b'}$ is a mean-preserving spread of f_2^b. From the equilibrium condition, we obtain*

$$\overline{\lambda}=1-\frac{bR+a\,(R-1)}{R+(1-\alpha)\,(R-1)}=1-\frac{R-1+b}{R+(1-\alpha)\,(R-1)},$$

Note that $\overline{\lambda}$ is decreasing in b. Furthermore, in this case, we have $\widetilde{\lambda}=\frac{1-\alpha}{2-\alpha}$ so that the constrained efficient level of lending is independent of mean-preserving spreads. Hence, as uncertainty about future liquidity shocks increase, modelled by a probability distribution that is a mean-preserving spread, the wedge between the constrained efficient lending and its equilibrium level increases.

5 Lender of Last Resort

We now describe an equilibrium in which the Central Bank acts as the sole supplier of liquidity. Our approach is constructive. We assume that $\alpha = 1$ and that the Bank chooses as its policy the constrained efficient policy (m_0^*, m_1^*, m_2^*). We define an equilibrium with a LoLR along the lines of the laisser-faire equilibrium. At date 2, there are no buyers, so the demand for liquidity comes from the $(1-\theta_1)\,\theta_2$ agents who have received a liquidity shock at date 2. Since the supply of money is $\max\{m_0^* - \theta_1, 0\}$, the market clearing price $p_2(\theta_1, \theta_2)$ is defined by

$$p_2(\theta_1, \theta_2) = \begin{cases} 1 & \text{if } (1-\theta_1)\,\theta_2 > \max\{m_0^* - \theta_1, 0\} \\ R^{-1} & \text{if } (1-\theta_1)\,\theta_2 < \max\{m_0^* - \theta_1, 0\}. \end{cases}$$

Similarly, at date 1, the demand for liquidity comes from the θ_1 agents who receive a liquidity shock at date 1 and the supply is at most m_0^*. If $\theta_1 > m_0^*$ the market clearing price must be $p_1(\theta_1) = 1$, but when $\theta_1 < m_0^*$ the price may lie anywhere between $E[p_2(\theta_1, \theta_2) \mid \theta_1]$ and 1. Since the Central Bank can control the price we assume that it sets $p_1(\theta_1) = E[p_2(\theta_1, \theta_2) \mid \theta_1]$, so that the $1-\theta_1$ agents who did not receive a shock are indifferent between hoarding and buying. Then the market clearing price is

$$p_1(\theta_1) = \begin{cases} 1 & \text{if } \theta_1 > m_0^* \\ E[p_2(\theta_1, \theta_2) \mid \theta_1] & \text{if } \theta_1 < m_0^*. \end{cases}$$

Market clearing at date 0 requires that it is optimal for agents to choose $\alpha = 1$. If an agent chooses to remain illiquid at date 0, his payoff in state (θ_1, θ_2) is

$$\theta_1 R\,(1 - p_1(\theta_1)) + (1-\theta_1)\,\theta_2 R\,(1 - p_2(\theta_1, \theta_2)) + (1-\theta_1)(1-\theta_2)(R-1), \tag{10}$$

since with probability θ_1 he receives a liquidity shock at date 1 and gives up $p_1(\theta_1)$ units of the asset for cash (or defaults in the case $p_1(\theta_1) = 1$), with probability $(1-\theta_1)\,\theta_2$ he receives a liquidity shock at date 2 and gives up $p_2(\theta_1, \theta_1)$ units of the asset for cash (or defaults in the case $p_2(\theta_1, \theta_1) = 1$), and with probability $(1-\theta_1)(1-\theta_2)$ he receives no liquidity shock and retains one unit of the asset. By comparison, if he decides to become liquid at date 0, his payoff in state (θ_1, θ_2) is

$$R + (1-\theta_1)(1-\theta_2)(p_2(\theta_1, \theta_2)\,R - 1) - c, \tag{11}$$

since the agent can keep his asset for certainty and in the event that he does not receive a liquidity shock, his one unit of cash is worth $p_2(\theta_1, \theta_2)\,R$ at date 2. Note that we are here using the fact that hoarding is optimal at date 1. The expected value of (10) is

$$\begin{aligned} & E[\theta_1 R\,(1 - p_1(\theta_1)) + (1-\theta_1)\,\theta_2 R\,(1 - p_2(\theta_1, \theta_2)) + (1-\theta_1)(1-\theta_2)(R-1)] \\ & = E[\theta_1 R\,(1 - p_2(\theta_1, \theta_1)) + (1-\theta_1)\,\theta_2 R\,(1 - p_2(\theta_1, \theta_2)) + (1-\theta_1)(1-\theta_2)(R-1)] \\ & = E[R - (\theta_1 + (1-\theta_1)\,\theta_2)\,p_2(\theta_1, \theta_2)\,R - (1-\theta_1)(1-\theta_2)]. \end{aligned}$$

Comparing this with the expected value of the payoff (11),

$$E\left[R + (1-\theta_1)(1-\theta_2)(p_2(\theta_1,\theta_2)R - 1)\right] - c,$$

we see that not holding liquidity is optimal if and only if

$$E\left[(1-\theta_1)(1-\theta_2)p_2(\theta_1,\theta_2)R\right] - c \leq E\left[-(\theta_1 + (1-\theta_1)\theta_2)p_2(\theta_1,\theta_2)R\right]$$

or

$$E\left[p_2(\theta_1,\theta_2)R\right] \leq c.$$

From the planner's problem, we have the first-order condition

$$R + 1 - R\int_0^{m_0} F_2\left(\frac{m_0-\theta_1}{1-\theta_1}\right) f_1(\theta_1)\,d\theta_1 = c.$$

Since

$$\begin{aligned} E\left[p_2(\theta_1,\theta_2)\,|\theta_1\right] &= R^{-1}F_2\left(\frac{m_0^*-\theta_1}{1-\theta_1}\right) + \left(1 - F_2\left(\frac{m_0^*-\theta_1}{1-\theta_1}\right)\right) \\ &= 1 - \left(1-R^{-1}\right)F_2\left(\frac{m_0^*-\theta_1}{1-\theta_1}\right), \end{aligned}$$

for $\theta_1 < m_0$ and 1 otherwise,

$$\begin{aligned} E\left[p_2(\theta_1,\theta_2)\right] &= \int_0^{m_0^*}\left\{1 - \left(1-R^{-1}\right)F_2\left(\frac{m_0^*-\theta_1}{1-\theta_1}\right)\right\} f_1(\theta_1)\,d\theta_1 + 1 - F_1(m_0^*) \\ &= 1 - \left(1-R^{-1}\right)\int_0^{m_0^*} F_2\left(\frac{m_0^*-\theta_1}{1-\theta_1}\right) f_1(\theta_1)\,d\theta_1. \end{aligned}$$

Then

$$\begin{aligned} E\left[p_2(\theta_1,\theta_2)R\right] &= R - (R-1)\int_0^{m_0^*} F_2\left(\frac{m_0^*-\theta_1}{1-\theta_1}\right) f_1(\theta_1)\,d\theta_1 \\ &\leq R + 1 - R\int_0^{m_0^*} F_2\left(\frac{m_0^*-\theta_1}{1-\theta_1}\right) f_1(\theta_1)\,d\theta_1 \\ &= c, \end{aligned}$$

as required.

6 Appendix: Proofs

Proof of Proposition 3 The buyers end date 1 with $(1+p_1)$ units of the asset and no cash; the hoarders end the period with one unit of the asset and one unit of cash. Consider the buyers first. A fraction θ_2 of the buyers receive a liquidity shock and have a payoff $(1+p_1-p_2)R$; a fraction $(1-\theta_2)$ do not receive a cash shock and have a payoff $(1+p_1)R-1$. Thus, the buyers' payoff at date 1 is

$$\begin{aligned}&\int_0^1 \{\theta_2(1+p_1-p_2)R+(1-\theta_2)((1+p_1)R-1)\} f_2(\theta_2)\,d\theta_2\\ =\ &\int_0^1 \{(1+p_1-\theta_2 p_2)R-(1-\theta_2)\} f_2(\theta_2)\,d\theta_2,\end{aligned}$$

where p_2 is a function of θ_2 (given θ_1). Now consider the hoarders. A fraction θ_2 of the hoarders receive a liquidity shock and have a payoff R and a fraction $(1-\theta_2)$ do not receive a shock and have a payoff $(1+p_2)R-1$. Thus, the hoarders' payoff at date 1 is

$$\int_0^1 \{\theta_2 R+(1-\theta_2)((1+p_2)R-1)\} f_2(\theta_2)\,d\theta_2 = \int_0^1 \{(1+(1-\theta_2)p_2)R-(1-\theta_2)\} f_2(\theta_2)\,d\theta_2,$$

where p_2 is, again, a function of θ_2. It is optimal to buy if and only if the buyers' payoff is at least as great as the hoarders, that is,

$$\int_0^1 \{(1+p_1-\theta_2 p_2)R\} f_2(\theta_2)\,d\theta_2 \ge \int_0^1 \{(1+(1-\theta_2)p_2)R\} f_2(\theta_2)\,d\theta_2,$$

or

$$p_1 \ge \int_0^1 p_2 f_2(\theta_2)\,d\theta_2.$$

Similarly, it will be optimal to hoard if and only if

$$p_1 \le \int_0^1 p_2 f_2(\theta_2)\,d\theta_2.$$

Now we can prove that equilibrium requires $0<\lambda<1$. From Proposition 2, we know that the distribution of the random variable p_2 is

$$p_2 = \begin{cases} R^{-1} & \text{w. pr. } F_2((1-\alpha)(1-\lambda)) \\ 1 & \text{w. pr. } F_2(1-\lambda)-F_2((1-\alpha)(1-\lambda)) \\ 1+p_1 & \text{w. pr. } 1-F_2(1-\lambda) \end{cases}$$

and the expected value of p_2 is

$$\begin{aligned}E[p_2] &= F_2((1-\alpha)(1-\lambda))R^{-1}+(F_2(1-\lambda)-F_2((1-\alpha)(1-\lambda)))+\\ &\qquad (1-F_2(1-\lambda))(1+p_1)\\ &= F_2((1-\alpha)(1-\lambda))\left(R^{-1}-1\right)-F_2(1-\lambda)p_1+1+p_1.\end{aligned}$$

Suppose that $\lambda = 0$. Then market clearing at date 1 requires $p_1 = 1$ and

$$E[p_2] = F_2(1-\alpha) R^{-1} + 1 - F_2(1-\alpha) < 1.$$

But optimality of hoarding at date 1 requires $p_1 \leq E[p_2]$. This contradiction establishes that $\lambda > 0$.

Next, suppose that $\lambda = 1$. Then market clearing at date 2 requires that

$$E[p_2] = 1 + p_1.$$

But the optimality of buying at date 1 requires that $p_1 \geq E[p_2]$, which is clearly impossible. This contradiction establishes that $\lambda < 1$.

Since $0 < \lambda < 1$, the liquid agents must be indifferent between hoarding and buying. From the optimality conditions derived earlier, it is obvious that $p_1 = E[p_2]$.

Proof of Proposition 4 From Proposition 3, we know what

$$\begin{aligned} p_1 &= E[p_2] \\ &= F_2((1-\alpha)(1-\lambda))\left(R^{-1}-1\right) - F_2(1-\lambda)\, p_1 + 1 + p_1 \end{aligned}$$

which implies that

$$p_1 = \frac{1 - F_2((1-\alpha)(1-\lambda))(1-R^{-1})}{F_2(1-\lambda)}.$$

Using this equation, we can define a function $\tilde{p}(\lambda)$ by putting

$$\tilde{p}(\lambda) = \frac{1 - F_2((1-\alpha)(1-\lambda))(1-R^{-1})}{F_2(1-\lambda)}$$

for any $\lambda \in (0,1)$. The function $\tilde{p}(\lambda)$ is increasing in λ and varies from $1 - F_2((1-\alpha))(1-R^{-1})$ to ∞ as λ varies from 0 to 1. Then there exists a unique value $\bar{\lambda}$ such that $\tilde{p}\left(\bar{\lambda}\right) = 1$ and $\tilde{p}(\lambda) < 1$ if and only if $\lambda < \bar{\lambda}$.

If $\tilde{p}(\lambda) < 1$ then market clearing requires

$$(1-\alpha)(1-\theta_1)\lambda = \alpha\theta_1$$

or

$$\lambda = \frac{\alpha\theta_1}{(1-\alpha)(1-\theta_1)}.$$

Let $\bar{\theta}_1$ be the unique value of θ_1 that satisfies

$$\bar{\lambda} = \frac{\alpha\theta_1}{(1-\alpha)(1-\theta_1)}.$$

Since the right hand side is increasing in θ_1 and varies from 0 to ∞ as θ_1 varies from 0 to 1 there is a unique solution to this equation and it satisfies $0 < \bar{\theta}_1 < 1$.

We claim that the equilibrium value of λ, call it $\lambda(\theta_1)$, satisfies

$$\lambda(\theta_1) = \min\left\{\frac{\alpha\theta_1}{(1-\alpha)(1-\theta_1)}, \bar{\lambda}\right\}$$

for any θ_1. If $\theta_1 < \bar{\theta}_1$ then

$$(1-\alpha)(1-\theta_1)\bar{\lambda} > \alpha\theta_1$$

and market clearing requires $\lambda < \bar{\lambda}$. Then $p_1 = \tilde{p}(\lambda) < 1$ implies that all illiquid agents who receive a liquidity must obtain liquidity, that is,

$$\lambda = \frac{\alpha\theta_1}{(1-\alpha)(1-\theta_1)} < \bar{\lambda}.$$

If $\theta_1 \geq \bar{\theta}_1$, then

$$(1-\alpha)(1-\theta_1)\bar{\lambda} \leq \alpha\theta_1$$

and equilibrium requires $\lambda = \bar{\lambda}$. To see this, recall that $\lambda > \bar{\lambda}$ implies that $\tilde{p}(\lambda) > 1$, which is impossible, and that $\lambda < \bar{\lambda}$ implies that $(1-\alpha)(1-\theta_1)\lambda < \alpha\theta_1$ and $\tilde{p}(\lambda) < 1$, a contradiction. This completes the proof of our claim. Hence,

$$\begin{aligned} p_1(\theta_1) &= \tilde{p}\left(\min\left\{\frac{\alpha\theta_1}{(1-\alpha)(1-\theta_1)}, \bar{\lambda}\right\}\right) \\ &= \min\left\{\tilde{p}\left(\frac{\alpha\theta_1}{(1-\alpha)(1-\theta_1)}\right), 1\right\}. \end{aligned}$$

Proof of Proposition 5 We can calculate the expected return of an agent who chooses to acquire cash at $t = 0$ and chooses to become a hoarder at $t = 1$ as follows. With probability θ_1 he is hit by the liquidity shock at $t = 1$ and uses his cash for his own investment so that his return is R. With probability $(1-\theta_1)\theta_2$ he is not hit by the liquidity shock at $t = 1$ but gets hit at $t = 2$, in which case, his return is again R. And with probability $(1-\theta_1)(1-\theta_2)$ he is not hit by the liquidity shock and can use his spare liquidity to acquire p_2 units of the asset at $t = 2$ and his return is $(1+p_2)R - 1$. Hence, the expected return of a liquid agent that chooses to become a hoarder at $t = 1$ can be written as:

$$
\begin{aligned}
& \int_0^1\int_0^1 \{\theta_1 R + (1-\theta_1)\theta_2 R + (1-\theta_1)(1-\theta_2)\left((1+p_2)R-1\right)\}\; f_1(\theta_1)f_2(\theta_2)d\theta_1 d\theta_2 - c \\
= \;& R + \int_0^1\int_0^1 \{(1-\theta_1)(1-\theta_2)\left(p_2 R - 1\right)\}\; f_1(\theta_1)f_2(\theta_2)d\theta_1 d\theta_2 - c \\
= \;& R + \int_0^1 (1-\theta_1)R\underbrace{\left[\int_0^1 p_2 f_2(\theta_2)d\theta_2\right]}_{=p_1} f_1(\theta_1)d\theta_1 - \\
& \int_0^1\int_0^1 (1-\theta_1)\left(\theta_2 p_2 R + (1-\theta_2)\right)\; f_1(\theta_1)f_2(\theta_2)d\theta_1 d\theta_2 - c \\
= \;& R + \int_0^1 (1-\theta_1)p_1 R\; f_1(\theta_1)d\theta_1 - \int_0^1\int_0^1 (1-\theta_1)\left(\theta_2 p_2 R + (1-\theta_2)\right)\; f_1(\theta_1)f_2(\theta_2)d\theta_1 d\theta_2 - c.
\end{aligned}
$$

In other words, a hoarder is always guaranteed to have a return of R from his own investment but in case he is not hit by a liquidity shock, he can make an additional return from acquiring assets at $t = 2$.

We can calculate the expected return of illiquid agents as follows. With probability $(1-\theta_1)(1-\theta_2)$ he is not hit by the liquidity shock and his return is $R-1$. With probability θ_1 he is hit by the liquidity shock at $t = 1$, and sells a fraction of his assets for cash so that his return is $(1-p_1)R$. With probability $(1-\theta_1)\theta_2$ he is not hit by the liquidity shock at $t = 1$ but gets hit at $t = 2$, in which case his return is $\max\{0,(1-p_2)R\}$. Hence, the expected return of an illiquid agent can be written as:

$$
\begin{aligned}
& \int_0^1\int_0^1 \{\theta_1(1-p_1)R + (1-\theta_1)(1-\theta_2)\left(R-1\right) + (1-\theta_1)\theta_2 \max\{0,(1-p_2)R\}\}\, f_1(\theta_1)f_2(\theta_2)d\theta_1 d\theta_2 \\
= \;& R - \int_0^1 \theta_1 p_1 R f_1(\theta_1)d\theta_1 - \int_0^1\int_0^1 (1-\theta_1)\left(\theta_2 p_2 R + (1-\theta_2)\right)\; f_1(\theta_1)f_2(\theta_2)d\theta_1 d\theta_2 + \\
& \int_0^1\int_{\theta_2>\theta_2^{**}} (1-\theta_1)\theta_2 p_1 R\; f_2(\theta_2)f_1(\theta_1)d\theta_2 d\theta_1,
\end{aligned}
$$

since $1-p_2 = -p_1$ for $\theta_2 > \theta_2^{**}$.

In equilibrium, illiquid agents and hoarders (therefore buyers) should have the same expected return. Note that the first and the third terms in the expected returns for a hoarder and an illiquid agent is common. Hence, in equilibrium, we obtain

$$
\int_0^1 p_1 f_1(\theta_1)d\theta_1 - \frac{c}{R} = \int_0^1 (1-\theta_1)p_1 \underbrace{\left[\int_{\theta_2>\theta_2^{**}}^1 \theta_2 f_2(\theta_2)d\theta_2\right]}_{=(1-F_2(\theta_2^{**}))E[\theta_2|\theta_2>\theta_2^{**}]} f_1(\theta_1)d\theta_1,
$$

which can be written as

$$
\int_0^1 p_1\left\{1-(1-\theta_1)(1-F_2(\theta_2^{**}))E\left[\theta_2\,|\theta_2>\theta_2^{**}\,]\right)\right\}\; f_1(\theta_1)d\theta_1 = \frac{c}{R}.
$$

Proof of Proposition 6 Let $(1-\alpha)(1-\theta_1)\lambda$ be the measure of buyers at $t=1$, which in equilibrium equals the number of illiquid agents that manage to borrow. There are three cases to consider at $t=2$.

i) For $\theta_2 < \theta_2^*$, there is enough liquidity at $t=2$ for all agents that got hit by the liquidity shock at $t=2$. In that case, there are $1-\alpha\theta_1+(1-\alpha)(1-\theta_1)\lambda$ units of the asset since all the assets except for the ones held by illiquid agents hit by the shock at $t=1$ who could not get the needed liquidity (a measure of $\alpha\theta_1-(1-\alpha)(1-\theta_1)\lambda$) are pursued until $t=3$. In that case, the assets have a return of $(1-\alpha\theta_1+(1-\alpha)(1-\theta_1)\lambda)R$ at $t=3$. Furthermore, the creditors received $(1-\alpha)(1-\theta_1)\lambda+(1-\alpha)\theta_1$ and $\theta_2(1-\theta_1)$ at $t=1$ and $t=2$, respectively. And, there are $(1-\alpha)-[(1-\alpha)(1-\theta_1)\lambda+(1-\alpha)\theta_1+\theta_2(1-\theta_1)]$ units of cash left with the hoarders. Hence, the total output at $t=3$ is

$$(1-\alpha\theta_1+(1-\alpha)(1-\theta_1)\lambda)R+(1-\alpha).$$

ii) For $\theta_2^* < \theta_2 < \theta_2^{**}$, there is enough liquidity for all buyers that get hit by the liquidity shock at $t=2$ but not enough for all illiquid agents that get hit at $t=2$. Hence, some of the long assets held by illiquid agents that got hit at $t=2$ need to be liquidated prematurely, in addition to the $\alpha\theta_1-(1-\alpha)(1-\theta_1)\lambda$ units that got liquidated at $t=1$.

At $t=2$, the supply of cash comes from the hoarders that did not get hit by the liquidity shock at $t=2$, which has a measure of $(1-\alpha)(1-\theta_1)(1-\lambda)(1-\theta_2)$. The buyers who got hit by the liquidity shock at $t=2$ are the ones to receive cash first so that only $(1-\alpha)(1-\theta_1)(1-\lambda)(1-\theta_2)-(1-\alpha)(1-\theta_1)\lambda\theta_2$ units of cash is left for illiquid agents hit by the shock at $t=2$. Hence, the measure of assets that get liquidated prematurely at $t=2$ can be calculated as:

$$\begin{aligned}
& \alpha(1-\theta_1)\theta_2-(1-\alpha)(1-\theta_1)(1-\lambda)(1-\theta_2)+(1-\alpha)(1-\theta_1)\lambda\theta_2 \\
= & \alpha(1-\theta_1)\theta_2-(1-\alpha)(1-\theta_1)[(1-\lambda)(1-\theta_2)-\lambda\theta_2] \\
= & \alpha(1-\theta_1)\theta_2-(1-\alpha)(1-\theta_1)(1-\theta_2-\lambda)
\end{aligned}$$

Hence, the total measure of assets that got liquidated prematurely (both at $t=1$ and $t=2$) is:

$$\begin{aligned}
& \alpha\theta_1-(1-\alpha)(1-\theta_1)\lambda+\alpha(1-\theta_1)\theta_2-(1-\alpha)(1-\theta_1)(1-\theta_2-\lambda) \\
= & \alpha\theta_1+\alpha(1-\theta_1)\theta_2-(1-\alpha)(1-\theta_1)(1-\theta_2) \\
= & \alpha-(1-\theta_1)(1-\theta_2).
\end{aligned}$$

Hence, the total output at $t=3$ is

$$(1-\alpha+(1-\theta_1)(1-\theta_2))R+(1-\alpha).$$

iii) For $\theta_2 > \theta_2^{**}$, there is not enough liquidity even for all buyers that got hit by the liquidity shock at $t = 2$. Hence, some of the long assets held by illiquid agents that got hit at $t = 2$ and all the assets held by illiquid agents that got hit at $t = 2$ need to be liquidated prematurely, in addition to the $\alpha\theta_1 - (1-\alpha)(1-\theta_1)\lambda$ units that got liquidated at $t = 1$.

At $t = 2$, the supply of cash comes from the hoarders that did not get hit by the liquidity shock at $t = 2$, which has a measure of $(1-\alpha)(1-\theta_1)(1-\lambda)(1-\theta_2)$. Hence, only a measure $(1-\alpha)(1-\theta_1)(1-\lambda)(1-\theta_2)$ of the buyers who got hit by the liquidity shock at $t = 2$ can get liquidity at $t = 2$, whereas the rest, which has a measure $(1-\alpha)(1-\theta_1)\lambda\theta_2 - (1-\alpha)(1-\theta_1)(1-\lambda)(1-\theta_2)$, gets liquidated. Hence, the measure of assets that gets liquidated prematurely at $t = 2$ can be calculated as:

$$\begin{aligned}
& \alpha(1-\theta_1)\theta_2 + \{(1-\alpha)(1-\theta_1)\lambda\theta_2 - (1-\alpha)(1-\theta_1)(1-\lambda)(1-\theta_2)\}(1+p_1) \\
= \; & \alpha(1-\theta_1)\theta_2 - (1-\alpha)(1-\theta_1)(1-\theta_2-\lambda)(1+p_1)
\end{aligned}$$

Hence, the total measure of assets that got liquidated prematurely (both at $t = 1$ and $t = 2$) is:

$$\begin{aligned}
& \alpha\theta_1 - (1-\alpha)(1-\theta_1)\lambda + \alpha(1-\theta_1)\theta_2 - (1-\alpha)(1-\theta_1)(1-\theta_2-\lambda)(1+p_1) \\
= \; & \alpha_1 - (1-\theta_1)(1-\theta_2) - (1-\alpha)(1-\theta_1)(1-\theta_2-\lambda)\,p_1.
\end{aligned}$$

Hence, the total output at $t = 3$ is

$$(1 - [\alpha_1 - (1-\theta_1)(1-\theta_2) - (1-\alpha)(1-\theta_1)(1-\theta_2-\lambda)\,p_1])R + (1-\alpha)\,.$$

Using the output for the three different regions of θ_2 given above, we can calculate the total expected output as

$$\begin{aligned}
E(\Pi) \; = \; & R + (1-\alpha) - \int_0^{\theta_2^*} [\alpha\theta_1 - (1-\alpha)(1-\theta_1)\lambda]\, R f_2(\theta_2) d\theta_2 \\
& - \int_{\theta_2^*}^{\theta_2^{**}} [\alpha - (1-\theta_1)(1-\theta_2)]\, R f_2(\theta_2) d\theta_2 - \int_{\theta_2^{**}}^{1} [\alpha - (1-\theta_1)(1-\theta_2)]\, R f_2(\theta_2) d\theta_2 \\
& + \int_{\theta_2^{**}}^{1} (1-\alpha)(1-\theta_1)(1-\theta_2-\lambda)\, p_1 R f_2(\theta_2) d\theta_2,
\end{aligned}$$

which can be written as

$$\begin{aligned}
E(\Pi) \; = \; & (1-\alpha)(R+1) + (1-\theta_1)(1 - E[\theta_2])R \\
& + R\int_0^{\theta_2^*} [\alpha(1-\theta_1) + (1-\alpha)(1-\theta_1)\lambda - (1-\theta_1)(1-\theta_2)]\, f_2(\theta_2) d\theta_2 + \\
& + R\int_{\theta_2^{**}}^{1} (1-\alpha)(1-\theta_1)(1-\theta_2-\lambda)\, p_1 f_2(\theta_2) d\theta_2.
\end{aligned}$$

In what follows, we restrict attention to the case where $p_1 \equiv 1$, for reasons we explain later. Using Leibniz's rule, we can obtain the effect on total output of a small change in λ as

$$R\left[\alpha(1-\theta_1)+(1-\alpha)(1-\theta_1)\lambda-(1-\theta_1)(1-\theta_2^*)\right]f_2(\theta_2^*)\frac{d\theta_2^*}{d\lambda}+R(1-\alpha)(1-\theta_1)F_2(\theta_2^*)-$$
$$R(1-\alpha)(1-\theta_1)(1-\theta_2^{**}-\lambda)p_1f_2(\theta_2^{**})\frac{d\theta_2^{**}}{d\lambda}-R(1-\alpha)(1-\theta_1)(1-F_2(\theta_2^{**})).$$

Using $\theta_2^{**}=1-\lambda$ and $\theta_2^*=(1-\alpha)(1-\lambda)$, we can show that

$$\begin{aligned}&[\alpha(1-\theta_1)+(1-\alpha)(1-\theta_1)\lambda-(1-\theta_1)(1-\theta_2^*)]=\\&\qquad(1-\theta_1)[\alpha+(1-\alpha)\lambda-(1-(1-\alpha)(1-\lambda))]\\&\qquad\qquad(1-\theta_1)[\alpha-(1-(1-\alpha))]=0,\end{aligned}$$

and

$$(1-\alpha)(1-\theta_1)(1-\theta_2^{**}-\lambda)=(1-\alpha)(1-\theta_1)(1-(1-\lambda)-\lambda)=0.$$

Hence, the derivative reduces to

$$\begin{aligned}&R(1-\alpha)(1-\theta_1)F_2(\theta_2^*)-R(1-\alpha)(1-\theta_1)(1-F_2(\theta_2^{**}))\\=\;&R(1-\alpha)(1-\theta_1)\{F_2(\theta_2^*)-(1-F_2(\theta_2^{**}))\}\end{aligned}$$

and the sign of the derivative is determined by the sign of

$$F_2(\theta_2^*)-(1-F_2(\theta_2^{**}))=F_2((1-\alpha)(1-\lambda))-(1-F_2(1-\lambda)).$$

Now we have to consider two cases, depending on whether θ_1 is greater or less than $\bar{\theta}_1$.
Case 1: Suppose that $\theta_1 > \bar{\theta}_1$. Then $p_1 \equiv 1$ and the equilibrium condition is

$$F_2\left(1-\bar{\lambda}\right)=1-F_2\left((1-\alpha)\left(1-\bar{\lambda}\right)\right)\left(1-R^{-1}\right).$$

But $1-R^{-1}<1$ implies that

$$F_2\left(1-\bar{\lambda}\right)>1-F_2\left((1-\alpha)\left(1-\bar{\lambda}\right)\right),$$

so

$$F_2\left((1-\alpha)\left(1-\bar{\lambda}\right)\right)-\left(1-F_2\left(1-\bar{\lambda}\right)\right)>0$$

and an increase in λ increases total output.
Case 2: Now suppose that $\theta_1 < \bar{\theta}_1$ so that all liquidity needs are met at date 1. Then λ cannot be increased. If λ is decreased a small amount, there will be excess demand for

liquidity and the price will jump to $p_1 = 1$. The effect of a small change in λ will correspond to our earlier calculation with $p_1 = 1$. Also, for $\theta_1 < \bar{\theta}_1$,

$$\lambda = \frac{\alpha\theta_1}{(1-\alpha)(1-\theta_1)} < \bar{\lambda},$$

so

$$\frac{d}{d\lambda}\left\{F_2\left((1-\alpha)(1-\lambda)\right) - (1 - F_2(1-\lambda))\right\} = -(1-\alpha) f_2\left((1-\alpha)(1-\lambda)\right) - f_2\left((1-\lambda)\right) < 0.$$

implies that

$$F_2\left((1-\alpha)(1-\lambda)\right) - (1 - F_2(1-\lambda)) > F_2\left((1-\alpha)\left(1-\bar{\lambda}\right)\right) - \left(1 - F_2\left(1-\bar{\lambda}\right)\right) > 0.$$

So if we increase hoarding a little bit at date 1, this result tells us that it is better to reduce hoarding, i.e., increase λ. In the limit, when λ reaches its equilibrium value, there will be a jump in the allocation, as the drop in p_1 triggers a non-negligible transfer of assets back to the illiquid agents. This will have a further positive impact on output, since the illiquid agents cannot receive another liquidity shock and so it is better for them to hold more assets. Thus, it is not optimal to reduce λ and it is not feasible to increase λ.

From the analysis of the two cases above, we can characterize the socially optimal level of λ as follows:

$$\lambda = \begin{cases} \frac{\alpha\theta_1}{(1-\alpha)(1-\theta_1)} & \text{if } \theta_1 < \widetilde{\theta}_1 \\ \widetilde{\lambda} & \text{if } \theta_1 > \widetilde{\theta}_1 \end{cases},$$

where $\widetilde{\lambda}$ is determined implicitly by the FOC

$$F_2\left((1-\alpha)\left(1-\widetilde{\lambda}\right)\right) + F_2\left(1-\widetilde{\lambda}\right) = 1.$$

Furthermore, we have

$$\widetilde{\theta}_1 = \frac{(1-\alpha)\widetilde{\lambda}}{\alpha + (1-\alpha)\widetilde{\lambda}} > \frac{(1-\alpha)\overline{\lambda}}{\alpha + (1-\alpha)\overline{\lambda}} = \overline{\theta}_1.$$

Proof of Proposition 7 We have the expected output as a function of θ_1 as follows:

$$\begin{aligned} E(\Pi(\theta_1)) &= (1-\alpha)(R+1-c) + (1-\theta_1)(1-E(\theta_2)) + \\ & (1-\theta_1)R\int_0^{\theta_2^*} (\alpha + (1-\alpha)\lambda - (1-\theta_2)) f_2(\theta_2)d\theta_2 - \\ & (1-\theta_1)R\int_{\theta_2^{**}}^1 \left\{(1-\alpha)(\lambda - (1-\theta_2))p_1\right\} f_2(\theta_2)d\theta_2. \end{aligned}$$

Using the Leibniz's rule, we obtain:

$$
\begin{aligned}
\frac{dE(\Pi(\theta_1))}{d\alpha} &= -(R+1-c) \\
&+(1-\theta_1)R\int_0^{\theta_2^*}\left(1-\lambda+(1-\alpha)\frac{d\lambda}{d\alpha}\right)f_2(\theta_2)d\theta_2 \\
&+(1-\theta_1)R\left(\alpha+(1-\alpha)\lambda-(1-\theta_2^*)\right)f_2(\theta_2^*)\left[\frac{d\theta_2^*}{d\alpha}\right] \\
&-(1-\theta_1)R\int_{\theta_2^{**}}^{1}\left((1-\theta_2)-\lambda+(1-\alpha)\frac{d\lambda}{d\alpha}\right)p_1\ f_2(\theta_2)d\theta_2 \\
&-(1-\theta_1)R\int_{\theta_2^{**}}^{1}\left(\frac{dp_1}{d\alpha}\right)\left\{(1-\alpha)\left(\lambda-(1-\theta_2)\right)\right\}f_2(\theta_2)d\theta_2 \\
&-(1-\theta_1)R(1-\alpha)\left(\lambda-(1-\theta_2^{**})\right)p_1\left[\frac{d\theta_2^{**}}{d\alpha}\right].
\end{aligned}
$$

Using $\theta_2^* = (1-\alpha)(1-\lambda)$ and $\theta_2^{**} = 1-\lambda$, we obtain $\lambda - (1-\theta_2^{**}) = 0$, and

$$
\alpha+(1-\alpha)\lambda-(1-\theta_2^*) = \alpha+(1-\alpha)\lambda-(1-(1-\alpha)(1-\lambda)) = 0,
$$

so that the 3rd and the 6th expressions disappear, which gives us

$$
\begin{aligned}
\frac{dE(\Pi(\theta_1))}{d\alpha} &= -(R+1-c) \\
&+(1-\theta_1)R\int_0^{\theta_2^*}\left(1-\lambda+(1-\alpha)\frac{d\lambda}{d\alpha}\right)f_2(\theta_2)d\theta_2 \\
&-(1-\theta_1)R\int_{\theta_2^{**}}^{1}\left((1-\lambda+(1-\alpha)\frac{d\lambda}{d\alpha}\right)p_1\ f_2(\theta_2)d\theta_2+(1-\theta_1)R\int_{\theta_2^{**}}^{1}\theta_2 p_1\ f_2(\theta_2)d\theta_2 \\
&-(1-\theta_1)R\int_{\theta_2^{**}}^{1}\left(\frac{dp_1}{d\alpha}\right)\left\{(1-\alpha)\left(\lambda-(1-\theta_2)\right)\right\}f_2(\theta_2)d\theta_2.
\end{aligned}
$$

From the equilibrium condition at $t=0$, we have

$$
R\int_0^1 p_1\ f_1(\theta_1)d\theta_1 - c = R\int_0^1(1-\theta_1)p_1\left[\int_{\theta_2^{**}}^{1}\theta_2\ f_2(\theta_2)d\theta_2\right]f_1(\theta_1)d\theta_1.
$$

Even though the condition holds on average over θ_1, we can still plug this in the above

derivative to get:

$$\begin{aligned}\frac{dE(\Pi(\theta_1))}{d\alpha} &= -(R+1-c) \\ &+(1-\theta_1)R\int_0^{\theta_2^*}\left(1-\lambda+(1-\alpha)\frac{d\lambda}{d\alpha}\right)f_2(\theta_2)d\theta_2 \\ &-(1-\theta_1)R\int_{\theta_2^{**}}^1\left(1-\lambda+(1-\alpha)\frac{d\lambda}{d\alpha}\right)p_1\ f_2(\theta_2)d\theta_2+(p_1R-c) \\ &-(1-\theta_1)R\int_{\theta_2^{**}}^1\left(\frac{dp_1}{d\alpha}\right)\{(1-\alpha)(\lambda-(1-\theta_2))\}f_2(\theta_2)d\theta_2.\end{aligned}$$

Case 1: $\theta_1 < \overline{\theta}_1$

For $\theta_1 < \overline{\theta}_1$, we have $\lambda = \frac{\alpha\theta_1}{(1-\alpha)(1-\theta_1)}$ so that $\frac{\partial\lambda}{\partial\alpha} = \frac{\theta_1}{(1-\alpha)^2(1-\theta_1)}$. Hence,

$$1-\lambda+(1-\alpha)\frac{d\lambda}{d\alpha} = 1+\frac{\theta_1}{1-\theta_1} = \frac{1}{1-\theta_1}.$$

Using this, we can obtain:

$$\begin{aligned}\frac{dE(\Pi(\theta_1))}{d\alpha} &= -(R+1-c) \\ &+RF_2(\theta_2^*)-Rp_1(1-F_2(\theta_2^{**}))+p_1R-c \\ &-(1-\theta_1)R\int_{\theta_2^{**}}^1\left(\frac{dp_1}{d\alpha}\right)\{(1-\alpha)(\lambda-(1-\theta_2))\}f_2(\theta_2)d\theta_2.\end{aligned}$$

Note that, in this region,

$$p_1 = \frac{1-F_2(\theta_2^*)\left(1-\frac{1}{R}\right)}{F_2(\theta_2^{**})},$$

so that $Rp_1F_2(\theta_2^{**}) = R\left[1-F_2(\theta_2^*)\left(1-\frac{1}{R}\right)\right]$. Using this, we obtain

$$\frac{dE(\Pi(\theta_1))}{d\alpha} = -(1-F_2(\theta_2^*))-R(1-\theta_1)\int_{\theta_2^{**}}^1\left(\frac{\partial p_1}{\partial\alpha}\right)\{(1-\alpha)(\lambda-(1-\theta_2))\}f_2(\theta_2)d\theta_2.$$

In this case, we know that

$$p_1 = \frac{1-F_2\left(\frac{1-\alpha-\theta_1}{1-\theta_1}\right)\left(1-\frac{1}{R}\right)}{F_2\left(\frac{1-\alpha-\theta_1}{(1-\alpha)(1-\theta_1)}\right)}.$$

Hence, we obtain:

$$\begin{aligned}\frac{dp_1}{d\alpha} &= \frac{f_2\left(\frac{1-\alpha-\theta_1}{1-\theta_1}\right)\left(\frac{1}{\theta_1}\right)\left(1-\frac{1}{R}\right)}{F_2\left(\frac{1-\alpha-\theta_1}{(1-\alpha)(1-\theta_1)}\right)} \\ &\quad + \frac{\left[1-F_2\left(\frac{1-\alpha-\theta_1}{1-\theta_1}\right)\left(1-\frac{1}{R}\right)\right] f_2\left(\frac{1-\alpha-\theta_1}{(1-\alpha)(1-\theta_1)}\right)\left(\frac{\theta_1}{(1-\alpha)^2(1-\theta_1)}\right)}{\left[F_2\left(\frac{1-\alpha-\theta_1}{(1-\alpha)(1-\theta_1)}\right)\right]^2}.\end{aligned}$$

Note that $\frac{dp_1}{d\alpha} > 0$. Hence, we obtain $\frac{dE(\Pi(\theta_1))}{d\alpha} < 0$.

Case 2: $\theta_1 > \overline{\theta}_1$

For $\theta_1 > \overline{\theta}_1$, we have $p_1 = 1$. Using this, we obtain:

$$\begin{aligned}\frac{dE(\Pi(\theta_1))}{d\alpha} &= -(R+1-c) \\ &\quad +(1-\theta_1)R\int_0^{\theta_2^*}\left(1-\lambda+(1-\alpha)\frac{d\lambda}{d\alpha}\right)f_2(\theta_2)d\theta_2 \\ &\quad -(1-\theta_1)R\int_{\theta_2^{**}}^1\left((1-\lambda+(1-\alpha)\frac{d\lambda}{d\alpha}\right)f_2(\theta_2)d\theta_2 + \underbrace{(1-\theta_1)R\int_{\theta_2^{**}}^1 \theta_2 p_1\ f_2(\theta_2)d\theta_2}_{=p_1R-c \text{ in equilibrium at } t=1}.\end{aligned}$$

We can write the above expression as:

$$\frac{dE(\Pi(\theta_1))}{d\alpha} = -1 + R\left[F_2(\theta_2^*) - 1 + F_2(\theta_2^{**})\right]\left(1-\lambda+(1-\alpha)\frac{d\lambda}{d\alpha}\right).$$

Furthermore, from the equilibrium at $t=1$, we have

$$F_2(\theta_2^*) - 1 + F_2(\theta_2^{**}) = F_2(\theta_2^*)\left(\frac{1}{R}\right).$$

Hence, we get

$$\frac{\partial E(\Pi)}{\partial\alpha} = -1 + \left(1-\lambda+(1-\alpha)\frac{d\lambda}{d\alpha}\right)\left[F_2(\theta_2^*)\right].$$

Using the implicit function theorem, we get

$$-\frac{d\overline{\lambda}}{d\alpha}f_2(1-\overline{\lambda}) = f_2((1-\alpha)(1-\overline{\lambda}))\left(1-\frac{1}{R}\right)\left[(1-\alpha)\frac{d\overline{\lambda}}{d\alpha} + (1-\overline{\lambda})\right],$$

so that

$$\frac{d\overline{\lambda}}{d\alpha} = -\frac{f_2((1-\alpha)(1-\overline{\lambda}))\left(1-\frac{1}{R}\right)(1-\overline{\lambda})}{f_2(1-\overline{\lambda}) + f_2((1-\alpha)(1-\overline{\lambda}))\left(1-\frac{1}{R}\right)(1-\alpha)} < 0.$$

This gives us

$$\frac{\partial E(\Pi)}{\partial \alpha} = \underbrace{-1 + (1-\lambda)F_2(\theta_2^*)}_{<0} + \underbrace{\left((1-\alpha)\frac{d\lambda}{d\alpha}\right)}_{<0}[F_2(\theta_2^*)] < 0.$$

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Figure 1: Timeline

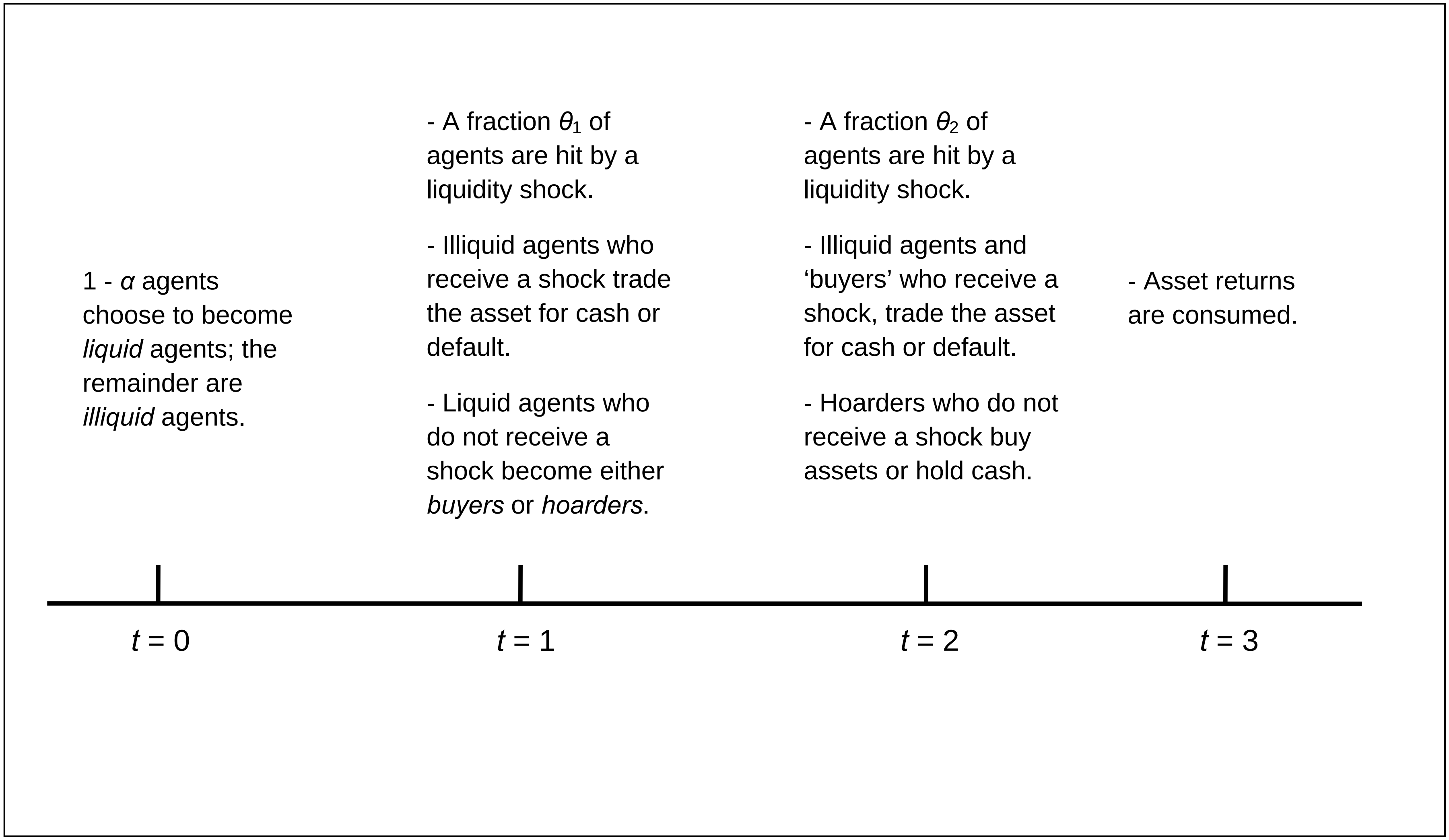

Figure 2: Allocations at dates 0 and 1

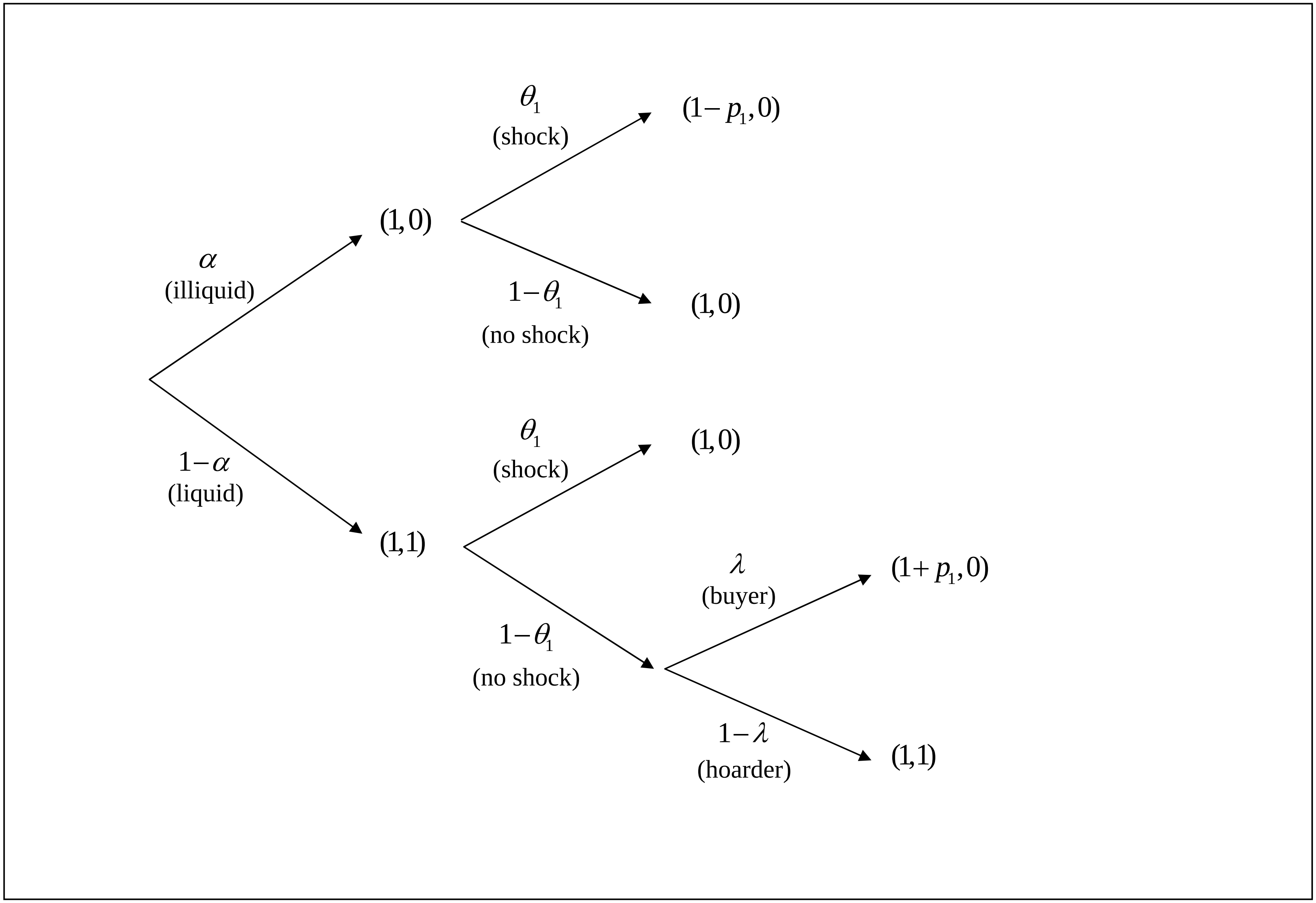

Figure 3a: Allocations at date 2

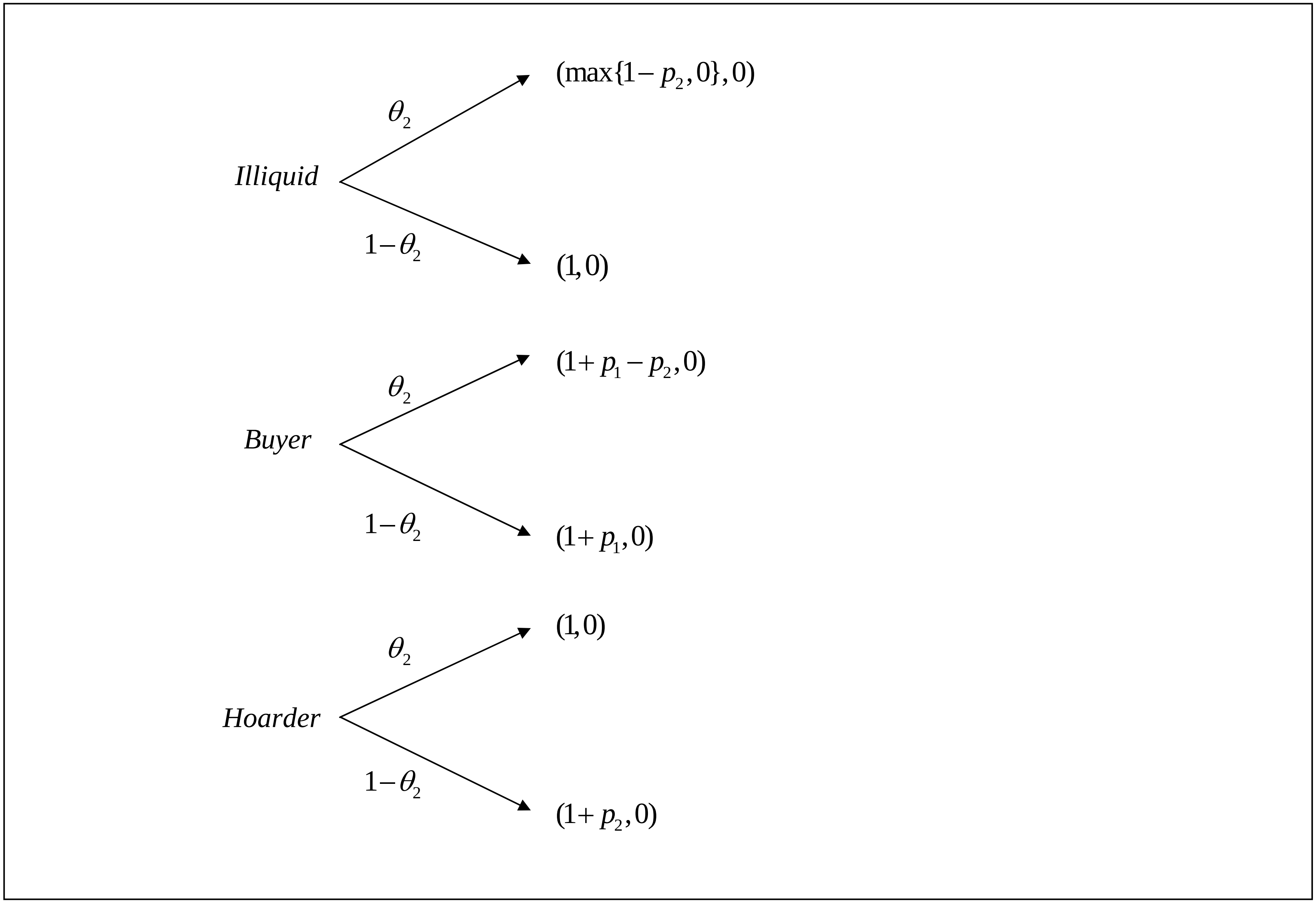

Figure 3b: Allocations at date 2

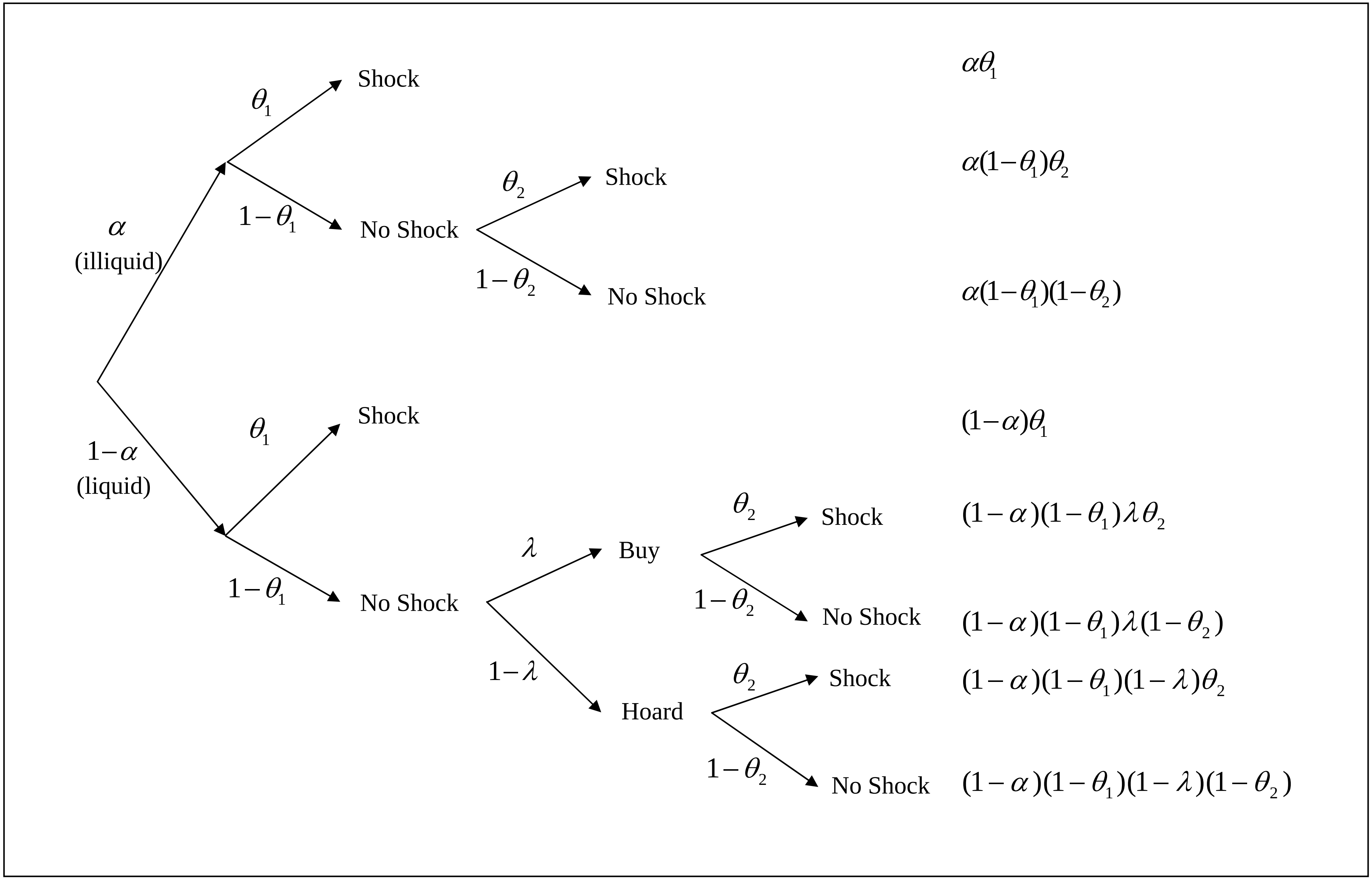

Figure 4: Terminal Payoffs

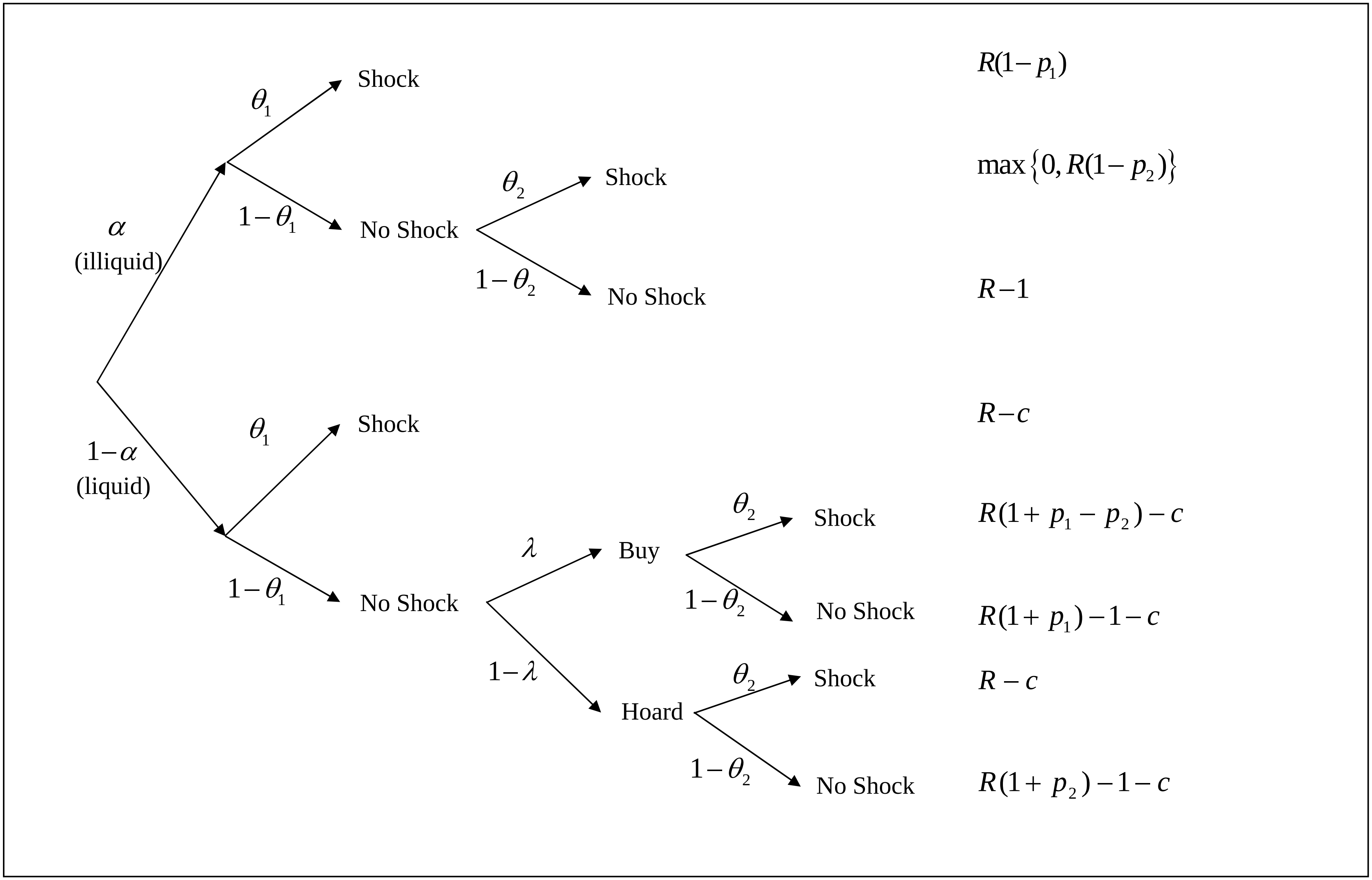

Figure 5C: Different demand and supply regimes

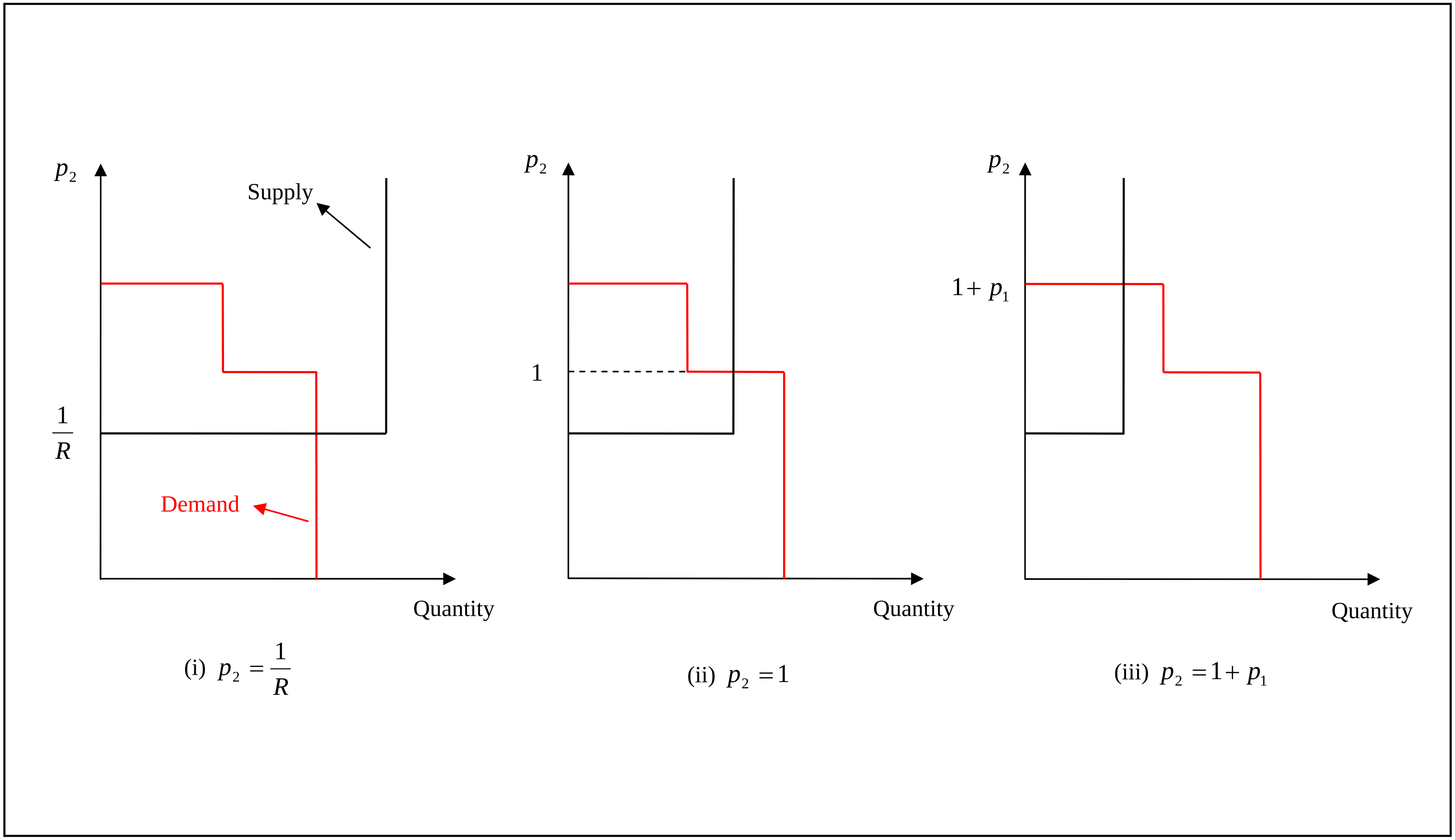

Figure 6a: Planner's choice m_0 as a function of c for R=3

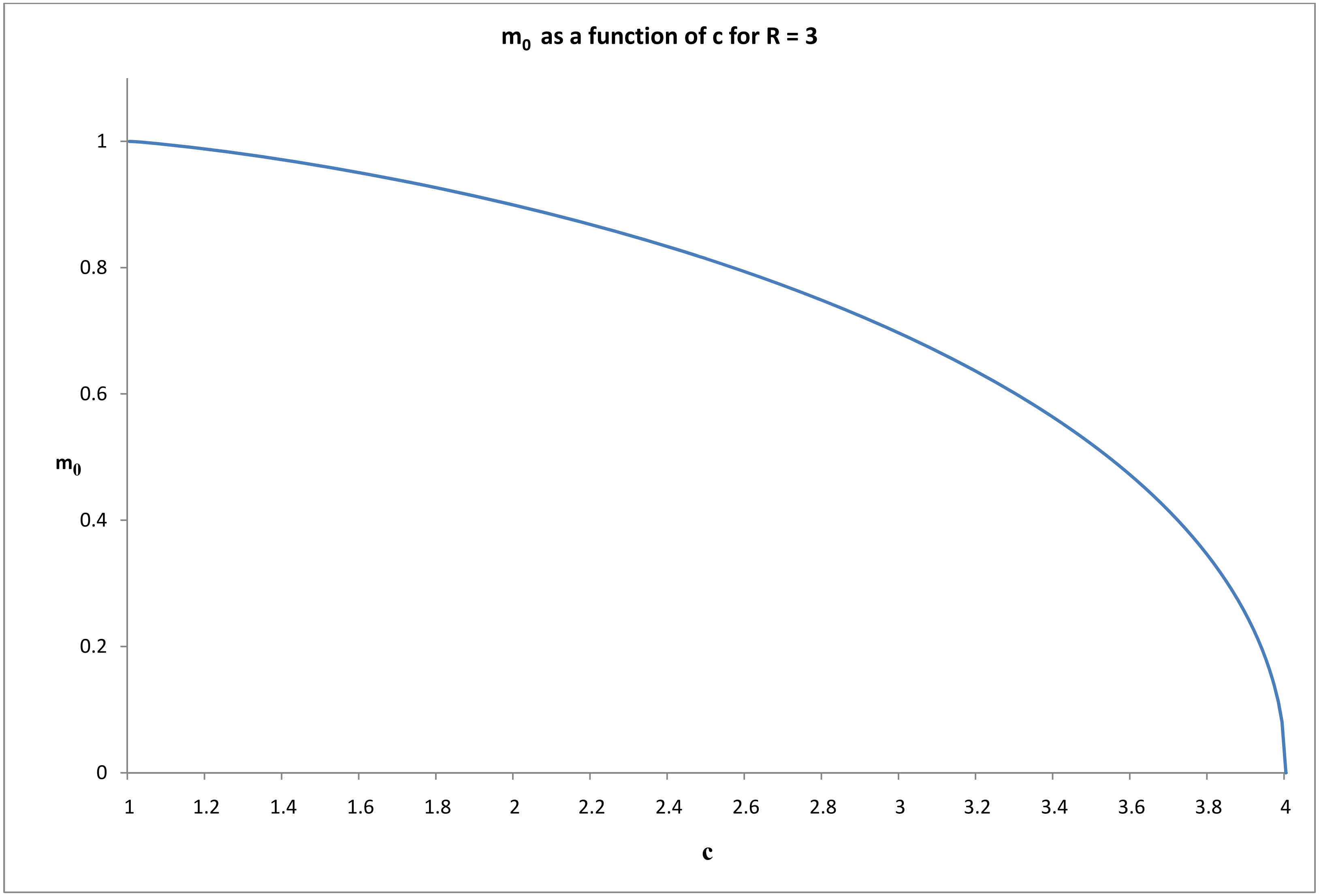

Figure 6b: Equilibrium and constrained efficient levels of λ as a function of θ_1 for R=3 and c=2

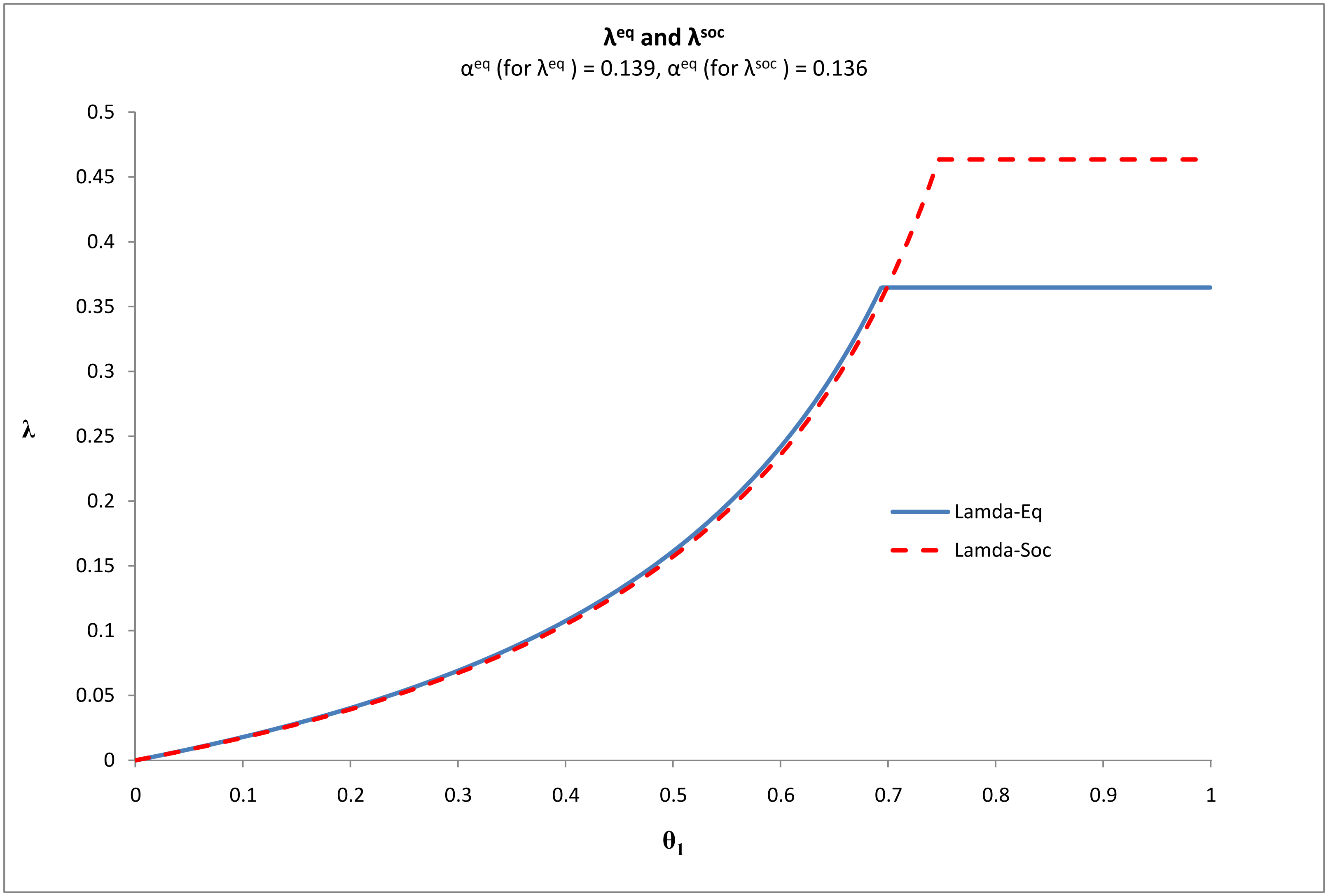

Figure 6c: Equilibrium and constrained efficient levels of α as a function of c for R=3

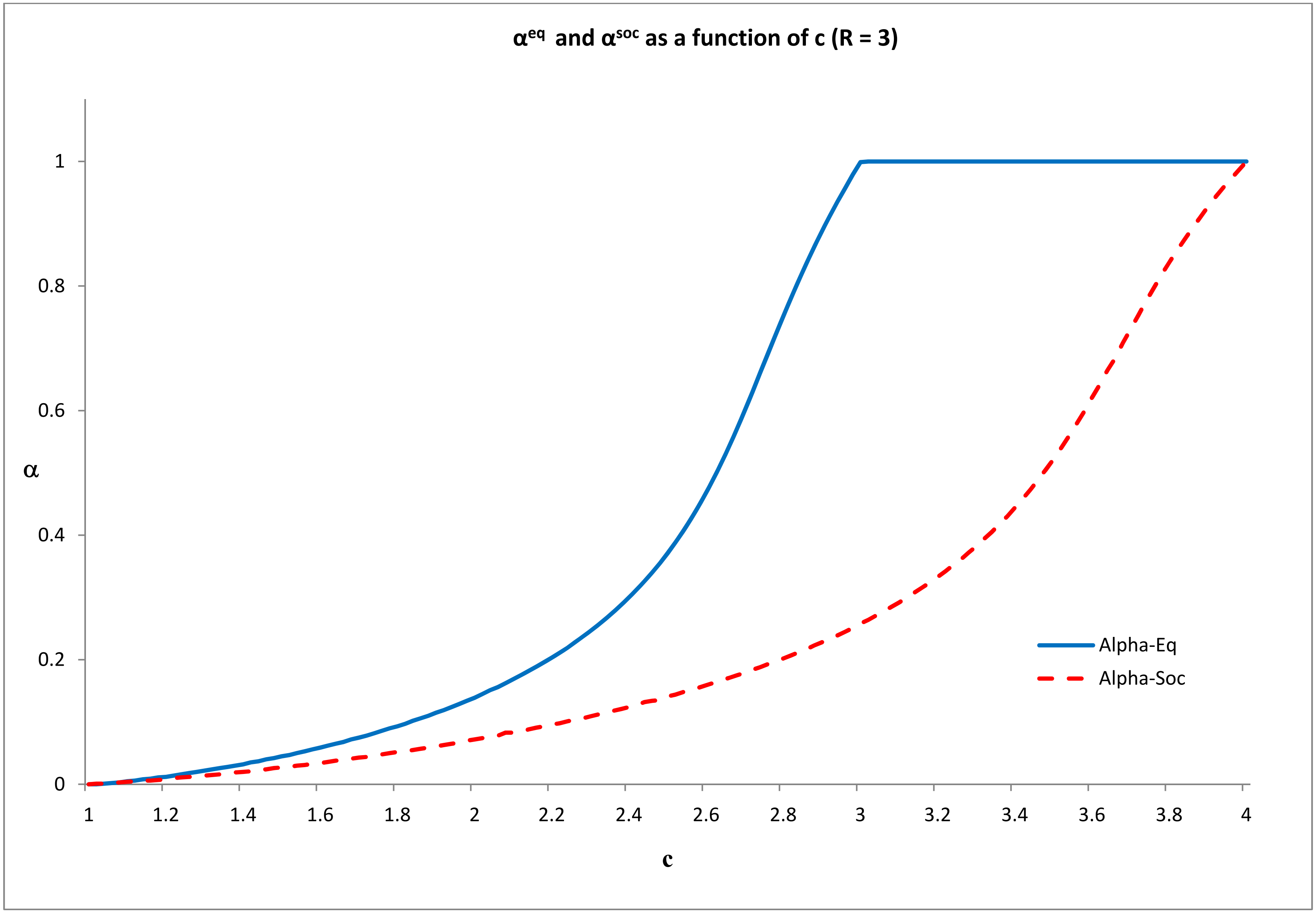

Figure 6d: Equilibrium and constrained efficient levels of α, and planner's choice ($1\text{-}m_0$) as a function of c for R=3