Abstract

The fundamental tradeoff between aversion to downside risk and the greater incentive power of punishments determines the form of compensation contracts. With quadratic utility, there is no aversion to downside risk, so concave contracts dominate convex contracts. When the third derivative of the utility function is positive, then convexity is valuable because it mitigates downside risk. Contrary to common wisdom, prudence and risk aversion have opposite effects on the curvature of optimal contracts. Prudence potentially renders some convex contracts optimal, whereas risk aversion alone makes them suboptimal. In a CARA-normal framework, numerical simulations and mathematical analysis both suggest that incentives should be confined to the tails of the distribution of performances. In a CRRA-lognormal setting, the prudence effect dominates the risk aversion effect for values of relative risk aversion smaller than one. However, option-like contracts are only optimal for very low values of relative risk aversion. Whereas the coefficient of RRA implied by the equity risk premium is very high, the coefficient of RRA implied by option-like compensation contracts is very low.

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This paper analyzes how the form of compensation contracts interacts with the form of the utility function to affect incentives, the valuation of implied transfer rules, and contractual efficiency. It derives optimality and dominance results, and explains salient features of observed CEO compensation.

In a special report on executive pay, The Economist noted that “the story behind the growth of pay in the 1990s is really the story of the option”, and concluded that “the chief mistake of the past 15 years was the granting of too many share options.” The first observation gives an idea of the recent importance of stock-options, a convex instrument, in managerial compensation. In 1980, CEOs’ remuneration packages mainly consisted of a cash salary and bonus, according to Hall and Liebman (1998). The median CEO did not receive any stock-options. Twenty years later, stock-options accounted for 51 percent of total pay of the median S&P 500 CEO, according to Murphy (2002). Stock-options are also common among rank-and-file employees, as noted by Hall and Murphy (2000).

Unfortunately, the convexity of most observed compensation profiles with respect to the appropriate performance measure (the stock price in the case of CEOs), is not satisfactorily explained by standard models of efficient contracting, such as those of Jenter (2002), Hall and Murphy (2002), or Dittmann and Maug (2007). By contrast, this paper shows how and why convex contracts can be optimal in the standard contracting model with moral hazard. It is noteworthy that I do not need any of the assumptions generally used in the literature to generate convex contracts: risk neutrality, limited liability, loss aversion, moral hazard in risk, and taxation. This paper emphasizes the importance of the third derivative of the utility

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1 The Economist, January 20th 2007.
2 Using numerical simulations of a CRRA-lognormal model, Jenter (2002) compares granting call options to granting stocks, and concludes that stocks are more efficient. Dittmann and Maug (2007) take a CRRA-lognormal model to the data, and conclude that U.S. CEOs should receive stocks and a short position in call options - which is equivalent to making them long stocks and short puts. They also solve for the general (nonlinear) contract, which is remarkably similar to the option contract I advocate in a CARA-normal setting. In a similar model, Hall and Murphy (2002) show that the optimal strike of stock-options is zero, thereby rejecting convexity in favor of restricted stocks.
3 First, risk neutrality and feasibility constraints (including limited liability) are crucial in Innes (1990). With a limited liability constraint and risk neutrality, Innes (1990) and to a lesser extent Lambert and Larcker (2004) obtain out-of-the-money option-like contracts. Second, my paper shows that neither loss aversion (a kink in the utility function) nor risk-seeking behaviors (a partly convex utility function) are required for convex contracts to be optimal. The model of executive compensation of de Meza and Webb (2007) features the former, while the model of Dittmann, Maug and Spalt (2008) include both. Instead, my framework nests loss aversion, but also allows for a smooth utility function. Third, I postulate that managerial effort affects the mean of returns, not its variance. Many papers, including Hirshleifer and Suh (1992), show that changing the curvature of the compensation profile affects the manager’s attitude toward the variance of the performance measure.
function - if this term is positive, the agent is called prudent. The main contribution of this paper is to identify and separate the effects of risk aversion and prudence on compensation design. I find that in a calibrated CRRA-lognormal model, option-like contracts are only optimal for very low values of relative risk aversion. However, the effect of prudence on compensation design dominates the effect of risk aversion for values of relative risk aversion smaller than one.

The model also shows that in equilibrium, observed low pay-performance sensitivities are not inconsistent with an adequate provision of incentives. Actually, for the same delivery of incentives, more efficient contracts tend to be characterized by lower pay-performance sensitivities. Small and moderate rewards are especially inefficient. In this regard, the steady increase in pay-performance sensitivities of CEO compensation throughout the late 1980s and 1990s advocated by Jensen and Murphy (1990) and documented by Hall and Liebman (1998), which was accompanied by the widespread diffusion of stock-options, may signal a move toward less efficient forms of remuneration. More specifically, the model explains the rise in observed pay-performance sensitivities and in compensation costs throughout the late 1980s and the 1990s. Since convex contracts provide less incentives, a rise in the average pay-performance sensitivity is required to maintain the same level of incentives. In addition, since small and

DeFusco, Johnson, and Zorn (1990) find that the implied volatility of a firm's stock price tends to increase after stock-options are granted. In Feltham and Wu (2001), stock-options become preferred to stocks when managerial effort affects the risk of the firm. However, Ross (2004) shows that a risk averse agent whose pay contract is convex in a given variable may still be averse to the volatility of this variable. Given this result, a fixed payment would be more effective at mitigating risk-avoiding behaviors on the part of the CEO (not to mention that it would also be less costly), a point already made by Hemmer, Kim and Verrecchia (2000). Fourth, tax considerations and accounting rules may also favor stock-options, as argued in Hall and Murphy (2003) and Jensen and Murphy (2004). Apart from the limited liability constraint, I do not consider these factors. I also assume that the efficient contractual paradigm is valid: in particular, I do not take the managerial power view of Bebchuk and Fried (2004), which argues that remuneration instruments are vehicles for rent extraction by entrenched executives.

An agent is risk averse where his marginal utility is decreasing in payments. He is prudent where his marginal utility is convex in payments. Prudence is implied by nonincreasing absolute risk aversion, and is required for agents to have a precautionary motive. Noticeably, HARA utility functions for which absolute risk tolerance is nondecreasing in wealth have a positive third derivative. There is evidence that individuals prefer positively skewed distributions with a small loss (relative to the expected payoff) with high probability and large gain with low probability - that is, with a long right tail. This is consistent with aversion to downside risk. An agent whose utility function has a positive third derivative has a preference for positive skewness. Such a preference has been extensively empirically documented - see for example Kraus and Litzenberger (1976). Finally, Scott and Horvath (1980) show that prudence is necessary for marginal utility to be positive for all wealth levels.
moderate rewards are associated with high agency costs, relying more on stock-options tends to result in higher average compensation costs.

When transfers from the principal to the agent are unconstrained (notably unbounded, which rules out limited liability for the principal or for the agent, or a credit constrained agent), it is well-known that moral hazard is an issue if and only if the agent is risk averse. For a risk neutral agent, any incentive-compatible and individually rational contract is optimal. On the contrary, when the agent’s marginal utility is not constant, the profile of his payments matters. Different contracts will be characterized by different agency costs. This paper identifies the two effects arising from individual preferences which determine the curvature of optimal compensation contracts.

The first is the impact of increasing effort on risk exposure, already recognized in Jenter (2002) and Chaigneau (2008). For any given performance, incentives are a function of the pay-performance sensitivity times the marginal utility of the payment associated to this performance. Clearly, when the agent is risk averse, a contract in which steep slopes correspond to low payments is well structured to maximize incentives: increasing effort decreases the variance of compensation. This suggests concave contracts and punishments.\(^5\)

The second is the effect of prudence, which can also be interpreted as aversion to downside risk, as a positive third derivative of the utility function reflects an increasing second derivative. It ensues that a prudent agent does not heavily discount upward variations from a given payment, but he strongly discounts downward variations. This is why incentives taking the form of punishments are more expensive than rewards. This suggests the utilization of convex contracts and rewards.

In brief, risk aversion, captured by a negative second derivative of the utility function, is conducive to concave contracts, whereas prudence, captured by a positive third derivative, is conducive to convex contracts. The analysis starts by only considering terms of the first and second-order, as in a quadratic utility function. This restriction is not innocuous, since a dominance argument then shows that the optimal contract cannot be convex, or cannot take the form of rewards.\(^6\) Then third-order terms are considered. When the third derivative of

\(^5\)Mirrlees (1975 and 1999) has shown that under certain conditions, paying the agent a fixed wage associated with a strong punishment for a very low performance is approximately optimal. However, I do not impose the Mirrlees conditions (a likelihood ratio which tends to minus infinity as the performance measure approaches minus infinity, and a utility function with unbounded support). In addition, the Mirrlees contract is not observed in practice, perhaps because it is not robust to limited liability.

\(^6\)The same result holds with mean-variance preferences, which capture pure risk aversion, or aversion to fluctuations in wealth. In a simple and general moral hazard setup with a symmetrically distributed probability density function and a performance additive in effort and noise, Chaigneau (2008) shows that convex contracts
the utility function is positive, then an agent values a convex payoff profile more highly than a symmetric concave profile with equal expected payoff. Far from being negligible, third-order terms need to be considered, lest convex contracts be suboptimal.

To obtain these results, the paper develops a new method to evaluate the effects of different compensation contracts. The challenge consists in separating the impact of prudence from the impact of risk aversion (with CARA and CRRA utility functions, prudence is intrinsically linked with risk aversion). The traditional approach, which maps the parameters of incentive-compatible and individually rational contracts into agency costs, is inevitably uni-dimensional, and therefore not appropriate for this purpose. The second section of this paper uses a different and complementary approach, in which contracts characterized by the same cost are compared in terms of the incentives they generate, and the expected utility they are associated with in equilibrium. Using a symmetric probability density function (with respect to the mean), symmetric contracts (with respect to a point) with the same expected payoff and the same average slope are compared. In this setting, only characteristics of the utility function may explain that two symmetric contracts deliver different incentives, and are associated with a different expected utility. More precisely, I show that risk aversion matters for relative incentives, and prudence matters for relative expected utility. All results in this part of the paper are obtained in a very general setting, with minimal hypotheses. They clearly identify the advantages associated with convexity (respectively rewards) and concavity (respectively punishments).7

These two effects are then combined in a CARA-normal framework, with step contracts and short puts. Numerical simulations show that agency costs are a non-monotonic and non-symmetric function of the option’s strike, or of the step contract’s cutoff, with incentive-compatible and individually rational contracts. I then prove that giving the agent either a short put with an extremely low strike, a terrible punishment for very poor performances, or a stupendous reward for superb performances, is approximately optimal. This is mainly due to the asymptotic properties of the likelihood ratio. On the contrary, small rewards are very inefficient - much more than small punishments.

While the relative importance of prudence and risk aversion is constant with CARA preferences, it is variable with CRRA preferences. In particular, risk aversion becomes negligible are suboptimal for an agent with mean-variance preferences.

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7In the same vein, Lambert, Larcker and Verrecchia (1991) assess separately the value of a compensation contract from the perspective of the manager, and the incentives it generates. Hall and Murphy (2002) numerically compare the executive’s valuation and the incentives associated with different compensation packages characterized by the same cost.
relative to prudence as the coefficient of relative risk aversion $\gamma$ approaches zero. In line with the implications of the previous results of the paper, I show that convex contracts are optimal in a calibrated CRRA-lognormal model for very low values of $\gamma$. This contrasts with the already referenced recent literature, which uses higher values of $\gamma$ and dismisses stock-options as suboptimal. However, in the perspective adopted in this paper, comparing convex contracts to concave contracts is only meaningful when they are functions of a symmetrically distributed random variable. To obtain results which fit in my framework, I reinterpret the curvature of contracts when it is expressed as a function of an asymmetrically distributed random variable. In effect, it is equivalent to consider a given contract and a lognormally distributed performance measure, or a convex transformation of this contract and a normally distributed performance measure. Thus, my framework can readily be applied to contracts which are contingent on a lognormally distributed performance measure such as the stock price. It turns out that the prudence effect is stronger than the risk aversion effect (so that the optimal contract is convex when plotted against a symmetric performance measure) for values of relative risk aversion less than one, which is low but plausible. As plotted against a symmetric performance measure, the optimal contract is linear if and only if the agent has a log utility.

This paper argues that convex contracts are more likely when the agent is more prudent (which implies aversion to downside risk) and less risk averse. Similarly, with a HARA utility function and a gamma distribution, Hemmer, Kim and Verrecchia (2000) obtain option contracts when managers have decreasing absolute risk aversion (which implies prudence) and low relative risk aversion. In accordance with my results, they also find that increasing the skewness of the performance distribution while holding the mean constant, which shrinks downside risk, diminishes the desirability of convexity. My relative contribution is to identify the advantages of convexity in a more general framework; to analyze the optimality of convex contracts in a more standard CRRA-lognormal model; to not restrict attention to stock-options-like contracts.

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8This is discussed on pages 15 and 16.

9Another related paper is Eeckhoudt and Gollier (2005). In a model of prevention where costly effort reduces the probability of a loss, it isolates the effect of prudence. It shows that, holding risk aversion constant, more prudent agents reduce their effort provision. Instead, they would rather accumulate more wealth, as prudence raises the marginal value of wealth. However, this effect heavily relies on the fact that the effort cost enters the utility function. As my model shows, with utility being separable in monetary transfers and effort cost, and with an additive effort structure, the same fact that prudence raises the marginal value of wealth makes effort increasing in risk exposure. In effect, the compensation contract can be structured in such a way that effort is indeed prevention against a loss: think about a step contract with a low transfer for bad outcomes.
The first section presents the model and some basic results, including the benchmark case of a risk averse agent who is not prudent. The second section turns to agents who are both risk averse and prudent over the whole domain, and disentangles the effects of risk aversion and prudence on the effectiveness of incentives and the valuation of different contracts. The efficiency of different contracts in a CARA-normal setting is studied in the third section. The fourth section calibrates a CRRA-lognormal model to a representative CEO, and identifies the coefficient of relative risk aversion implied by observed contracts. In light of these results, the fifth section discusses observed managerial compensation contracts. The sixth section concludes.

1 The model

We use a standard single period principal-agent model in the spirit of Holmstrom (1979). A risk neutral principal wants to implement a given level of effort. He makes a take-it-or-leave-it contract offer to the agent before the period starts. We only consider the first step of optimal contracting in Grossman and Hart (1983), which consists in minimizing the agency cost of implementing a given effort $e^*$.

Technology. By exerting a nonobservable effort $e$ at the beginning of the period, the agent displaces the mean of the distribution of the contractible performance measure $\tilde{\pi}$, which decomposes as $\tilde{\pi} = e + \tilde{\epsilon}$ and is realized at the end of the period. The random variable $\tilde{\epsilon}$ is distributed according to the probability density function $\varphi$, which is symmetric around the mean of zero, and the continuous c.d.f. $\Phi$. Its support can be unbounded, unless otherwise stated. Given effort $e^*$, the p.d.f. of $\tilde{\pi}$ is $\vartheta$. We assume that the monotone likelihood ratio is uninformative. The distribution of bargaining power is irrelevant to the efficiency of contracting, which is related to the compensation’s structure, not its level. Formally, the model can accommodate any allocation of bargaining power by adjusting the reservation utility of the agent. This does not affect any of the results.

Any optimal contract being a solution to the first step problem for a given level of effort, a contract which violates our results cannot be optimal.


Preferences. The agent’s welfare is separable in effort and wealth. The cost of effort is $\psi(e)$, where $\psi$ is an increasing and convex function mapping $[0, \infty)$ to $[0, \infty)$; $\psi$ satisfies the Inada conditions $\lim_{e \to 0} \psi'(e) = 0$ and $\lim_{e \to \infty} \psi'(e) = \infty$. The initial wealth of the agent is normalized at zero. The agent maximizes his expected welfare, which is equal to the expected utility of his end-of-period wealth minus the effort cost. His utility function, $u$, is three times differentiable, and increasing in end-of-period wealth $W$. Marginal utility is bounded above everywhere, except possibly in the limit, as $W$ approaches minus infinity. An agent is globally risk averse if and only if $u''(x) < 0$ for any $x$ on the domain. An agent is globally prudent if and only if $u'''(x) > 0$ for any $x$ on the domain. Unless otherwise specified, we assume that the agent is globally strictly risk averse ($u'' < 0$) and globally prudent ($u''' \geq 0$). A Taylor expansion of the utility function $u$ around any given payment $W^0$ highlights these two factors:

$$u(W) \approx u(W^0) + u'(W^0)(W - W^0) + \frac{1}{2} u''(W^0)(W - W^0)^2 + \frac{1}{6} u'''(W^0)(W - W^0)^3$$

Let $\bar{U} \geq u[0]$ denote the agent’s reservation utility.

Compensation contracts. The principal designs a contract that makes end-of-period payments to the agent contingent on the performance measure $\tilde{\pi}$. This compensation contract is defined by the transfer function $W(\pi)$, which maps the support of $\tilde{\pi}$ into $(-\infty, \infty)$. When contracts are evaluated at the equilibrium effort $e^\star$, we work for notational convenience with payments as a function of $\epsilon$ rather than $\pi$.

The analysis will often involve step contracts. A step contract is a triplet $\{q, w, \bar{w}\}$. It pays $w$ for $\epsilon \in [-\infty, q]$, and $\bar{w}$ for $\epsilon \in [q, \infty]$. Its wedge is $\bar{w} \equiv \bar{w} - w$. Letting the cutoff $q$ be strictly positive, a punishment contract, denoted by $P$, is a step contract that pays a “punishment” $\overline{w}_P$ for $\epsilon \in [-\infty, -q]$, and a wage $\tilde{w}_P$ for $\epsilon \in [-q, \infty]$, while a reward contract, denoted by $R$, is a step contract that pays a wage $\overline{w}_R$ for $\epsilon \in [-\infty, q]$, and a “reward” $\tilde{w}_R$ for $\epsilon \in [q, \infty]$. Because $q$ is positive and $\Phi(0) = 0.5$ with a symmetric p.d.f., a punishment or a reward (depending on the contract) occur relatively rarely in equilibrium, with probability less than one.

Given that the performance measure is additive in effort and noise, assuming that

$$\left(\varphi_x(\pi - e)\right)^2 \geq \varphi(\pi - e)\varphi_x(\pi - e)$$

for every $\pi$ ensures the monotone likelihood ratio property: a higher effort increases the likelihood of a high performance (see the appendix for details). It guarantees that the compensation profile be increasing in the performance on the whole domain.

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half. That is why the punishment contract can be viewed as a fixed wage associated with an occasional punishment, and the reward contract can be viewed as a fixed wage associated with an occasional reward.

**Constraints on contracting.** (not applicable in section 2, see the footnotes below)

To be accepted, the principal’s offer must satisfy the agent’s participation constraint at the equilibrium effort:

\[
\int_{-\infty}^{\infty} u[W(\pi)]\varphi(\epsilon)d\epsilon = U + \psi(e^{*})
\]  

(1)

If it does, the contract is said to be individually rational.\(^{13}\)

Using the first-order approach (see the next paragraph) and Leibniz’s theorem, the incentive constraint, as evaluated at the equilibrium effort, is

\[
\int_{-\infty}^{\infty} W'(\pi)u'[W(\pi)]\varphi(\epsilon)d\epsilon = \psi'(e^{*})
\]  

(2)

A contract that satisfies this constraint is said to be incentive-compatible.\(^{14}\)

**The first-order approach.** This paper uses the first-order approach, which allows to replace a continuum of incentive constraints by the local incentive constraint if the maximization problem of the agent is concave in effort. I show in the appendix that the first-order approach is always valid with step contracts in the setting used, as long as marginal utility is decreasing.

The first-order approach is always valid with concave contracts, as the second derivative of the agent’s objective function is then negative:

\[
E\left[W''(\tilde{\pi})u'[W(\tilde{\pi})] + (W'(\tilde{\pi}))^2u''[W(\tilde{\pi})]\right] - \psi''(e^{*}) < 0
\]

since \(W''(\tilde{\pi})<0\), \(u'[W(\tilde{\pi})]>0\), \(u''[W(\tilde{\pi})]<0\), and \(\psi''(e^{*})>0\).

Only convex contracts are potentially problematic. However, for any given convex contract \(W\) such that \(E[W''(\tilde{\pi})u'[W(\tilde{\pi})]]\) is a finite constant, there exists a sufficiently convex cost function such that the first-order approach is valid. Put differently, the set of convex contracts for which the first-order approach is valid can be enlarged as needed by increasing \(\psi''(e^{*})\), i.e., by increasing the convexity of the cost function. Since in this paper we consider given convex contracts, it is always possible to ensure that the first-order approach holds in each case. Notice that this approach would not be appropriate for selecting the most efficient

\(^{13}\)We replace this constraint with an expected cost constraint in section 2 of the paper.

\(^{14}\)The equilibrium effort which solves this equation may not be \(e^{*}\) in section 2 of the paper.
contract in an unbounded set of convex contracts.

**Agency costs.** Denote by $W^*$ the first-best payment that corresponds to effort $e^*$. That is,

$$u[W^*] \equiv \bar{U} + \psi(e^*) \quad (3)$$

As by assumption $\bar{U} \geq u[0]$, the utility function is increasing and the effort cost is nonnegative, we have

$$W^* \geq 0 \quad (4)$$

The agency cost corresponds to the cost of inducing effort $e^*$ for a given compensation contract $W$, beyond compensation for effort and for the reservation utility. Denote the agency cost by $AC_{u,W}$: it is a function of the utility function $u(W)$ and of the compensation contract $W(\pi)$. By definition, the expected cost of compensation with a contract $W$ is the agency cost associated with it plus the first-best cost:

$$E[W(\tilde{\pi})] \equiv W^* + AC_{u,W} \quad (5)$$

**Definition:** An optimal contract minimizes agency costs, given $e^*$ and the agent’s preferences. Any contract with nonpositive agency costs is optimal.

The problem of the principal is to minimize the agency cost of inducing effort $e^*$. Note that in a given setup, the optimal contract is not necessarily unique. In particular, several contracts can in certain settings achieve the required effort $e^*$ at zero agency costs.

To start with, suppose that the preferences of the agent are such that his marginal utility is constant. Let the principal offer any feasible contract $W$ that satisfies both the agent’s participation constraint and his incentive constraint.\(^{15}\) Because the first derivative of a risk neutral agent’s utility is constant and all higher-order derivatives are zero,

$$u(W(\pi)) = u(W^*) + u'(W^*)(W(\pi) - W^*)$$

Taking expectations,

$$E[u(W(\tilde{\pi}))] = u(W^*) + u'(W^*)(E[W(\tilde{\pi})] - W^*)$$

Besides, using (??) and (??),

$$E[u(W(\tilde{\pi}))] = u(W^*)$$

\(^{15}\)As previously indicated, the effort level to be elicited is given, and set at $e^*$. It is not necessarily the first-best effort, which explains why the (risk-neutral) agent is not necessarily the residual claimant.
So that

\[ u'(W^*) (E[W(\tilde{\pi})] - W^*) = 0 \]

Marginal utility being positive,

\[ E[W(\tilde{\pi})] = W^* \]

Substituting into (??), the agency cost of any contract satisfying both (??) and (??) when the
agent is risk neutral is zero.

Consider the more general case of an agent whose marginal utility is not constant. We
are going to derive a measure of agency costs, for any contract \( W \). A second-order Taylor
expansion of the utility function around \( W^0 \) writes as

\[ u(W) \approx u(W^0) + u'(W^0)(W - W^0) + \frac{1}{2} u''(W^0)(W - W^0)^2 \]

We also know that if \( u \) is concave, there exists a \( v_W \) in-between \( W^0 \) and \( W \) such that\(^{16}\)

\[ u(W) = u(W^0) + u'(W^0)(W - W^0) + \frac{1}{2} u''(v_W)(W - W^0)^2 \quad (6) \]

Setting \( W^0 = W^* \) in (??), taking expectations and rearranging, we get

\[ E[W(\tilde{\pi})] - W^* = \frac{1}{u'(W^*)} \left( E[u(W(\tilde{\pi})) - u(W^*)] - \frac{1}{2} E\left[u''(v_W)(W(\tilde{\pi}) - W^*)^2\right] \right) \quad (7) \]

Where \( E[u(W(\tilde{\pi})) = u(W^*) \) because of (??) and (??). Moreover, the left-hand side of (??)
being equal to the agency cost, we can write:

\[ AC_{u,W} = -\frac{1}{u'(W^*)} E\left[u''(v_W)(W(\tilde{\pi}) - W^*)^2\right] \quad (8) \]

which is positive if the utility function is concave, and where \( v_W \) is increasing in \( W \). Plugging
in (??) yields

\[ AC_{u,W} = \frac{1}{u'(W^*)} \int_{-\infty}^{\infty} \left( u(W^*) + u'(W^*)[W(\pi) - W^*] \right) - u[W(\pi)] \varphi(\epsilon) d\epsilon \quad (9) \]

The agency cost is the sum over all possible states of nature, as weighted by their respective
probabilities in equilibrium, of the difference between the valuation of compensation in this
state by a risk neutral and a risk averse agent.

As a benchmark case, assume that the second derivative of \( u \) is constant, equal to \( 2k \)
(where \( k \) is negative), on the whole domain. Using (??), it appears that the agency cost is
then proportional to the sum over all possible states of nature, as weighted by their respective
probabilities in equilibrium, of the weighted squared deviation between \( W(\pi) \) and \( W^* \):

\[ AC_{u,W} = \kappa \int_{-\infty}^{\infty} (W(\pi) - W^*)^2 \varphi(\epsilon) d\epsilon \quad (10) \]

\(^{16}\)This is an application of the mean value theorem, and is proved for example in Simon and Blume (1994),
p.828.
where $\kappa$ is a positive constant equal to $\frac{-k}{u'(W^*)}$. Notice that this expression is not proportional to the variance of $W(\tilde{\pi})$ - unless $E[W(\tilde{\pi})] = W^*$.

Finally, we identify instances where agency costs can easily be eliminated, even though the utility function of the agent is concave. The idea is to concentrate payments in the interval where the agent is risk neutral, which is only possible if he is risk neutral around the first-best payment. Divide the range of payments into three intervals: $(-\infty, W_1)$, $(W_1, W_2)$, and $(W_2, \infty)$, where $W_1 < W^* < W_2$. Then

**Claim 1:**

- If $W_2 - W_1$ is large enough and if marginal utility is constant on $(W_1, W_2)$, then an optimal contract exists, and induces $e^*$ at zero agency cost.
- If marginal utility is constant on $(-\infty, W_2)$, then an optimal contract exists, and induces $e^*$ at zero agency cost.
- If marginal utility is constant on $(W_1, \infty)$, then an optimal contract exists, and induces $e^*$ at zero agency cost.

In these cases, individually rational and incentive-compatible contracts can be constructed such that the agent’s marginal utility is constant over the range of all possible payments he may receive in equilibrium, so that agency costs are zero.

**Optimal contracting with risk aversion.** We now study optimal contract design with quadratic utility,\(^{17}\) which captures risk aversion and excludes prudence: by construction, quadratic utility is the only concave utility function with $u''' = 0$. This constitutes the benchmark case. In this setting, we show that convex contracts and step contracts with rewards are suboptimal. Roughly speaking, agents who are exclusively risk averse should not be offered upside participation.

**Proposition 1a** (benchmark case): If $u$ is quadratic, any reward contract with $\bar{w}$ inferior

\(^{17}\)I construct a utility function with a negative and constant second derivative in the appendix. Quadratic utility and the mean-variance criterion provide a useful benchmark, although they are questionable: the agent is just as averse to upside variations as to downside variations, which does not concur with either economic intuition or empirical and experimental studies. This unsatisfactory symmetry does not matter only if payoffs are linear in a symmetrically distributed random variable.
to an arbitrarily large constant is dominated by a punishment contract.

Risk aversion makes it more efficient to offer punishments rather than rewards for incentive purposes. A risk averse agent with a constant second derivative discounts as heavily downward and upward deviations from the optimal risk sharing rule. But for the same deviation with respect to the first-best payment, a punishment offers more incentives than a symmetric reward. The intuition is that the variance of the agent’s pay is decreasing in effort when punishments are used, whereas it is increasing in effort when rewards are used. Punishments can therefore be smaller and still be incentive-compatible. This diminishes the discount applied to the transfer rule, and therefore reduces agency costs.

We now consider concave and convex contracts. For technical reasons, the support of \( \tilde{\epsilon} \) cannot be unbounded in the following proposition, although the bounds can be arbitrarily large. See Chaigneau (2008) for a proof of the same result as in proposition 1b with an agent who maximizes a linear mean-variance criterion, in which case the support of \( \tilde{\epsilon} \) can be unbounded. Proposition 1b below exactly mirrors proposition 2a. This is because concave contracts share essential properties with punishments, as do convex contracts with rewards.

**Proposition 1b** (benchmark case): If \( u \) is quadratic, any compensation contract convex in the performance measure is dominated by a concave contract.

Risk aversion makes it more efficient to offer concave contracts rather than convex contracts. The intuition is as in the case with step contracts.\(^{18}\)

We also show in Claim 2 in the appendix that payments to an agent whose utility function is capped should be capped. Notice that CEO compensation contracts are typically not capped.\(^{18}\)

\(^{18}\) Quadratic utility implies mean-variance analysis. For any convex contract \( W \), the proof of proposition 1b constructs a concave contract \( V \) with the same pay-performance sensitivity, the same expected payoff, and the same variance as the original contract \( W \). But with a concave contract, the covariance between the pay-performance sensitivity and the marginal utility turns positive, which delivers further incentives for effort. The proof proceeds by flattening the contract \( V \) until it elicits the same effort as \( W \). The same effort is hence induced at the cost of a smaller deviation from \( W^* \), and therefore at a lower agency cost, as defined in (??). This demonstrates that \( W \) is suboptimal.
2 Disentangling the effects of risk aversion and prudence

If a positive and decreasing marginal utility were sufficient to describe individual preferences, the convexity of most observed managerial contracts would be puzzling. In effect, we have seen that it is inefficient to offer convex contracts to agents who are exclusively risk averse. However, the literature commonly accepts with good reasons that agents are both globally risk averse and globally prudent. In this case, results of the previous section do not apply. This section adds prudence (a convex marginal utility) to the description of preferences. It identifies the channel through which prudence potentially re-establishes the optimality of convex contracts.

Since risk aversion and prudence pull loosely speaking the curvature of the optimal contract in opposite directions, it is impossible to get an optimality result in the general case (the next sections present optimality results in the CARA-normal and CRRA-lognormal cases). To get around this obstacle, this section disentangles the interaction between risk aversion and the contract curvature on the one hand, and between prudence and the contract curvature on the other hand. In a sense to be defined below, risk aversion ensures that concave contracts or punishment contracts generate more incentives, while prudence ensures that convex contracts or reward contracts are more valued by the agent.

As in the quadratic utility case, risk averse agents have a decreasing marginal utility, which tends to render concave contracts and punishments more apt at providing incentives. However, with a positive $u'''$ which implies aversion to downside risk,\(^\text{19}\) downward deviations from $W^\star$ are more costly in terms of agency costs than upward deviations of the same magnitude. Valuation considerations alone therefore suggest the utilization of convex contracts and rewards. The following propositions formalize this tradeoff.

In the usual approach used elsewhere in this paper, the agent’s expected utility is constant across admissible contracts. In the approach used in this section, the expected payment from the principal is constant across contracts that have different effects on the agent. For a given cost of the contract for the principal, we separately evaluate the agent’s valuation of the transfer rule implied by the contract, and the effort incentives it generates. In other words, we

\(^{19}\)The case with $u''$ negative and decreasing is quite uncommon – especially because it represents an agent more averse to upside variations than to downside variations in his wealth. Furthermore, it cannot be resolved in general, because the sign of $E[u'(V(\hat{\pi}))] - E[u'(W(\hat{\pi}))]$ is indeterminate when neither $u$ nor $W$ are specified - the linear marginal utility implied by a constant $u''$ was crucial to obtain an equality between these two terms in the benchmark case. Let us just mention that $u''$ is still decreasing, which implies that concave contracts tend to provide more incentives than convex contracts for the same pay-performance sensitivity; and the fact that $u''$ is decreasing makes downward deviations from $W^\star$ less costly in terms of agency costs than upward deviations - which makes punishments more cost-effective than rewards: convex contracts are suboptimal.
do not require contracts to be individually rational or to implement \( e^* \). Instead, we separate the impact of the contract’s curvature on the agent’s valuation of the transfer rule and on his effort. In doing so we emphasize the tradeoff between downside risk protection provided by convex contracts and additional effort inducements provided by concave contracts. The former is valuable if the agent is averse to downside risk, whereas the latter comes into play with risk averse agents.

To start with, we compare any punishment contract \( W_p \) to a reward contract \( W_r \), which is symmetric to \( W_p \) with respect to the point \((0, E[W_p])\) in the \((\epsilon, W)\) plane. This has the following implications. First, the respective cutoffs of the contracts, \( \epsilon = -q \) and \( \epsilon = q \), are equidistant from \( \epsilon = 0 \). Second, both contracts have the same wedge:

\[
\bar{w}_p - w_p = \bar{w}_r - w_r = \hat{w}
\]  

(11)

Third, since the p.d.f. is symmetric around \( \epsilon = 0 \), both step contracts have the same expected payoff in equilibrium:

\[
E[W_p] = \Phi(-q)w_p + (1 - \Phi(-q))\bar{w}_p = \Phi(q)w_r + (1 - \Phi(q))\bar{w}_r = E[W_r] \equiv \alpha
\]  

(12)

This implies that these two contracts are as costly to the principal.

Along the same lines, we compare any given convex contract \( W_E \) to a concave contract \( W_A \) defined by

\[
W_A(\pi) = 2\bar{W} - W_E(-\pi + 2e^*)
\]  

(13)
Figure 2: Comparing a concave contract to a convex contract.

and conversely. Set $\hat{W}$ such that

$$E[W_i(\pi)|e^*] = \hat{W}$$

for $i = A, E$. This implies that

$$E[W_A(\pi)|e^*] = E[W_E(\pi)|e^*]$$

We also equalize the average slope across contracts: $E[W'_i(\pi)|e^*]$ is a constant, $i = A, E$. All these contracts would implement the same effort if the agent were risk neutral.

The agent’s valuation of a contract and the incentives it delivers are a function of three factors: the contract itself, the probability distribution, and the utility function. With a symmetric p.d.f., symmetrical contracts with the same expected payment will be valued differently and induce different effort levels only if the utility function is nonlinear. Opting for a symmetric p.d.f. and comparing symmetrical contracts enables to isolate the impact that derivatives of different orders of the utility function have on valuations and incentives. More precisely, transforming a reward contract into a punishment contract by turning it by 180 degrees leaves the value of a symmetric p.d.f. unchanged at the cutoff. Any effect on incentives will therefore be uniquely attributable to the utility function. The transformation also leaves the absolute value of the likelihood ratio unchanged at the cutoff. Holmstrom (1979) shows that the optimal contract satisfies

$$\frac{1}{u'(W(\pi))} = \lambda + \mu \frac{\partial e(\pi)}{\partial \pi}$$

(15)
In view of this condition which defines the optimal contract, the only factor that prevents the optimal compensation contract from being symmetric with respect to the point of coordinates \((e^\star, W(e^\star))\) (and therefore neither convex nor concave) is the concavity of the utility function.\(^{20}\)

We begin by comparing the agent’s valuation of the structure of payments of any two step contracts satisfying the relations described above, i.e., characterized by the same expected cost, the same wedge \(\hat{w}\), and opposite cutoffs. We momentarily ignore effort to focus only on risk sharing. The first proposition shows that the agent’s valuation of such symmetrical transfer rules is independent of the concavity of his utility function. What matters is the convexity of his marginal utility.

**Proposition 2a:** Suppose that \(e = e^\star\). An agent with preferences characterized by \(u''' = 0\) derives the same expected utility from this reward contract and from the corresponding punishment contract. An agent with preferences characterized by a constant and positive \(u'''\) derives a higher expected utility from a reward contract than from the corresponding punishment contract. An agent with preferences characterized by a variable \(u''\) derives a higher expected utility from a reward contract than from the corresponding punishment contract if \(u(2i-1)\) is positive for all \(i \geq 2\).

Notice that this last condition is satisfied by CARA and CRRA functions.\(^{21}\) This is the first central result of the paper. Agents who are prudent (with \(u''' > 0\)) are averse to downside risk, i.e., they apply a heavier discount to downward variations than to upward variations from a given payoff level. Proposition 2a claims that incentive effects aside, upward variations from the first-best payment are more cost-effective than downward variations. When agent are prudent, risk sharing considerations therefore favor rewards rather than punishments. The same result holds when we compare any convex contract to the corresponding concave contract described above.

**Proposition 2b:** Suppose that \(e = e^\star\). For any given convex contract \(W_E\), an agent with preferences characterized by \(u''' = 0\) derives the same expected utility from this transfer

\(^{20}\)This consideration alone suggests that typical 80%/120% incentive plans, as described in Murphy (1999), are suboptimal for risk averse agents.

\(^{21}\)More generally, Scott and Horvath (1980) derive positive preferences for odd order central moments, and negative preferences for even order central moments as necessary consequences of a positive and decreasing marginal utility - with the requirement that agents have consistent preferences, in the sense that the sign of their preferences is independent of the wealth level.
rule and from the corresponding concave transfer rule $W_A$ defined in (??). An agent with preferences characterized by a constant and positive $u''$ derives a higher expected utility from $W_E$ than from $W_A$. An agent with preferences characterized by a variable $u''$ derives a higher expected utility from $W_E$ than from $W_A$ if $u^{(2i-1)}$ is positive for all $i \geq 2$.

The intuition is the same as in proposition 2a.

Now turn to effort inducement, by again comparing two step contracts satisfying the relations described above.

**Proposition 3a:** If $u'' < 0$, a given punishment contract delivers more incentives than the corresponding reward contract. If $u'' = 0$, both contracts deliver equal incentives.

The same result holds for concave versus convex contracts.

**Proposition 3b:** If the agent’s preferences are characterized by $u'' < 0$ and $u''' = 0$, any given convex contract $W_E$ delivers strictly less incentives than the corresponding concave contract $W_A$ defined in (??). If the agent’s preferences are such that $u'' < 0$ and $u^{(2i)}$ is nonpositive for all $i \geq 2$, a convex contract $W_E$ delivers strictly less incentives than the corresponding concave contract $W_A$.

This is the second central result of the paper. Downward variations from the first-best payment generate more incentives than symmetrical upward variations of the same magnitude. This result holds regardless of the sign of $u'''$. For a given contract, the extent of effort incentives depends on the concavity of the utility function. Notice that the last conditions in propositions 2b and 3b are satisfied by both CARA and CRRA utility functions.\(^\text{22}\)

It is worth underlining that although prudence increases effort (in the sense that, everything else being equal, an agent with a positive $u'''$ exerts more effort than an agent with a zero $u'''$, as will be shown later), it does not per se increases the relative incentives provided by either convex or concave contracts. Only the fourth and higher-order even derivatives matter in this respect (this is apparent in (??) in the appendix).

Propositions 2a and 2b on the one hand, and 3a and 3b on the other hand, highlight the

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\(^{22}\)More generally, they are satisfied by HARA utility functions, as defined by

$$\frac{u''(x)}{u'(x)} = \frac{1}{ax + b} > 0$$

as long as $a$ is nonnegative (a negative $a$ would imply an increasing absolute risk aversion).
essential tradeoff in compensation contract design. Protecting prudent agents against downside risk by offering rewards rather than punishments must be traded off against the property of punishments to make the variance of income a decreasing function of effort in equilibrium, which elicits more effort if the agent is averse to the variance of his income. Risk aversion does not result in any tradeoff in this setting: with a constant and negative $u''$, punishments dominate rewards and concavity dominate convexity, as stated in propositions 2a and 2b. It is aversion to downside risk, captured by a positive $u'''$, that potentially reinstates the relative efficiency of rewards.

In general, we cannot conclude which effect will dominate. We can expect them to interact along the range of payments, and the optimal contract to trade off these two forces. Nevertheless, optimality results can be obtained, but they require more assumptions. For example, Mirrlees (1975 and 1999) has shown that contracts featuring a fixed wage and an extreme punishment approximately reach the first-best in a certain setting: the postulated asymptotic property of the likelihood ratio makes aversion to downside risk irrelevant. More on this in the next section.

3 The relative efficiency of different contracts in a CARA-normal setting

Since the advantages of concavity (respectively punishments) depend on risk aversion, and the advantages of convexity (respectively rewards) depend on prudence, no optimality or dominance result can be obtained with global risk aversion combined with global prudence. But we can be more specific. At the cost of specifying the agent’s utility function and the probability distribution, this section measures the agency costs of individually rational and incentive-compatible step contracts and short put contracts in a CARA-normal framework, and identifies quasi-optimal contracts within these two classes of contracts.23

Agency costs measure (in)efficiency. With step contracts, we measure the efficiency of concentrating incentives at any given point of the PM distribution. Given that rewards attenuate downside risk (see proposition 2a) and punishments are more effective at providing incentives

23In the widely used CARA-normal framework nested in our model, the curvature is not a trivial issue. It is well known that we can use mean-variance analysis when the contract is linear (intuitively, the level of effort does not affect the variance of compensation, and the agent is as exposed to downside risk as to upside risk). But we know from Chaigneau (2008) that with a mean-variance criterion, the optimal contract cannot be convex, and a linear contract is dominated by the same contract to which a cap is applied, which renders it concave. Therefore, in a CARA-normal framework the optimal contract cannot be linear.
(see proposition 3a), it is a priori unclear which type of contract is more efficient. However, it turns out that the latter effect dominates: except at the tails of the distribution, punishments tend to be more efficient. Next, we study short put contracts. They are the simplest concave contracts, and they can easily be constructed with put options. Mapping the strikes of options into agency costs allows to determine whether it is preferable to have small incentives along a large subinterval of performances, or strong incentives along a small subinterval. Long call contracts are discussed in footnote 28.

3.1 Step contracts

We start with step contracts. Assume that the probability distribution is normal and the agent has CARA utility. He is therefore both globally risk averse and globally prudent. Figure 3 reports the results of a numerical simulation in which the standard deviation is 1, and the coefficient of absolute risk aversion is 1. It displays the agency cost, as a percentage of the first-best cost, of an incentive-compatible and individually rational contract, as a function of the cutoff $q$ (further details on the method used can be found in the appendix).

We observe that (i) the cost of the contract converges toward a lower bound, the first-best cost, for a low enough cutoff $q$: the absolute value of the likelihood ratio is so high at extreme outcomes that these outcomes are almost perfectly informative. (ii) Agency costs are then increasing in $q$. The major force at play in the interval comprising intermediate values of $q$
is the greater effectiveness of punishments at motivating effort, as highlighted in propositions 1a, 1b, 3a, and 3b. Agency costs are highest for \( q \approx 2 \): small rewards are inefficient. (iii) Agency costs then decline, and quickly converge toward their lower bound of zero as \( q \) becomes large enough, for the same reason as in (i).\(^{24}\) The downside risk is mitigated by the fact that payments are bounded below with a step contract. It is nevertheless present, which explains why the agency cost is slightly higher at \( q = -10 \) than at \( q = 10 \), which can only be attributed to downside risk aversion.\(^{25}\) Thus, the convergence towards the first-best cost happens on both tails of the distribution, but it is faster to the right. This is attributable to the lingering effect of downside risk aversion on the left. Yet, the difference vanishes altogether for significantly higher absolute values of \( q \): the likelihood ratio effect then becomes predominant, as observed in (i) and (iii).

For small absolute values of the cutoff, (small) punishments beat (small) rewards. Agency costs represent 7.2\% of the first-best cost of the contract for \( q = -2 \), against 23.9\% for \( q = 2 \). This is due to the greater incentive power of punishments. For intermediate absolute values of the cutoff, (moderate) rewards beat (moderate) punishments. Agency costs represent an infinitesimal fraction of the first-best cost of the contract for \( q = 10 \), against 0.1\% for \( q = -10 \). This is due to aversion to downside risk. For very large absolute values of the cutoff, both (extreme) punishments and (extreme) rewards are approximately optimal. This is due to the asymptotic properties of the likelihood ratio. The result below proves that concentrating transfers at either tail of the distribution is quasi-optimal, and is discussed in the appendix.

**Proposition 4:** In a CARA-normal setting, the agency cost of an incentive-compatible and individually rational step contract converges toward zero as its cutoff \( q \) approaches plus or minus infinity.

As a robustness check, figure 4 reports the results of the same numerical simulation in a CRRA-lognormal setup, with a coefficient of relative risk aversion of 2.\(^{26}\) The results are qualitatively extremely similar, which suggests that they are not specific to one particular

\(^{24}\)Consider the following thought experiment. In the absence of asymmetric incentives and without aversion to downside risk (or prudence), but with an agent who is not risk neutral (this is inconsistent with the previous postulates, but is only for the sake of the argument), the agency costs of step contracts would only reflect variations of the likelihood ratio, and be symmetric around the origin.

\(^{25}\)Likelihood ratios are exactly symmetrical, and the greater incentive power of punishments pushes agency costs in the other direction.

\(^{26}\)By definition, \( \ln(\tilde{c}) \) is normally distributed. As a normalization, its mean is 0 and its variance is 1.
3.2 Short put contracts

For step contracts whose cutoff falls in the range of relatively likely performances, punishments are more efficient than rewards. Since punishments and concave contracts share the same essential properties, we now study the efficiency of short puts with the same approach.\textsuperscript{28} The principal offers the agent a fixed wage $w$ and $s$ short positions in a put ($s$ may also be viewed as the slope of the contract on the relevant interval), with strike $k$ as a function of $\epsilon$. These

\textsuperscript{27}Only one detail stands out: the slow convergence to the first-best cost on the “rewards” side. This is because $\exp\{10\} \approx 22,000$. The cutoff must therefore be extremely high for asymptotic properties of the likelihood ratio to come into play.

\textsuperscript{28}The first-order approach is not guaranteed to work with long calls, especially for high strike prices. Existence problems and multiple equilibria issues erupt, which is why results for call contracts with positive strikes are not reported. With a long call contract, raising the slope of the contract may either augment or diminish the equilibrium effort (on the one hand it increases the slope, but on the other hand it reduces marginal utility at every performance) and increases expected utility at the equilibrium effort; raising $w$ reduces the equilibrium effort (by decreasing marginal utility) and increases expected utility. There may be zero, one or more equilibria, some of which may be invalid. However, it is worth pointing out that as expected, an incentive-compatible and individually rational long call with a very low strike (for which the first-order approach is therefore in all likelihood valid) approximates a linear contract, and so incurs the same agency cost as the symmetric short put (with a very high strike), or 7.8%. An at-the-money long call contract (with strike $k = 0$) is characterized by agency costs of 23.7%, a result nevertheless dependent on the validity of the first-order approach.
The contract parameters are displayed in figure 5. The agency cost of the contract is then reported in figure 6, as a percentage of the first-best cost. The lower it is, the more efficient the contract is.

One striking lesson is that the agency cost is a non-monotonic function of the strike of a short put contract. As depicted in figure 6, the agency cost is (i) increasing in the strike for very low strikes, (ii) decreasing in the strike for negative strikes close to zero, then (iii) increasing in the strike for positive and moderate strikes, and finally (iv) decreasing in the strike for very high strikes. Observation (i) suggests that the optimal short put contract may

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29 This couple is unique. In effect, with any short put contract, raising $s$ augments the equilibrium effort (by increasing both the slope and marginal utility at every performance) but reduces expected utility at the equilibrium effort; raising $w$ has the opposite effect of reducing the equilibrium effort (by decreasing marginal utility) and of increasing expected utility.
Figure 6: Agency cost of short puts as a percentage of the first-best cost.

have a very low strike and a steep slope - this conjecture is made formal in the proposition below.

**Proposition 5**: In a CARA-normal setting, the agency cost of an incentive-compatible and individually rational short put contract converges toward zero as its strike approaches minus infinity.

Short put contracts with a very low strike are approximately optimal. This is attributable to the combined effect of a likelihood ratio of high magnitude which guarantees that agents who exert high effort are approximately insured (in particular, the downside risk is literally negligible at the equilibrium effort, given the asymptotic properties of the likelihood ratio), and to the greater incentive power of punishments. Observation (ii) indicates that aversion to downside risk is a powerful force for negative strikes not too distant from zero. A smaller strike entails a steeper slope to maintain incentive-compatibility; very low payments then occur with a non-negligible probability. Observation (iii) underlines the undesirability of a positive and significant pay-performance sensitivity around and to the right of the expected performance (at $\epsilon = 0$) in equilibrium, for incentives reasons. Observation (iv) suggest that capping compensation reduces efficiency: for a sufficiently high strike, a linear contract dominates a short put contract - the linear contract being a short put contract with an infinite strike. This is due
to the likelihood ratio effect already identified by Mirrlees (and which notably applies to the normal distribution, as shown in Bolton and Dewatripont (2005)): given a very high performance, it is almost certain that effort was adequate; therefore, it is desirable to reward agents for very high performances, and conversely to punish them for very low performances. The suboptimality of capping a linear compensation contract (at a relatively high level) glaringly contrasts with the fact that uncapped contracts are dominated (by specific capped contracts) when the agent has mean-variance preferences, as in Chaigneau (2008).

4 Optimal contracts in a CRRA-lognormal setting

The CRRA-lognormal setting is commonly used in models of executive compensation. Calibrating it is straightforward: the parameters of the distribution of stock prices, which is assumed to be lognormal, can be estimated. The only unknown, in the utility function, is the coefficient of relative risk aversion \( \gamma \). Picking the correct value is all the more difficult that estimates of \( \gamma \) vary widely. In particular, experiments yield values close to 1, while values around 50 are required to generate the historical equity risk premium of the U.S. market in a standard asset pricing model.\(^{30}\) This is why we can be relatively agnostic on this issue. Nevertheless, we can ask if there is any value of \( \gamma \) that approximately generates the compensation contract of a typical CEO in a calibrated CRRA-lognormal model of efficient contracting.

Current managerial compensation contracts heavily use stock-options, and are generally convex. Dittmann and Maug (2007) use values of \( \gamma \) between 0.5 and 10, and conclude that the optimal contract is concave for an overwhelming majority of CEOs (except for very low performances, because of limited liability). What about smaller values of \( \gamma \)? The results in propositions 2b and 3b indicate that prudence must matter “more” than risk aversion for convex contracts to be optimal. But with CRRA utility, risk aversion becomes negligible relative to prudence as the coefficient of relative risk aversion \( \gamma \) approaches zero. More precisely, at \( u(x) \), the index of absolute risk aversion is \( \frac{\gamma}{x} \), while the index of absolute prudence is \( \frac{\gamma+1}{x} \).\(^{31}\)

\(^{30}\)The literature on the equity risk premium was originated by Mehra and Prescott (1985). Arrow (1971) argues that relative risk aversion should be close to 1, for theoretical reasons. Kydland and Prescott (1982) need a relative risk aversion between 1 and 2 to replicate the observed fluctuations in consumption and investment. Finally, Campbell, Lo and MacKinlay (1997), as well as Ait-Sahalia and Lo (2000) summarize estimates of relative risk aversion obtained in the macroeconomic literature. In experiments, Harrison, Lau and Rutstrom (2007) obtain an average value of relative risk aversion of 0.67, while Bombardini and Trebbi (2007) obtain an average value of 1. The latter also review the experimental literature.

\(^{31}\)The index of absolute prudence, as introduced by Kimball (1990), is defined at every \( x \) as \(-\frac{u''(x)}{u''(x)}\). With a CARA utility function with coefficient of absolute risk aversion \( \alpha \), the index of absolute risk aversion is \( \alpha \).
I apply Dittmann and Maug’s approach to their representative CEO, as described in figure 1 of their paper, with a limited liability constraint imposing nonnegative payments (more details can be found in the appendix). As shown in figure 7, low values of the coefficient of relative risk aversion $\gamma$ generate optimal contracts whose shape is convex ($P_T/P_0$ is the end-of-period stock-price as a percentage of the initial stock price). Optimal contracts are mostly concave for $\gamma \geq 0.2$, and mostly convex for $\gamma \leq 0.1$.\(^{32}\) In line with my results, in Dittmann and Maug (2007) the proportion of CEOs for whom holding a positive amount of stock-options is optimal is decreasing in risk aversion, and reaches 17.5% when $\gamma = 0.5$.

It is remarkable that the optimal payment with $\gamma = 0.1$ can approximately be replicated with a zero fixed wage, an option with a strike equal to 135% of the initial stock-price, and no shares. Perhaps the preponderance of stock-options in the late 1990s was predicated on the assumption that CEOs have a very low risk aversion in a booming and stable economy, especially if they are entrenched. If this is accurate, then the model predicts that compensation and so is the index of absolute prudence.

\(^{32}\)This being said, these precise values are of limited interest, since they are dependent on the other parameters of the model, which differ across CEOs. But the qualitative finding that optimal contracts become convex for sufficiently low values of $\gamma$ is robust.
sation practices will switch from stock-options to restricted stocks and other “less convex”
instruments as the economic prospects deteriorate.

Whatever the value of $\gamma$, all optimal contracts have one thing in common: punishments for
failure. The limited liability constraint, which rules out negative payments, is always binding
for very low stock-prices. It is remarkable that even for low values of $\gamma$, when prudence
and aversion to downside risk dominate risk aversion, punishments for failure are optimal.
In effect, when $\gamma = 0.1$, optimal transfers are zero for 59% of performances. These results
imply that most CEOs should not be paid any fixed wage. However, most current managerial
compensation contracts typically feature positive transfers for the whole range of performances.

Since optimal contracts are concave for plausible values of relative risk aversion, it is
tempting to conclude that risk aversion matters more than prudence in compensation contract
design. Nevertheless, it is unclear what the convexity or concavity of a contract mean when
the p.d.f. is not symmetric around the mean.\textsuperscript{33} Crucially, this is the case of the lognormal
distribution. Fortunately, we know that a logarithmic transformation of a lognormally dis-
tributed variable is normally distributed. Operating such a transformation enables us to plot
a given contract as a function of a normal variable, which is symmetrically distributed and
therefore fits in our model. This transformation “convexifies” any contract. For instance, a
contract which is linear as a function of a lognormally distributed variable is convex as a function of the

\textsuperscript{33}The importance of a symmetric p.d.f. is explained on pages 15 and 16.

\textsuperscript{34}If this is unclear, consider a contract which is linear when a random variable, say $\varepsilon$, is lognormally dis-
tributed. For instance, assume that in the $(\varepsilon, W)$ plane, the contract goes through the points $(0, 0), (1, 1)$ and
$(e, e)$. By definition, the variable $ln(\varepsilon)$ is normally distributed. The same contract in the $(ln(\varepsilon), W)$ plane
goes through the points $(-\infty, 0), (0, 1)$ and $(1, e)$. It is identical to the exponential function, which is convex.
Note that on the relevant intervals, it is equivalent to represent the $W$ contract in the $(ln(\varepsilon), W)$ plane, or the
$exp(W)$ contract in the $(\varepsilon, W)$ plane. Either way, this transformation convexifies the form of the contract.
Conversely, a contract which is linear when plotted against a normally distributed random variable plots as an
affine transformation of the logarithm function against a lognormally distributed random variable.
As explained in the appendix, without limited liability\textsuperscript{36} the optimal contract in a CRRA-lognormal setting takes this form (by definition, $\tilde{\pi}$ is lognormally distributed):

$$W(\pi) = (\alpha_0 + \alpha_1 \ln(\pi))^{\frac{1}{\gamma}}$$

where $\alpha_1$ is positive. The same contract as a function of a normally distributed random variable $\tilde{x}$, with $x \equiv \ln(\pi)$, is

$$W(x) = (\alpha_0 + \alpha_1 x)^{\frac{1}{\gamma}}$$

Given relative risk aversion $\gamma$, the curvature of the optimal contract looks very different when plotted as a function of $x$:

**Proposition 6**: As a function of a normally distributed random variable, the optimal contract is concave if and only if $\gamma > 1$, convex if and only if $\gamma < 1$, and linear for a log utility.

It is remarkable that this result holds for any values of the parameters other than relative risk aversion. The intuition is that for any wealth level, the ratio of the index of absolute prudence over the index of absolute risk aversion, $\frac{\gamma + 1}{\gamma}$, is decreasing in $\gamma$. As proposition 6 indicates, prudence is stronger than risk aversion for values of relative risk aversion less than 1, and conversely for values of relative risk aversion greater than 1. For a log utility, the two effects cancel out and the optimal contract is linear. Note that where the limited liability constraint is not binding, the optimal contract with $\gamma = 1$ on figure 7 has the form of a multiple of the log function, and is linear when plotted in the plane of figure 8.

This result shows that prudence is more important to understanding the curvature of optimal contracts than is suggested by a naive analysis of calibrated CRRA-lognormal models. Nevertheless, the fact that observed contracts are convex as a function of a variable whose distribution is approximately lognormal means that they are even more convex when plotted against the corresponding normally distributed variable, as shown in figure 8. This implies at least one of the following: either the relative risk aversion of CEOs is very low; or the CRRA utility function does not accurately represent the preferences of CEOs; or the standard efficient contracting model is inappropriate; or observed contracts are inefficient.

\textsuperscript{35}This observation calls for a reinterpretation of the result of Kim, Hemmer and Verrecchia (2000): increasing skewness may not diminish the “desirability” of convexity because it shrinks downside risk, but simply because it alters the properties of the plane on which the contract is plotted.

\textsuperscript{36}Adding a limited liability constraint would not change the curvature of the contract on the subset of performances where the limited liability constraint does not bind.
5 Rationalizing observed contracts

This penultimate section applies the results of the paper to explain facts and puzzles of CEO compensation.

5.1 Low pay-performance sensitivities

At least two factors make average pay-performance sensitivity a bad measure of incentive provision. First, as propositions 3a and 3b indicate, all else equal, a given slope of the compensation contract delivers more incentives to a risk averse agent when it corresponds to a low payment. Second, changes in payment associated to more informative performances yield more incentives.\(^\text{37}\)

In the CARA-normal setting of section 3, a measure of expected pay-performance sensitivity for every incentive-compatible and individually rational step contract with cutoff \(q\) is

\(^{37}\)To get the intuition, compare the inexistent incentive effect of a positive slope in a region where effort does not change the probability distribution on outcomes to the powerful incentive effect of a negative payment for outcomes that are only obtained with low effort used in tandem with a positive payment for outcomes that are only obtained with high effort.
Figure 9: Expected pay-performance sensitivity of step contracts: CARA-normal.

Figure 10: Expected pay-performance sensitivity of step contracts: CRRA-lognormal.
obtained by multiplying the value of the p.d.f. at \( q \) by the wedge of the associated contract. Results are presented in figure 9 for the CARA-normal case, and in figure 10 for the CRRA-lognormal case. As figure 9 (respectively 10) indicates when coupled with figure 3 (respectively 4), the more efficient the contract, the lower its expected pay-performance sensitivity tends to be. The only exception concerns contracts with moderately high cutoffs, which are quite efficient but are nevertheless characterized by high pay-performance sensitivities.\(^{38}\)

This may explain why a move towards contracts with reward-like features such as stock-options throughout the 1990s was accompanied by a rise in observed pay-performance sensitivities, as documented in Hall and Liebman (1998). It seems that low observed pay-performance sensitivities may indicate that incentives are efficiently provided, rather than not adequate, as the seminal paper of Jensen and Murphy (1990) argued. This finding may also contribute to explaining the puzzling observation made by Brick, Palmon, and Wald (2008). In a paper entitled “Too much pay performance sensitivity?”, they show that future stock returns are negatively associated with option compensation.

5.2 Tournaments, promotions, dismissals, and golden parachutes

Limited liability may prevent the use of punishments. Besides, the fact that very high and unlikely rewards are more efficient than moderately high and relatively likely rewards in widely used frameworks indicates that winner-take-all tournaments may be efficient even for risk averse agents (in Lazear and Rosen (1981), tournaments are efficient when agents are risk neutral). With sufficiently many agents entering the “competition”, a winner-take-all tournament is approximately equivalent to giving all agents a step contract - a related point has been made by Green and Stokey (1983). Crucially, with a continuum of identical agents who are all offered incentive-compatible and individually rational contracts, the distribution of performances is deterministic. There will be a proportion \( 1 - \Phi(q) \) of agents with a performance higher than \( \pi = e^* + q \). For instance, if the technology is such that performance is normally distributed with a standard deviation of 1, and with an arbitrarily large number of agents,\(^{38}\) this is due to the fact that even though the wedge of an unlikely reward must be very large for a risk averse agent (much more than the wedge of a symmetrical punishment), a prudent agent only moderately discounts upside variations in his pay, so that the agency cost of such a contract is low. The fact that the convergence of the expected pay-performance sensitivity towards zero as punishments get more extreme is faster than the convergence of agency costs towards zero is similarly explained: the wedge of a punishment does not need to be so large to motivate a risk averse agent, but a prudent agent heavily discounts downward variations from his pay, which slows down the fall in agency costs. The sharp fall of expected pay-performance sensitivity for extreme realizations of the performance measure is attributable to the likelihood ratio effect.
then there will be only 1 agent out of 10,000 to have a performance that surpasses the mean of the distribution by 3.719.

This suggests that companies should let everyone compete for the CEO job. With sufficiently many contenders, it is almost certain that the one with the highest performance exerted high effort - or, in an adverse selection framework, has high ability. An arbitrarily high likelihood ratio for very high performances means that good luck is necessary but not sufficient to attain these outcomes. In this context, agents are adequately compensated via the fixed payment $w$, and they are incentivized with the prize $\hat{w}$, which can in practice take the form of a big promotion.

Such tournaments will be more valued by prudent agents who are not risk averse on the upside. These agents are also those for whom providing incentives is cheap. That is, the agents who will enter a tournament for the CEO position are precisely those whose interests as CEO can be aligned with shareholders’ at a low cost. Thus, tournaments are also an effective selection mechanism.

As the Mirrlees mechanism and CARA-normal simulation have shown, dismissals (i.e., a very low payment for a terrible performance, where the punishment can involve career concerns), are an efficient incentive mechanism. However, Dittmann and Maug (2007) convincingly argue that taking dismissals into account hardly modifies the shape of observed compensation contracts: nowadays, punishments for CEOs are barely in use. Either they are unnecessary, because CEOs are prudent and approximately risk neutral for high payments. In this case, not using severance payments, i.e., not insuring a CEO against a possible dismissal, is unnecessarily costly to the firm ex ante, and therefore best avoided. Or observed contracts are inefficient, and corporate governance may be at fault. There is also a third possibility. It is noteworthy that the efficiency of punishments seems to be especially inconsistent with the existence of golden parachutes. Except that the prospect of getting the CEO job would not be such a big prize if it means getting fired and irremediably tarnishing one’s reputation with a significant but uncertain probability. Roughly speaking, with uncertainty regarding the firm-CEO matching, the CEO job must pay well in all circumstances to remain the strong inducement to work hard that it is. Hence the golden parachutes: their purpose would thus consist in motivating fellow employees. If this is true, CEO compensation should not only be regarded with the CEO in place in mind: golden parachutes may be part of an efficient incentive package (with the large reward being the CEO job) for aspiring CEOs.

32
5.3 Stock-options

Supposing that our technological and contractual framework captures the essential traits of the agency problem involving CEOs, are there some CEO preferences that would make adopting stock-options optimal? Within the standard model, there are at least two possibilities.

First, in a CRRA-lognormal model with a very low coefficient of relative risk aversion, prudence trumps risk aversion, and the optimal contract is convex - as shown in section 4. Hence, a very low level of risk aversion is needed if the standard model is to generate convex contracts. Part of it may be explained by a selection effect: relative to the average individual, entrepreneurs and CEOs are arguably either less risk averse, or tend to underestimate risk.

Second, agents with CARA or CRRA utility functions become risk neutral in the limit: as \( W \) tends to infinity, \( u''(W) \) approaches zero. Suppose instead that the agent becomes risk neutral or risk loving for payments greater than a given \( W \). Then a compensation package consisting of a fixed wage \( W \) and stock-options with slope \( s \) and strike \( e^* + k \) generates nonpositive agency costs as long as \( W \) is lower than the first-best payment \( W^* \). With such preferences, this condition is most likely to be satisfied when the first-best payment to the agent is high. This may contribute to explaining why stock-options and other convex incentive mechanisms are confined to well-paid employees. However, the first-best optimality of stock-options rests on the assumption of prudence, and on risk aversion being limited to low payments.

5.4 Relative performance evaluation and pay-for-luck

The presence of pay for luck in compensation contracts has long been perplexing, as in Bertrand and Mullainathan (2000). However, the prevalence of mechanisms with rewards and convex contracts in managerial compensation indicates that prudence may be more relevant than risk aversion.

\(^{39}\) According to Dittmann and Maug (2007), taking into account bonus payments and salary changes makes observed compensation contracts even more convex. Convexity is a defining and inescapable feature of CEO pay.\(^ {40}\)

Even with these preferences, stock-options may not be the most efficient contract: stock-options might be optimal in the sense that they generate zero agency costs, but they may still not minimize agency costs. In effect, should the agent be globally prudent (with \( u''' \) strictly positive on the whole domain), a stock-options contract is dominated by the step contract with very large rewards. The intuition is that the principal wants to offer a step contract with a \( w \) as close as possible to \( W^* \) to lessen as much as possible the agency cost of risk aversion, and a \( \bar{w} \) as high as possible to benefit from risk appetite on the upside. For the stock-option package defined above to be optimal, preferences must take the following form: \( u'' \) can be negative for payments lower than \( W \), then \( u'' \) must be nonnegative and constant for payments higher than \( W \).\(^ {41}\) This is very stringent, and quite unlikely to accurately describe the agent’s preferences. All things considered, it seems that stock-options may be approximately optimal, but they are at best a rough instrument for incentivizing prudent managers.
aversion to describe the preferences of the top management (see above). If this is true, and if the CEO is risk loving for sufficiently high payments, then shareholders will reduce agency costs further by offering him some lottery on the upside. This is precisely what pay-for-luck provides. To the extent that it is a bounded below zero-mean risk (this is exactly what it is when stock market performance and other signals are not filtered out of stock-options), pay-for-luck shifts probability mass away from high incomes, and toward moderately high and very high incomes, which is valuable to the agent only if he is risk loving for high incomes. Relative performance evaluation is only valuable for globally risk averse agents. For agents averse to losses relative to a given reference income but risk neutral or risk loving on the upside, relative performance evaluation should only apply on the downside - for instance for dismissal decisions.

All else equal, more prudence (a higher $u''''$) attenuates the adverse effects of pay-for-luck, and can even make it optimal on the upside. Besides, more prudence exclusively increases the advantages associated with convexity. The model therefore predicts that the degree of convexity of compensation contracts should be positively correlated with the extent of pay-for-luck. For example, the stock market performance should be more filtered out of restricted stocks than out of stock-options.42

Pay-for-luck is not only potentially valued by a prudent agent. It also provides him with further incentives. It is well known that with a positive $u''''$, the precautionary motive for savings arises out of the Euler equation associated to the intertemporal savings problem. Similarly, for any given compensation contract, the incentives received by a prudent agent are stronger if a zero-mean risk is added to compensation.

**Proposition 7**: If marginal utility is convex, adding a zero-mean risk to compensation increases the agent’s effort provision.

When the agent is prudent, adding uncontrollable risks to a given compensation contract results in higher effort.43 Pay-performance sensitivities and the precautionary motive are two

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42The best econometric strategy to test this prediction would use panel data, so as to control for firm-specific and CEO-specific effects. It would determine whether, for a given firm-CEO match, switching from stock-options to restricted stocks increases relative performance evaluation, and conversely. To account for the time-varying awareness of the advantages associated with relative performance evaluation, a year dummy should be included in the regression.

43Imagine that the agent’s contract stipulates that he should gamble part of his compensation at an actuarily fair premium. The possibility of losing, coupled with his aversion to downside risk, raises his optimal effort, to forestall very bad outcomes. This is the notion of precautionary effort.
levers to induce effort. However, the latter’s effectiveness is directly related to the intensity of
the former. In this sense, they are complements overall, although they are substitutes at the
margin. This implies that, everything else being equal, the extent of pay-for-luck should be
nondecreasing in the sensitivity of the agent’s compensation to his performance. Controlling for
other effects, salesmen and top executives should have more pay-for-luck than other employees
- bearing in mind that “more” can still be none.

The informativeness principle unveiled by Holmstrom (1979), according to which an unin-
formative variable should not be included in the compensation contract, relies on the validity
of the first-order approach, which in turn is compromised if the agent is, say, risk loving over
a subset of payments. The assumption of a globally risk averse agent is at the core of the
informativeness principle. Instead, with a prudent agent who is not risk averse for payments
higher than a given $W$, it can be optimal to introduce, or not filter out, uninformative risks.

**Corollary 1**: Given $W$, making compensation more noisy in the $[W, \infty)$ interval without
changing its expected value relaxes both the participation constraint and the incentive con-
straint.

Put differently, a more noisy pay in the region where the agent is risk loving increases
incentives and augments the agent’s valuation of the contract, at zero cost to the principal.
Once again, given a set of preferences, pay-for-luck will only be observed if the agent is paid
more than a threshold level above which risk aversion ceases ($W$ in corollary 1) in certain
states of nature - otherwise the informativeness principle applies. The model therefore predicts
that pay-for-luck will be more common among top managers than among baseline employees.
Furthermore, it will be concentrated on the set of high payments, whereas it will not apply on
the downside.

**6 Conclusion**

With moral hazard and unconstrained transfers, the tradeoff between risk sharing and incen-
tives determines the level of effort. The agent is given incentives at the cost of a deviation from
the optimal risk sharing rule. This paper has shown that designing a compensation contract
to induce a given level of effort involves another tradeoff, between valuation and the effective-
ness of incentives. Whereas risk sharing leads to protection against downside risk and upside
participation for prudent agents, incentive considerations lead to contracts with a relatively
flat wage for most outcomes coupled with strong punishments for poor performances. Risk aversion alone makes convex contracts suboptimal, while prudence potentially makes some convex contracts optimal. Given the form of managerial compensation contracts, and the weakness of punishments for failure, it seems that prudence, rather than risk aversion, is key to understanding the curvature of CEO pay.

The model generates a number of predictions. First, efficient incentive schemes will typically be characterized by observed low pay-performance sensitivities: in equilibrium, the agent is approximately insured, but still adequately incentivized. The model notably predicts that for a given level of incentives, compensation schemes such as stock-options-based ones which use small and moderate rewards will be characterized by a higher average pay-performance sensitivity, and will be more costly on average. Second, in the standard CRRA-lognormal setting, supposing that the principal offers efficient contracts, the lower the agent’s relative risk aversion, the more convex is his compensation contract. More precisely, convex contracts which concentrate variations in pay on the right tail of performances will be offered to agents whose relative risk aversion is smaller than a given cutoff. Concave contracts which concentrate variations in pay on the left tail (on the subset of performances where the limited liability constraint is not binding) will be offered to agents whose relative risk aversion is larger than the cutoff. If we accept that CEOs and entrepreneurs tend to be less risk averse than rank-and-file employees, it makes sense that the former be incentivized with convex contracts, and the latter with punishments such as dismissal.

This being said, for widely used utility functions and plausible values of risk aversion, contracts which are concave as a function of the stock price and punishments for failure seem to be more efficient than convex contracts and rewards for success. Besides, nonpositive transfers for poor performances are robust to different specifications of preferences. Conditional on the standard moral hazard model for executive compensation being appropriate, the fact that they are not observed in practice either implies that CARA and CRRA preferences do not accurately reflect individual preferences, or that current CEO contracts are inefficient.

7 Appendix

How the MLRP is obtained

If $\bar{\pi} = e + \tilde{e}$, then $\vartheta(\pi|e) = \varphi(\pi - e)$, and $\vartheta_\pi(\pi|e) = -\varphi_\pi(\pi - e)$.

The MLRP is

$$\frac{d}{d\pi} \left\{ \frac{\vartheta_\pi(\pi|e)}{\vartheta(\pi|e)} \right\} \geq 0 \quad \forall \pi$$
Which can be rewritten as
\[
\frac{d}{d\pi} \left\{ \frac{-\varphi(\pi - e)}{\varphi(\pi - e)} \right\} \geq 0 \quad \forall \pi
\]
The p.d.f. of \( \tilde{\pi} \) therefore satisfies the MLRP if and only if
\[
\left( \varphi(\pi - e) \right)^2 > \varphi(\pi - e)\varphi_{\pi\pi}(\pi - e) \quad \forall \pi
\] (16)
Since the left-hand side reaches its minimum at \( \epsilon = 0 \), this condition is more likely to be satisfied when the cumulative distribution function of \( \tilde{e} \) is weakly convex to the left of the mean and weakly concave to the right, and when the corresponding p.d.f. is concave around the mean. In such a case, \( \varphi_{\pi\pi} \) is negative when \( \varphi(\pi - e) \) is large, and positive values of \( \varphi_{\pi\pi} \) are weighted by low values of \( \varphi(\pi - e) \). The condition (16) is satisfied by the normal distribution as long as the variance is not infinite.

**Proof of claim 1:**

Let \( \Theta \) be the c.d.f. of \( \tilde{\pi} \). The local incentive constraint (16) can also be written in the following form:
\[
\int_{-\infty}^{\infty} u[W(\pi)] \frac{\partial}{\partial e} \vartheta(\pi) d\pi = \psi'(e^*)
\]
Consider the compensation contract defined by \( W'(\pi) = \tilde{W} \) for \( \pi \in (-\infty, q) \) and \( W(\pi) = \bar{W} \) for \( \pi \in [q, \infty) \). The corresponding incentive constraint is
\[
\int_{-\infty}^{q} u[W] \frac{\partial}{\partial e} \vartheta(e^* + \epsilon) d\epsilon + \int_{q}^{\infty} u[\bar{W}] \frac{\partial}{\partial e} \vartheta(e^* + \epsilon) d\epsilon = \psi'(e^*)
\] (17)
where we use the fact that \( d\pi = d(e + \epsilon) = de \). The participation constraint is
\[
\int_{-\infty}^{q} u[W] \vartheta(e^* + \epsilon) d\epsilon + \int_{q}^{\infty} u[\bar{W}] \vartheta(e^* + \epsilon) d\epsilon = \hat{U} + \psi(e^*)
\] (18)
If \( W \geq W_1 \) and \( \bar{W} \leq W_2 \), then, marginal utility being constant,
\[
u(W) = u[W^\star] + u'[W^\star](W - W^\star) \quad \text{and} \quad \nu(\bar{W}) = u[W^\star] + u'[W^\star](\bar{W} - W^\star)
\] (19)
Plugging into (18), the agency cost of this contract is zero, which implies that \( q, \tilde{W} \text{ and } \bar{W} \) can be set so that
\[
\Theta(q)\tilde{W} + (1 - \Theta(q))\bar{W} = W^\star
\] (20)
Plugging the terms in (18) into the participation constraint, (18) is now
\[
u(W^\star) + \nu'[W^\star]\left( \Theta(q)\tilde{W} + (1 - \Theta(q))\bar{W} - W^\star \right) = \hat{U} + \psi(e^*)
\] (21)
Comparing with (18), the participation constraint is satisfied and binding if and only if (18) holds. Take the payments \( \tilde{W} \) and \( \bar{W} \) as given. Since by definition \( \Theta(q) \) is an increasing and
continuous function equal to zero for the smallest possible value of \( q \) and to one for the highest possible value of \( q \), and since \( W_1 < W^* < W_2 \), for any admissible value of \( W \) and \( \bar{W} \), there exists a \( q \) that satisfies the participation constraint.

Finally, the payments must be adjusted to satisfy the incentive constraint. Given \( W < W^* \) and \( q \), there exists \( \bar{W} \) such that the incentive constraint (??) is satisfied. Alternatively, given \( W > W^* \) and \( q \), there exists \( W \) such that the incentive constraint (??) is satisfied. In the first case, this necessitates that \( W_2 \) be sufficiently larger than \( W^* \). In the second case, this necessitates that \( W_1 \) be sufficiently smaller than \( W^* \). One way or another, the wedge between \( W_1 \) and \( W_2 \) must be “large enough”.

The first point in claim 1 is proved. The proofs of the following two points follow by respectively setting \( W_1 = -\infty \) and \( W_2 = \infty \).

**A utility function with a constant second derivative**

Consider the function \( u(x) = -x^2 \) on the interval \((-1, 0)\). This interval has the power of the continuum, and hence of the set of all real numbers (which means that there exists a one-to-one correspondence between these two sets). We are going to extend it to \((a, b)\), where \((a, b) \in \mathbb{R}^2\). Denote the new function by \( U \). For \( \lambda \in [0, 1] \), it is defined by

\[
\lim_{y \to \lambda a + (1-\lambda)b} U'(y) = \lim_{x \to -\lambda} u'(x)
\]

This equation establishes a one-to-one correspondence between \( y \) and \( \lambda \), and between \( \lambda \) and \( x \). It therefore establishes a one-to-one correspondence between \( x \) and \( y \). Given \( a \) and \( b \), we can express the argument \( y \) of \( U \) as a function of \( \lambda \): \( U''(y) = u''(-\lambda) \). Importantly, \( U''(y) \) is positive on \((a, b)\). Let \( a = -b \), so that \( U'(0) = u'(-0.5) = 1 \). The function \( U \) is also characterized by

\[
U''(y) \equiv u''(x) \equiv -2 \quad \forall y \in (a, b), x \in (-1, 0)
\]

\[
U^n(y) \equiv 0 \quad \forall y \in (a, b), n \geq 3
\]

Set \( U(0) = 0 \). For \( y \in (0, b) \), use the fundamental theorem of calculus to get:

\[
U(y) = \int_0^y U'(z)dz
\]

Likewise for \( y \in (-b, 0) \),

\[
U(y) = -\int_y^0 U'(z)dz
\]

Finally let \( b \) be arbitrarily large.

**The first-order approach with step contracts**
The incentive constraint\(^{44}\) is
\[
\lim_{a \to 0} \int_{q-a}^{q+a} s u'[W(\epsilon)] \varphi(\epsilon) d\epsilon = \psi'(e^*)
\]
where \( s = \frac{w - w}{2a} \). Or
\[
\varphi(-q) \hat{w} \int_{-\hat{w}}^{\hat{w}} u'[W] dW = \psi'(e^*) \tag{22}
\]

The left-hand side of the incentive constraint (??) is independent of \( e \). Since the effort cost \( \psi(e) \) is convex, only one \( e \) satisfies (??) given the parameters of the contract (which are in turn adjusted so that the \( e \) in question is \( e^* \)). Moreover, the agent’s optimization problem does not have a corner solution: at \( e = 0 \), the agent increases his expected utility by marginally increasing his effort, since \( \psi'(0) = 0 \); as \( e \) approaches infinity, the agent increases his expected utility by marginally decreasing his effort, since \( \lim_{e \to \infty} \psi'(e) = \infty \) and the marginal utility of monetary transfers is bounded above. As a consequence, the first-order approach is valid if and only if the second-order condition is verified at \( e^* \), i.e., if the second derivative of the agent’s objective function at \( e^* \) is negative.

It writes as
\[
E \left[ W''( \tilde{\epsilon} ) u'[W(\tilde{\epsilon})] \right] + E \left[ (W'( \tilde{\epsilon} ))^2 u''[W(\tilde{\epsilon})] \right] - \psi''(e^*) \tag{23}
\]
Start with the first term. Special treatment is needed, since \( W'' \) is undefined for a step contract. But we know that a step contract is obtained as the limit of a floored and capped contract with a linear slope centered around the cutoff \( q \), as the slope \( \frac{w - w}{2a} \) approaches infinity. This latter contract is convex for low performances around the low kink at \( q - a \), and convex for high performances around the high kink at \( q + a \). Furthermore, the curvature is exactly symmetric: \( W''(q - a) = -W''(q + a) \equiv W''(q) > 0 \). For \( 0 < x \leq \frac{w + w}{2} \),
\[
E \left[ W''( \tilde{\epsilon} ) u'[W(\tilde{\epsilon})] \right] \approx \lim_{a \to 0} \left\{ P(q - a < \tilde{\epsilon} < q + a) \right. \\
\left. \left( P(\tilde{\epsilon} < q - a < \tilde{\epsilon} < q + a) W''(q - a) \int_{-\hat{w}}^{\hat{w} + x} u'[W] dW + P(\tilde{\epsilon} > q - a < \tilde{\epsilon} < q + a) W''(q + a) \int_{\hat{w} - x}^{\hat{w}} u'[W] dW \right) \right\}
\approx \lim_{a \to 0} \left\{ \varphi(q) 2a \frac{1}{2} W''(q) (u[w + x] - u[w] - u[\hat{w}] + u[\hat{w} + x]) \right\}
\]
With a positive and decreasing marginal utility, this term is positive.

The second term in (??) is
\[
E \left[ (W'( \tilde{\epsilon} ))^2 u''[W(\tilde{\epsilon})] \right] = \int_{q-a}^{q+a} \frac{\hat{w}^2}{(2a)^2} u''[W(\epsilon)] \varphi(\epsilon) d\epsilon
\]
\[
= \hat{w}^2 \varphi(q) (u'[w] - u'[\hat{w}]) \lim_{a \to 0} \frac{1}{2a}
\]
\(^{44}\)A more detailed derivation is available in the proof of proposition 3a.
With a decreasing marginal utility, this expression is negative. The second derivative of the agent’s objective function evaluated at $e^\ast$ is

$$\varphi(q)W''(q)(u[w + x] - u[w] - u[\bar{w}] + u[\bar{w} + x]) \lim_{a \to 0} \{a\} + \bar{w}^2 \varphi(q)(u'[w] - u'[\bar{w}]) \lim_{a \to 0} \frac{1}{2a} - \psi''(e^\ast)$$

Since marginal utility is decreasing, the first term is positive, and the second term is negative. However, the former becomes arbitrarily small as $a$ approaches zero, whereas the absolute value of the latter explodes. Effort cost being convex, the last term is negative. The second derivative of the agent’s objective function is always negative at $e^\ast$, and the first-order approach is valid.

**Proof of proposition 1a:**

Assume that the second derivative of $u$ is constant on the whole domain. Since $\bar{w}_R$ is bounded, we can construct a quadratic utility function whose marginal utility is positive for the set of eligible payments - even when $\tilde{\epsilon}$ has an unbounded support.

Given the reward contract $W$, defined by $\{ q, w_R, \bar{w}_R \}$ and $q > 0$ that satisfies the participation constraint of the agent and induces him to exert effort $e^\ast$, consider the contract $V$ defined by

$$\{-q, w_P, \bar{w}_P\} \equiv \{-q, 2E[W(\tilde{\pi}) - \bar{w}_R, 2E[W(\tilde{\pi})] - \bar{w}_R\}$$

By construction, $V$ is a punishment contract.

Rewrite the agency cost in (??) for a utility function with a constant second derivative, i.e., of the form $u(x) = kx^2 + hx + a$, where $k < 0$, $h > 0$:

$$AC_{u,W} = \int_{-\infty}^{\infty} \left[ kW'^2 + hW'^2 + a + (kW'^2 + h)[W(\pi) - W^\ast] \right] - kW(\pi)^2 - hW(\pi) - a \varphi(\epsilon)d\epsilon \quad (24)$$

Taking expectations term by term:

$$AC_{u,W} = kW'^2 + hW'^2 + a + (kW'^2 + h)[E[W(\tilde{\pi})] - W^\ast] - kE[W(\tilde{\pi})^2] - hE[W(\tilde{\pi})] - a \quad (25)$$

Substituting for the expression of the second moment about the origin:

$$AC_{u,W} = kW'^2 + kW'^2 [E[W(\tilde{\pi})] - W^\ast] - k\left[ Var[W(\tilde{\pi})] + E[W(\tilde{\pi})]^2 \right] \quad (26)$$

Notice that if $k = 0$, there are no agency costs, as expected. In this equation, the two terms that differ across contracts are the expectation and the variance of compensation. With quadratic utility, two contracts with the same expectation and the same variance have the same agency cost. Should one satisfy the participation constraint of the agent, then the other
satisfies it too, as the expected quadratic utility of any random variable $x$ only involves the expected value of $x$ and its variance:

$$E[u(\tilde{x})] = E[k\tilde{x}^2 + h\tilde{x} + a] = kE[\tilde{x}^2] + hE[\tilde{x}] + a = k\text{var}[\tilde{x}] + k(E[\tilde{x}])^2 + hE[\tilde{x}] + a$$

Bearing that in mind, first note that the expected payment of $V$ is equal to the expected payment of $W$:

$$E[V(\pi)] = \Phi(-q)w_P + (1 - \Phi(-q))\bar{w}_P = (1 - \Phi(q))(2E[W(\tilde{x})] - \bar{w}_R) + \Phi(q)(2E[W(\tilde{x})] - \bar{w}_P)$$

$$= (1 - \Phi(q))\bar{w}_R - \Phi(q)\bar{w}_P + 2E[W(\tilde{x})] = E[W(\tilde{x})]$$

where the symmetry of the probability distribution was used. Second, the variance of $V$ is equal to the variance of $W$:

$$\text{var}[V(\tilde{x})] = \Phi(-q)(w_P - E[V(\tilde{x})])^2 + (1 - \Phi(-q))(\bar{w}_P - E[V(\tilde{x})])^2$$

$$= (1 - \Phi(q))(2E[W(\tilde{x})] - \bar{w}_R - E[V(\tilde{x})])^2 + \Phi(q)(2E[W(\tilde{x})] - \bar{w}_R - E[V(\tilde{x})])^2$$

$$= (1 - \Phi(q))(E[W(\tilde{x})] - \bar{w}_R)^2 + \Phi(q)(E[W(\tilde{x})] - \bar{w}_P)^2 = \text{var}[W(\tilde{x})]$$

Now consider the incentives provided by $W$ and $V$. The left-hand side of the incentive constraint with a reward contract is

$$\int_{-\infty}^{\infty} W'(\pi)u'[W(\pi)]\varphi(\epsilon)d\epsilon = \lim_{a \to 0} \int_{q-a}^{q+a} \frac{1}{2a}(\bar{w}_R - \bar{w}_P)u'[W(\pi)]\varphi(\epsilon)d\epsilon$$

$$\approx \varphi(q)\bar{w} \int_{\Xi_R}^{\bar{w}_R} u'[W]dW$$

(27)

The left-hand side of the incentive constraint with a punishment contract is likewise

$$\int_{-\infty}^{\infty} V'(\pi)u'[V(\pi)]\varphi(\epsilon)d\epsilon \approx \varphi(-q)\bar{w} \int_{\Xi_P}^{\bar{w}_P} u'[W]dW$$

(28)

The facts that $\Phi(0) = 0.5$ and $q > 0$, together with the fact that both contracts have the same expected payment, imply that

$$\bar{w}_P + w_P < \bar{w}_R + \bar{w}_R$$

In turn, this inequality, when coupled with decreasing marginal utility and the fact that both contracts have the same wedge between payments, implies that the left-hand side of the incentive constraint of a punishment contract is higher than the left-hand side of the incentive constraint of a reward contract. The wedge between $\bar{w}_P$ and $\bar{w}_P$ is too large: the punishment contract induces an effort higher than $e^*$.

Maintaining the cutoff $-q$ fixed, we raise $w_P$ while simultaneously reducing $\bar{w}_P$ in order for the expected payment to remain unchanged, until the wedge between the two payments
is sufficiently reduced for the punishment contract to be incentive-compatible. For any unit increase in \( w_P \), \( \bar{w}_P \) changes by
\[
\begin{align*}
\frac{d\bar{w}_P}{dw_P} &= -\frac{\Phi(-q)}{1 - \Phi(-q)} < 0
\end{align*}
\] (29)

Given this condition, the increase in \( w_P \) is determined for \((??)\) to be equal to \((??)\). There exists a unique solution to this problem. In effect, the change in the left-hand side of the incentive constraint is
\[
d\left\{ \varphi(-q)\tilde{w}\int_{w_P}^{\bar{w}_P} u'[W]dW \right\} = \left[ -u'[w_P]dw_P + u'[\bar{w}_P]d\bar{w}_P + d\tilde{w}(u[\bar{w}_P] - u[w_P]) \right]\varphi(-q)
\]
Marginal utility and \( dw_P \) being positive, the first term is negative. As \( d\bar{w}_P \) is negative, the second term is negative as well. The third term is also negative: utility is increasing, and the wedge between payments shrinks. That is, the left-hand side of the incentive constraint is reduced by an increase in \( w_P \) and the corresponding decrease in \( \bar{w}_P \). What is more, with \( w_P = \bar{w}_P \), it is equal to zero. Therefore, there exists only one couple \( \{w_P, \bar{w}_P\} \) that preserves the expected payment and induces effort \( e^* \).

At this point, \( W \) and \( V \) implement the same effort at the same agency cost (since they share the same expected payment, see \((??)\)). However, reducing the wedge between \( w_P \) and \( \bar{w}_P \) while maintaining the expected payment fixed has reduced the variance of \( V \) (this is trivial), which relaxes the participation constraint. The last stage of the proof consists in reducing both \( w_P \) and \( \bar{w}_P \) in order for the left-hand side of the incentive constraint to remain unchanged, until the participation constraint binds. Downward translations in \( w_P \) and \( \bar{w}_P \) must be such that
\[
d\left\{ \varphi(-q)\tilde{w}\int_{w_P}^{\bar{w}_P} u'[W]dW \right\} = 0
\]
Or
\[
\frac{d\bar{w}_P}{dw_P} = \frac{u'([w_P]) + u[\bar{w}_P] - u[w_P]}{u'([w_P]) + u[\bar{w}_P] - u[w_P]}
\]
The right-hand side of this equation is positive. Hence, both payments must diminish. Since marginal utility is decreasing, we must have \( d\bar{w}_P < dw_P \). The “fixed wage” \( \bar{w}_P \) must be reduced more than the “punishment” \( w_P \). Reducing both \( w_P \) and \( \bar{w}_P \) obviously reduces the expected payment, and therefore agency costs. The new contract, which the agent accepts, induces \( e^* \) at a lower agency cost than \( W \). It therefore dominates \( W \).

**Proof of proposition 1b:**

Assume that the second derivative of \( u \) is constant on the whole domain.
Given the convex contract $W$ that satisfies the participation constraint of the agent and induces him to exert effort $e^*$, consider the contract

$$V(\pi) \equiv -\delta W(-\pi + 2e^*) + w$$  \hspace{1cm} (30)$$

with $\delta$ strictly positive. Since $W(\pi)$ is convex in $\pi$, $\delta W(-\pi + 2e^*)$ is convex in $\pi$, and $V(\pi)$ is concave in $\pi$.

In equation (30), the two terms that differ across contracts are the expectation and the variance of compensation. With quadratic utility, two contracts with the same expectation and the same variance have the same agency cost. If one satisfies the participation constraint, then the other does too, as the expected quadratic utility of a random variable $\tilde{x}$ is only a function of the variance and the expectation of $x$:

$$E[u(\tilde{x})] = E[k\tilde{x}^2 + h\tilde{x} + a] = kE[\tilde{x}^2] + hE[\tilde{x}] + a = k\text{var}[\tilde{x}] + k(E[\tilde{x}])^2 + hE[\tilde{x}] + a$$

For any value of $w$, the functions $V(\pi)$ and $W(\pi)$ have the same variance for $\delta = 1$. In effect, consider the function $Y$, symmetric to $W$ with respect to the horizontal line going through the point $(e^*, W(e^*))$:

$$Y(\pi) = -W(\pi) + 2W(e^*)$$

Its variance writes

$$\text{var}[Y(\tilde{x})] \equiv E\left[\left(Y(\tilde{\pi}) - E[Y(\tilde{\pi})]\right)^2\right] = E\left[\left(-W(\tilde{\pi}) + 2W(e^*) + E[W(\tilde{\pi})] - 2W(e^*)\right)^2\right]$$

$$= E\left[\left(W(\tilde{\pi}) - E[W(\tilde{\pi})]\right)^2\right] \equiv \text{var}[W(\tilde{\pi})]$$  \hspace{1cm} (31)$$

Consider the function $V_{\delta=1}$, symmetric to $Y$ with respect to the vertical line going through the point $(e^*, W(e^*))$. By definition, it writes as

$$V_{\delta=1}(\pi) = Y(\pi - 2(\pi - e^*)) = Y(-\pi + 2e^*)$$

The expectation of $V_{\delta=1}(\pi)$ is

$$E[V_{\delta=1}(\tilde{\pi})] \equiv \int_{-\infty}^{\infty} V_{\delta=1}(\pi)\vartheta(\pi)d\pi = \int_{-\infty}^{\infty} Y(-\pi + 2E[\tilde{\pi}])\vartheta(\pi)d\pi$$

$$= \int_{-\infty}^{\infty} Y(\pi)\varphi(-\pi + 2E[\tilde{\pi}])d\pi = \int_{-\infty}^{\infty} Y(\pi)\varphi(\pi)d\pi \equiv E[Y(\tilde{\pi})]$$  \hspace{1cm} (32)$$

where the second equality uses the definition of $\varphi$, the third involves a change of variable (and the fact that the variable of integration is immaterial), and the fourth uses the symmetry of
Combining these two results, we get

$$\varphi \text{ unveiled above. Equalities below involve the same steps, plus the fact that } E[Y(\tilde{\pi})] \text{ is a constant.}$$

$$\var{V_{\delta=1}(\pi)} \equiv \int_{-\infty}^{\infty} \left(V_{\delta=1}(\pi) - E[V_{\delta=1}(\pi)]\right)^2 \var{\pi} d\pi = \int_{-\infty}^{\infty} \left(Y(-\pi + 2e^*) - E[Y(\tilde{\pi})]\right)^2 \var{\pi} d\pi = \int_{-\infty}^{\infty} (Y(\pi) - E[Y(\tilde{\pi})])^2 \var{\pi} d\pi \equiv \var{Y(\tilde{\pi})}$$

Eventually, combining (??) with (??),

$$\var{V_{\delta=1}(\pi)} = \var{Y(\tilde{\pi})} = \var{W(\tilde{\pi})}$$

Set \( w \) such that \( E[W(\tilde{\pi})] = E[V(\tilde{\pi})] \). Then \( W \) and \( V \) have the same agency cost, and they both satisfy the participation constraint.

Now consider the incentives these two contracts deliver. For any contract \( f \),

$$E\left[f'(\tilde{\pi})u'[f(\tilde{\pi})]\right] = \text{cov}(f'(\tilde{\pi}), u'[f(\tilde{\pi})]) + E[f'(\tilde{\pi})]E[u'[f(\tilde{\pi})]]$$

First, compare the sign of the first term in the equation above for \( f = W \) and \( f = V \).

Combining

$$\frac{\partial}{\partial \pi} W'(\pi) = W''(\pi) > 0 \quad \text{and} \quad \frac{\partial}{\partial \pi} u'[W(\pi)] = W'(\pi)u''[W(\pi)] < 0$$

To get

$$\text{cov}(W'(\tilde{\pi}), u'[W(\tilde{\pi})]) < 0$$

Then combine

$$\frac{\partial}{\partial \pi} V'(\pi) = V''(\pi) < 0 \quad \text{and} \quad \frac{\partial}{\partial \pi} u'[V(\pi)] = V'(\pi)u''[V(\pi)] < 0$$

To get

$$\text{cov}(V'(\tilde{\pi}), u'[V(\tilde{\pi})]) > 0$$

Second, compare both components of the second term in (??) for \( f = W \) and \( f = V_{\delta=1} \). On the one hand, using the function \( Y \) as defined above, remembering that \( W(e^*) \) is a constant, we get \( Y'(\pi) = -W'(\pi) \), and \( E[Y'(\tilde{\pi})] = E[W'(\tilde{\pi})] \). The derivative of \( V_{\delta=1} \) with respect to \( \pi \) is \( V'_{\delta=1}(\pi) = -Y'(\pi + 2e^*) \). Going once again through the same steps,

$$E[V'_{\delta=1}(\pi)] \equiv \int_{-\infty}^{\infty} V'_{\delta=1}(\pi) \var{\pi} d\pi = \int_{-\infty}^{\infty} -Y'(-\pi + 2E[\tilde{\pi}]) \var{\pi} d\pi = \int_{-\infty}^{\infty} -Y'(\pi) \var{\pi} d\pi = -E[Y'(\tilde{\pi})]$$

Combining these two results,

$$E[V'_{\delta=1}(\pi)] = -E[Y'(\tilde{\pi})] = E[W'(\tilde{\pi})]$$
On the other hand, since the second derivative of the utility function is constant, its first
derivative is of the form \( u'(x) = 2kx + h \), where \( k \) and \( h \) are constants. Because agency costs
are identical under \( W \) and \( V \), we have from (??):

\[
E[W(\tilde{\pi})] = E[V(\tilde{\pi})]
\]

Substituting for the expression of \( V \),

\[
E[W(\tilde{\pi})] = E[-\delta W(-\tilde{\pi} + 2e*) + w]
\]

Multiplying both sides by \( 2k \) then adding \( h \), for \( \delta = 1 \),

\[
E[2kW(\tilde{\pi}) + h] = E[2k(-W(-\tilde{\pi} + 2e*) + w) + h]
\]

Identifying the form of the utility function,

\[
E[u'(W(\tilde{\pi}))] = E[u'(-W(-\tilde{\pi} + 2e*) + w)]
\]

That is,

\[
E[u'(W(\tilde{\pi}))] = E[u'(V(\tilde{\pi}))]
\]  (40)

Using (??) and (??), it appears that the second term on the right-hand side in (??) is identical
for \( W \) and \( V \). Using (??) and (??), the contract \( V \) creates more incentives than \( W \): \( V \) elicits
a higher effort than \( W \), for the same agency cost.

The proof proceeds by showing that there exists a \( \delta \in (0, 1) \) such that \( V \) induces the same
effort as \( W \) for a lower agency cost.

First, incentives delivered by \( V \), \( E[V'(\tilde{\pi})u'[V(\tilde{\pi})]] \), are zero for \( \delta = 0 \), are monotonically
increasing in \( \delta \), and are greater than incentives delivered by \( W \) for \( \delta = 1 \), as shown above.
The first claim is trivial: with \( \delta = 0 \), \( V(\pi) = w \) for all \( \pi \), so that \( V'(\pi) \) is identically zero. As
for the second claim:

\[
\frac{\partial}{\partial \delta} E \left[ V'(\tilde{\pi})u'[V(\tilde{\pi})] \right] = \frac{\partial}{\partial \delta} E \left[ \delta W'(-\tilde{\pi} + 2e*)u'[-\delta W(-\tilde{\pi} + 2e*) + w] \right] = E \left[ \frac{\partial}{\partial \delta} \left\{ \delta W'(-\tilde{\pi} + 2e*)u'[-\delta W(-\tilde{\pi} + 2e*) + w] \right\} \right] = E \left[ W'(-\tilde{\pi} + 2e*)u'[-\delta W(-\tilde{\pi} + 2e*) + w] + \delta W'(-\tilde{\pi} + 2e*)(-W(-\tilde{\pi} + 2e*) + w)u'[-\delta W(-\tilde{\pi} + 2e*) + w] \right] \]  (41)

Because \( W \) and \( u \) are increasing, the first term is positive. Because the second derivative of
the utility function is constant and negative, the second term is of the same sign as

\[
E \left[ W'(-\tilde{\pi} + 2e*)W(-\tilde{\pi} + 2e*) \right] = \text{cov} \left( W'(-\tilde{\pi} + 2e*), W(-\tilde{\pi} + 2e*) \right) + E[W'(-\tilde{\pi} + 2e*)]E[W(-\tilde{\pi} + 2e*)]
\]

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The covariance is positive, as $W$ is increasing and convex:
\[
\frac{\partial}{\partial \pi} W(-\pi + 2e^*) > 0 \quad \text{and} \quad \frac{\partial}{\partial \pi} W'(-\pi + 2e^*) > 0
\]

Because $W$ is increasing, $E[W'(-\pi + 2e^*)]$ is positive. Combining (44) with nonnegative agency costs, $E[(W(-\pi + 2e^*))$ is positive. The second term in (44) is therefore positive.

To summarize, we have shown that
\[
E[V'(\tilde{\pi}) u'[V(\tilde{\pi})]] = \begin{cases} 0 & \text{if } \delta = 0 \\ E[W'(\tilde{\pi}) u'[W(\tilde{\pi})]] & \text{if } \delta = 1 \end{cases}
\]

and that the term $E[V'(\tilde{\pi}) u'[V(\tilde{\pi})]]$ is monotonically increasing in $\delta$ on the whole domain.

It follows that there exists a $\delta$ smaller than one that implements the same effort $e^*$ as the contract $W$.

Second, the agency cost of $V$ is equal to the agency cost of $W$ for $\delta = 1$, as previously proved. Furthermore, using the definition of $V$ and the measure of agency costs in the quadratic utility case in (44),
\[
AC_{u,V} = \kappa \int_{-\infty}^{\infty} \left( -\delta W(-\pi + 2e^*) + w - W^* \right)^2 \varphi(\epsilon) d\epsilon
\]

Taking the derivative with respect to $\delta$,
\[
\frac{\partial}{\partial \delta} AC_{u,V} = 2\kappa E \left[ \left( -\delta W(-\pi + 2e^*) + \frac{\partial w}{\partial \delta} \right) \left( -\delta W(-\pi + 2e^*) + w - W^* \right) \right] \quad (42)
\]

Dropping the positive constant $2\kappa$ for notational convenience and rearranging,
\[
\frac{\partial}{\partial \delta} AC_{u,V} = \left( E \left[ -W(-\pi + 2e^*) \right] + \frac{\partial w}{\partial \delta} \right) E \left[ -\delta W(-\pi + 2e^*) + w - W^* \right] + \text{cov} \left( W(-\pi + 2e^*), \delta W(-\pi + 2e^*) \right) \quad (43)
\]

The “fixed wage” $w$ adjusts so that the participation constraint (39) holds. Applying the implicit function theorem to the participation constraint with contract $V$,
\[
\frac{\partial w}{\partial \delta} = -\frac{E \left[ -W(-\pi + 2e^*) u'[\delta W(-\pi + 2e^*) + w] \right]}{E \left[ u'[\delta W(-\pi + 2e^*) + w] \right]} \quad (44)
\]

Plugging into (43),
\[
\frac{\partial}{\partial \delta} AC_{u,V} = E \left[ -\delta W(-\pi + 2e^*) + w - W^* \right] \left( -E[W(-\pi + 2e^*)] + \frac{\text{cov} \left( W(-\pi + 2e^*), u'[\delta W(-\pi + 2e^*) + w] \right)}{E[u'[\delta W(-\pi + 2e^*) + w]]} \right)
\]
\[
+ E[W(-\pi + 2e^*)] + \delta \text{cov} \left( W(-\pi + 2e^*), W(-\pi + 2e^*) \right) \quad (45)
\]
With quadratic utility, marginal utility is linear. Removing constants out of the covariance,
\[
\frac{\partial}{\partial \delta} AC_{u,V} = -2k\delta\text{cov}(W(-\tilde{\pi} + 2e^*), W(-\tilde{\pi} + 2e^*)) - \frac{E[-\delta W(-\tilde{\pi} + 2e^*) + w - W^*]}{2kE[-\delta W(-\tilde{\pi} + 2e^*) + w]}
\]
\[+ \delta\text{cov}(W(-\tilde{\pi} + 2e^*), W(-\tilde{\pi} + 2e^*))\]

As agency costs are nonnegative,
\[E[-\delta W(-\tilde{\pi} + 2e^*) + w - W^*] \geq 0\]

With (??), we know that \(W^* \geq 0\), which yields
\[\frac{E[-\delta W(-\tilde{\pi} + 2e^*) + w - W^*]}{E[-\delta W(-\tilde{\pi} + 2e^*) + w]} \leq 1\]

It immediately follows that the derivative of the agency cost as a function of \(\delta\) is positive.

As a consequence, a concave contract \(V\) defined in (??) with \(\delta < 1\) implements \(e^*\) at a lower agency cost than the convex contract \(W\). For any convex contract \(W\), there exists a concave contract \(V\) which dominates \(W\).

**Capped utility function and capped payments:**

Very risk averse agents may not value payments in excess of a threshold. In this case their compensation should be bounded above.

**Claim 2:** If there exists \(W_4\) such that \(u(W_5) = u(W_4)\), for any \(W_5 > W_4\), then \(W(\tilde{\pi}) \leq W_4\) for all \(W(\tilde{\pi})\).

Increasing transfers above the threshold \(W_4\) is worthless to the agent, and it does not have any incentive value. This implies that the transfer function to agents whose utility function is capped should be capped.

**Proof:**

Compare any contract \(\tilde{W}\) characterized by \(\tilde{W}(\pi) > W_4\) on a subinterval \([p, \infty)\) to one where \(W(\pi) = W_4\) on the same subinterval. On the one hand, the latter is less costly to the principal. On the other hand, the agent is indifferent between these two contracts, and they generate the same effort. In effect, the participation constraint is unaffected:

\[
\int_{p}^{\infty} u(\tilde{W}(\pi))\phi(\epsilon)d\epsilon = \int_{p}^{\infty} u(W_4)\phi(\epsilon)d\epsilon
\]

The incentive constraint is unaffected as well, as \(u'(\tilde{W}(\pi)) = 0\) for \(\tilde{W}(\pi) \geq W_4\):

\[
\int_{p}^{\infty} \tilde{W}'(\pi)u'(\tilde{W}(\pi))\phi(\epsilon)d\epsilon = 0
\]
Therefore, any contract with \( \hat{W}(\tilde{\pi}) > W_4 \) on a subinterval is dominated by a contract where \( W(\pi) = W_4 \) on this subinterval.

**Proof of proposition 2a:**

As a normalization, define \( \hat{w} \) to be the wedge necessary to induce a risk neutral agent to exert adequate effort. A risk neutral agent has a constant marginal utility, say \( u'(W^*) \). His incentive constraint writes as

\[
\hat{w}(q)u'(W^*)\varphi(q) = \psi'(e^*)
\]

For any \( q \), \( \hat{w} \) solves the equation above:

\[
\hat{w}(q) = \frac{1}{\varphi(q) u'(W^*)} \psi'(e^*)
\] (46)

Denote the expected utility provided by a contract with punishments by \( EU_P \), and by \( EU_R \) for a contract with rewards. By definition,

\[
EU_P = \Phi(-q)u[\hat{w}_P] + (1 - \Phi(-q))u[\hat{w}_P]
\]

\[
EU_R = \Phi(q)u[\hat{w}_R] + (1 - \Phi(q))u[\hat{w}_R]
\]

Using (??) and (??),

\[
EU_P = \Phi(-q)u\left[ \alpha - \frac{1 - \Phi(-q)}{\varphi(-q)} \frac{\psi'(e^*)}{u'(W^*)} \right] + (1 - \Phi(-q))u\left[ \alpha + \frac{\Phi(-q)}{\varphi(-q)} \frac{\psi'(e^*)}{u'(W^*)} \right]
\]

\[
EU_R = \Phi(q)u\left[ \alpha - \frac{1 - \Phi(q)}{\varphi(q)} \frac{\psi'(e^*)}{u'(W^*)} \right] + (1 - \Phi(q))u\left[ \alpha + \frac{\Phi(q)}{\varphi(q)} \frac{\psi'(e^*)}{u'(W^*)} \right]
\]

Rewriting,

\[
EU_P = au[\alpha - \beta b] + bu[\alpha + \beta a]
\]

\[
EU_R = bu[\alpha - \beta a] + au[\alpha + \beta b]
\]

where \( b > a > 0 \), and \( \beta > 0 \). We use a third order Taylor expansion around \( \alpha \).

\[
u(x) = u(\alpha) + u'(\alpha)(x - \alpha) + \frac{1}{2} u''(\alpha)(x - \alpha)^2 + \frac{1}{6} u'''(\alpha)(x - \alpha)^3
\]

We know that there exists \( y_b^+ \in [\alpha - \beta b, \alpha], y_a^- \in [\alpha, \alpha + \beta a], y_a^+ \in [\alpha - \beta a, \alpha], y_b^+ \in [\alpha, \alpha + \beta b].

\[
\begin{align*}
u(\alpha - \beta b) &= u(\alpha) + u'(\alpha)(-\beta b) + \frac{1}{2} u''(\alpha)(-\beta b)^2 + \frac{1}{6} u'''(\alpha)(y_b^+)(-\beta b)^3 \\
u(\alpha + \beta a) &= u(\alpha) + u'(\alpha)(\beta a) + \frac{1}{2} u''(\alpha)(\beta a)^2 + \frac{1}{6} u'''(\alpha)(y_a^-)(\beta a)^3 \\
u(\alpha - \beta a) &= u(\alpha) + u'(\alpha)(-\beta a) + \frac{1}{2} u''(\alpha)(-\beta a)^2 + \frac{1}{6} u'''(\alpha)(y_a^+)(-\beta a)^3 \\
u(\alpha + \beta b) &= u(\alpha) + u'(\alpha)(\beta b) + \frac{1}{2} u''(\alpha)(\beta b)^2 + \frac{1}{6} u'''(\alpha)(y_b^+)(\beta b)^3
\end{align*}
\]
Substituting,

\[ EU_P = (a + b)u[\alpha] + u'[\alpha](-a\beta b + b\beta a) + \frac{1}{2}u''[\alpha](a(-\beta b)^2 + b(\beta a)^2) \]
\[ + \frac{1}{6}au'''(y_+^b)(-\beta b)^3 + \frac{1}{6}bu'''(y_+^a)(\beta a)^3 \]

\[ EU_R = (a + b)u[\alpha] + u'[\alpha](-b\beta a + a\beta b) + \frac{1}{2}u''[\alpha](b(-\beta a)^2 + a(\beta b)^2) \]
\[ + \frac{1}{6}bu'''(y_+^a)(-\beta a)^3 + \frac{1}{6}au'''(y_+^b)(\beta b)^3 \]

First, if \( u'' \) is a constant, so that \( u''' = 0 \), then \( EU_P = EU_R \). Second, if \( u''' \) is a positive constant, then mobilize \( b > a > 0 \) to get \( EU_P < EU_R \). Third, if \( u^{(4)} \neq 0 \), then \( EU_P < EU_R \) if and only if

\[ -u'''(y_+^b)b^2 + u'''(y_+^a)a^2 < -u'''(y_+^a)a^2 + u'''(y_+^b)b^2 \]

Equivalently,

\[ (u'''(y_+^a) + u'''(y_+^b))a^2 < (u'''(y_+^b) + u'''(y_+^a))b^2 \]

This condition will be satisfied if \( u^{(i)} \) is positive for odd \( i \geq 3 \). In effect, if \( u \) is continuously differentiable an infinite number of times, then Taylor’s theorem gives

\[ EU_P = \sum_{i=0}^{\infty} \frac{1}{(2i)!}u^{(2i)}(\alpha)(a(-\beta b)^{2i} + b(\beta a)^{2i}) + \sum_{i=1}^{\infty} \frac{1}{(2i-1)!}u^{(2i-1)}(\alpha)(a(-\beta b)^{2i-1} + b(\beta a)^{2i-1}) \]
\[ EU_R = \sum_{i=0}^{\infty} \frac{1}{(2i)!}u^{(2i)}(\alpha)(b(-\beta a)^{2i} + a(\beta b)^{2i}) + \sum_{i=1}^{\infty} \frac{1}{(2i-1)!}u^{(2i-1)}(\alpha)(b(-\beta a)^{2i-1} + a(\beta b)^{2i-1}) \]

As can be easily checked, all even order terms in \( EU_P \) are equal to those in \( EU_R \); on the contrary, if \( u^{(2i-1)} \) is positive (respectively negative) for all \( i \geq 2 \), then all odd order terms are larger (respectively smaller) in \( EU_R \), except for \( i = 1 \) for which there is equality. Consequently, if \( u^{(2i-1)} \) is positive for all \( i \geq 2 \), then \( EU_R \) is larger than \( EU_P \).

**Proof of proposition 2b:**

Denote a concave contract by \( W_A \), and its expected utility by \( EU_A \). Denote a convex contract by \( W_E \), and its expected utility by \( EU_E \). By definition,

\[ EU_A \equiv \int_{-\infty}^{\infty} u[W_A(\pi)]\varphi(\epsilon)\,d\epsilon \]
\[ EU_E \equiv \int_{-\infty}^{\infty} u[W_E(\pi)]\varphi(\epsilon)\,d\epsilon \]

We henceforth adopt the same approach as in the proof of proposition 2a, except that we integrate the agent’s utility over the whole range of payments, instead of only two payments.
Using a Taylor expansion around $\hat{W}$ for a concave contract, there exists a function $y(W)$ such that

$$EU_A = \int_{-\infty}^{\infty} \left[ u(\hat{W}) + u'[\hat{W}](W_A(\pi) - \hat{W}) + \frac{1}{2} u''[\hat{W}](W_A(\pi) - \hat{W})^2 ight. \
+ \left. \frac{1}{6} u'''[y(W)](W_A(\pi) - \hat{W})^3 \right] \varphi(\epsilon) d\epsilon$$

$$= u(\hat{W}) + u'[\hat{W}] \left( \int_{-\infty}^{\infty} W_A(\pi) \varphi(\epsilon) d\epsilon - \hat{W} \right) + \frac{1}{2} u''[\hat{W}] \int_{-\infty}^{\infty} \left( W_A(\pi) - \hat{W} \right)^2 \varphi(\epsilon) d\epsilon \
+ \frac{1}{6} u'''[y(W)] \int_{-\infty}^{\infty} \left( W_A(\pi) - \hat{W} \right)^3 \varphi(\epsilon) d\epsilon$$

where the second term is zero because of (??). Similarly for a convex contract defined by $W_E = 2\hat{W} - W_A(-\pi + 2\epsilon^*),$

$$EU_E = u(\hat{W}) + \frac{1}{2} u''[\hat{W}] \int_{-\infty}^{\infty} \left( \hat{W} - W_A(-\pi + 2\epsilon^*) \right)^2 \varphi(\epsilon) d\epsilon \
+ \frac{1}{6} u'''[y(W)] \int_{-\infty}^{\infty} \left( \hat{W} - W_A(-\pi + 2\epsilon^*) \right)^3 \varphi(\epsilon) d\epsilon$$

Adding up all higher order terms and using Taylor’s theorem,

$$EU_A = \sum_{i=0}^{\infty} \frac{1}{(2i)!} u^{(2i)}(\hat{W}) \int_{-\infty}^{\infty} \left( W_A(\pi) - \hat{W} \right)^{2i} \varphi(\epsilon) d\epsilon$$

$$+ \sum_{i=1}^{\infty} \frac{1}{(2i-1)!} u^{(2i-1)}(\hat{W}) \int_{-\infty}^{\infty} \left( W_A(\pi) - \hat{W} \right)^{2i-1} \varphi(\epsilon) d\epsilon$$

$$EU_E = \sum_{i=0}^{\infty} \frac{1}{(2i)!} u^{(2i)}(\hat{W}) \int_{-\infty}^{\infty} \left( \hat{W} - W_A(-\pi + 2\epsilon^*) \right)^{2i} \varphi(\epsilon) d\epsilon$$

$$+ \sum_{i=1}^{\infty} \frac{1}{(2i-1)!} u^{(2i-1)}(\hat{W}) \int_{-\infty}^{\infty} \left( \hat{W} - W_A(-\pi + 2\epsilon^*) \right)^{2i-1} \varphi(\epsilon) d\epsilon$$

All even order terms in $EU_E$ are equal to those in $EU_A$ since for any integer $i \geq 0,$

$$\int_{-\infty}^{\infty} \left( W_E(\pi) - \hat{W} \right)^{2i} \varphi(\epsilon) d\epsilon = \int_{-\infty}^{\infty} \left( \hat{W} - W_A(-\pi + 2\epsilon^*) \right)^{2i} \varphi(\epsilon) d\epsilon$$

$$= \int_{-\infty}^{\infty} \left( \hat{W} - W_A(\pi) \right)^{2i} \varphi(\epsilon) d\epsilon = \int_{-\infty}^{\infty} \left( W_A(\pi) - \hat{W} \right)^{2i} \varphi(\epsilon) d\epsilon$$

(47)

where the second equality follows from $E[\hat{\pi}] = \epsilon^*$ and the fact that the probability distribution is symmetric around the mean. The first moment is zero, as previously observed. The signs of higher moments are still left to determine. I prove below, in the last part of the proof, that their sign is positive for a convex $W$. The proof that they are negative for a concave $W$ follows the exact same lines. Combining this result with the condition that $u^{(2i+1)}$ is positive for $i \geq 1$ implies that $EU_E \geq EU_A$, which proves proposition 2b. Also, it immediately follows that if
$u'''$ is a positive constant (so that all higher-order derivatives are zero), then $EU_E \geq EU_A$.

Lastly, if $u''' = 0$, we have $EU_E = EU_A$.

Consider any increasing and convex function $f$ with argument $\pi$. Denote its mean by $m_1$. We are going to replicate the function $f$ as the limit of a sequence of piecewise linear functions $g_j$. Set up a linear function $g_0$ with any positive slope such that $g_0(e^*) = m_1$. Call the intersection between $f$ and $m_1$ a “node”, denote it by $n_1$, and set $\hat{\pi}_1 \equiv f^{-1}(m_1)$. Clearly, since $m_1$ is the mean of $f$,

$$\int_{-\infty}^{\hat{\pi}_1} (m_1 - f(\pi)) \vartheta(\pi) d\pi = \int_{\hat{\pi}_1}^{\infty} (f(\pi) - m_1) \vartheta(\pi) d\pi$$

Next, set up a piecewise linear function $g_1$ defined by its origin $n_1$ and its slopes $s_1^-$ and $s_1^+$ such that

$$g_1(\hat{\pi}_1) = m_1$$

$$\int_{-\infty}^{\hat{\pi}_1} (m_1 - s_1^- (\pi - \hat{\pi}_1)) \vartheta(\pi) d\pi = \int_{-\infty}^{\hat{\pi}_1} f(\pi) \vartheta(\pi) d\pi$$

$$\int_{\hat{\pi}_1}^{\infty} (m_1 + s_1^+ (\pi - \hat{\pi}_1)) \vartheta(\pi) d\pi = \int_{\hat{\pi}_1}^{\infty} f(\pi) \vartheta(\pi) d\pi$$

These last two equalities imply that the function $g_1$ has the same mean $m_1$ as $f$. The transformation of $f$ into $g_1$ is mean-preserving.

This is illustrated in figure 11. We now repeat this procedure on each sub-interval.
Denote the mean of \( f \) on \((-\infty, \hat{\pi}_1)\) by \( m_2^- \):

\[
\int_{-\infty}^{\hat{\pi}_1} f(\pi) d\pi = m_2^-
\]

Denote the mean of \( f \) on \((\hat{\pi}_1, \infty)\) by \( m_2^+ \):

\[
\int_{\hat{\pi}_1}^{\infty} f(\pi) d\pi = m_2^+
\]

Denote the intersections between \( f \) and \( m_2^- \) by \( n_2^- \), and set \( \hat{\pi}_2^- \equiv f^{-1}(m_2^-) \) (where \( \hat{\pi}_2^- \in (-\infty, \hat{\pi}_1) \)). Denote the intersections between \( f \) and \( m_2^+ \) by \( n_2^+ \), and set \( \hat{\pi}_2^+ \equiv f^{-1}(m_2^+) \) (where \( \hat{\pi}_2^+ \in (\hat{\pi}_1, \infty) \)). Clearly, since \( f \) has a mean of \( m_2^- \) on \((-\infty, \hat{\pi}_1)\), and of \( m_2^+ \) on \((\hat{\pi}_1, \infty)\),

\[
\int_{-\infty}^{\hat{\pi}_2^-} (m_2^- - f(\pi)) d\pi = \int_{\hat{\pi}_2^-}^{\hat{\pi}_1} (f(\pi) - m_2^-) d\pi
g\]

\[
\int_{\hat{\pi}_2^-}^{\hat{\pi}_1} (m_2^- - f(\pi)) d\pi = \int_{\hat{\pi}_1}^{\hat{\pi}_2^+} (f(\pi) - m_2^-) d\pi
\]

Next, at each of the nodes \( n_2^- \) and \( n_2^+ \), set up a piecewise linear function \( g_2 \) defined by its origin - either \( n_2^- \) or \( n_2^+ \) - and its slopes - respectively \( s_2^-, s_2^+, s_2^-, \) and \( s_2^+ \) - such that, for the node of coordinates \((\hat{\pi}_2^-, m_2^-)\),

\[
g_2(\hat{\pi}_2^-) = m_2^-
\]

\[
\int_{\hat{\pi}_1}^{\hat{\pi}_2^-} (m_2^- - s_2^-(\pi - \hat{\pi}_2^-)) d\pi = \int_{\hat{\pi}_2^-}^{\hat{\pi}_1} f(\pi) d\pi
\]

\[
\int_{\hat{\pi}_2^-}^{\hat{\pi}_1} (m_2^- + s_2^+(\pi - \hat{\pi}_2^+)) d\pi = \int_{\hat{\pi}_1}^{\hat{\pi}_2^+} f(\pi) d\pi
\]

And likewise from the node of coordinates \((\hat{\pi}_2^+, m_2^-)\). To summarize, on the interval \((-\infty, \hat{\pi}_1)\) both \( g_1 \) and \( g_2 \) have a mean of \( m_2^- \); on the interval \((\hat{\pi}_1, \infty)\) both \( g_1 \) and \( g_2 \) have a mean of \( m_2^+ \).

This second step on the interval \((-\infty, \hat{\pi}_1)\) is illustrated in figure 12.

Repeat the same procedure on the four intervals \((-\infty, \hat{\pi}_2^-), [\hat{\pi}_2^-, \hat{\pi}_1], [\hat{\pi}_1, \hat{\pi}_2^+], \) and \([\hat{\pi}_2^+, \infty)\), and so on.

Eventually, for any \( \pi \),

\[
\lim_{j \to \infty} g_j(\pi) = f(\pi)
\]

Departing from a linear function \( g_0 \), we constructed \( f \) step by step, where each step involved a mean-preserving transformation of \( g_j \) into \( g_{j+1} \), for \( j \geq 0 \). That is, for any \( j \),

\[
E[g_j(\pi)] = E[f(\pi)] = m_1
\]
Figure 12: A mean-preserving transformation: step 2.

Notice that $g_j$ is a discontinuous, piecewise linear function. Transforming $g_j$ into $g_{j+1}$ approximates the convex function $f$ more closely.

We are now going to show that for a convex $f$, each step of this transformation increases the $i$-th moment, where $i \geq 2$.

Consider the interval $(\bar{\pi}, \hat{\pi})$, on which $g_j$ has slope $s_j$ and

$$\int_\pi^{\bar{\pi}} g_j(\pi) \vartheta(\pi) d\pi \equiv m_{j+1}$$

Since by construction we must also have

$$\int_\pi^{\hat{\pi}} f(\pi) \vartheta(\pi) d\pi \equiv m_{j+1}$$

We know that $g_{j+1}$ on $(\bar{\pi}, \hat{\pi})$ is such that

$$g_{j+1}(\hat{\pi}_{j+1}^+) = m_{j+1}$$

and

$$\int_\pi^{\hat{\pi}_{j+1}} \left( m_{j+1} - s_{j+1}^- (\pi - \hat{\pi}_{j+1}) \right) \vartheta(\pi) d\pi = \int_{\hat{\pi}_{j+1}}^{\pi} \left( s_{j+1}^+ (\hat{\pi}_{j+1} - \pi) - m_{j+1} \right) \vartheta(\pi) d\pi$$

where, because $f$ is convex, $\hat{\pi}_{j+1} \equiv g_j^{-1}(m_{j+1}) < g_{j+1}^{-1}(m_{j+1}) \equiv \hat{\pi}_{j+1}$. This is illustrated in figure 13.
We focus on the action above $m_{j+1}$ - the proof for the part under $m_{j+1}$ is exactly symmetric, and therefore omitted. By construction, the mean-preserving transformation of $g_j$ into $g_{j+1}$ is such that

$$\int_{\hat{\pi}_{j+1}}^{\bar{\pi}} \left( s_j(\pi - \hat{\pi}_{j+1}) - m_{j+1} \right) \vartheta(\pi) d\pi = \int_{\hat{\pi}_{j+1}}^{\bar{\pi}} \left( s_{j+1}^+ (\pi - \hat{\pi}_{j+1}) - m_{j+1} \right) \vartheta(\pi) d\pi$$

Based on this equality - which implies that $s_{j+1}^+ > s_j$, we must show that, for any integer $i \geq 2$,

$$\int_{\hat{\pi}_{j+1}}^{\bar{\pi}} \left( s_j(\pi - \hat{\pi}_{j+1}) - m_{j+1} \right)^i \vartheta(\pi) d\pi < \int_{\hat{\pi}_{j+1}}^{\bar{\pi}} \left( s_{j+1}^+ (\pi - \hat{\pi}_{j+1}) - m_{j+1} \right)^i \vartheta(\pi) d\pi$$

I prove this result heuristically. The function $g_j$ is turned into $g_{j+1}$ by conducting a series of incremental changes, each of which is mean-preserving by construction. Where $\pi$ is higher than $\hat{\pi}_{j+1}$ and such that $g_{j+1}(\pi) < g_j(\pi)$, the area in-between $g_j(\pi)$ and $\max\{m_{j+1}, g_{j+1}(\pi)\}$ is successively transferred to exactly cover the area in-between $g_{j+1}(\pi)$ and $g_j(\pi)$, where $g_{j+1}(\pi) > g_j(\pi)$. This shift transforms $g_j$ into $g_{j+1}$. This transformation is mean-preserving by construction, but we shall determine its impact on the $i$-th moment, for $i \geq 2$.

Given a function $g$ (prior to the first step, $g \equiv g_j$, and after the last step, $g \equiv g_{j+1}$), reduce $g$ at $x$ by a small $A(x)$ in a neighborhood of $x$ of length $u$, where $x$ is such that $g_j(x) > g_{j+1}(x)$, and increase $g$ at $x'$ by a small $A(x')$ in a neighborhood of $x'$ of length $u$, where $x'$ is such that $g_j(x') < g_{j+1}(x')$. This implies that $x' > x$, and $g(x') > g(x)$. Repeat this procedure as needed to transform $g_j$ into $g_{j+1}$. At any point during this transformation, the mean of the
function \( g \) is
\[
\int_{\pi_{j+1}}^{\pi} \left( g(\pi) - m_{j+1} \right) \theta(\pi) d\pi
\]
(50)

Given \( dg(x) = -A(x) \) and \( dg(x') = A(x') \), applying the first difference operator to this expression gives
\[
d \int_{\pi_{j+1}}^{\pi} \left( g(\pi) - m_{j+1} \right) \theta(\pi) d\pi = \int_{\pi_{j+1}}^{\pi} \left( dg(\pi) - m_{j+1} \right) \theta(\pi) d\pi \approx -A(x) \varphi(x) u + A(x') \varphi(x') u
\]
The change in the mean is the sum of a loss of \( A(x) \varphi(x) u \) and of a gain of \( A(x') \varphi(x') u \). The \( i \)-th moment around \( m_{j+1} \) of the function \( g \), \( i \geq 2 \), is
\[
\int_{\pi_{j+1}}^{\pi} \left( g(\pi) - m_{j+1} \right)^i \theta(\pi) d\pi
\]
(51)

Similarly applying the first difference operator to this expression,
\[
d \int_{\pi_{j+1}}^{\pi} \left( g(\pi) - m_{j+1} \right)^i \theta(\pi) d\pi = \int_{\pi_{j+1}}^{\pi} d \left( g(\pi) - m_{j+1} \right)^i \theta(\pi) d\pi = \int_{\pi_{j+1}}^{\pi} dg(\pi) \ i \ \theta(\pi) \left( g(\pi) - m_{j+1} \right)^{i-1} d\pi
\]
\[
\approx -A(x) i (g(x) - m_{j+1})^{i-1} \varphi(x) u + A(x') i (g(x') - m_{j+1})^{i-1} \varphi(x') u
\]
(52)
The change in the \( i \)-th moment is the sum of a loss of \( A(x) i (g(x) - m_{j+1})^{i-1} \varphi(x) u \) and of a gain of \( A(x') i (g(x') - m_{j+1})^{i-1} \varphi(x') u \). The mean-preserving condition which relates \( A(x') \) to \( A(x) \) is
\[
\frac{A(x') \varphi(x')}{A(x) \varphi(x)} = 1
\]
This implies that
\[
\frac{A(x')(g(x') - m_{j+1})^{i-1} \varphi(x')}{A(x)(g(x) - m_{j+1})^{i-1} \varphi(x)} > 1
\]
for any \( i \geq 0 \), because \( g(x') > g(x) > m_{j+1} \). This means that the expression in (52) is positive: the \( i \)-th moment, \( i \geq 2 \), is increased by this mean-preserving transformation.

Below \( m_{j+1} \), the converse holds for symmetrical reasons, and the absolute value of the (negative for an odd-order moment, positive for an even-order moment) \( i \)-th moment about the mean, \( i \geq 2 \), is decreased by the mean-preserving transformation. It follows that the net effect of the transformation on odd-order moments is positive.\(^{45}\)

All in all, with a convex \( f \), each transformation of \( g_j \) into \( g_{j+1} \), for \( j \geq 0 \), preserves the mean but increases any \( i \)-th moment of odd-order greater or equal to 3. Since \( g_0 \) is linear, any odd-order \( i \)-th moment around the mean of \( g_0 \) is zero. Starting from \( g_0 \) and combining all these steps leaves the mean unchanged and increases any \( i \)-th moment, \( i = 3, 5, \ldots \). But an infinite number of such steps yields the function \( f \), whose \( i \)-th moment, \( i = 3, 5, \ldots \), is

\(^{45}\)As we have (52), we are not concerned about the effect on even-order moments.
therefore positive.

**Proof of proposition 3a:**

The left-hand side of the incentive constraint with a punishment contract is

\[
\int_{-\infty}^{\infty} W'(\pi) u'[W(\pi)] \varphi(\epsilon) d\epsilon = \lim_{a \to 0} \int_{-q-a}^{-q+a} \frac{1}{2a} (\bar{w}_P - w_P) u'[W(\pi)] \varphi(\epsilon) d\epsilon
\]

\[
\approx (\bar{w}_P - w_P) \lim_{a \to 0} \left\{ \frac{\Phi(-q + a) - \Phi(-q - a)}{2a} \left[ u'[W(\pi)] \right]_{-q+a}^{-q-a} \right\} = \varphi(-q) \bar{w} \int_{\bar{w}_P}^{w_P} u'[W] dW \quad (53)
\]

where we used the fact that \( \varphi \) becomes approximately flat as \( a \) becomes arbitrarily small.

The left-hand side of the incentive constraint with a reward contract is likewise

\[
\int_{-\infty}^{\infty} W'(\pi) u'[W(\pi)] \varphi(\epsilon) d\epsilon \approx \varphi(q) \bar{w} \int_{\bar{w}_R}^{w_R} u'[W] dW
\]

Since \( q > 0 \), and \( \Phi(0) = 0 \).

Using \( \varphi(-q) = \varphi(q) \) in conjunction with (53), (53) and a decreasing marginal utility, we obtain the desired result.

With a constant marginal utility, the effort is identical under both contracts.

**Proof of proposition 3b:**

The left-hand side of the incentive constraint is

\[
E \left[ W'(\tilde{\pi}) u'[W(\tilde{\pi})] \right] = \text{cov} \left( W'(\tilde{\pi}), u'[W(\tilde{\pi})] \right) + E[W'(\tilde{\pi})] E[u'[W(\tilde{\pi})]]
\]

Assume that \( W \) is concave (respectively convex). Then \( W' \) is decreasing in \( \pi \) (resp. increasing), while \( u' \) is decreasing and \( W' \) is positive, so that the covariance term is positive (resp. negative). As already shown in the proof of proposition 2b, \( E[W'(\tilde{\pi})] \) is a constant.

We now show that \( E[u'[W(\tilde{\pi})]] \) is larger (respectively identical) with a concave contract \( W_A \) than with the corresponding convex contract \( W_E \) if \( u^{(2)}[\bar{W}] < 0 \) (resp. \( u^{(2)}[\bar{W}] = 0 \)), for \( i = 2, 3, \ldots \):

\[
E[u'[W(\tilde{\pi})]] = u'[\bar{W}] + u''[\bar{W}] E[W(\pi) - \bar{W}] + \frac{1}{2} u'''[\bar{W}] E[W(\pi) - \bar{W}]^2
\]

\[
+ \frac{1}{6} u''''[\bar{W}] E[W(\pi) - \bar{W}]^3 + \ldots
\]

where \( \bar{W} \equiv E[W(\pi)]. \) First, \( E[W(\pi) - \bar{W}] = 0 \). Second, \( u^{(i)}[\bar{W}], \) for \( i = 1, 2, \ldots, \) is identical across contracts. Third, the even order terms \( E[W(\pi) - \bar{W}]^{(2i)}, \) for \( i = 1, 2, \ldots, \) are the
same whether with a convex contract $W_E$ or with the corresponding concave contract $W_A$, as already shown in the proof of proposition 2b. Fourth, the odd order terms $E[W(\pi) - \tilde{W}]^{(2i-1)}$, for $i = 2, 3, \ldots$, are positive with a convex contract, and negative with a concave contract, as shown in the proof of proposition 2b. Putting all these elements together, $E[u'(W(\tilde{\pi}))]$ is larger with a concave contract than with the corresponding convex contract if $u^{(2i)}[\tilde{W}] < 0$, for $i \geq 2$. If $u^{(2i)}[\tilde{W}] = 0$, for $i \geq 2$, then the term $E[u'(W(\tilde{\pi}))]$ is the same for a concave contract and the corresponding convex contract.

In summary, the left-hand side of the incentive constraint is higher with a concave contract than with a convex contract if the utility function verifies $u^{(2i)} \leq 0$, for $i \geq 2$.

**Step contracts in a CARA-normal setting**

Using the notations already defined in section 1, the participation constraint is

$$\Phi(q)(-\exp\{-\alpha w\}) + (1 - \Phi(q))(-\exp\{-\alpha \bar{w}\}) = \bar{U} + \psi(e^*)$$

Or

$$(-\exp\{-\alpha \bar{w}\}) \left[1 + (1 - \Phi(q))\exp\{-\alpha \hat{w}\}\right] = \bar{U} + \psi(e^*)$$

(56)

The incentive constraint is

$$\lim_{a \to 0} \int_{q-a}^{q+a} \alpha s \exp\{-\alpha W(e)\} \varphi(e) de = \psi'(e^*)$$

where $s = \frac{w - \bar{w}}{2a}$. In the interval $(q - a, q + a)$ under consideration, as $a$ approaches zero, then, loosely speaking, $W$ follows a Bernoulli distribution with probability $1/2$, since $\varphi$ is approximately flat on an arbitrarily small interval.\(^{46}\) Using this insight, the incentive constraint rewrites as

$$\alpha \hat{w} \varphi(q) \int_{\hat{w}}^{\bar{w}} \exp\{-\alpha W\} dW = \psi'(e^*)$$

Or

$$\hat{w} \left[-\exp\{-\alpha \bar{w}\} + \exp\{-\alpha \hat{w}\}\right] \varphi(q) = \psi'(e^*)$$

(57)

For any given $q$, and for admissible values of $\bar{U}$ and $e^*$, the solution to the system of two equations (56) and (57) in two unknowns $\hat{w}$ and $\bar{w}$ exists and is unique.\(^{47}\)

\(^{46}\)More precisely, $e$ follows a Bernoulli distribution, with $w = \hat{w} + \tilde{w} e$.

\(^{47}\)The left-hand side of the participation constraint is increasing in $\hat{w}$ and in $\bar{w}$. It is equal to minus infinity for $\hat{w} = -\infty$ and $\bar{w} = -\infty$, and to zero for $\hat{w} = \infty$ and $\bar{w} = \infty$. The left-hand side of the incentive constraint is decreasing in $\hat{w}$ and increasing in $\bar{w}$. It is equal to $0$ for $\hat{w} = \tilde{w}$, and to infinity for a finite $\hat{w}$ and $\bar{w}$ = $-\infty$. If the LHS of the incentive constraint is too low, and so is the LHS of the participation constraint, then raise $\tilde{w}$. If the LHS of the incentive constraint is too low, and the LHS of the participation constraint is too high, then
Proof of proposition 4:

The ratio of the left-hand side of (??) over the right-hand side is equal to one, and similarly for (??). We therefore have

\[
\frac{-\exp\{-\alpha w\} \left[ \Phi(q) + (1 - \Phi(q))\exp\{-\alpha \hat{w}\} \right]}{\hat{w}\exp\{-\alpha \hat{w}\}|1 + \exp\{-\alpha \hat{w}\}|\varphi(q)} = \frac{\bar{U} + \psi(e^*)}{\psi'(e^*)}
\]

Or

\[
\frac{\Phi(q) + (1 - \Phi(q))\exp\{-\alpha \hat{w}\}}{\hat{w}[1 + \exp\{-\alpha \hat{w}\]}\varphi(q)} = -\frac{\bar{U} + \psi(e^*)}{\psi'(e^*)} \tag{58}
\]

As \( q \) approaches \(-\infty\), \( 1 - \Phi(q) \) is approximately equal to 1. The equilibrium condition (??) is

\[
\Phi(q) + \exp\{-\alpha \hat{w}\} \xrightarrow{q \to -\infty} -\bar{U} + \psi(e^*) \tag{59}
\]

where \( \Phi(q) \) is approximately zero. Rearranging this equation,

\[
\frac{\exp\{-\alpha \hat{w}\}}{1 + \exp\{-\alpha \hat{w}\}} \xrightarrow{\bar{w} \varphi(q)} \frac{\bar{w} \varphi(q)}{\psi'(e^*)} \tag{60}
\]

Equating to the incentive constraint (??) and performing some algebra yields

\[
-\exp\{-\alpha \hat{w}\} \xrightarrow{q \to -\infty} \bar{U} + \psi(e^*) \tag{61}
\]

Using the definition of \( W^* \) in (??), the only solution to this equation is

\[
\hat{w} \xrightarrow{q \to -\infty} W^*
\]

The cost of the contract is

\[
\Phi(q)w + (1 - \Phi(q))\hat{w}
\]

It is bounded above by \( \hat{w} \). Therefore, agency costs are approximately zero.

Now consider the case where \( q \) approaches infinity. The equilibrium condition (??) is

\[
\frac{1}{\hat{w}} \frac{1 + (1 - \Phi(q))\exp\{-\alpha \hat{w}\}}{\varphi(q)(1 + \exp\{-\alpha \hat{w}\})} = -\frac{\bar{U} + \psi(e^*)}{\psi'(e^*)} > 0 \tag{62}
\]

Dividing both the numerator and the denominator of the second fraction by \( \varphi(q) \),

\[
\frac{1}{\hat{w}} \frac{1 - \Phi(q)}{\varphi(q)} \frac{\exp\{-\alpha \hat{w}\}}{1 + \exp\{-\alpha \hat{w}\}} = -\frac{\bar{U} + \psi(e^*)}{\psi'(e^*)} \tag{63}
\]

As the ratio \( \frac{1 - \Phi(q)}{\varphi(q)} \) approaches zero for \( q \) sufficiently large, and \( \varphi(q) \) approaches zero as \( q \) tends to infinity, this implies

\[
\hat{w} \xrightarrow{q \to \infty} 0
\]

diminish \( \hat{w} \). If the LHS of the incentive constraint is too high, and the LHS of the participation constraint is too low, then raise \( \hat{w} \). If the LHS of the incentive constraint is too high, and so is the LHS of the participation constraint, then diminish \( \hat{w} \).
Using this fact, (??) becomes

\[
\frac{1}{\hat{w}} \left[ \frac{1}{\varphi(q)} + \frac{1 - \Phi(q)}{\varphi(q)} \frac{\exp(-\alpha \hat{w})}{1 + \exp(-\alpha \hat{w})} \right] = -\frac{\hat{U} + \psi(e^*)}{\psi'(e^*)} 
\]

(64)

Or

\[
\frac{1}{\hat{w}} \left[ \frac{1}{1 - \Phi(q)} + \frac{\exp(-\alpha \hat{w})}{1 + \exp(-\alpha \hat{w})} \right] = -\frac{\varphi(q)}{1 - \Phi(q)} \frac{\hat{U} + \psi(e^*)}{\psi'(e^*)} 
\]

(65)

The fraction of exponentials approaches zero as \( \hat{w} \) tends to infinity. Rearranging,

\[
(1 - \Phi(q)) \hat{w} = -\frac{1 - \Phi(q)}{\varphi(q)} \frac{\psi'(e^*)}{\hat{U} + \psi(e^*)} 
\]

(66)

But for the normal distribution, the ratio \( \frac{1 - \Phi(q)}{\varphi(q)} \) approaches zero for \( q \) sufficiently large. Therefore,

\[
(1 - \Phi(q)) \hat{w} \to_{q \to \infty} 0
\]

The cost of the contract is then

\[
w + (1 - \Phi(q)) \hat{w} \approx \hat{w}
\]

Finally, satisfying the participation constraint requires that \( \hat{w} < W^* \). Thus, the expected cost of the contract is approximately equal to the first-best cost, and agency costs are approximately zero.

**The CARA utility function and the optimality of extreme rewards**

The second derivative of the utility function of a globally risk averse agent converges to zero as \( W \) approaches infinity. In effect, a function cannot be increasing and concave at the limit of an unbounded domain: it becomes linear in the limit, which implies asymptotic risk-neutrality - this is consistent with prudence. A discussion relegated to the appendix shows that while this property drives the result in proposition 4, it is not a sufficient condition for extreme rewards to be optimal.

As explained in the core of the article,

\[
\lim_{q \to \infty} u''(\bar{w}) = 0 
\]

(67)

We know from (??) that agency costs are proportional to the sum over equilibrium-probability-weighted payments of the square of the difference between the payment and the first-best payment multiplied by the second derivative of the utility function. Thus, we may at first glance believe that property (??) suffices to make agency costs arbitrarily small, as long as \( q \) is large enough.

59
However, the second derivative in question is not evaluated at the payment \( \bar{w} \), but at a payment \( v_W \) comprised in-between the payment \( \bar{w} \) and the first-best payment \( W^* \). With \( u'' > 0 \), \( v_W \) is strictly increasing in the payment \( \bar{w} \).

Rearranging the equation (??) with a CARA utility function yields

\[
\frac{1}{2} u''(v_W)(\bar{w} - W^*) = \frac{-exp(-\alpha \bar{w}) + exp(-\alpha W^*)}{\bar{w} - W^*} - \alpha exp(-\alpha W^*)
\]

Since CARA utility is concave, the right-hand side is strictly negative, and does not approach zero as \( \bar{w} \) becomes arbitrarily large. We must therefore always have \( u''(v) < 0 \), whereas \( \lim_{w \to \infty} u''(w) = 0 \).

CARA utility is therefore not sufficient to make extreme rewards optimal, regardless of the distribution function. Appropriate asymptotic properties of the likelihood ratio are also necessary to make a huge reward for a very good performance approximately optimal.

**CARA-normal simulation with puts and calls:**

In a CARA-normal setting, we are going to compare the relative incentive effects and relative agent’s valuation of option contracts featuring either short puts or long calls that are as costly to the principal, as in section 2. A contract is defined by its strike \( k \) and its slope \( s \), which are exogenously set. Given these values, a coefficient of absolute risk aversion of 1 and a standard deviation of 1, the fixed wage \( w \) is calculated by Mathematica in order to equate the cost of any contract to 2. For each combination of \( k \) and \( s \), and the associated \( w \), figure 14 reports the agent’s valuation of the contract at the equilibrium effort (the left-hand side of the participation constraint), and the effort incentives created by this contract (the left-hand side of the incentive constraint).

The results are consistent with propositions 2b and 3b. For example, compare two contracts with the same expected cost: a short put contract with a strike of 1 and a slope of 1 to a long call contract with a strike of \(-1\) and a slope of 1 (both are in red in figure 14). As expected, the long call contract yields a higher expected utility, but the short put contract elicits greater effort. Also note that a short put contract with a slope of 1, a high strike (10) and a fixed wage of 12 is approximately valued as a symmetric long call contract with a slope of 1, a low strike \((-10)\) and a fixed wage of 8. This is because both are approximations of a linear contract of slope 1 that cuts the \( y \) axis at 2. They constitute in this sense the benchmark case.

Finally, long call contracts with high strikes seem quite inefficient, as they provide extremely low incentives. However, the opposite holds for symmetric short put contracts. Indeed, for a short put contract with \( k \) low enough, multiplying both the strike and the slope by a constant
larger than 1 only negligibly affects the agent’s valuation of the contract, but provides much higher incentives.

**Proof of proposition 5:**

From (??), the first-best payment $W^\ast$ is such that

$$-\exp\{-\alpha W^\ast\} = \bar{U} + \psi(e^\ast)$$  \hfill (68)

At the second-best, the pay schedule must satisfy both the participation constraint and the incentive constraint:

$$\int_{-\infty}^{k} -\alpha (w + s(e - k)) \varphi(e) de + \int_{-\infty}^{\infty} -\exp\{-\alpha w\} \varphi(e) de = \bar{U} + \psi(e^\ast)$$  \hfill (69)

so\text{ }\int_{-\infty}^{k} \exp\{-\alpha (w + s(e - k))\} \varphi(e) de = \psi'(e^\ast)$$  \hfill (70)

The left-hand side of the incentive constraint is the average pay-performance sensitivity multiplied by the expected marginal utility conditional on $e$ being lower than the strike $k$. These two equations rewrite as

$$-\exp\{-\alpha w\} \left[ \int_{-\infty}^{k} \exp\{-\alpha (e - k)\} \varphi(e) de + 1 - \Phi(k) \right] = \bar{U} + \psi(e^\ast)$$  \hfill (71)

so\text{ }\exp\{-\alpha w\} \int_{-\infty}^{k} \exp\{-\alpha (e - k)\} \varphi(e) de = \psi'(e^\ast)$$  \hfill (72)
Call the integral in both equations $A$. It is equal to

$$A = \frac{1}{\sqrt{2\pi} \sigma} \int_{-\infty}^{k} \exp\{-\alpha s (\epsilon - k)\} \exp\left\{-\frac{\epsilon^2}{2\sigma^2}\right\} d\epsilon = \frac{1}{\sqrt{2\pi} \sigma} \int_{-\infty}^{k} \exp\left\{-\frac{2\alpha s (\epsilon - k)\sigma^2 + \epsilon^2}{2\sigma^2}\right\} d\epsilon$$

$$= \frac{1}{\sqrt{2\pi} \sigma} \int_{-\infty}^{k} \exp\left\{-\frac{(\epsilon - (-\alpha s^2))^2 - 2\alpha s k \sigma^2 - \alpha^2 s^2 \sigma^4}{2\sigma^2}\right\} d\epsilon$$

$$= \frac{1}{\sqrt{2\pi} \sigma} \int_{-\infty}^{k} \exp\left\{\alpha sk + 0.5\alpha^2 s^2 \sigma^2\right\} \int_{-\infty}^{\epsilon} \exp\left\{-\frac{(\epsilon - (-\alpha s^2))^2}{2\sigma^2}\right\} d\epsilon$$

$$= \Phi(k + \alpha s^2) \exp\{\alpha sk + 0.5\alpha^2 s^2 \sigma^2\}$$

We perform a change of variable to express $A$ as a function of the standard normal c.d.f., $F$.

Let

$$y \equiv \frac{\epsilon - (-\alpha s^2)}{\sigma}$$

So that

$$dy = \frac{d\epsilon}{\sigma}$$

And

$$b \equiv \frac{k - (-\alpha s^2)}{\sigma}$$

Then

$$A = \exp\left\{\alpha sk + 0.5\alpha^2 s^2 \sigma^2\right\} \int_{-\infty}^{b} \exp\left\{-\frac{y^2}{2}\right\} \sigma dy$$

$$= \sigma \exp\{\alpha sk + 0.5\alpha^2 s^2 \sigma^2\} F(b)$$

The ratio of the left-hand side of (??) over the right-hand side is equal to one, and similarly for (??). We therefore have

$$-\exp\{-\alpha w\} \left[A + 1 - \Phi(k)\right] = \frac{\bar{U} + \psi(e^*)}{\psi'(e^*)}$$

where the right-hand side of this equation is a constant, in the sense that it is invariant across admissible contracts. Rearranging,

$$\frac{1}{\sigma \alpha} \left(1 + \frac{1 - \Phi(k)}{A}\right) = -\frac{\bar{U} + \psi(e^*)}{\psi'(e^*)} \quad (73)$$

Let the strike $k$ of the short put be arbitrarily small. Then, for the contract to be incentive compatible and satisfy (??), the slope $s$ needs to be arbitrarily large. This contract features extremely large punishments for extremely bad performances - which occur very rarely. Furthermore, as $s$ becomes arbitrarily large, $\frac{1 - \Phi(k)}{A}$ must become arbitrarily large as well for (??) to be satisfied. Since $1 - \Phi(k)$ approaches 1 as $k$ becomes arbitrarily small, this mean
that $A$ must approach zero. As $k$ is arbitrarily small and $s$ arbitrarily large, the participation constraint (73) is therefore approximately

$$-\exp\{-\alpha w\} \approx \bar{U} + \psi(e^*) \quad (74)$$

Comparing with (73), the fixed wage is approximately equal to the first-best wage $W^*$. At the equilibrium effort, the optimal risk sharing rule (full insurance of the agent) is almost in place, but not completely, the agent will therefore be paid a fixed wage $w$ higher than $W^*$. The first-best is arbitrarily closely approximated, but not fully attained, with this short put contract. Since the cost of a short put contract with an arbitrarily small $k$ is lower than the cost of its fixed wage $w$, and since $w$ is arbitrarily close to $W^*$, agency costs are arbitrarily small.

**Optimal contracts in a CRRA-lognormal model**

The Holmstrom condition (72) describes the optimal contract $W$ when the principal is risk-neutral. Dittmann and Maug (2007) study a model in which an agent with a CRRA utility function controls at a cost the mean of the performance measure distribution, which is lognormally distributed. They show that with nonnegative transfers, the optimal contract takes the form:

$$W(\bar{\pi}) = \begin{cases} 
(\alpha_0 + \alpha_1 \ln(\pi))^\frac{1}{\gamma} - W_0 \exp\{rfT\} & \text{if } \pi \geq \bar{\pi} \\
0 & \text{if } \pi < \bar{\pi}
\end{cases}$$

where $\bar{\pi} \equiv \exp\{(W_0 \exp\{rfT\})^{1−\alpha_0}\}/\alpha_1$, $\alpha_0$ and $\alpha_1$ are two constants which are determined to satisfy the participation constraint and the incentive constraint using the method of Dittmann and Maug, $\gamma$ is the coefficient of relative risk aversion, $W_0$ is the initial wealth of the CEO, $rf$ is the risk-free interest rate on an annual basis, and $T$ is the length of the period.

According to Dittmann and Maug, the representative CEO has a fixed wage of $1.2m$, is endowed with 0.42% of his company’s equity and 0.50% in stock-options, and has an initial wealth unrelated to his company of $9.1m$. The market value of equity of his company is $3.7bn$, the options’ strike amounts to 63% of the time 0 value of the stocks, the time period is 8.5 years, the risk-free rate is 6.6%, and the dividend yield is 2.3%.

**Convex transformations and skewness**

Assume that the random variable $\tilde{\varepsilon}$ is symmetrically distributed. Its skewness is equal to

$$\frac{E[(\tilde{\varepsilon} - E[\tilde{\varepsilon}])^3]}{E[(\tilde{\varepsilon} - E[\tilde{\varepsilon}])^2]^\frac{3}{2}}$$

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Since $\tilde{\epsilon}$ is symmetrically distributed, the numerator is zero, and therefore the skewness is zero.

The skewness of $f(\tilde{\epsilon})$ is

$$\frac{E[(f(\tilde{\epsilon}) - E[f(\tilde{\epsilon})])^3]}{E[(f(\tilde{\epsilon}) - E[f(\tilde{\epsilon})])^2]^{\frac{3}{2}}}$$

(75)

The denominator in (??) is positive. In addition, if $f$ is convex, the proof of proposition 2b shows that the numerator in (??) is positive (just identify $f(\epsilon)$ with $W_E(\pi)$ and $E[f(\tilde{\epsilon})]$ with $\hat{W} = E[W_E(\hat{\pi})]$). We have shown that a convex transformation of any symmetrically distributed random variable increases has a positive skewness. As a corollary, since the skewness of any symmetrically distributed random variable is zero, a convex transformation of this variable increases the skewness.

**Proof of proposition 7:**

I need to show that the left-hand side of the incentive constraint (??) is larger if a zero-mean risk is added to compensation.

The LHS of (??) with a zero-mean risk is

$$E_{\tilde{\pi}} \left[ E_{\tilde{\bar{x}}} \left[ W'(\tilde{\pi})u'[W(\tilde{\pi}) + \tilde{x}] \right] \right] = E_{\tilde{\pi}} \left[ W'(\tilde{\pi})E_{\tilde{\bar{x}}} \left[ u'[W(\tilde{\pi}) + \tilde{x}] \right] \right]$$

$$> E_{\tilde{\pi}} \left[ W'(\tilde{\pi})u'[W(\tilde{\pi}) + E_{\tilde{\bar{x}}}[\tilde{x}]] \right] = E_{\tilde{\pi}} \left[ W'(\tilde{\pi})u'[W(\tilde{\pi})] \right]$$

which is the LHS of (??) without a zero-mean risk. The inequality follows from Jensen inequality and the fact that $u'$ is convex.

**Proof of corollary 1:**

Given any contract $W$, the effect of adding a pure (zero-mean) bounded noise that alters probabilities in the $[W^N, \infty)$ interval is twofold.

First, a direct application of proposition 7 shows that it will increase incentives.

Second, it will increase the agent’s valuation of the contract. Suppose that $\tilde{x}$ is distributed on $[-c, \infty)$, with $E[\tilde{x}] = 0$. Then, with $\phi$ being the c.d.f. of the original contract $W$ conditional on the equilibrium effort $e^\ast$, adding $\tilde{x}$ to the contract $W$ increases the expected utility of the agent:

$$\int_{-\infty}^{W^N+c} u[W]d\phi(W) + \int_{W^N+c}^{\infty} E_{\bar{x}}u[W + \tilde{x}]d\phi(W)$$

$$> \int_{-\infty}^{W^N+c} u[W]d\phi(W) + \int_{W^N+c}^{\infty} u[W + E_{\bar{x}}[\tilde{x}]]d\phi(W) = \int_{-\infty}^{\infty} u[W]d\phi(W)$$

The inequality follows from Jensen inequality, with a convex $u$ on the relevant interval.

Thus, both the participation constraint and the incentive constraint are relaxed.
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