Capital Requirements and Asset Prices*

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Abstract

We consider a pure-exchange general equilibrium economy populated by investors with heterogeneous preferences and beliefs. The investors can potentially default on their risky positions unless these positions are backed by collateral. Each investor receives labor income, only fraction of which is pledgeable. We study the effects of a constraint that requires investors to keep their pledgeable financial capital above a certain threshold to provide sufficient collateral. We show that mere possibility of a crisis significantly decreases interest rates and increases Sharpe ratios. Constraints increase stock price-dividend ratios and generate spikes, crashes, and clustering of stock return volatilities. Stock price has a large liquidity premium over non-pledgeable labor incomes. The equilibrium is stationary, and all investors survive in the long run. The asset prices are found in closed form.

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1. Introduction

Financial markets play a key role in facilitating risk sharing and efficient allocation of assets among investors. However, trading in financial assets often entails moral hazard due to investors’ incentives to default on their risky positions. The moral hazard can be alleviated by collateralized trades whereby risky positions are backed by financial capital that can be seized in the event of default. The latter arrangement restores the functionality of the financial markets at a cost of restricting risk sharing among investors. In this paper, we develop a parsimonious model which sheds light on the economic effects of such restrictions on asset prices, their moments, and the distribution of consumption and wealth in the economy. Our analysis is facilitated by closed-form solutions of the model and the stationarity of equilibrium.

We consider a pure exchange economy with one consumption good produced by a tree, similar to Lucas (1978). The economy is populated by two representative investors with heterogeneous constant relative risk aversion (CRRA) preferences over consumption and heterogeneous beliefs about the growth rate of the output. The investors receive non-tradable labor incomes proportional to output and invest their wealth in financial assets such as bonds and stocks. The investors have limited liability and can re-enter the financial market following defaults on their risky positions, such as short-sales and debt. In the event of default the financial capital can be seized by counterparties but only a fraction of labor income can be expropriated. The arising moral hazard problem is resolved by requiring risky positions to be backed by collateral in such a way that the next-period’s value of financial capital stays above a certain threshold. We label the latter constraint as capital requirement. A special case of zero threshold arises when the labor income is non-pledgeable. A constraint with zero threshold requires the financial capital to be positive so that losses on one asset are offset by gains on the other assets. Negative threshold arises when part of the labor income can be pledged. A positive threshold is similar to loss absorbing capital in capital requirements imposed on banks by regulators.

The aggregate consumption growth rates are independent and identically distributed (i.i.d.) but may occasionally experience large negative transitory shocks during low-probability production crises in the economy. These shocks help us explore how mere anxiety about the possibility of a production crisis affects the economy by making capital requirements binding. We solve the model in closed form for general risk aversions and beliefs, and explore the effects of capital requirements on interest rates, Sharpe ratios, price-dividend ratios, stock return volatilities, and distributions of investors’ consumption.
shares in the aggregate output.

Our main results are as follows. First, we show that mere possibility of a large drop in the aggregate consumption next period decreases interest rates and increases Sharpe ratios relative to the unconstrained economy when the irrationally optimistic investor has low wealth. Hence, rare production crises and capital requirements amplify the effects of each other. These effects occur because investors “fly to quality” by buying riskless bonds when they expect constraints to bind next period.

Next, we show that the capital requirements increase stock price-dividend ratio relative to the unconstrained benchmark. The effects of constraints are stronger when investors are close to their default boundaries, which makes price-dividend ratio a U-shaped function of one of the investor’s share of the aggregate consumption. The price-dividend ratio spikes upwards in response to small economic shocks near default boundaries giving rise to cycles of inflating and deflating stock prices in the economy.

Our intuition for the results on price-dividend ratios is as follows. Absent any frictions, the investors’ consumption shares gradually approach zero or one and, accordingly, the economic impact of one of the investors vanishes in the long-run (e.g., Blume and Easley, 2006; Yan, 2008; Chabakauri, 2015, among others). The capital requirements restrict financial losses and protect investors from losing their consumption shares. The result is that consumption shares are bounded away from zero and one. When the constraint of one investor is binding the constraint of the other is loose. The unconstrained investor’s marginal utility of consumption is proportional to the prices of Arrow-Debreu securities. This marginal utility is expected to be higher in the economy with constraints because the unconstrained investor’s consumption share has an upper bound below one. In the economy without constraints the marginal utility can fall further. Consequently, the prices of Arrow-Debreu securities, and hence, also the stock price, are higher in the constrained economy. Two additional factors inflate stock prices. First, capital requirements restrict the short-selling by pessimists, which inflates stock prices (e.g., Harrison and Kreps, 1978). Second, stock has an additional value as collateral, in contrast to labor income.

The dynamics of price-dividend ratio determines the effect of constraints on volatilities. We show that capital requirements dampen volatilities in bad times, when aggregate consumption is low, and amplify them in good times, when aggregate consumption is high. The latter effect makes capital requirements a useful tool for curbing excessive volatility in bad times. The explanation is that the U-shape makes price-dividend ratio procyclical in good and countercyclical in bad times. As a result, the price-dividend ratio and the
dividend move in the same direction in good times and in opposite directions in bad times. Because the stock price is the product of the price-dividend ratio and the dividend, stock return volatility increases in good times and decreases in bad times. We find that the volatility exhibits clustering and is very sensitive to economic shocks when investors are close to hitting their constraints, which gives rise to spikes and crashes of volatility.

We also derive the distributions of investors’ consumption shares in analytic form, and show that they are stationary. The stationary implies that all investors, including those with incorrect beliefs, survive in the long run. Thus, the model answers a question posed by Friedman (1953) and further explored by Blume and Easley (2006) and others on whether irrational investors in financial markets can survive in the long-run. The stationarity arises because constraints prevent investors from losing their consumption shares.

Finally, we measure the illiquidity of labor incomes due to their non-tradability and non-pledgeability. First, we derive shadow prices of claims to labor incomes such that exchanging marginal units of these claims for consumption good at shadow prices does not affect investors’ welfare. Then, we construct portfolios of stocks that replicate labor incomes. Such portfolios exist because labor incomes and stock dividends are proportional in our model. We define the liquidity premia for stocks as the percentage differences in the values of replicating portfolios and shadow prices. The stock liquidity premium from the view of a particular investor widens close to that investor’s default boundary and ranges from 0% to 35% in our calibration, depending on the distance to the boundary.

The paper makes several methodological contributions. First, it develops a tractable way of inducing the stationary of equilibrium, and derives the stationary distributions of investors’ consumption shares in closed form for general risk aversions and beliefs. Second, it characterizes the equilibrium in terms of linear differential-difference equations, which have a term with a delayed argument that is additionally restricted to exceed the lower reflecting boundary. The paper solves these equations in closed form. Finally, the paper introduces a tractable discrete-time framework that makes exposition less technical and permits taking continuous-time limits. The tractability and stationarity make our model a convenient benchmark for asset pricing research that can be extended in various directions.

**Related Literature.** Closest to us are papers that study economies where investors have limited liability and face solvency constraints. Deaton (1990) considers a partial equilibrium model in which investors trade in a riskless asset with an exogenous interest rate and face a non-negativity constraint on their financial wealth. Detemple and Serrat (2003) also study non-negative wealth constraint in a model where investors have het-
erogeneous beliefs but identical risk aversions. They mainly focus on interest rates and Sharpe ratios. Chien and Lustig (2010) study a similar constraint in an economy with a continuum of investors that receive non-pledgeable labor incomes affected by idiosyncratic shocks. Lustig and van Nieuwerburg (2005) study the role of housing collateral when labor income is non-pledgeable. Related works with similar constraints include Geanakoplos (2003, 2009), Fostel and Geanakoplos (2008, 2014), and Blume et al (2015), among others. The constraints in our paper are similar to those in this literature but we focus on heterogeneity in preferences and beliefs, and develop new tools to solve the model in closed form. Our approach yields new economic insights about the dynamics of consumption shares, price-dividend ratios and volatilities.

Kehoe and Levine (1993), Kocherlakota (1996), and Osambela (2015) study economies where investors face participation constraints. Investors in these models are weakly better off not defaulting and are permanently excluded from securities markets if they default. Alvarez and Jermann (2000) show that such constraints can be implemented by imposing certain “not too tight” solvency portfolio constraints. Alvarez and Jermann (2001) find that such constraints help explain equity premia in the U.S. economy.

Our constraint restricts borrowing and short-selling in equilibrium. Consequently, the paper is also related to the literature on economic effects of borrowing, margin, short-sale and position limit constraints (e.g., Detemple and Murthy, 1997; Basak and Cuoco, 1998; Basak and Croitoru, 2000, 2006; Gromb and Vayanos, 2002, 2010; Pavlova and Rigobon, 2008; Brunnermeier and Pedersen, 2009; Gårleanu and Pedersen, 2011; Buss et al, 2013; Chabakauri, 2013, 2015; Rychkov, 2014; Brumm et al, 2015), portfolio insurance (e.g., Basak, 1995) and VaR constraints (e.g., Basak and Shapiro, 2001). It is also related to macro-finance, financial intermediation, and banking literatures that study economies with frictions (Kiyotaki and Moore, 1997; Brunnermeier and Sannikov, 2014; He and Krishnamurthy, 2012, 2013; Kondor and Vayanos, 2015; Klimenko, Pfeil, Rochet, and De Nicolo, 2016) and to the literature on frictionless economies with heterogeneous investors (e.g., Basak, 2005; Chan and Kogan, 2002; Yan, 2008; Longstaff and Wang, 2012; Bhamra and Uppal, 2014; Borovička, 2015; Gårleanu and Panageas, 2015, among others).

2. Economic setup

We consider a pure-exchange infinite-horizon economy with one consumption good produced by an exogenous Lucas (1978) tree. The economy is populated by two representative
heterogeneous investors $A$ and $B$ that hold shares in the tree and receive labor income each period. To facilitate the exposition, we start with a discrete-time economy with dates $t = 0, \Delta t, 2\Delta t, \ldots$, and later take a continuous-time limit.

2.1. Aggregate output and securities markets

At each point of time $t = 0, \Delta t, 2\Delta t, \ldots$ the economy is in one of the three states: $\omega_1$, $\omega_2$, and $\omega_3$. With probability $1 - \lambda \Delta t$ the economy is either in state $\omega_1$ or state $\omega_2$, which we call normal states, and with probability $\lambda \Delta t$ in state $\omega_3$, which we call the crisis state. Parameter $\lambda > 0$ is the crisis intensity. States $\omega_1$ and $\omega_2$ have probabilities $1/2$ conditional on the economy being in a normal state. Figure 1 depicts the structure of uncertainty.

At date $t$ the tree produces $D_t \Delta t$ units of aggregate output, where $D_t$ follows a process

$$\Delta D_t = D_t[\mu_D \Delta t + \sigma_D \Delta w_t + J_D \Delta j_t],$$

where $\mu_D \geq 0$, $\sigma_D > 0$, and $J_D \leq 0$ are output growth mean, volatility, and drop during a crisis, respectively, and $\Delta D_t = D_{t+\Delta t} - D_t$ is the change in output. Processes $w_t$ and $j_t$ are discrete-time analogues of a Brownian motion and Poisson processes, respectively. These processes follow dynamics $w_{t+\Delta t} = w_t + \Delta w_t$ and $j_{t+\Delta t} = j_t + \Delta j_t$, where increments $\Delta w_t$ and $\Delta j_t$ are i.i.d. random variables given by:

$$\Delta w_t = \begin{cases} + \sqrt{\Delta t}, & \text{in state } \omega_1, \\ - \sqrt{\Delta t}, & \text{in state } \omega_2, \\ 0, & \text{in state } \omega_3, \end{cases} \quad \Delta j_t = \begin{cases} 0, & \text{in state } \omega_1, \\ 0, & \text{in state } \omega_2, \\ 1, & \text{in state } \omega_3. \end{cases}$$

1Chabakauri (2014) shows that process (1) converges to a continuous-time Lévy process as $\Delta t \to 0$. 

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Figure 1
States of the Economy

After time $t$ the economy moves to a normal state with probability $1 - \lambda \Delta t$ and to a crisis state with probability $\lambda \Delta t$. Conditional on being in a normal state the economy moves to either $\omega_1$ or $\omega_2$ with equal probabilities.
It can be easily verified that \( \mathbb{E}_t[\Delta w_t|\text{normal}] = 0 \) and \( \text{var}_t[\Delta w_t|\text{normal}] = \Delta t \), similar to a Brownian motion, where \( \mathbb{E}_t[\cdot] \) and \( \text{var}_t[\cdot] \) are expectation and variance conditional on time-\( t \) information. Parameters \( \mu_D, \sigma_D, \) and \( J_D \) are such that \( D_t > 0 \) for all \( t \).

The economy is populated by two representative price-taking investors \( A \) and \( B \). Each investor stands for a continuum of identical investors of unit mass. Fractions \( l_A \) and \( l_B \) of the aggregate output \( D_t\Delta t \) are paid to investors \( A \) and \( B \) as their labor incomes, respectively. Labor incomes are non-tradable. Fractions \( l_A \) and \( l_B \) can be also interpreted as non-tradable shares in the aggregate output such as holdings of illiquid assets. The remaining fraction \( 1 - l_A - l_B \) is paid as a dividend to the shareholders.

The investors can trade three securities at each date \( t \): 1) a riskless bond in zero net supply, which pays one unit of consumption at date \( t + \Delta t \); 2) one stock in net supply of one unit, which is a claim to the stream of dividends \((1 - l_A - l_B)D_t\Delta t\); 3) a one-period insurance contract in zero net supply, which pays one unit of consumption in the crisis state \( \omega_3 \) and zero otherwise. Absent any frictions the market is complete. Time-\( t \) bond, stock, and insurance prices \( B_t, S_t, \) and \( P_t \), respectively, are determined in equilibrium.

### 2.2. Investor heterogeneity and optimization problems

The investors have heterogeneous CRRA preferences over consumption, given by

\[
    u_i(c) = \begin{cases} 
        c^{1 - \gamma_i}, & \text{if } \gamma_i \neq 1, \\
        \ln(c), & \text{if } \gamma_i = 1,
    \end{cases} 
\]

where \( i = A, B \). The investors agree on time-\( t \) asset prices and the aggregate output but disagree on the probabilities of states. Investor \( A \) is rational and has correct probabilities

\[
    \pi_A(\omega_1) = \frac{1 - \lambda D_t}{2}, \quad \pi_A(\omega_2) = \frac{1 - \lambda D_t}{2}, \quad \pi_A(\omega_3) = \lambda D_t, 
\]

where \( \lambda \) is such that probabilities (4) are positive. Investor \( B \) has biased probabilities

\[
    \pi_B(\omega_1) = \frac{1 - \lambda_B D_t}{2}(1 + \delta \sqrt{\Delta t}), \quad \pi_B(\omega_2) = \frac{1 - \lambda_B D_t}{2}(1 - \delta \sqrt{\Delta t}), \quad \pi_B(\omega_3) = \lambda_B D_t, 
\]

where crisis intensity \( \lambda_B \) and disagreement parameter \( \delta \) are such that probabilities (5) are positive. It is immediate to verify that \( \pi_B(\omega_1) + \pi_B(\omega_2) + \pi_B(\omega_3) = 1 \), and hence, \( \pi_B(\omega) \) is a probability measure. Throughout the paper, by \( \mathbb{E}_t[i][\cdot] \) and \( \text{var}_t[i][\cdot] \) we denote conditional expectations and variances under the probability measure of investor \( i \).
It can be easily verified that time-\(t\) conditional expected output growth rate in normal times under the beliefs of investor \(B\) is given by:

\[
\mathbb{E}^B_t \left[ \frac{\Delta D_t}{D_t} \right]_{\text{normal}} = (\mu_D + \delta \sigma_D) \Delta t,
\]

(6)

Therefore, parameter \(\delta\) measures the extent of the investor disagreement about the expected output growth during normal times. For tractability, we assume that investor \(B\) does not update probabilities over time. We also assume that investor \(B\) is weakly less risk averse and more optimistic than investor \(A\): \(\gamma_A \geq \gamma_B\), \(\lambda \geq \lambda_B\) and \(\delta \geq 0\).

At date 0 the investors have certain endowments of financial assets. The total time-\(t\) disposable wealth of investor \(i\) is given by

\[
W_{it} + l_i \Delta t = c_{it} \Delta t + b_{it} B_t + n_{it}(S_t, P_t)^T,
\]

(8)

\[
W_{i,t+\Delta t} = b_{it} + n_{it}(S_{t+\Delta t} + (1 - l_A - l_B) D_{t+\Delta t} \Delta t, 1_{(\omega_{i+\Delta t}=\omega_3)})^T,
\]

(9)

and the capital requirement constraint with tightness \(k_i\):

\[
W_{i,t+\Delta t} \geq k_i (S_{t+\Delta t} + (1 - l_A - l_B) D_{t+\Delta t} \Delta t).
\]

(10)

The threshold in constraint (10) can be seen as the value of the pledgeable part of the labor income multiplied by -1. Let \(\hat{l}_i D_{t+\Delta t}\) be the pledgeable income of investor \(i\), where \(\hat{l}_i < 7\)
l. This income is proportional to dividends \((1 - l_A - l_B)D_t \Delta t\), and hence, can be replicated by a portfolio of \(\hat{n}_i = \hat{l}_i / (1 - l_A - l_B)\) units of stock with cum-dividend value \(\hat{n}_i(S_t + (1 - l_A - l_B)D_t \Delta t)\). The investors can circumvent the non-tradability of claims to pledgeable income by shorting stocks against these claims. Hence, these claims are, effectively, tradable and have the same value as the replicating portfolio. The requirement to have positive pledgeable wealth then becomes \(W_{i,t + \Delta t} + \hat{n}_i(S_{t + \Delta t} + (1 - l_A - l_B)D_{t + \Delta t} \Delta t) \geq 0\), which is equivalent to constraint (10) with \(k_i = -\hat{n}_i\). Lemma A.1 in the Appendix demonstrates that the case \(k_i \neq 0\) reduces to the case with \(k_i = 0\) by a change of variable.

In line with the discussion above, constraint (10) with \(k_i = 0\) corresponds to non-pledgeable labor income. Such constraint requires investors to fully collateralize their asset holdings so that losses on one asset are offset by gains on their other assets. The case \(k_i < 0\) arises when part of the labor income can be pledged. The unconstrained economy is a special case when \(k_i = -\infty\). The constraint with \(k_i > 0\) can be interpreted as a minimum capital requirement, and its threshold as loss absorbing capital. Such constraints are imposed on banks by financial regulators (e.g., Klimenko, Pfeil, Rochet, and De Nicolo, 2016).

2.3. Equilibrium

**Definition.** An equilibrium is a set of asset prices \(\{B_t, S_t, P_t\}\) and of consumption and portfolio policies \(\{c_{it}^*, b_{it}^*, n_{it}^*\}_{i \in \{A, B\}}\) that solve optimization problem (7) for each investor, given processes \(\{B_t, S_t, P_t\}\), and consumption and securities markets clear:

\[
    c_{A}^* + c_{B}^* = D_t, \quad b_{A}^* + b_{B}^* = 0, \quad n_{A, st}^* + n_{B, st}^* = 1, \quad n_{A, pt}^* + n_{B, pt}^* = 0. \tag{11}
\]

In addition to asset prices, we derive price-dividend and wealth-aggregate consumption ratios \(\Psi = S / ((1 - l_A - l_B)D)\) and \(\Phi_t = W_t^* / D\), respectively. We also derive annualized \(\Delta t\)-period riskless interest rates \(r_t\), stock mean-returns \(\mu_t\) and volatilities \(\sigma_t\) in normal times, and the percentage change of the stock price in the crisis state, denoted by \(J_t\).

We derive the equilibrium in terms of state variable \(v_t\) given by the log-ratio of marginal utilities of investors evaluated at their shares of the aggregate consumption \(c_{it}^* / D_t\):

\[
v_t = \ln \left( \frac{(c_{A}^* / D_t)^{-\gamma_A}}{(c_{B}^* / D_t)^{-\gamma_B}} \right). \tag{12}
\]

Substituting consumption shares of investors \(A\) and \(B\), denoted by \(s_t = c_{A}^* / D_t\) and \(1 - s_t = c_{B}^* / D_t\), into equation (12), we express \(v_t\) as a function of \(s_t\):

\[
v_t = \gamma_B \ln(1 - s_t) - \gamma_A \ln(s_t). \tag{13}
\]
Variable $v_t$ is a decreasing function of $s_t$, and hence, $s_t$ is an alternative state variable.

We assume that the process for the aggregate consumption is such that the investors’ value functions evaluated at aggregate consumption are finite:

$$\mathbb{E}_0 \left[ \sum_{t=0}^{\infty} e^{-\rho t} u_t(D_t) \Delta t \right] < +\infty.$$  \hfill (14)

### 3. Characterization of equilibrium

First, we derive the investors’ state price densities (SPD) $\xi_{it}$ and $\xi_{it}$. Then, we find asset prices from the following standard equations of asset pricing (e.g., Duffie (2001, p.23)):

$$B_t = \mathbb{E}_t^i \left[ \frac{\xi_{it,t+\Delta t}}{\xi_{it}} \right],$$  \hfill (15)

$$S_t = \mathbb{E}_t^i \left[ \frac{\xi_{it,t+\Delta t}}{\xi_{it}} \left( S_{t+\Delta t} + (1 - l_A - l_B)D_{t+\Delta t}\Delta t \right) \right],$$  \hfill (16)

$$P_t = \mathbb{E}_t^i \left[ \frac{\xi_{it,t+\Delta t}}{\xi_{it}} 1_{\{\omega_{t+\Delta t} = \omega_3\}} \right],$$  \hfill (17)

where $i = A, B$. The state price density $\xi_{it}$ exists for each investor $i$ due to the absence of arbitrage opportunities in our economy.\(^2\) The investors can eliminate arbitrage because strategies with zero investment and nonnegative payoffs are feasible given constraints (8)–(10). To ensure the uniqueness of SPD $\xi_{it}$ for each investor $i$, we assume and later verify that the matrix of asset payoffs is invertible in equilibrium. The SPDs $\xi_{it}$ and $\xi_{bt}$ differ due to heterogeneity in beliefs and are linked by the change of measure equation\(^3\)

$$\frac{\xi_{b,t+\Delta t}}{\xi_{bt}} = \frac{\xi_{A,t+\Delta t}}{\xi_{At}} \frac{\pi_A(\omega_{t+\Delta t})}{\pi_B(\omega_{t+\Delta t})}. $$  \hfill (18)

We find the SPDs from the first order conditions in terms of investors’ marginal utilities of consumption and Lagrange multipliers for capital requirements (10). First, we rewrite the budget equations (8)–(9) in a static form that expresses the current wealth in terms of current consumption and the expected discounted future wealth (e.g., Cox and Huang, 1989). Then, we solve investor optimizations by dynamic programming and the method of Lagrange multipliers. Lemma 1 below reports the results.

\(^2\)The proof of existence of the SPD in arbitrage-free economies can be found in Duffie (2001, p.4).

\(^3\)Three equations (15)–(17) can be rewritten as equations for three unknowns $\pi_i(\omega_k)\xi_{i,t+\Delta t}(\omega_k)/\xi_{it}$, where $k = 1, 2, 3$ and $i$ is set to either $A$ or $B$. The solution of these equations is unique when the matrix of asset payoffs is invertible, and hence, $\pi_B(\omega_{t+\Delta t})\xi_{b,t+\Delta t}/\xi_{bt} = \pi_A(\omega_{t+\Delta t})\xi_{A,t+\Delta t}/\xi_{At}$ for all states.
Lemma 1 (Dynamic programming and the first order condition).

1) Let $V_i(W_t, v_t; l_i)$ denote the value function of investor $i$, where $v_t$ is the state variable. Then, the value function solves the following Hamilton-Jacobi-Bellman equation:

$$V_i(W_t, v_t; l_i) = \max_{c_{it}} \left\{ u_i(c_{it}) \Delta t + e^{-\rho \Delta t} \mathbb{E}_t^i [V_i(W_{t+\Delta t}, v_{t+\Delta t}; l_i)] \right\},$$  

subject to the static budget constraint and capital requirement

$$W_{it} + l_i D_i \Delta t = c_{it} \Delta t + \mathbb{E}_t^i \left[ \frac{\xi_{it+\Delta t}}{\xi_{it}} W_{it+\Delta t} \right],$$

$$W_{it+\Delta t} \geq k_i (S_{t+\Delta t} + (1-l_A-l_B)D_{t+\Delta t}\Delta t).$$

2) Value function $V_i(W_t, v_t; l_i)$ is a concave function of wealth $W_{it}$.

3) The SPDs $\xi_{it}$ and optimal consumptions $c_{it}^*$ satisfy the first order conditions

$$\frac{\xi_{it+\Delta t}}{\xi_{it}} = e^{-\rho \Delta t} \left( c_{it+\Delta t}^{*-\gamma_i} + \ell_{it+\Delta t} \right) \left( c_{it}^{-\gamma_i} \right),$$

where $\ell_{it+\Delta t} \geq 0$ is the Lagrange multiplier for capital requirement (21) satisfying the complementary slackness condition $\ell_{it+\Delta t} \left( W_{it+\Delta t}^* - k_i (S_{t+\Delta t} + (1-l_A-l_B)D_{t+\Delta t}\Delta t) \right) = 0$.

We use Lemma 1 to derive the dynamics of state variable $v_t$. First, suppose constraints do not bind. In this case, Lagrange multipliers $\ell_{it+\Delta t}$ vanish and the first order conditions (22) are the same as in the unconstrained economy. The dynamics of the state variable $v_t$ in the unconstrained region of the state-space is then the same as in the unconstrained economy, and is found in closed form below, similar to Chabakauri (2015). Next, let $\bar{\sigma}$ and $\underline{\sigma}$ be the values of the state variable $v_t$ when constraints (10) of investors $A$ and $B$ bind, respectively. We guess and verify below that state variable $v_t$ stays within boundaries $\underline{\sigma} \leq v_t \leq \bar{\sigma}$. Intuitively, binding capital requirements restrict the investors’ losses of wealth and consumption, which traps the state variable in the interval $[\underline{\sigma}, \bar{\sigma}]$. The boundaries $\bar{\sigma}$ and $\underline{\sigma}$ are found from the condition that the constraints bind: $W_{it+\Delta t} = k_i (S_{t+\Delta t} + (1-l_A-l_B)D_{t+\Delta t}\Delta t)$. Dividing these constraints by $D_{t+\Delta t}$, we obtain equations for $\bar{\sigma}$ and $\underline{\sigma}$:

$$\Phi_A(\bar{\sigma}) = k_A (1-l_A-l_B) (\Psi(\bar{\sigma}) + \Delta t), \quad \Phi_B(\underline{\sigma}) = k_B (1-l_A-l_B) (\Psi(\underline{\sigma}) + \Delta t).$$

Proposition 1 below reports the dynamics of $v_t$, and Appendix contains the proof.

**Proposition 1 (Closed-form dynamics of state variable $v_t$).**

Given the boundaries $\bar{\sigma}$ and $\underline{\sigma}$, the equilibrium dynamics of state variable $v_t$ is given by:

$$v_{t+\Delta t} = \max \left\{ \underline{\sigma}; \min \left\{ \bar{\sigma}; v_t + \mu_v \Delta t + \sigma_v \Delta w_t + J_v \Delta \bar{f} \right\} \right\},$$

where

$$J_v = \frac{\partial}{\partial v} \left( \Phi_B(\underline{\sigma}) - \Phi_A(\bar{\sigma}) \right).$$
where drift $\mu_v$, volatility $\sigma_v$, and jump $J_v$ are given in closed form by:

$$
\mu_v = \frac{1}{2\Delta t} \left( (\gamma_A - \gamma_B) \ln[(1 + \mu_D \Delta t)^2 - \sigma_D^2 \Delta t] + \ln \left( \frac{1 - \lambda_B \Delta t}{1 - \lambda \Delta t} \right)^2 + \ln(1 - \delta^2 \Delta t) \right), \hspace{1cm} (25)
$$

$$
\sigma_v = \frac{1}{2\sqrt{\Delta t}} \left( (\gamma_A - \gamma_B) \ln \left( \frac{1 + \mu_D \Delta t + \sigma_D \sqrt{\Delta t}}{1 + \mu_D \Delta t - \sigma_D \sqrt{\Delta t}} \right) + \ln \left( \frac{1 + \delta \sqrt{\Delta t}}{1 - \delta \sqrt{\Delta t}} \right) \right), \hspace{1cm} (26)
$$

$$
J_v = (\gamma_A - \gamma_B) \ln(1 + \mu_D \Delta t + J_D) + \ln \left( \frac{\lambda v}{\lambda} \right) - \mu_v \Delta t. \hspace{1cm} (27)
$$

Boundaries $\tau$ and $\underline{v}$ are reflecting when $\Delta t$ is sufficiently small; that is, $v_t$ does not stay at the boundaries forever: $\text{Prob}(\tau > v_t+\Delta t|v_t = \tau) > 0$ and $\text{Prob}(v_t+\Delta t > \underline{v}|v_t = \underline{v}) > 0$.

Our next step is to find the process for SPD and asset prices. When the constraint of investor $A$ is not binding, the Lagrange multiplier in (22) is zero. Equation (22) then gives the SPD in the unconstrained region: $\xi_{A,t+\Delta t}/\xi_{A,t} = e^{-p \Delta t} (c_{A,t+\Delta t}^*/c_{A,t}^*)^{-\gamma_A} = e^{-\rho \Delta t} \left( s(v_{t+\Delta t}) D_{t+\Delta t}/(s(v_t) D_t) \right)^{-\gamma_A}$. When investor $A$ is constrained, investor $B$ is unconstrained, and hence, the state price density can be obtained analogously in terms of investor $B$’s marginal utility, and then converted to correct beliefs of investor $A$ using the change of measure equation (18). After obtaining the SPD, we use equation (16) for stock prices and equation (20) for investors’ wealths to derive price-dividend and wealth-consumption ratios. Proposition 2 reports the results.

**Proposition 2 (Characterization of equilibrium in discrete time).**

1) The state price density under the beliefs of investor $A$ is given by:

$$
\xi_{A,t+\Delta t}/\xi_{A,t} = e^{-p \Delta t} \left( s(v_{t+\Delta t}) D_{t+\Delta t}/s(v_t) D_t \right)^{-\gamma_A} \exp \left( \max \{0; v_t + \mu_v \Delta t + \sigma_v \Delta w_t + J_v \Delta j_t - \tau \} \right), \hspace{1cm} (28)
$$

where investor $A$’s time-$t$ consumption share $s(v_t)$ solves equation (13).

2) The price-dividend ratio $\Psi$ and wealth-aggregate consumption ratios $\Phi_i$ are functions of the state variable $v$, and satisfy equations:

$$
\Psi(v_t) = E^A_t \left[ \frac{\xi_{A,t+\Delta t}}{\xi_{A,t}} \frac{D_{t+\Delta t}}{D_t} (\Psi(v_{t+\Delta t}) + \Delta t) \right], \hspace{1cm} (29)
$$

$$
\Phi_i(v_t) = E^A_t \left[ \frac{\xi_{A,t+\Delta t}}{\xi_{A,t}} \frac{D_{t+\Delta t}}{D_t} \Phi_i(v_{t+\Delta t}) \right] + (1_{i=A}s(v_t) + 1_{i=B}(1 - s(v_t)) - l_i) \Delta t, \hspace{1cm} (30)
$$

Boundaries $\underline{v}$ and $\overline{v}$ solve equations (23). The matrix of asset payoffs is invertible if and only if $\sigma_t(1 + r_t \Delta t) \neq 0$, where $\sigma_t$ and $r_t$ are the stock return volatility in normal times and the interest rate, respectively.
3) The price-dividend ratio in the constrained economy is higher than in the unconstrained economy for the same value of state variable $v_t$ in the two economies.

### 3.1. Closed-form solution in a continuous-time limit

Next, we take continuous-time limit $\Delta t \to 0$ and derive the equilibrium in closed form. Taking the limit allows rewriting equations (29) and (30) for the price-dividend and wealth-consumption ratios, $\Psi_t$ and $\Phi_t$, as differential-difference equations. For tractability, we derive ratios $\Psi_t$ and $\Phi_t$ in terms of a transformed ratio $\hat{\Psi}(v; \theta)$, which satisfies a simpler equation reported in Lemma 2 below.

**Lemma 2 (Differential-difference equation).** In the limit $\Delta t \to 0$, the price-dividend ratio $\Psi$ and wealth-aggregate consumption ratios $\Phi_i$ are given by:

$$
\Psi(v) = \hat{\Psi}(v; -\gamma_A)s(v)^{\gamma_A},
$$

$$
\Phi_i(v) = \left((1_{i=A} - 1_{i=B})\hat{\Psi}(v; 1 - \gamma_A) + (1_{i=B} - l_i)\hat{\Psi}(v; -\gamma_A)\right)s(v)^{\gamma_A},
$$

where $s(v)$ solves equation (13) and $\hat{\Psi}(v; \theta)$ satisfies a differential-difference equation

$$
\frac{\sigma_v^2}{2}\hat{\Psi}''(v; \theta) + \left(\hat{\mu}_v + (1 - \gamma_A)\sigma_D\hat{\sigma}_v\right)\hat{\Psi}'(v; \theta)
$$

$$
- \left(\lambda + \rho - (1 - \gamma_A)\mu_D + \frac{(1 - \gamma_A)\gamma_A\sigma_D^2}{2}\right)\hat{\Psi}(v; \theta)
$$

$$
+ \lambda(1 + J_D)^{1-\gamma_A}\hat{\Psi}\left(\max\{v; v + \hat{J}_v\}; \theta\right) + s(v)^\theta = 0,
$$

subject to the reflecting boundary conditions

$$
\hat{\Psi}'(\bar{v}; \theta) = 0, \quad \hat{\Psi}'(\underline{v}; \theta) - \hat{\Psi}(\bar{v}; \theta) = 0,
$$

where $\hat{\mu}_v$, $\hat{\sigma}_v \geq 0$, and $\hat{J}_v \leq 0$ are constants given by:

$$
\hat{\mu}_v = (\gamma_A - \gamma_B)\left(\mu_D - \frac{\sigma_D^2}{2}\right) + \lambda - \lambda_B - \frac{\sigma^2}{2},
$$

$$
\hat{\sigma}_v = (\gamma_A - \gamma_B)\sigma_D + \delta,
$$

$$
\hat{J}_v = (\gamma_A - \gamma_B)\ln(1 + J_D) + \ln\left(\frac{\lambda_B}{\lambda}\right).
$$

The boundaries $\bar{v}$ and $\underline{v}$ solve the following equations:

$$
\hat{\Psi}(\bar{v}; 1 - \gamma_A) = l_A(1 - k_A) + (1 - l_B)k_A,
$$

$$
\hat{\Psi}(\underline{v}; -\gamma_A) = l_Ak_B + (1 - l_B)(1 - k_B).
$$
We observe that equation (33) is linear, in contrast to economies with constraints directly imposed on trading strategies of investors (e.g., Gârleanu and Pedersen, 2012; Chabakauri, 2013, 2015; Rytchkov, 2014). This equation is a differential-difference equation with a “delayed” argument in the fourth term on the left-hand side of the equation because $\tilde{J}_v \leq 0$. This term is further complicated by the fact that the delayed argument is restricted to stay above the lower boundary $\underline{v}$, which gives rise to the dependence of the fourth term on a peculiar argument $\max\{v; v + \tilde{J}_v\}$. This term captures investors’ decisions in anticipation of hitting their wealth constraint.

Before deriving the equilibrium in the general case, in Corollary 1 below, we provide analytical price-dividend ratios when there is no crisis and investors have log preferences. 

**Corollary 1 (Analytical asset prices in a special case).** Suppose, investors $A$ and $B$ have logarithmic preferences and there is no production crisis, that is, $\lambda = \lambda_B = 0$. Then, price-dividend ratio $\Psi(v)$ is given by:

$$\Psi(v) = \frac{1}{\rho} + \frac{C_1 e^{\varphi^+} + C_2 e^{\varphi^-}}{1 + e^v},$$

(39)

where $\varphi_{\pm} = 0.5(1 \pm \sqrt{1 + 8\rho/\delta^2})$, and constants $C_1$ and $C_2$ are given by equations (A41) and (A42) in the Appendix, respectively.

In Section 4 below, we argue that price-dividend ratio (39) captures some important properties of price-dividend ratio that hold in the general case with arbitrary risk aversions and crises. Hence, this special case can be used as a tractable benchmark in asset pricing research. Nevertheless, we undertake a comprehensive investigation of equilibrium in the general case. Our results are reported in Proposition 3 below.

**Proposition 3 (Closed-form solutions).**

1) In the limit $\Delta t \rightarrow 0$ the price-dividend ratio $\Psi$ and wealth-consumption ratios $\Phi_i$ are given by equations (31) and (32), where function $\tilde{\Psi}(v; \theta)$ is given by:

$$\tilde{\Psi}(v; \theta) = \int_v^\varpi s(y)^\theta \tilde{\psi}(v - y) dy + \int_\varpi^v s(y)^\theta \left[ \tilde{\psi}(v - y) - \tilde{\psi}(\varpi - y) \right] dy$$

$$+ \int_\varpi^v \frac{s(y)^\theta \left[ \tilde{\psi}(v - y) - \tilde{\psi}(\varpi - y) \right] dy}{1 + H \left( \tilde{\psi}(\varpi - v) - \int_v^{\varpi - \varpi} \tilde{\psi}(y) dy \right)} \left( 1 - H \int_0^{\varpi - \varpi} \tilde{\psi}(y) dy \right),$$

(40)
where \( s(y) \) solves equation\(^4\) (13), and \( \hat{\psi}(x) \), \( H \) and some auxiliary variables are given by:

\[
\hat{\psi}(x) = \frac{2}{\delta_v^2} \sum_{n=0}^{\infty} \left[ \left( \frac{2\lambda(1 + J_D)^{1-\gamma_A}}{\delta_v^2} \right) n \exp \left( \frac{(\zeta_+ + \zeta_-)(x + n\hat{J}_v)}{2} \right) \right. \\
\left. \frac{\exp \left( \frac{(\zeta_+ - \zeta_-)(x + n\hat{J}_v)}{2} \right)}{\frac{1}{2} \frac{(\zeta_+ - \zeta_-)(x + n\hat{J}_v)}{2}} \right] Q_n \right) 
\]

\[
Q_n(x) = \exp(-x) \sum_{m=0}^{n} (2x)^{n-m} \frac{(n+m)!}{m!(n-m)!} - \exp(x) \sum_{m=0}^{n} (-2x)^{n-m} \frac{(n+m)!}{m!(n-m)!} 
\]

\[
H = \lambda + \rho - (1 - \gamma_A)\mu_D + \frac{(1 - \gamma_A)\gamma_A^2\sigma_D^2 - \lambda(1 + J_D)^{1-\gamma_A}}{2} 
\]

\[
\zeta_+ = -\frac{\hat{\mu}_v + (1 - \gamma_A)\hat{\sigma}_v\sigma_D + \sqrt{(\hat{\mu}_v + (1 - \gamma_A)\hat{\sigma}_v\sigma_D)^2 + 2\hat{\sigma}_v^2 \left( \lambda + \rho - (1 - \gamma_A)\mu_D + \frac{(1 - \gamma_A)\gamma_A^2\sigma_D^2}{2} \right)}}{\hat{\sigma}_v^2} 
\]

2) Stock return volatility in normal times and the jump size \( J_t \) are given by:

\[
\sigma_t = \sigma_D + \frac{(\hat{\psi}(v_t; -\gamma_A) - \gamma_A(1 - s(v_t)))}{\hat{\psi}(v_t; -\gamma_A)} \frac{\gamma_A(1 - s(v_t))}{\gamma_A(1 - s(v_t)) + \gamma_Bs(v_t)} \hat{\sigma}_v, 
\]

\[
J_t = \frac{(1 + J_D)\hat{\psi} \left( \max \{v; v_t + \hat{J}_v\}, -\gamma_A \right) s \left( \max \{v; v_t + \hat{J}_v\} \right) \gamma_A }{\hat{\psi}(v_t; -\gamma_A)s(v_t)\gamma_A} - 1. 
\]

Numbers of shares \( n_{i, st}^* \) and leverage \( L_{it} = -\beta_{it}B_t \) to market price \( S_t \) ratio are given by:

\[
n_{i, st}^* = \frac{\Phi_i(v_t)\sigma_D + \Phi'_i(v_t)\hat{\sigma}_v}{\hat{\psi}(v_t)\sigma_t}, \quad L_{it} = \frac{\Phi_i(v_t)}{\hat{\psi}(v_t)(1 - l_A - l_B)}. 
\]

The closed-form solution (40) is the unique solution of the boundary value problem (33)–(34). We note that infinite summation in expression (42) for function \( \hat{\psi}(x) \) has only finite number of terms for a fixed \( x \) because \( \hat{J}_v \leq 0 \) and indicators \( 1_{\{x + n\hat{J}_v \geq 0\}} \) are zero for sufficiently large \( n \). We double-checked the expression for \( \hat{\psi}(v; \theta) \) by solving problem (33)–(34) using the method of finite differences and verifying that it gives the same result. We use expression (40) to prove the existence of boundaries \( \tilde{\nu} \) and \( \tau \). This proof, however, is technically complicated, and is presented in Appendix B.

We call the interval \( v \in [\tilde{\nu}, \nu - \hat{J}_v] \) in the state-space a period of anxious economy, similar to Fostel and Geanakoplos (2008).\(^5\) When the economy falls into this state, even a small

\(^4\)Although \( s(y) \) is not in closed form, we observe from equation (13) that its inverse is given by \( s^{-1}(x) = \gamma_n \ln(x) - \gamma_A \ln(1 - x) \). The change of variable \( x = s(y) \) eliminates implicit functions, similar to Chabakauri (2015). We keep all integrals in terms of \( s(y) \) because \( s(y) \) is intuitive and easily computable from (13).

\(^5\)However, in contrast to Fostel and Geanakoplos (2008), the disagreement about the consumption growth dynamics in our economy does not increase during these periods.
possibility of a crisis renders the capital requirement binding and leads to deleveraging in the economy. To explore the economic effects of the anxious economy, we provide closed-form expressions for the interest rates $r_t$ and risk premia in normal times $\mu_t - r_t$, which can be easily obtained using previously derived equations for asset prices and the state price density. Proposition 4 below reports the results.

**Proposition 4 (Interest rates and risk premia in the limit).** For a sufficiently small interval $\Delta t$, the interest rate $r_t$ and risk premium $\mu_t - r_t$ in normal times are given by:

$$
\begin{align*}
    r_t &= \left\{ \begin{array}{l}
        \lambda + \rho + \gamma_A \mu_D - \frac{\gamma_A (1 + \gamma_A)}{2} \sigma_D^2 + \left( \frac{\gamma_A \sigma_D \hat{\sigma}_v - \hat{\mu}_v}{\gamma_B} \right)(1 - s_t) \Gamma_t \\
        - \sigma_v^2 \left( \frac{1}{2 \gamma_B} (1 - s_t)^2 \Gamma_t^2 + \frac{1}{2 \gamma_A \gamma_B^2} s_t (1 - s_t) \Gamma_t^3 \right) \\
        - \lambda (1 + J_D)^{-\gamma_A} \left( \frac{s \left( \max \{ v; v_t + \hat{J}_v \} \right)}{s_t} \right)^{-\gamma_A} + O(\Delta t), \text{ for } v < v_t < v,
        \\
        \frac{(1 - s_t) \Gamma_t (1_{v=v} - 1_{v=v}) - \gamma_A}{2 \gamma_B \sqrt{\Delta t}} \hat{\sigma}_v + O(1), \text{ for } v = v \text{ or } v = v.
    \end{array} \right.
\end{align*}
$$

$$
\begin{align*}
    \mu_t - r_t &= \left( \gamma_A \sigma_D - \frac{(1 - s_t) \Gamma_t \hat{\sigma}_v}{\gamma_B} + \frac{(1 - s_t) \Gamma_t \hat{\sigma}_v (1_{v=v} + 1_{v=v}) - \gamma_B \hat{\sigma}_v 1_{v=v}}{2 \gamma_B} \right) \sigma_t \\
    &- \lambda (1 + J_D)^{-\gamma_A} J_t \left( \frac{s \left( \max \{ v; v_t + \hat{J}_v \} \right)}{s_t} \right)^{-\gamma_A} + O(\sqrt{\Delta t}),
\end{align*}
$$

where drift $\hat{\mu}_v$, volatility $\hat{\sigma}_v$, and $\hat{J}_v$ of the state variable $v$ are given by equations (35)–(37), volatility $\sigma_t$ and jump size $J_t$ are given by equations (46)–(47), respectively, and $\Gamma_t \equiv \gamma_A \gamma_B / \left( \gamma_A (1 - s_t) + \gamma_B s_t \right)$ is the risk aversion of a representative investor.

The effects of capital requirements on interest rates and risk premia arise due to the investors’ concern that a potential crisis may render the constraint binding next period when the economy is close to boundary $v$. The last term in the first equation in (49) for the interest rate quantifies the impact of capital requirements on precautionary savings due to a downward jump in the aggregate consumption, which we further discuss in Section 4.

Equations (49) and (50) also feature terms with indicator functions $1_{v=v}$ and $1_{v=v}$, which are non-zero only at the boundaries $v$ and $v$. For the interest rate $r_t$ these terms have the order of magnitude proportional to $1/\sqrt{\Delta t}$, and hence, the interest rate has singularities at the boundaries $v$ and $v$ when $\Delta t \to 0$. Similar singularities arise in a continuous-time
model of Detemple and Serrat (2003). Our discrete-time analysis sheds new light on these singularities by uncovering their order of magnitude $1/\sqrt{\Delta t}$. In particular, although the annualized interest rate $r_t$ is unbounded when $\Delta t \to 0$, the per-period rate $r_t \Delta t$ is finite and has an order of magnitude $O(\sqrt{\Delta t})$.

The intuition for the singularity is that near the boundaries $\underline{v}$ and $\overline{v}$ even a small shock $\Delta w_t$ may lead to a default. Consequently, when the capital requirement of an investor binds at time $t$, the investor allocates larger fraction of labor income to bond than in the interior region $\underline{v} < v_t < \overline{v}$ and requires higher risk premium. Therefore, the interest rate decreases and Sharpe ratio increases at the boundaries.

### 3.2. Stationary distribution of consumption share

Absent any frictions, state variable $v$ follows an arithmetic Brownian motion with a jump. This process is non-stationary and induces non-stationarity of the unconstrained equilibrium where one of the investors’ share of consumption gradually converges to zero. As we explain in Section 4, the labor incomes alone are not sufficient for stationarity. The capital requirements (10) make the processes for $v$ and consumption share $s$ stationary because they protect investors against losing their shares of aggregate consumption beyond certain limits. We compute the transition densities and the stationary probability density function (PDF) of consumption share $s$ in closed form in the continuous-time limit assuming there is no crisis risk, that is, $\lambda = \lambda_B = 0$. Proposition 5 reports the results.

**Proposition 5 (Stationary distribution of consumption share).** Suppose, $\lambda = \lambda_B = 0$. Then, the PDF $f(s, \tau; s_t; \tau)$ of consumption share $s$ at time $\tau$ conditional on observing share $s_t$ at time $t$ is given in closed form by expression (A80) in the Appendix. Furthermore, the stationary PDF of consumption share $s$ is given by:

$$f(s) = \frac{2\mu_v}{\sigma_v^2} \left( \frac{\gamma_A}{s} + \frac{\gamma_B}{1-s} \right) \left( \frac{(1-s)^{\gamma_B/s}\gamma_A}{(1-s)^{\gamma_B/\sigma_v^2}} - \left(1-\overline{s}\right)^{\gamma_B/\sigma_v^2} \right)^{2\mu_v/\sigma_v^2} 1_{\{s \leq \underline{s} \leq s \leq \overline{s}\}},$$

(51)

where $\mu_v = (\gamma_A - \gamma_B)(\mu_D - \sigma_D^2/2) - \delta^2/2$, $\sigma_v = (\gamma_A - \gamma_B)\sigma_D + \delta$, $1_{\{s \leq \underline{s} \leq s \leq \overline{s}\}}$ is an indicator function and $\underline{s}$ and $\overline{s}$ are the bounds on the consumption share $s$, which solve equation (23) for $\overline{v}$ and $\underline{v}$, respectively.

One important economic implication of the stationarity of equilibrium is that both investors survive in the long-run. This finding is in contrast to frictionless economies where only one investor survives, depending on beliefs and risk aversions (e.g., Blume and
Figure 2: Convergence to stationary distribution of consumption share $s_t = c^*_{s,t}/D_t$

The Figure shows transition densities $f(s, t; s_0, 0)$ for the starting point $s_0 = 0.2$ and the stationary distribution $f(s)$ (i.e., density for $t = \infty$). We set $\gamma_A = 2$, $\gamma_B = 1.5$, $\mu_D = 0.018$, $\sigma_D = 0.032$, $\lambda = \lambda_B = 0$, $\rho = 0.02$, $\delta = 0.1125$, $k_A = k_B = 0$, $\bar{s} = 0.1$, $\bar{s} = 0.9$, $l_A = 0.123$, and $l_B = 0.14$.

Easley, 2001; Yan, 2008; Chabakauri, 2015; among others). Figure 2 plots the stationary PDF (51) and transition densities $f(s, t; s_0, 0)$, for parameters described in the legend and explained in Section 4 below. The stationary distribution on Figure 2 is bimodal, and hence, both rational and irrational investors can occasionally have large consumption shares. The distribution has a larger mass around $s = 0.1$ because the labor share $l_B = 0.14$ of investor $B$ exceeds the labor share $l_A = 0.123$ of investor $A$ in this example.

4. Analysis of Equilibrium

In this section, we study the equilibrium for calibrated parameters. We set the parameters of the aggregate consumption process to $\mu_D = 0.018$, $\sigma_D = 0.032$, $J_D = -0.2$, and the crisis intensities of investors $A$ and $B$ to $\lambda = 0.017$ and $\lambda_B = 0.01$, respectively. The risk aversions are $\gamma_A = 2$ and $\gamma_B = 1.5$. The disagreement parameter is $\delta = 0.1125$, which corresponds to the mean growth rate (6) under investor $B$’s beliefs equal to $1.2\mu_D$. The constraint tightness parameters in (10) are $k_A = k_B = 0$, so that labor incomes are non-pledgeable. The shares of labor income $l_A = 0.123$ and $l_B = 0.14$ are chosen to generate

Drift $\mu_D$ and volatility $\sigma_D$ are within the ranges considered in the literature (e.g., Basak and Cuoco, 1998; Chan and Kogan, 2002; Gârleanu and Panageas, 2015), intensity $\lambda = 0.017$ is from Barro (2009).
Figure 3
Leverage and stock holdings of optimistic and less risk averse investor $B$
Panels (a) and (b) depict optimistic and less risk averse investor $B$’s leverage/market price ratio $L_t/S_t$ and the number of shares $n_{st}$, respectively, as functions of consumption share $s_t = c_{At}/D_t$. The solid and dashed lines correspond to constrained and unconstrained economies, respectively.

symmetric bounds on investor $A$’s consumption share: $\underline{s} = 0.1$ and $\bar{s} = 0.9$.

We plot the equilibrium distributions and processes as functions of consumption share $s_t = c_{At}/D_t$ because $s$ lies in the interval $[0, 1]$ and is more intuitive than variable $v$. We observe that consumption share $s$ is countercyclical in the sense that $\text{corr}_t(ds_t, dD_t) < 0$. Intuitively, the aggregate wealth and consumption shift to pessimist $A$ (optimist $B$) following negative (positive) shocks to output. We call a process procyclical (countercyclical) if that process is a decreasing (increasing) function of $s$.

4.1. Equilibrium processes

Figure 3 depicts investor $B$’s leverage/market ratio $L_t/S_t$ and stock holdings $n_{st}$ in the constrained (solid line) and unconstrained (dashed line) economies. Panel (a) demonstrates the cyclicality of leverage. The leverage is lowest when either investor $A$ or investor $B$ bind on their constraints. Intuitively, when $s = \bar{s}$, investor $B$’s financial wealth is zero, and hence, $B$ lacks collateral and cannot borrow. When $s = \underline{s}$, investor $A$’s financial wealth is zero and the labor income $l_A D_t \Delta t$ is infinitesimally small in the continuous-time limit. The liquidity dries up because investor $A$ cannot supply credit.

Panel (b) presents the number of stocks held by investor $B$. Consider first the unconstrained economy where the labor income is pledgeable. From panel (b) we observe that in

\footnote{To avoid finding bounds $\underline{s}$ and $\bar{s}$ numerically, we set them exogenously to $\underline{s} = 0.1$ and $\bar{s} = 0.9$ and then recover the shares of labor incomes $l_A = 0.123$ and $l_B = 0.14$ that imply these bounds in equilibrium. First, we find $v$ and $\bar{s}$ from equation (13) for $v$, and then find $l_A$ and $l_B$ from equations (38).}
this economy investor $B$ shorts stocks despite being more optimistic than investor $A$ when consumption share $s$ is close to 1. The intuition is that in bad times, following a sequence of negative shocks to output, investor $B$ shorts stocks to finance consumption and backs short positions by the pledgeable labor income. The stream of labor income $l_B D_t \Delta t$ is equivalent to dividends from holding $\hat{n}_B = l_B/(1 - l_A - l_B)$ units of non-tradable shares in the Lucas tree. Short-selling allows the investor to circumvent the non-tradability of labor income and freely adjust the effective share $\hat{n}_B + n_{B, st}$ in the Lucas tree. Overcoming the non-tradability of labor incomes makes this economy similar to the non-stationary unconstrained economy where investors can freely trade shares in the Lucas tree. The financial wealth can then become negative. The capital requirement imposes non-negative wealth constraint, which precludes investor $B$ from shorting. The trading strategy of investor $A$ equals $1 - n^*_{B t}$ in equilibrium and can be analyzed similarly. Investor $A$ also has an additional motive to short stocks due to being more pessimistic than investor $B$.

Figure 4 depicts the interest rate $r_t$, Sharpe ratio $(\mu_t - r_t)/\sigma_t$ in normal times, price-dividend ratio $\Psi$, and excess stock return volatility in normal times $(\sigma_t - \sigma_D)/\sigma_D$ in the constrained (solid line) and unconstrained (dashed line) economies. Panel (a) shows the interest rates $r_t$. The interest rate declines sharply when the economy enters into an anxious state close to the boundary $\bar{s}$ where even a small possibility of a crisis next period makes the constraint of investor $B$ binding. The intuition is as follows. In the unconstrained economy, a crisis around state $\bar{s}$ generates wealth transfer to the pessimistic and more risk averse investor $A$ and increases her consumption share $s$ above $\bar{s}$. In the constrained economy, consumption share $s$ is capped by $\bar{s}$. Consequently, following a crisis, investor $A$’s marginal utility $\left(c^*_A\right)^{-\gamma_A}$ is higher in the constrained than in the unconstrained economy. As a result, investor $A$ is more willing to smooth consumption in the constrained economy, and hence, the interest rate declines due to the precautionary savings motive. In particular, the investor buys more bonds, which drives interest rates down.

Panel (b) of Figure 4 shows that the Sharpe ratio increases to compensate investor $A$ for purchasing risky assets from investor $B$. Our results on interest rates and Sharpe ratios show that the rare crises and capital requirements reinforce the effects of each other. In particular, the decreases in interest rates and increases in Sharpe ratios during anxious times arise only when both the crises and the constraints (10) are simultaneously present.

From panel (c), we observe that the capital requirements give rise to higher price-

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8We exclude the singularities in the dynamics of $r_t$ and focus on the dynamics in the unconstrained region because the economy spends an infinitesimal amount of time at the boundaries.
Figure 4
Equilibrium processes
Panels (a)–(d) show interest rate $r_t$, Sharpe ratio $(\mu_t - r_t)/\sigma_t$, price-dividend ratio $\Psi_t$, and excess volatility $(\sigma_t - \sigma_D)/\sigma_D$ as functions of $s_t = c_{A_t}^*/D_t$ for the constrained (solid lines) and unconstrained (dashed lines) economies.

dividend ratio $\Psi$ than in the unconstrained economy, $\Psi_{t}^{\text{constr}} - \Psi_{t}^{\text{unc}} > 0$, as proven in Proposition 2. The increases in price-dividend ratio $\Psi$ are larger around the boundaries $s$ and $\bar{s}$, which makes ratio $\Psi$ a U-shaped function of $s$ sensitive to small shocks close to boundaries. The U-shape is not specific to heterogeneity in risk aversions. Ratio $\Psi$ has the same shape when both investors are logarithmic and $\Psi$ is given by equation (39).

The intuition for the U-shape is as follows. Suppose, consumption share $s$ is close to the boundary $\bar{s}$, where investor $B$’s constraint is likely to bind but investor $A$ is unconstrained. Because investor $A$’s constraint is loose the state price density $\xi_t^A$ is proportional to investor $A$’s marginal utility $(c_{A_t}^*)^{-\gamma_A}$. In the constrained economy the consumption share of investor $A$ is capped by $s < 1$ whereas in the unconstrained economy it can increase above $\bar{s}$. Therefore, the marginal utility of investor $A$ and, hence, the state price density are expected to be higher in the constrained than in the unconstrained economy. Consequently, stocks are more valuable in the constrained economy around the boundary $\bar{s}$. The intuition around $\underline{s}$ can be analyzed in a similar way.
There are two additional economic forces contributing to higher stock prices in the constrained economy. First, the constraint curbs short-selling by pessimist $A$, which may increase stock prices (e.g., Harrison and Kreps, 1978). Second, the stock can be used as collateral that helps relax the constraint, which gives rise to a premium.

The results on panel (d) demonstrate that the constraint makes volatility more procyclical, reduces volatility in bad times (around $\underline{s}$) and increases it in good times (around $\bar{s}$). This is because U-shaped price-dividend ratio in the constrained economy is more procyclical in good times (i.e., around $\bar{s}$) and more countercyclical in bad times (i.e., around $\underline{s}$) than in the unconstrained economy. Stock price $S_t = \Psi_t D_t$ is more volatile in good times (around $\bar{s}$) because both $\Psi$ and $D_t$ change in the same direction, and is less volatile in bad times (around $\underline{s}$) because $\Psi$ and $D_t$ change in opposite directions and partially offset the effects of each other. Lower volatility in bad times is in line with the previous literature on the effects of portfolio constraints on asset prices (e.g., Chabakauri, 2013, 2015; Brunnermeier and Sannikov, 2014, among others). The empirical literature finds that the volatility tends to be higher in bad times (e.g., Schwert, 1989). However, high volatility can be explained by high uncertainty about the economic growth and learning effects in bad times (e.g., Veronesi, 1999), which are absent in our model.

Boundary conditions (34) allow us to explore volatility $\sigma_t$ near the boundaries $\underline{s}$ and $\bar{s}$ using closed form expressions in Corollary 2 below.\footnote{We observe that $\sigma_t(1 + r_t \Delta t) \neq 0$, and hence, as shown in Proposition 3, the matrix of asset payoffs is invertible under our calibration, which is assumed in the beginning of Section 3.}

**Corollary 2 (Stock return volatility at the boundaries).** *Stock return volatility in normal times $\sigma_t$ satisfies the following boundary conditions:

\[
\sigma(\underline{s}) = \sigma_D + \frac{\underline{s} \gamma_A \bar{\sigma}_v}{\gamma_A (1 - \underline{s}) + \gamma_B \bar{s}} > \sigma_D, \tag{52}
\]

\[
\sigma(\bar{s}) = \sigma_D - \frac{(1 - \bar{s}) \gamma_A \bar{\sigma}_v}{\gamma_A (1 - \bar{s}) + \gamma_B \bar{s}} < \sigma_D. \tag{53}
\]

By continuity, inequalities (52) and (53) also hold in a vicinity of the boundaries. Panel (d) shows that volatility $\sigma_t$ is very steep at the boundaries: it spikes close to $\underline{s}$ and crashes close to $\bar{s}$, consistent with Corollary 2. It also evolves in three regimes of low, medium, and high volatility, which resembles volatility clustering documented in the empirical literature (e.g., Bollerslev, 1987). The distribution of consumption share $s$ on Figure 2 implies that the economy persists in these clusters for some time.
4.2. Measuring illiquidity of non-pledgeable labor incomes

In this section, we measure the illiquidity of labor incomes due to their non-tradability and non-pledgeability. We consider a marginal representative investor \( i \) that does not affect asset prices and characterize this investor’s shadow indifference price \( \hat{S}_{it} \) of labor income. We define \( \hat{S}_{it} \) as the price such that exchanging marginal \( \Delta l_i \) units of labor income for \( \hat{S}_{it} \Delta l_i \) units of wealth leaves the investor’s utility unchanged. More formally, consider the investor’s value functions \( V_i(W_{it}, v_t; l_i) \) satisfying the dynamic programming equation (19) subject to constraints (20) and (21). Price \( \hat{S}_{it} \) is the solution of equation \( V_i(W_{it}^*, v_t; l_i) = V_i(W_{it}^* + \hat{S}_{it} \Delta l_i, v_t; l_i - \Delta l_i) \) when \( \Delta l_i \to 0 \). Taking the limit, we obtain:

\[
\hat{S}_{it} = \frac{\partial V_i(W_{it}^*, v_t; l_i)}{\partial l_i} / \frac{\partial V_i(W_{it}^*, v_t; l_i)}{\partial W_{it}}. \tag{54}
\]

The definition of shadow indifference price \( \hat{S}_{it} \) comes from the literature on the valuation of derivative securities in incomplete markets (e.g., Davis, 1997).

The labor incomes \( l_i D_t \Delta t \) are proportional to dividends \( (1 - l_A - l_B) D_t \Delta t \). Therefore, if claims on labor incomes were tradable and pledgeable, shadow price \( \hat{S}_{it} \) would have been equal to \( S_t/(1 - l_A - l_B) \). However, labor incomes are non-tradable and non-pledgeable. Hence, from the view of investor \( i \), the stock enjoys liquidity premium, which we define as

\[
\Lambda_{it} = \frac{S_t/(1 - l_A - l_B) - \hat{S}_{it}}{S_t/(1 - l_A - l_B)}. \tag{55}
\]

We find derivatives in equation (54) using the envelope theorem. Then, we derive prices \( \hat{S}_{it} \) and show that premia (55) are positive and large. Proposition 6 reports our results.

**Proposition 6 (Shadow prices and the liquidity premium).** In the limit \( \Delta t \to 0 \), investor \( i \)'s shadow price of a unit of labor income is given by:

\[
\hat{S}_{it} = \hat{\Psi}_i(v; -\gamma_A) s(v)^{\gamma_A} D_t, \quad i = A, B, \tag{56}
\]

where \( \hat{\Psi}_i(v; \theta) \) satisfies differential-difference equation (33) subject to the following boundary conditions for investors \( A \) and \( B \)

\[
\hat{\Psi}_A'(v; \theta) = 0, \quad \hat{\Psi}_A(v; \theta) = 0, \tag{57}
\]

\[
\hat{\Psi}_B'(v; \theta) = \hat{\Psi}_B(v; \theta), \quad \hat{\Psi}_B(\overline{v}; \theta) = \hat{\Psi}_B(\overline{v}; \theta). \tag{58}
\]

The investors’ liquidity premia for stocks \( \Lambda_A \) and \( \Lambda_B \) are positive, and hence,

\[
S_t/(1 - l_A - l_B) > \hat{S}_{At}, \quad S_t/(1 - l_A - l_B) > \hat{S}_{Bt}. \tag{59}
\]
The shadow prices and liquidity premia can be found in closed form, similar to stock prices in Section 3. We do not present the closed form solutions for brevity and solve for shadow prices using the method of finite differences. Figure 5 plots the liquidity premia (55) for the same calibrated parameters as in Section 4.1. We observe that investors A and B have different valuations of their labor incomes due to differences in preferences and beliefs. Moreover, their stock liquidity premia $\Lambda_i$ are close to zero when the investors are far away from the boundaries where their respective capital requirements become binding. The premia increase up to 35% close to the boundaries where the liquid stock is more valuable for the purposes of relaxing the constraints.

5. Conclusion

We develop a parsimonious and tractable theory of asset pricing under capital requirement constraints. We show that requiring investors to collateralize their trades has significant effects on asset prices and their moments. The constraints decrease interest rates and increase Sharpe ratios when optimistic investors are close to default boundaries. They also increase price-dividend ratios, amplify volatilities in good states and dampen them in bad states. Hence, capital requirements emerge as viable instruments for stabilizing markets in bad times. The tractability of our model allows us to obtain asset prices and the distributions of consumption shares in closed form.
References


Chabakauri, G., 2015, “Asset Pricing with Heterogeneous Preferences, Beliefs, and Portfolio Constraints,” *Journal of Monetary Economics* 75, 21-34.


Appendix A: Proofs

Lemma A.1 (Change of variable). Maximization of expected discounted utility (7) subject to budget constraints (8) and (9), and capital requirement constraint (10) is equivalent to maximizing (7) with respect to \( c_{it} \), \( b_{it} \) and \( \tilde{n}_{it} \) subject to the following set of constraints:

\[
\begin{align*}
\tilde{W}_{it} + l_i D_i \Delta t &= c_{it} \Delta t + b_{it} B_t + \tilde{n}_{it} (S_t, P_t)^\top, \\
\tilde{W}_{i,t+\Delta t} &= b_{it} + \tilde{n}_{it} (S_{t+\Delta t} + (1 - l_A - l_B) D_{t+\Delta t}, 1_{\omega_{i+\Delta t}=\omega_{it}})^\top, \\
\tilde{W}_{i,t+\Delta t} &\geq 0,
\end{align*}
\]

where \( \tilde{W}_{it} = W_{it} - k_i S_t \) and \( \tilde{W}_{i,t+\Delta t} = W_{i,t+\Delta t} - k_i (S_{t+\Delta t} + (1 - l_A - l_B) D_{t+\Delta t}) \).

Proof of Lemma A.1. Substituting \( n_{it} = \tilde{n}_{it} + (k_i, 0) \) into constraints (8) and (9), we obtain constraints (A1) and (A2). Rewriting constraint (10) in terms of variable \( \tilde{W}_{i,t+\Delta t} \), we obtain constraint (A3). Finally, we note that wealth \( \tilde{W}_{it} = W_{it} - k_i S_t \) is worth \( \tilde{W}_{i,t+\Delta t} \) next period. Hence, (A1) and (A2) can be seen as self-financing budget constraints.

Proof of Lemma 1.

1) We start by demonstrating the equivalence of the dynamic (8)–(9) and static budget constraints (20). Multiplying equation (9) by \( \xi_{i,t+\Delta t}/\xi_{it} \), taking expectation operator \( \mathbb{E}_i[] \) on both sides, and using equations (15)–(17) for asset prices, we obtain:

\[
\mathbb{E}_i^t \left[ \frac{\xi_{i,t+\Delta t}}{\xi_{it}} W_{i,t+\Delta t} \right] = b_{it} B_t + n_{it} (S_t, P_t)^\top.
\]

(A4)

From the budget constraint equation (8), we observe that the right-hand side of (A4) equals \( W_{it} + l_i D_i \Delta t \), and hence, we obtain the static budget constraint (20). Conversely, if there exists \( W_{i,t+\Delta t} \) satisfying constraints (20) and (21) there exist trading strategies \( b_{it} \) and \( n_{it} \) that replicate \( W_{i,t+\Delta t} \) because the underlying market is effectively complete (i.e., the payoff matrix is invertible). Then, rewriting the optimization problem (7) in a recursive form, we obtain the dynamic programming equation (19) for the value function.

2) Consider wealth levels \( W_{it} \) and \( \tilde{W}_{it} \). Let \( \{c^*_it, b^*_it, n^*_it\} \) and \( \{\tilde{c}^*_it, \tilde{b}^*_it, \tilde{n}^*_it\} \) be optimal consumptions and portfolios that correspond to \( W_{it} \) and \( \tilde{W}_{it} \), respectively, and satisfy constraints (8)–(10). For any \( \alpha \in [0, 1] \), policies \( \{\alpha \tilde{c}^*_it + (1 - \alpha) c^*_it, \alpha \tilde{b}^*_it + (1 - \alpha) b^*_it, \alpha \tilde{n}^*_it + (1 - \alpha) n^*_it\} \) would be optimal for any \( \alpha \in [0, 1] \).
\( \alpha n^*_t \) are admissible for wealth \( \alpha W_{it} + (1 - \alpha) \tilde{W}_{it} \). By concavity of CRRA utilities:
\[
V_i(\alpha W_{it} + (1 - \alpha) \tilde{W}_{it}, v_t; l_t) \geq \sum_{\tau=t}^{\infty} u_i(\alpha \tilde{c}_{it}^* + (1 - \alpha)c_{it}^*) \\
\geq \sum_{\tau=t}^{\infty} (\alpha u_i(\tilde{c}_{it}^*) + (1 - \alpha)u_i(c_{it}^*)) \\
= \alpha V_i(W_{it}, v_t; l_t) + (1 - \alpha)V_i(\tilde{W}_{it}, v_t; l_t).
\]

Therefore, \( V_i(W_{it}, v_t; l_t) \) is a concave function of wealth.

3) Consider the following Lagrangian:
\[
\mathcal{L} = u_i(c_{it})\Delta t + e^{-\rho\Delta t}\mathbb{E}^t_t\left[V_i(W_{it+\Delta t}, v_{t+\Delta t}; l_t)\right] \\
+ \eta_{it}\left(W_{it} + l'D_{it}\Delta t - c_{it}\Delta t - \mathbb{E}^t_t\left[\xi_{i,t+\Delta t}W_{i,t+\Delta t}\right]\right) \\
+ \mathbb{E}^t_t\left[e^{-\rho\Delta t}\xi_{i,t+\Delta t}\left(W_{i,t+\Delta t} - k_i(S_{it+\Delta t} + (1 - l_a - l_b)D_{t+\Delta t}\Delta t)\right)\right], \quad (A6)
\]
where multiplier \( \xi_{i,t+\Delta t} \) satisfies the complementary slackness condition \( \xi_{i,t+\Delta t}\left(W_{i,t+\Delta t} - k_i(S_{it+\Delta t} + (1 - l_a - l_b)D_{t+\Delta t}\Delta t)\right) = 0 \). Differentiating the Lagrangian (A6) with respect to \( c_{it} \) and \( W_{i,t+\Delta t} \), we obtain:
\[
\begin{align*}
\frac{\partial V_i(W_{it+\Delta t}, v_{t+\Delta t}; l_t)}{\partial W} &= u_i'(c_{it}^*) = \eta_{it}, \\
\frac{\partial V_i(W_{it+\Delta t}, v_{t+\Delta t}; l_t)}{\partial W} &= \eta_{it}\xi_{i,t+\Delta t}. \quad (A7)
\end{align*}
\]
By the envelope theorem (e.g., Back (2010, p.162)):
\[
\frac{\partial V_i(W_{it+\Delta t}, v_{t+\Delta t}; l_t)}{\partial W} = u_i'(c_{it}^*). \quad (A8)
\]

Substituting the partial derivative of the value function (A9) and the marginal utility (A7) into equation (A8), and then dividing both sides of the equation by \( u_i'(c_{it}^*) \), we obtain the expression for the SPD (22). ■

**Proof of Proposition 1.**

**Step 1.** Consider the case when constraints do not bind, and hence, \( \ell_{i,t+\Delta t} = 0 \). Then, using equation (12) for the state variable \( v_t \) and the first order conditions (22), we obtain:
\[
v_{t+\Delta t} - v_t = \ln \left( \frac{(c_{A,t+\Delta t}/c_{it}^*)^{-\gamma_A}}{(c_{B,t+\Delta t}/c_{it}^*)^{-\gamma_B}} \right) \left( \frac{D_{t+\Delta t}}{D_t} \right)^{\gamma_A - \gamma_B} = \ln \left( \frac{\xi_{A,t+\Delta t}/\xi_{it} t_{A,t+\Delta t}/\xi_{it}}{(D_{t+\Delta t}/D_t)^{\gamma_A - \gamma_B}} \right).
\]
From the above equation and the change of measure equation (18), which relates SPDs \( \xi_{A,t+\Delta t} \) and \( \xi_{B,t+\Delta t} \), we obtain the dynamics of \( v_t \) when constraints do not bind:
\[
v_{t+\Delta t} - v_t = \ln \left( \frac{\pi_B(\omega_{it+\Delta t})}{\pi_A(\omega_{it+\Delta t})} \right) \left( \frac{D_{t+\Delta t}}{D_t} \right)^{\gamma_A - \gamma_B}. \quad (A10)
\]
Let $\overline{v}$ and $\underline{v}$ be the boundaries satisfying Equations (23), at which the constraints of investors $A$ and $B$ bind, respectively. Let investor $A$’s constraint be binding so that $v_{t+\Delta t} = \overline{v}$, and hence, $\ell_{A,t+\Delta t} \geq 0$. Using Equation (12) for $v_t$, first order conditions (22), and $\ell_{A,t+\Delta t} \geq 0$, we obtain:

$$
\overline{v} - v_t \leq \ln \left( \frac{(c_{A,t+\Delta t})^{-\gamma_A} + \ell_{A,t+\Delta t}/c_{At}^{-\gamma_A}}{(c_{B,t+\Delta t}^{-\gamma_B} - c_{BT}^{-\gamma_B} - \gamma_B)} \left( \frac{D_{t+\Delta t}}{D_t} \right)^{\gamma_A - \gamma_B} \right)
= \ln \left( \frac{\xi_{A,t+\Delta t}/\xi_{At}}{(D_{t+\Delta t}/D_t)^{\gamma_A - \gamma_B}} \right) = \ln \left( \frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})} \left( \frac{D_{t+\Delta t}}{D_t} \right)^{\gamma_A - \gamma_B} \right). \tag{A11}
$$

Similarly, for $v_{t+\Delta t} = \underline{v}$ we obtain that $\underline{v} - v_t \geq \ln \left( \frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})} \left( \frac{D_{t+\Delta t}}{D_t} \right)^{\gamma_A - \gamma_B} \right)$.

The latter two inequalities imply that when the constraint binds $v_{t+\Delta t}$ is given by:

$$
v_{t+\Delta t} = \max \left\{ \underline{v}; \min \left\{ \overline{v}; v_t + \ln \left( \frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})} \left( \frac{D_{t+\Delta t}}{D_t} \right)^{\gamma_A - \gamma_B} \right) \right\} \right\}. \tag{A12}
$$

We observe that (A12) is also satisfied in the unconstrained case where $\underline{v} < v_{t+\Delta t} < \overline{v}$. It remains to prove that $v_t$ does not escape $[\underline{v}, \overline{v}]$ interval. Consider a marginal investor of type $A$. We guess that $v_t$ follows dynamics (A12) and verify that the consumption choice of investor $A$ indeed implies this dynamics. The analysis for investor $B$ is similar.

We have shown above that $v_t$ satisfies inequality (A11) when investor $A$ is constrained. Now, we show the opposite: investor $A$ is constrained when $v_t$ satisfies (A11). Hence, $v_{t+\Delta t}$ cannot exceed $\overline{v}$. Consider $v_t$ such that $v_t + \ln \left( \frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})} \left( \frac{D_{t+\Delta t}}{D_t} \right)^{\gamma_A - \gamma_B} \right) > \overline{v}$ for some $\omega_{t+\Delta t}$ and $v_t \in (\underline{v}, \overline{v})$. Because $\underline{v} < v_t < \overline{v}$, investor $A$ consumes $c_{At}^* = s(v_t)D_t$, as shown above. We show that the constraint of investor $A$ binds and $c_{A,t+\Delta t}^* = s(\overline{v})D_{t+\Delta t}$.

This consumption level confirms that $v_{t+\Delta t} = \overline{v}$ is indeed an equilibrium outcome.

Consider the constraint of investor $A$ at date $t$ in the state $\omega_{t+\Delta t}$ where $v_{t+\Delta t} = \overline{v}$:

$$
W_{A,t+\Delta t} \geq k_A(S_{t+\Delta t} + (1 - l_A - l_B)D_{t+\Delta t} \Delta t) = \Phi_A(\overline{v}) D_{t+\Delta t}, \tag{A13}
$$

where the last equality holds by the definition of $\overline{v}$. Using the concavity of the value function, proven in Lemma 1, and condition (A9) from the envelope theorem, we obtain:

$$
u'_A(c_{A,t+\Delta t}) = \frac{\partial V_A(W_{A,t+\Delta t}, \omega_{t+\Delta t}; l_A)}{\partial W} \leq \frac{\partial V_A(\Phi_A(\overline{v}) D_{t+\Delta t}, \omega_{t+\Delta t}; l_A)}{\partial W} = u'_A(s(\overline{v}) D_{t+\Delta t}). \tag{A14}
$$

Because $u'_i(c)$ is a decreasing function, we find that $c_{A,t+\Delta t}^*/D_{t+\Delta t} \geq s(\overline{v})$.

Investor $B$ is unconstrained when $v_{t+\Delta t} = \overline{v}$, and hence, has SPD

$$
\frac{\xi_{B,t+\Delta t}}{\xi_{Bt}} = e^{-\rho \Delta t} \left( \frac{c_{B,t+\Delta t}^*}{c_{Bt}^*} \right)^{-\gamma_B} = e^{-\rho \Delta t} \left( \frac{(1 - s(\overline{v})) D_{t+\Delta t}}{(1 - s(v_t)) D_t} \right)^{-\gamma_B}. \tag{A15}
$$
From the change of measure equation (18) and the FOC (22), the SPD of investor $A$ is
\[
\frac{\xi_{A,t+\Delta t}}{\xi_{At}} = \frac{\xi_{B,t+\Delta t}}{\xi_{Bt}} \frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})} = e^{-\rho\Delta t} \left( \frac{c^*_{A,t+\Delta t}}{(c^*_A)^{-\gamma_A}} \right) .
\]
(A16)

From (A16) and (A15), we find the Lagrange multiplier:
\[
\frac{l_{A,t+\Delta t}}{(c^*_{A,t+\Delta t})^{-\gamma_A}} = \left( \frac{c^*_{A,t+\Delta t}}{c^*_A} \right)^{\gamma_A} \left( \frac{(1 - s(\pi))D_{t+\Delta t}}{(1 - s(v_t))D_t} \right)^{\gamma_B} \frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})} - 1
\geq \left( \frac{s(\pi)D_{t+\Delta t}}{s(v_t)D_t} \right)^{\gamma_A} \left( \frac{(1 - s(\pi))D_{t+\Delta t}}{(1 - s(v_t))D_t} \right)^{\gamma_B} \pi_A(\omega_{t+\Delta t}) - 1
= \left( \frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})} \left( \frac{D_{t+\Delta t}}{D_t} \right)^{\gamma_A^{-\gamma_B}} \right) e^{v_t - \gamma} - 1 > 0.
\]

The first inequality follows from the fact that $c^*_{A,t+\Delta t} \geq s(\pi)D_{t+\Delta t}$ we proved above. The second equality holds by the definition of state variable (12). The second inequality comes from the assumption that $v_t + \ln \left( \frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})} \left( \frac{D_{t+\Delta t}}{D_t} \right)^{\gamma_A^{-\gamma_B}} \right) > \gamma$. Hence, the Lagrange multiplier $l_{A,t+\Delta t}$ is strictly positive. From the complementary slackness condition, the constraint (A13) must be binding. Therefore, inequality (A14) becomes an equality, and hence, $c^*_{A,t+\Delta t} = s(\pi)D_{t+\Delta t}$.

**Step 2.** We now look for coefficients $\mu_v$, $\sigma_v$ and $J_v$ such that:
\[
\mu_v \Delta t + \sigma_v \Delta w_t + J_v \Delta j_t = \ln \left( \frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})} \left( \frac{D_{t+\Delta t}}{D_t} \right)^{\gamma_A^{-\gamma_B}} \right).
\]
(A17)

We write identity (A17) in each of the states $\omega_{t+\Delta t} \in \{\omega_1, \omega_2, \omega_3\}$ and obtain the following system of three linear equations with three unknowns $\mu_v$, $\sigma_v$ and $J_v$:
\[
\mu_v \Delta t + \sigma_v \sqrt{\Delta t} = \ln \left( \frac{(1 - \lambda_B \Delta t)(1 + \delta \Delta t)}{1 - \lambda \Delta t} \right) + (\gamma_A - \gamma_B) \ln(1 + \mu_D \Delta t + \sigma_D \sqrt{\Delta t}),
\]
\[
\mu_v \Delta t - \sigma_v \sqrt{\Delta t} = \ln \left( \frac{(1 - \lambda_B \Delta t)(1 - \delta \Delta t)}{1 - \lambda \Delta t} \right) + (\gamma_A - \gamma_B) \ln(1 + \mu_D \Delta t - \sigma_D \sqrt{\Delta t}),
\]
\[
\mu_v \Delta t + J_v = \ln \left( \frac{\lambda_B}{\lambda} \right) + (\gamma_A - \gamma_B) \ln(1 + \mu_D \Delta t + J_D).
\]
(A18)

Solving the above system, we obtain $\mu_v$, $\sigma_v$ and $J_v$ reported in Proposition 1.
Step 3. Finally, we show that the boundaries are reflecting for a sufficiently small $\Delta t$. Suppose, two conditions are satisfied: $\mu_v \Delta t - \sigma_v \sqrt{\Delta t} < 0$ and $\mu_v \Delta t + \sigma_v \sqrt{\Delta t} > 0$. Then, the boundaries are reflecting: 1) if $v_t = \overline{v}$, then $v_{t+\Delta t} = v_t + \mu_v \Delta t - \sigma_v \sqrt{\Delta t} < \overline{v}$ with positive probability; 2) if $v_t = \underline{v}$, then $v_{t+\Delta t} = v_t + \mu_v \Delta t + \sigma_v \sqrt{\Delta t} > \underline{v}$ with positive probability. It can be easily verified that as $\Delta t \to 0$, $\mu_v \to \hat{\mu}_v$ and $\sigma_v \to \hat{\sigma}_v$, where $\hat{\mu}_v$ and $\hat{\sigma}_v$ are constants given by equations (35) and (36), respectively. Because $\sigma_v > 0$ and $\sqrt{\Delta t}$-terms dominate $\Delta t$-terms for small $\Delta t$, we find that $\mu_v \Delta t - \sigma_v \sqrt{\Delta t} < 0$ and $\mu_v \Delta t + \sigma_v \sqrt{\Delta t} > 0$ for all sufficiently small $\Delta t$. Hence, the boundaries are reflecting.

Proof of Proposition 2. 1) First, we derive the SPD $\xi_{At}$ under the correct beliefs of investor $A$. When investor $A$’s constraint does not bind, substituting $c^*_A = s(v_t)D_t$ into the first order condition (22) we find that

$$\frac{\xi_{A,t+\Delta t}}{\xi_{At}} = e^{-\rho \Delta t} \left( \frac{s(v_{t+\Delta t})}{s(v_t)} \right)^{-\gamma_A} \left( \frac{D_{t+\Delta t}}{D_t} \right)^{-\gamma_A}. \tag{A19}$$

Equation (A19) is consistent with SPD (28) because when the constraint does not bind $v_{t+\Delta t} = v_t + \mu_v \Delta t + \sigma_v \Delta w_t + J_v \Delta j_t < \overline{v}$, and hence the exponential term in (28) vanishes.

When the constraint of investor $A$ binds, the constraint of investor $B$ is loose: the constraints cannot bind simultaneously lest to violate the market clearing conditions. Therefore, the ratio $\xi_{B,t+\Delta t}/\xi_{Bt}$ is given by FOC (22) for investor $B$ with $\ell_B = 0$. Using equation (18), we rewrite the latter SPD under the correct beliefs of investor $A$:

$$\frac{\xi_{A,t+\Delta t}}{\xi_{At}} = e^{-\rho \Delta t} \left( \frac{1 - s(v_{t+\Delta t})}{1 - s(v_t)} \right)^{-\gamma_B} \left( \frac{D_{t+\Delta t}}{D_t} \right)^{-\gamma_B} \frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})}. \tag{A20}$$

Next, from equation (13) for consumption share $s$ we find that $(1 - s_t)^{-\gamma_B} = e^{-\gamma_A s_t^{-\gamma_A}}$. Substituting the latter equality into equation (A20), and also using equation (A17) for the increment $v_{t+\Delta t} - v_t$, we obtain:

$$\frac{\xi_{A,t+\Delta t}}{\xi_{At}} = e^{-\rho \Delta t} \left( \frac{s(v_{t+\Delta t})}{s(v_t)} \right)^{-\gamma_A} \left( \frac{D_{t+\Delta t}}{D_t} \right)^{-\gamma_A} e^{\gamma_A - \gamma_B} \frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})} \left( \frac{D_{t+\Delta t}}{D_t} \right)^{\gamma_A - \gamma_B} \exp\{v_t - v_{t+\Delta t} + \mu_v \Delta t + \sigma_v \Delta w_t + J_v \Delta j_t\} \tag{A21}$$

The fact that the constraint of investor $A$ is binding means that $v_{t+\Delta t} = \overline{v}$ and $v_t + \mu_v \Delta t + \sigma_v \Delta w_t + J_v \geq \overline{v}$ (because otherwise $v_{t+\Delta t} < \overline{v}$, and hence, the constraint does not bind). Therefore, the exponential term $\exp(v_t - v_{t+\Delta t})$ in equation (A21) can be replaced with $\exp(\max\{0, v_t + \mu_v \Delta t + \sigma_v \Delta w_t + J_v \Delta j_t - \overline{v}\})$. When the constraint of investor $A$ does not
bind the latter term vanishes and we obtain equation (A19). Therefore, both equations (A19) and (A21) are summarized by equation (28) for $\xi_{\lambda,t+\Delta t}/\xi_{\lambda,t}$.

2) The equation (29) for the price-dividend ratio can be easily obtained by substituting $S_t = (1 - l_A - l_B)\Psi_t$ into equation (16) for stock prices in terms of SPD and then dividing both sides by $D_t$. The equation (30) for the wealth-aggregate consumption ratio can be obtained by substituting $W_t = D_t \Phi_t$ into the equation for the static budget constraint (20) and dividing both sides by $D_t$.

To derive the matrix of asset returns, we rewrite the stock price dynamics as follows:

$$
\Delta S_t + D_t \Delta t = \mu_t \Delta t + \sigma_t \Delta w_t + J_t \Delta j_t.
$$

Therefore, the matrix of time-$(t + \Delta t)$ bond, stock and insurance returns is given by:

$$
\begin{pmatrix}
1 + r_t \Delta t & 1 + \mu_t \Delta t + \sigma_t \sqrt{\Delta t} & 0 \\
1 + r_t \Delta t & 1 + \mu_t \Delta t - \sigma_t \sqrt{\Delta t} & 0 \\
1 + r_t \Delta t & J_t & 1/P_t
\end{pmatrix}.
$$

It is easy to see that the determinant of the above matrix is given by $-2\sigma_t \Delta t (1 + r_t \Delta t)/P_t$. Therefore, the matrix is non-degenerate when $\sigma_t (1 + r_t \Delta t) \neq 0$.

3) In the unconstrained economy, the state variable $v_{t+\Delta t}^{unc}$ follows dynamics:

$$
v_{t+\Delta t}^{unc} = \mu_v \Delta t + \sigma_v \Delta w_t + J_v \Delta j_t.
$$

Define processes $U_{t+\Delta t} = U_t + \Delta U_t$ and $V_{t+\Delta t} = V_t + \Delta V_t$, where increments are given by:

$$
\Delta U_t = \max\{0; v_t + \mu_v \Delta t + \sigma_v \Delta w_t + J_v \Delta j_t - \bar{v}\}, \quad \Delta V_t = \max\{0; v_t - \mu_v \Delta t - \sigma_v \Delta w_t - J_v \Delta j_t\}.
$$

The process for the state variable in the constrained economy can be rewritten as

$$
v_{t+\Delta t} = v_t + \mu_v \Delta t + \sigma_v \Delta w_t + J_v \Delta j_t + \Delta V_t - \Delta U_t.
$$

If the state variables have the same value at time 0, i.e., $v_0 = v_0^{unc}$, we obtain:

$$
v_t = v_t^{unc} + V_t - U_t
$$

Next, we prove that the SPD is higher in the constrained economy.

$$
\frac{\xi_{\lambda,t+\Delta t}}{\xi_{\lambda,t}} = e^{-\rho \Delta t} \left( \frac{s(v_{t+\Delta t})}{s(v_t)} \frac{D_{t+\Delta t}}{D_t} \right)^{-\gamma_A} \exp(\Delta U_t),
$$

$$
\frac{\xi_{\lambda,t+\Delta t}^{unc}}{\xi_{\lambda,t}^{unc}} = e^{-\rho \Delta t} \left( \frac{s(v_{t+\Delta t}^{unc})}{s(v_t^{unc})} \frac{D_{t+\Delta t}}{D_t} \right)^{-\gamma_A}.
$$

33
Iterating the above equations, we obtain:

\[
\frac{\xi_M}{\xi_M} = e^{-pt} \left( \frac{s(v_t)}{s(v_0)} \right)^{-\gamma_A} \exp(U_t),
\]

\[
\frac{\xi^\text{unc}_M}{\xi^\text{unc}_M} = e^{-pt} \left( \frac{s(v^\text{unc}_t)}{s(v_0)} \right)^{-\gamma_A}.
\]

By the definition of \(s(v)\) in equation (13), we have \(e^v = (1 - s(v))^{\gamma_B} \cdot s(v)^{-\gamma_A}\). Hence,

\[
\frac{\xi_M/\xi_M}{\xi^\text{unc}_M/\xi^\text{unc}_M} = \left( \frac{s(v_t)}{s(v^\text{unc}_t)} \right)^{-\gamma_A} \exp(U_t)
\]

\[
= \left( \frac{s(v^\text{unc}_t + V_t - U_t)}{s(v^\text{unc}_t)} \right)^{-\gamma_A} e^{v^\text{unc}_t} e^\left( -v^\text{unc}_t - U_t \right)
\]

\[
\geq s(v^\text{unc}_t - U_t)^{-\gamma_A} e^\left( -v^\text{unc}_t - U_t \right) \cdot s(v^\text{unc}_t)^{\gamma_A} e^{v^\text{unc}_t}
\]

\[
= (1 - s(v^\text{unc}_t - U_t))^{-\gamma_B} \cdot (1 - s(v^\text{unc}_t))^{\gamma_B} \geq 1.
\]

Therefore, we conclude that \(\xi_M/\xi_M > \xi^\text{unc}_M/\xi^\text{unc}_M\). The latter inequality and the equation for stock prices (16) imply that \(\Psi(v_0) \geq \Psi^\text{unc}(v_0)\). The proof for the case when time-\(t\) variables in the constrained and unconstrained economies coincide is analogous. 

**Proof of Lemma 2.** Define the following function in discrete time:

\[
\tilde{\Psi}(v_t; \theta) = \mathbb{E}_t^A \left[ e^{-\rho_\Delta t + \Delta U_t} \left( \frac{D_{t+\Delta t}}{D_t} \right)^{1-\gamma_A} \tilde{\Psi}(v_{t+\Delta t}; \theta) \right] + s(v_t)^\theta \Delta t,
\]

where \(\Delta U_t\) is given by equation

\[
\Delta U_t = \max\{0; v_t + \mu_u \Delta t + \sigma_u \Delta w_t + J_u \Delta j_t - \tau\}.
\]

Comparing equation (A29) with equations (29) and (30) for \(\Psi\) and \(\Phi\) and using the linearity of equation (A29), it easy to observe that \(\Psi(v_t)\) and \(\Phi_t(v_t)\) are given by the following equations in terms of \(\tilde{\Psi}(v_t; \theta)\):

\[
\Psi(v_t) = \tilde{\Psi}(v_t, -\gamma_A) s(v_t)^{\gamma_A} - \Delta t,
\]

\[
\Phi(v_t) = \left( \left(1_{\{i=A\}} - 1_{\{i=B\}} \right) \tilde{\Psi}(v_t; 1 - \gamma_A) + (1_{\{i=B\}} - l_t) \tilde{\Psi}(v_t; -\gamma_A) \right) s(v)^{\gamma_A}.
\]

Taking limit \(\Delta t \to 0\), we obtain equations (31) and (32) for \(\Psi(v_t)\) and \(\Phi_t(v_t)\).

First, we derive the equation for \(\tilde{\Psi}(v_t; \theta)\) when \(v_t\) belongs to the interior \((\underline{v}, \overline{v})\). For a sufficiently small \(\Delta t\) we have \(\Delta U_t = 0\), where \(\Delta U_t\) is given by (A30). Then, we rewrite
the expectation of \((D_{t+\Delta t})/D_t\)^{1-\gamma_A} \hat{\Psi}(v_t; \theta) as follows:

\[
\mathbb{E}_t^A \left[ \left( \frac{D_{t+\Delta t}}{D_t} \right)^{1-\gamma_A} \hat{\Psi}(v_{t+\Delta t}; \theta) \right] = (1 - \lambda \Delta t) \mathbb{E}_t^A \left[ \left( \frac{D_{t+\Delta t}}{D_t} \right)^{1-\gamma_A} \hat{\Psi}(v_{t+\Delta t}; \theta) \right]_{\text{normal}} + \lambda \Delta t \mathbb{E}_t^A \left[ \left( \frac{D_{t+\Delta t}}{D_t} \right)^{1-\gamma_A} \hat{\Psi}(v_{t+\Delta t}; \theta) \right]_{\text{crisis}}.
\]

\[\text{Noting that in the crisis } D_{t+\Delta t}/D_t = 1 + \mu_v \Delta t + J_v \text{ and } v_{t+\Delta t} = \max\{v; v_t + \mu_v \Delta t + J_v\} \text{ and in the normal state } D_{t+\Delta t}/D_t = 1 + \mu_v \Delta t + \sigma_d \Delta w_t \text{ and } v_{t+\Delta t} = v_t + \mu_v \Delta t + \sigma_d \Delta w_t, \]

\[
\text{using Taylor expansions for } (D_{t+\Delta t}/D_t)^{1-\gamma_A} \text{ and } \hat{\Psi}(v_{t+\Delta t}; \theta), \text{ we find:}
\]

\[
\mathbb{E}_t^A \left[ \left( \frac{D_{t+\Delta t}}{D_t} \right)^{1-\gamma_A} \hat{\Psi}(v_{t+\Delta t}; \theta) \right]_{\text{crisis}} = (1 + J_D)^{1-\gamma_A} \hat{\Psi}\left(\max\{v; v_t + J_v\}\right) \hat{\Psi}(v_t; \theta) \]

\[
\mathbb{E}_t^A \left[ \left( \frac{D_{t+\Delta t}}{D_t} \right)^{1-\gamma_A} \hat{\Psi}(v_{t+\Delta t}; \theta) \right]_{\text{normal}} = \left[ 1 + \left( (1 - \gamma_A) \mu_D + \frac{(1 - \gamma_A) \gamma_A \sigma_d^2}{2} \Delta t \right) J_D \right] \hat{\Psi}(v_t; \theta) + \left( \mu_v + (1 - \gamma_A) \sigma_d \sigma_v \right) \hat{\Psi}'(v_t; \theta) \Delta t + \frac{\sigma_v^2}{2} \hat{\Psi}''(v_t; \theta) \Delta t + o(\Delta t).
\]

Substituting (A32)-(A33) into (A29), we obtain:

\[
\hat{\Psi}(v_t; \theta) = \left[ 1 - \left( \lambda + \rho - (1 - \gamma_A) \mu_D + \frac{(1 - \gamma_A) \gamma_A \sigma_d^2}{2} \Delta t \right) J_D \right] \hat{\Psi}(v_t; \theta) + \left( \mu_v + (1 - \gamma_A) \sigma_d \sigma_v \right) \hat{\Psi}'(v_t; \theta) \Delta t + \frac{\sigma_v^2}{2} \hat{\Psi}''(v_t; \theta) \Delta t + \lambda (1 + J_D)^{1-\gamma_A} \hat{\Psi}\left(\max\{v; v_t + J_v\}\right) \Delta t + s(v) \Delta t + o(\Delta t).
\]

Canceling similar terms, diving by \(\Delta t\), taking limit \(\Delta t \to 0\), and noting that \(\mu_v, \sigma_v\) and \(J_v\) converge to \(\hat{\mu}_v, \hat{\sigma}_v\) and \(\hat{J}_v\) given by (35)-(37), we obtain equation (33) for \(\hat{\Psi}(v_t; \theta)\).

Next, we derive the boundary conditions for \(\hat{\Psi}(v_t; \theta)\). From equation (24), the state variable dynamics at lower bound is \(v_{t+\Delta t} = \omega + \max\{0, \mu_v \Delta t + \sigma_v \Delta w_t + J_v \Delta j_t\}\). We use \(\Delta v_t\) to denote the difference of \(v_{t+\Delta t}\) and \(v_t\). In this case,

\[
\Delta v_t = \max\{0, \mu_v \Delta t + \sigma_v \Delta w_t + J_v \Delta j_t\}.
\]

For sufficiently small \(\Delta t\) increment \(\Delta v_t\) is positive only in state \(\omega_1\) and is zero otherwise. In state \(\omega_1\), \(\Delta v_t = \mu_v \Delta t + \sigma_v \sqrt{\Delta t}\). Therefore, the order of \(\mathbb{E}_t^A[\Delta v_t]\) is \(\sqrt{\Delta t}\), but second order terms involving \(\Delta v_t\) have lower order:

\[
\lim_{\Delta t \to 0} \frac{\mathbb{E}_t^A[\Delta v_t]}{\sqrt{\Delta t}} = \frac{\sigma_v}{2},
\]

\[
\lim_{\Delta t \to 0} \frac{\mathbb{E}_t^A[(\Delta v_t)^2]}{\sqrt{\Delta t}} = \lim_{\Delta t \to 0} \frac{\mathbb{E}_t^A[\Delta v_t \Delta w_t]}{\sqrt{\Delta t}} = \lim_{\Delta t \to 0} \frac{\mathbb{E}_t^A[\Delta v_t \Delta j_t]}{\sqrt{\Delta t}} = 0.
\]
Taylor expansion of $\hat{\Psi}(v_{t+\Delta t}; \theta)$ at $v_t = \bar{v}$ is given by

$$
\hat{\Psi}(v_{t+\Delta t}; \theta) = \hat{\Psi}(\bar{v}; \theta) + \hat{\Psi}'(\bar{v}; \theta) \Delta v_t + \frac{1}{2} \hat{\Psi}''(\bar{v}; \theta) \Delta v_t^2 + o(\sqrt{\Delta t}). \quad (A37)
$$

In subsequent calculations we keep terms with order of $\sqrt{\Delta t}$. Using the above results, we obtain the following expansion:

$$
E_t^A \left[ \left( \frac{D_t + \Delta t}{D_t} \right)^{1-\gamma_A} \hat{\Psi}(v_{t+\Delta t}; \theta) \right] = E_t^A \left[ (1 + \mu_d \Delta t + \sigma_d \Delta w_t + J_d \Delta j_t)^{1-\gamma_A} \left( \hat{\Psi}(\bar{v}; \theta) + \hat{\Psi}'(\bar{v}; \theta) \Delta v_t + \frac{1}{2} \hat{\Psi}''(\bar{v}; \theta) \Delta v_t^2 \right) \right] \quad (A38)
$$

Substituting (A38) into (A29), taking into account that $\Delta U_t = 0$ at $v_t = \bar{v}$, and canceling $\hat{\Psi}(\bar{v}; \theta)$ on both sides, we obtain the first boundary condition $\hat{\Psi}'(\bar{v}; \theta) = 0$.

At the upper bound $v_t = \bar{v}$ investor $A$ is constrained, and hence, $\Delta U_t$ in (A30) is positive. From (24) the state variable at the upper bound is

$$
v_{t+\Delta t} = \min\{\bar{v}, v_t + \mu_s \Delta t + \sigma_s \Delta w_t + J_s \Delta j_t\} = v_t + \mu_s \Delta t + \sigma_s \Delta w_t + J_s \Delta j_t - \Delta U_t. \quad (A39)
$$

The order of $E_t^A[\Delta U_t]$ is $\sqrt{\Delta t}$, but second order terms involving $\Delta U_t$ have order $o(\sqrt{\Delta t})$. Proceeding in the same way as (A36)-(A38), we arrive at

$$
\hat{\Psi}(\bar{v}; \theta) = \hat{\Psi}(\bar{v}; \theta) + \left[ \hat{\Psi}(\bar{v}; \theta) - \hat{\Psi}'(\bar{v}; \theta) \right] E_t^A[\Delta U_t] + o(\sqrt{\Delta t}).
$$

Canceling similar terms, taking limit $\Delta t \to 0$, we obtain condition $\hat{\Psi}(\bar{v}; \theta) - \hat{\Psi}'(\bar{v}; \theta) = 0$.

Finally, we derive the equations for $\tau$ and $\underline{v}$. Taking limit $\Delta t \to 0$ in equations (23), we find that these equations become: $\Phi_A(\tau) = k_A(1 - l_A - l_b)\Psi(\tau)$, $\Phi_B(\underline{v}) = k_B(1 - l_A - l_b)\Psi(\underline{v})$. Substituting $\Phi_A(\tau)$ and $\Phi_B(\underline{v})$ in terms of $\hat{\Psi}(\tau; \theta)$ from equations (32) into the latter equations for the boundaries, after some algebra, we obtain equations (38).

**Proof of Corollary 1.** Consider the case $\lambda = \lambda_b = 0$ and $\gamma_A = \gamma_B = 1$. Then, $s(v)$ solving equation (24) is given by $s(v) = 1/(1 + e^v)$, $\Psi(v) = \hat{\Psi}(v)s(v)$, where $\hat{\Psi}(v)$ solves a special case of equation (33) given by:

$$
\frac{\delta^2}{2} \hat{\Psi}''(v) - \frac{\delta^2}{2} \hat{\Psi}'(v) - \rho \hat{\Psi}(v) + 1 + e^v = 0, \quad (A40)
$$

subject to boundary conditions (34). It can be easily verified that $\hat{\Psi}(v) = C_1 e^{\varphi_- v} + C_2 e^{\varphi_+ v} + (1 + e^v)/\rho$ satisfies (A40). Substituting $\hat{\Psi}(v)$ into boundary conditions (34), we obtain the following system for coefficients $C_1$ and $C_2$:

$$
C_1 \varphi_- e^{\varphi_- v} + C_2 \varphi_+ e^{\varphi_+ v} + e^v/\rho = 0; \quad C_1(\varphi_- - 1)e^{\varphi_- v} + C_2(\varphi_+ - 1)e^{\varphi_+ v} - 1/\rho = 0.
$$
Applying the transform to equation (A44), we arrive at the following equation:

\[
C_1 = \frac{1}{\rho \varphi_+(\varphi_- - 1)e^{\rho \varphi_+ - \varphi_+ - \rho \varphi_- - \varphi_-}} (\varphi_+ - 1)e^{\rho \varphi_+ + \varphi_+} + \varphi_+ e^{\rho \varphi_+ + \varphi_+} \quad (A41)
\]

\[
C_2 = -\frac{1}{\rho \varphi_+(\varphi_- - 1)e^{\rho \varphi_+ - \varphi_+ - \rho \varphi_- - \varphi_-}} (\varphi_- - 1)e^{\rho \varphi_+ + \varphi_+ - \varphi_- - \varphi_-} - \varphi_- (\varphi_+ - 1)e^{\rho \varphi_+ + \varphi_+ - \varphi_- - \varphi_-}. \quad (A42)
\]

**Proof of Proposition 3.** 1) First, we solve the differential-difference equation in Lemma 2. We denote \( g(x) = \tilde{\Psi}(x + \nu; \theta) \) and apply the following changes of variables:

\[
x = v - \nu, \quad \tilde{\sigma} = \tilde{\sigma}_v, \quad \tilde{\mu} = \tilde{\mu}_v + (1 - \gamma_\lambda)\sigma_D \tilde{\sigma}_v, \quad \tilde{J} = -\tilde{J}_v, \quad \tilde{\lambda} = \lambda(1 + J_D)^{1-\gamma_\lambda},
\]

\[
\tilde{\rho} = \lambda + \rho - (1 - \gamma_\lambda)\mu_D + \frac{(1 - \gamma_\lambda)\gamma_\lambda}{2} \sigma_D^2.
\]

Equations (33) and (34) with new variables now become:

\[
\frac{\tilde{\sigma}^2}{2} g''(x) + \tilde{\mu} g'(x) - \tilde{\rho} g(x) + \tilde{\lambda} g(\max\{x - \tilde{J}, 0\}) + s(x + \nu)^\theta = 0, \quad (A44)
\]

\[
g'(0) = 0, \quad g(\nu) - g'(0) = 0. \quad (A45)
\]

Let \( \mathcal{L}[g(x)] = \int_0^\infty e^{-xz}g(x)dx \) be the Laplace transform of \( g(x) \), and similarly for other functions. The Laplace transforms of \( g'(x) \), \( g''(x) \) and \( g(\max\{x - \tilde{J}, 0\}) \) are given by:

\[
\mathcal{L}[g'(x)] = z\mathcal{L}[g(x)] - g(0),
\]

\[
\mathcal{L}[g''(x)] = z^2\mathcal{L}[g(x)] - zg(0) - g'(0),
\]

\[
\mathcal{L}[g(\max\{x - \tilde{J}, 0\})] = \int_0^\infty e^{-xz}g(\max\{x - \tilde{J}, 0\})dx
\]

\[
= \int_0^\tilde{J} e^{-xz}g(0)dx + \int_\tilde{J}^\infty e^{-xz}g(x - \tilde{J})dx
\]

\[
= \frac{1}{z}(1 - e^{-Jz})g(0) + e^{-Jz}\mathcal{L}[g(x)].
\]

Applying the transform to equation (A44), we arrive at the following equation:

\[
\frac{\tilde{\sigma}^2}{2} \left(z^2\mathcal{L}[g(x)] - zg(0) - g'(0)\right) + \tilde{\mu} \left(z\mathcal{L}[g(x)] - g(0)\right) - \tilde{\rho}\mathcal{L}[g(x)]
\]

\[
+ \tilde{\lambda} \left(e^{-Jz}\mathcal{L}[g(x)] + \frac{1}{z}(1 - e^{-Jz})g(0)\right) + \mathcal{L}\left[s(x + \nu)^\theta\right] = 0. \quad (A47)
\]

Applying boundary condition \( g'(0) = 0 \) and solving for \( \mathcal{L}[g(x)] \), we obtain:

\[
\mathcal{L}[g(x)] = \frac{\mathcal{L}\left[s(x + \nu)^\theta\right]}{\tilde{\rho} - \tilde{\mu}z - \frac{\tilde{\rho}}{z^2}z^2 - \lambda e^{-Jz}} + g(0) \left(\frac{1}{z} - \frac{\tilde{\rho} - \tilde{\lambda}}{\tilde{\rho} - \tilde{\mu}z - \frac{\tilde{\rho}}{z^2}z^2 - \lambda e^{-Jz}} \cdot \frac{1}{z}\right). \quad (A48)
\]
Now define a new function \( \hat{\psi}(x) \) through inverse Laplace transform

\[
\hat{\psi}(x) = \mathcal{L}^{-1} \left[ \frac{1}{\hat{\rho} - \hat{\mu}z - \frac{\hat{\sigma}^2}{2}z^2 - \hat{\lambda}e^{-Jz}} \right].
\]

(A49)

Next, we apply inverse transform to each term in (A48). Noting that \( \mathcal{L}^{-1}[1/z] = 1 \) and using the theorem which states that Laplace transform of a convolution is the product of Laplace transforms, we derive the following inverse transforms:

\[
\mathcal{L}^{-1} \left[ \frac{\mathcal{L} [s(x + y)^\theta]}{\hat{\rho} - \hat{\mu}z - \frac{\hat{\sigma}^2}{2}z^2 - \hat{\lambda}e^{-Jz}} \right] = \int_0^x s(y + x)^\theta \cdot \hat{\psi}(x - y)dy,
\]

(A50)

\[
\mathcal{L}^{-1} \left[ \frac{\mathcal{L} \left[ \frac{1}{\hat{\rho} - \hat{\mu}z - \frac{\hat{\sigma}^2}{2}z^2 - \hat{\lambda}e^{-Jz}} \right] \cdot \frac{1}{z}}{\hat{\rho} - \hat{\mu}z - \frac{\hat{\sigma}^2}{2}z^2 - \hat{\lambda}e^{-Jz}} \right] = \int_0^x \mathbf{1}_{\{v \geq 0\}} \cdot \hat{\psi}(x - y)dy = \int_0^x \hat{\psi}(y)dy.
\]

The linearity of the Laplace transform gives the following equation:

\[
g(x) = \mathcal{L}^{-1} [\mathcal{L} [g(x)]] = \int_0^x s(y + x)^\theta \cdot \hat{\psi}(x - y)dy + g(0) \left[ 1 - \left( \hat{\rho} - \hat{\lambda} \right) \int_0^x \hat{\psi}(y)dy \right].
\]

(A51)

We calculate \( g(0) \) below, and then after changing the variable back from \( x \) to \( v = x + y \), substituting in expressions for \( \hat{\rho} \) and \( \hat{\lambda} \) from (A43), we obtain (40).

Next, we solve for \( \hat{\psi}(x) \) in closed form. We expand \( \mathcal{L} [\hat{\psi}(x)] \) as series, and sum up the inverse transforms of each term in the summation to get \( \hat{\psi}(x) \).

\[
\mathcal{L} [\hat{\psi}(x)] = \frac{1}{\hat{\rho} - \hat{\mu}z - \frac{\hat{\sigma}^2}{2}z^2 - \hat{\lambda}e^{-Jz}} = (\hat{\rho} - \hat{\mu}z - \frac{\hat{\sigma}^2}{2}z^2)^{-1} \cdot \frac{\hat{\lambda}e^{-Jz}}{\hat{\rho} - \hat{\mu}z - \frac{\hat{\sigma}^2}{2}z^2} = \sum_{n=0}^{\infty} \frac{\hat{\lambda}^n e^{-nJz}}{(\hat{\rho} - \hat{\mu}z - \frac{\hat{\sigma}^2}{2}z^2)^{n+1}}.
\]

(A52)

The above series converges for \( z \) such that \( |\hat{\rho} - \hat{\mu}z - (\hat{\sigma}^2/2)z^2| > |\hat{\lambda}\exp(-Jz)| \). This holds if the real part of \( z \) is sufficiently large, e.g., \( \Re(z) > 4|\hat{\mu}|/\hat{\sigma}^2 + (2/\hat{\sigma})\sqrt{\hat{\rho} + \hat{\lambda}} \). The inverse Laplace transform can then be calculated along the line \((\bar{z} - i\infty, \bar{z} + i\infty)\) in the complex domain where \( \bar{z} > 4|\hat{\mu}|/\hat{\sigma}^2 + (2/\hat{\sigma})\sqrt{\hat{\rho} + \hat{\lambda}} \), and hence, the inequality for \( \Re(z) \) is satisfied.

Let \( \zeta_- < \zeta_+ \) be roots of \( \hat{\rho} - \hat{\mu}z - \hat{\sigma}^2 z^2/2 = 0 \), given by (45). We use the following inversion formula for \( 1/|z(\zeta_+ - \zeta_-)|^{n+1} \) from Gradshteyn and Ryzhik (2007, p. 1117):

\[
\mathcal{L}^{-1} \left[ \frac{1}{|z(\zeta_+ - \zeta_-)|^{n+1}} \right] = \frac{\sqrt{\pi}}{\Gamma(n+1)} \frac{x^{n+\frac{1}{2}}}{(\zeta_+ - \zeta_-)^{n+\frac{1}{2}}} I_{n+\frac{1}{2}} \left( \frac{\zeta_+ - \zeta_-}{2} x \right).
\]

(A53)
Function $e^{-nJz}$ in the complex domain corresponds to a shift from $x$ to $x - n\tilde{J}$. Therefore,

$$
\mathcal{L}^{-1} \left[ \frac{\tilde{\lambda}^n e^{-nJz}}{(\tilde{\rho} - \tilde{\mu} z - \frac{\tilde{\sigma}^2}{2} z^2)^n + 1} \right] = \tilde{\lambda}^n \left( -\frac{\tilde{\sigma}^2}{2} \right)^{-n-1} 1_{x > n\tilde{J}}
$$

(A54)

$$
\times \frac{\sqrt{\pi}}{\Gamma(n + 1)} (x - n\tilde{J})^{n+\frac{1}{2}} e^{\frac{\zeta_+ + \zeta_-}{2} (x - n\tilde{J})} I_{n + \frac{1}{2}} \left( \frac{\zeta_+ - \zeta_-}{2} (x - n\tilde{J}) \right).
$$

Consequently, the explicit expression for $\hat{\psi}(x)$ is given by:

$$
\hat{\psi}(x) = \sum_{n=0}^{\infty} \tilde{\lambda}^n \left( -\frac{\tilde{\sigma}^2}{2} \right)^{-n-1} 1_{x \geq n\tilde{J}} \sqrt{\pi} (x - n\tilde{J})^{n+\frac{1}{2}} \frac{\zeta_+ + \zeta_-}{2} (x - n\tilde{J}) I_{n + \frac{1}{2}} \left( \frac{\zeta_+ - \zeta_-}{2} (x - n\tilde{J}) \right),
$$

(A55)

where function $I_{n + \frac{1}{2}}(\cdot)$ is a modified Bessel function of the first kind, $\zeta_- < \zeta_+$ are given by (45) and $\tilde{\rho}, \tilde{\mu}, \tilde{\sigma}, \tilde{\lambda}$, and $\tilde{J}$ are defined in (A43). Bessel function $I_{n + \frac{1}{2}}(\cdot)$ is given by (see equation 8.467 in Gradshteyn and Ryzhik (2007)):

$$
I_{n + \frac{1}{2}}(z) = \frac{1}{\sqrt{2\pi z}} \left[ e^z \sum_{m=0}^{n} \frac{(-1)^m (n + m)!}{m! (n - m)!(2z)^m} + (-1)^{n+1} e^{-z} \sum_{m=0}^{n} \frac{(n + m)!}{m! (n - m)!(2z)^m} \right].
$$

(A56)

Substituting (A56) into (A55), after minor algebra, we obtain expression (42) for $\hat{\psi}(x)$. The infinite series (A55) has only finite number of non-zero terms because for a fixed $x$ indicators $1_{x \geq n\tilde{J}}$ vanish for sufficiently large $n$, and hence, (A55) is well-defined.

To find $g(0)$ in equation (A51), we first evaluate $\hat{\psi}(0)$. From the above formula (A55), because $1_{0 \geq n\tilde{J}} = 0$ for all $n > 0$, we obtain

$$
\hat{\psi}(0) = -\frac{2}{\tilde{\sigma}^2} \cdot \frac{e^{\zeta_-} - e^{\zeta_+}}{\zeta_+ - \zeta_-} = 0.
$$

(A57)

Differentiating (A51) and using $\hat{\psi}(0) = 0$, we find:

$$
g'(x) = \int_0^x s(y + \psi)^{\theta} \cdot \hat{\psi}'(x - y) dy - g(0) \cdot (\tilde{\rho} - \tilde{\lambda}) \hat{\psi}(x),
$$

(A58)

We solve for $g(0)$ from the boundary condition $g(\bar{\psi} - \psi) - g'((\bar{\psi} - \psi) = 0$ and obtain:

$$
g(0) = \frac{\int_0^{\bar{\psi} - \psi} s(y + \psi)^{\theta} \cdot \left[ \hat{\psi}'(\bar{\psi} - \psi - y) - \hat{\psi}(\bar{\psi} - \psi - y) \right] dy}{1 - (\tilde{\rho} - \tilde{\lambda}) \int_0^{\bar{\psi} - \psi} \hat{\psi}(y) dy + (\tilde{\rho} - \tilde{\lambda}) \hat{\psi}(\bar{\psi} - \psi)}.
$$

(A59)

Substituting (A59) into (A51), we derive equation (40) for $\tilde{\Psi}(v; \theta)$. 

39
2) Next we solve for stock volatility and jump size. In the unconstrained region \( v < v_t < \overline{v} \), stock price \( S_t \), dividend \( D_t \) and state variable \( v_t \) follow processes:

\[
dS_t = S_t[\mu_t dt + \sigma_t dw_t + J_t dj_t],
\]

\[
dD_t = D_t[\mu_t dt + \sigma_t dw_t + J_t dj_t],
\]

\[
dv_t = \mu_v dt + \sigma_v dw_t + \left( \max\{v_t; v_t + \tilde{J}_t\} - v_t \right) dj_t.
\]

Applying Ito's lemma to \( S_t = (1 - l_A - l_B)\Phi(v_t; -\gamma_A)s(v_t)^{\gamma_A}D_t \), and matching \( dw_t \) and \( dj_t \) terms, after some algebra, we obtain \( \sigma_t \) and \( J_t \) in Proposition 3.

Equation equation (9) for \( W_{i,t+\Delta t} \), implies the following expressions for \( n_{i,st}^* \) and \( b_{it}^* \):

\[
n_{i,st}^* = \left[ \frac{\text{var}_t[W_{i,t+\Delta t} - W_{it} | \text{normal}]}{\text{var}_t[\Delta S_t + (1 - l_A - l_B)D_t \Delta t | \text{normal}]^2} \right]^{-1},
\]

\[
b_{it}^* = \mathbb{E}_t[W_{i,t+\Delta t} | \text{normal}] - n_{it}^* \mathbb{E}_t[\Delta S_t + (1 - l_A - l_B)D_t \Delta t | \text{normal}].
\]

Taking limit \( \Delta t \to 0 \) in the above expressions and using expansions similar to those in the proof of Lemma 2, we obtain the number of stocks and the leverage per the market value of stocks in equation (48).

**Proof of Proposition 4.** From equation (15) for the bond price and the fact that \( 1 = B_t(1 + r_t \Delta t) \) we find that the riskless interest rate \( r_t \) is given by:

\[
r_t = \frac{1 - \mathbb{E}_t[\xi_{A,t+\Delta t}/\xi_{A,t}]}{\mathbb{E}_t[\xi_{A,t+\Delta t}/\xi_{A,t}] \Delta t} = \frac{1 - (1 - \lambda \Delta t)\mathbb{E}_t[\xi_{A,t+\Delta t}/\xi_{A,t} | \text{normal}] - \lambda \Delta t \mathbb{E}_t[\xi_{A,t+\Delta t}/\xi_{A,t} | \text{crisis}]}{\mathbb{E}_t[\xi_{A,t+\Delta t}/\xi_{A,t}] \Delta t},
\]

where \( \xi_{A,t+\Delta t}/\xi_{A,t} \) is given by equation (28). We separately calculate \( \mathbb{E}_t[\xi_{A,t+\Delta t}/\xi_{A,t} | \text{normal}] \) and \( \mathbb{E}_t[\xi_{A,t+\Delta t}/\xi_{A,t} | \text{crisis}] \), and then take the limit \( \Delta t \to 0 \).

We start with the derivation of \( \mathbb{E}_t[\xi_{A,t+\Delta t}/\xi_{A,t} | \text{normal}] \) when \( v < v_t < \overline{v} \), and hence, by continuity, for a sufficiently small \( \Delta t \) the economy is unconstrained next period, so that \( v < v_{t+\Delta t} < \overline{v} \). In the unconstrained region \( \Delta v_t = \mu_v \Delta t + \sigma_v \Delta w_t \) and the SPD is given by (A19). From the expression for the SPD, using expansions (A70) and (A72), we obtain:

\[
\mathbb{E}_t \left[ \frac{\xi_{A,t+\Delta t}}{\xi_{A,t}} \right]_{\text{normal}} = \mathbb{E}_t \left[ (1 + a_t \Delta v_t + b_t (\Delta v_t)^2) \left( 1 - r_A \Delta t - \kappa_A \Delta w_t \right)_{\text{normal}} + o(\Delta t) \right] = 1 + a_t \mu_v \Delta t + b_t \sigma_v^2 \Delta t - r_A \Delta t - \kappa_A a_t \sigma_v \Delta t + o(\Delta t).
\]

(A62)
Conditioning on the crisis state, we have:

\[ E_t \left[ \frac{\xi_{t+\Delta t}}{\xi_t} \right]_{\text{crisis}} = (1 - \rho \Delta t)(1 + \mu \Delta t + J_D) - \gamma \left( s(\max\{v_t, v_t + \mu \Delta t + J_v\}) \right)^{-\gamma A} \]

\[ = (1 + J_D)^{-\gamma A} \left( s(\max\{v_t, v_t + \mu \Delta t + J_v\}) \right)^{-\gamma A} + o(\Delta t). \]

(A63)

Substituting \( a_t \) and \( b_t \) from (A71) into equation (A62), and then substituting (A62) and (A63) into equation (A61), after simple algebra, we obtain \( r_t \) in (49) for the case \( v < v_t < \overline{v} \).

Now, we derive \( r_t \) at the boundaries \( v \) and \( \overline{v} \). The SPD is given by (28). Using expansions (A70) and (A72), we obtain the following expansion:

\[ E_t \left[ \frac{\xi_{t+\Delta t}}{\xi_t} \right]_{\text{normal}} = E_t \left[ ((1 + a_t \Delta v_t + b_t(\Delta v_t)^2)(1 - r_A \Delta t - \kappa_A \Delta w_t) \times (1 + \Delta U_t + 0.5(\Delta U_t)^2) \right]_{\text{normal}} + o(\Delta t) \]

\[ = E_t \left[ 1 + a_t \Delta v_t + b_t(\Delta v_t)^2 - r_A \Delta t - \kappa_A \Delta w_t - \kappa_A a_t \Delta v_t \Delta w_t + \Delta U_t - \kappa_A \Delta w_t \Delta U_t + a_t \Delta U_t \Delta v_t + 0.5(\Delta U_t)^2 \right]_{\text{normal}} + O(\Delta t), \]

(A64)

where \( \Delta U_t \) is given by equation (A30). Using equation (24) for the process \( v_t \) and equation (A30) for \( \Delta U_t \), for a fixed \( v_t \) and sufficiently small \( \Delta t \), we find that \( \Delta v_t \) and \( \Delta U_t \) at the boundaries are given by:

\[ \Delta v_t = \begin{cases} 
\min(0, \mu_v \Delta t + \sigma_v \Delta w_t), & \text{if } v_t = \overline{v}, \\
\max(0, \mu_v \Delta t + \sigma_v \Delta w_t), & \text{if } v_t = v,
\end{cases} \]

(A65)

\[ \Delta U_t = \begin{cases} 
0, & \text{if } v_t < \overline{v}, \\
\max(0, \mu_v \Delta t + \sigma_v \Delta w_t), & \text{if } v_t = \overline{v},
\end{cases} \]

(A66)

We note that for a sufficiently small \( \Delta t \) the sign of \( \mu_v \Delta t + \sigma_v \Delta w_t \) is solely determined by the second term \( \sigma_v \Delta w_t \) because it has the order of magnitude \( \sqrt{\Delta t} \). Volatility \( \sigma_v \) is positive because under our assumptions investor \( A \) is more risk averse and more pessimistic. Using the latter observation, substituting equations (A65) and (A66) into equation (A64) and
computing the expectation, we obtain:

$$
E_t \left[ \frac{\xi_{A,t+\Delta t}}{\xi_{A,t}} \right]_{\text{normal}} = 1 + \left\{ \begin{array}{ll}
\left( a_t (\mu_v - \kappa_A \sigma_v) + b_t \sigma_v^2 + \frac{\mu_v + \kappa_A \sigma_v + \sigma_v^2}{2} - r_A \right) \Delta t \\
+ \frac{\sigma_v (1 - a_t)}{2} \sqrt{\Delta t} + O(\Delta t), & \text{if } v_t = v,
\end{array} \right.
\right.
\right.
\left. \left\{ \begin{array}{ll}
\left( a_t \mu_v - a_t \kappa_A \sigma_v + b_t \sigma_v^2 \right) \Delta t + \frac{a_t \sigma_v}{2} \sqrt{\Delta t} + O(\Delta t), & \text{if } v_t = \overline{v}.
\end{array} \right. \right\}
(A67)

Substituting (A67) and (A63) into equation (A61) for the interest rate $r_t$, we obtain $r_t$ in (49) for the case when $v_t$ is at the boundary.

To obtain the risk premium, we first decompose stock returns as follows:

$$
\Delta S_t + D_{t+\Delta t} = \mu_t \Delta t + \sigma_t \Delta w_t + J_t \Delta j_t.
(A68)
$$

Multiplying both sides of (A68) by $\xi_{A,t+\Delta t}/\xi_{A,t}$ and taking expectations, we obtain:

$$
E_t \left[ \frac{\xi_{A,t+\Delta t}}{\xi_{A,t}} \Delta S_t + D_{t+\Delta t} \Delta t}{S_t} \right] = \mu_t \Delta t E_t \left[ \frac{\xi_{A,t+\Delta t}}{\xi_{A,t}} \Delta w_t \right] + J_t E_t \left[ \frac{\xi_{A,t+\Delta t}}{\xi_{A,t}} \Delta j_t \right].
$$

On the other hand, from the equation for stock price (16) we find that:

$$
E_t \left[ \frac{\xi_{A,t+\Delta t}}{\xi_{A,t}} \Delta S_t + D_{t+\Delta t} \Delta t}{S_t} \right] = 1 - E_t \left[ \frac{\xi_{A,t+\Delta t}}{\xi_{A,t}} \right].
$$

Combining the last two equations and the equation (A61) for the interest rate, we obtain:

$$
\mu_t - r_t = - \left( \sigma_t \left[ \frac{\xi_{A,t+\Delta t}}{\xi_{A,t}} \Delta w_t \right] + J_t \left[ \frac{\xi_{A,t+\Delta t}}{\xi_{A,t}} \Delta j_t \right] \right) 1 + r_t \frac{\Delta t}{\Delta t}.
(A69)
$$

Then, proceeding in the same way as with the calculation of interest rates and using similar expansions, we obtain equation (50) for the risk premium. 

**Lemma A.2 (Useful expansions).**

1) For small increment $\Delta v_t = v_{t+\Delta t} - v_t$ the ratio $(s(v_{t+\Delta t})/s(v_t))^{-\gamma_A}$ has expansion:

$$
\left( \frac{s(v_{t+\Delta t})}{s(v_t)} \right)^{-\gamma_A} = 1 + a_t \Delta v_t + b_t (\Delta v_t)^2 + o(\Delta t),
(A70)
$$

where coefficients $a_t$ and $b_t$ are given by:

$$
a_t = \frac{(1 - s_t) \Gamma_t}{\gamma_B}, \quad b_t = \frac{1}{2 \gamma_B^2} (1 - s_t)^2 \Gamma_t^2 + \frac{1}{2 \gamma_B^3} s_t (1 - s_t) \Gamma_t^3,
(A71)
$$

42
\( \Gamma_t = \gamma_A \gamma_B / (\gamma_A (1 - s) + \gamma_B s) \) is the risk aversion of the representative investor and \( s_t \) is consumption share of investor A that solves equation (13).

2) For the case \( J_D = 0 \), the SPD in a one-investor economy can be expanded as follows:

\[
e^{-\rho \Delta t} \left( \frac{D_{t+\Delta t}}{D_t} \right)^{-\gamma_A} = 1 - r_A \Delta t - \kappa_A \Delta w_t + o(\Delta t), \tag{A72}
\]

where \( r_A \) and \( \kappa_A \) are the riskless rate and the Sharpe ratio in an economy populated only by investor A, given by:

\[
r_A = \rho + \frac{\gamma_A (1 + \gamma_A)}{2} \sigma_D^2, \quad \kappa_A = \gamma_A \sigma_D. \tag{A73}
\]

**Proof of Lemma A.2.** 1) We expand the ratio on the left-hand side of (A70) using Taylor’s formula, and observe that \( a_t = (s(v_t)^{-\gamma_A})'/s(v_t)^{-\gamma_A} \) and \( b_t = 0.5(s(v_t)^{-\gamma_A})''/s(v_t)^{-\gamma_A} \). Differentiating, we obtain the following expressions for \( a_t \) and \( b_t \):

\[
a_t = -\gamma_A s'(v_t)/s(v_t), \quad b_t = \frac{\gamma_A (1 + \gamma_A)}{2} \left( \frac{s'(v_t)}{s(v_t)} \right)^2 - \frac{\gamma_A s''(v)}{2} s(v). \tag{A74}
\]

To find derivatives \( s'(v) \) and \( s''(v) \), we differentiate equation (13) twice with respect to \( v \), and obtain two equations for the derivatives:

\[
1 = -\left( \frac{\gamma_A}{s_t} + \frac{\gamma_B}{1 - s_t} \right) s'(v_t), \tag{A75}
\]

\[
0 = \left( \frac{\gamma_A}{s_t^2} - \frac{\gamma_B}{(1 - s_t)^2} \right) (s'(v_t))^2 - \left( \frac{\gamma_A}{s_t} + \frac{\gamma_B}{1 - s_t} \right) s''(v_t). \tag{A76}
\]

Finding \( s'(v) \) and \( s''(v) \) from the system (A75)–(A76) and substituting them into expressions (A74) for coefficients \( a_t \) and \( b_t \), after some algebra, we obtain expressions (A71).

2) Substituting \( D_{t+\Delta t}/D_t \) from (1) into equation (A72), after some algebra, we obtain:

\[
e^{-\rho \Delta t} \left( \frac{D_{t+\Delta t}}{D_t} \right) = e^{\gamma_A \mu_D \Delta t - \gamma_A \sigma_D^2 \Delta w_t} \tag{A77}
\]

\[
= (1 - \rho \Delta t) \left( 1 - \left( \gamma_A \mu_D - \frac{\gamma_A (1 + \gamma_A)}{2} \sigma_D^2 \right) \Delta t - \gamma_A \sigma_D \right) + o(\Delta t)
\]

\[
= 1 - r_A \Delta t - \kappa_A \Delta w_t + o(\Delta t). \tag{A77}
\]

**Proof of Proposition 5.** Consider a reflected arithmetic Brownian motion with boundaries \( \underline{v} \) and \( \overline{v} \) and dynamics \( dv_t = \mu_v dt + \sigma_v dw_t \) when \( \underline{v} < v_t < \overline{v} \), where \( w_t \) is a Brownian
motion. The transition density for this process is given by (see Veestraeten, 2004):

\[
 f_v(v, \tau; v_t, t) = \frac{1}{\sqrt{2\pi\sigma_v^2(\tau-t)}} \sum_{n=-\infty}^{+\infty} \left[ \exp \left( -\frac{2\tilde{\mu}_v n(\bar{v} - v)}{\sigma_v^2} - \frac{(v - v_t - \tilde{\mu}_v(\tau-t) + 2n(\bar{v} - v))^2}{2\sigma_v^2(\tau-t)} \right) \right]
 + \exp \left( -\frac{2\tilde{\mu}_v}{\sigma_v^2} (v_t - v + n(\bar{v} - v)) \right) - \frac{(v - v_t - \tilde{\mu}_v(\tau-t) + 2(v_t - v + n[\bar{v} - v]))^2}{2\sigma_v^2(\tau-t)} \right)
 + 2\tilde{\mu}_v \frac{+\infty}{\sum_{n=0}^{+\infty}} \exp \left( -\frac{2\tilde{\mu}_v}{\sigma_v^2} (\bar{v} - v + n[\bar{v} - v]) \right) \mathcal{N} \left( \frac{v_t + \tilde{\mu}_v(\tau-t) - v + 2(v_t - v + n[\bar{v} - v])}{\sigma_v\sqrt{\tau-t}} \right)
 - \exp \left( \frac{2\tilde{\mu}_v}{\sigma_v^2} (v - v + n[\bar{v} - v]) \right) \left( 1 - \mathcal{N} \left( \frac{v_t + \tilde{\mu}_v(\tau-t) - v + 2(v_t - v + n[\bar{v} - v])}{\sigma_v\sqrt{\tau-t}} \right) \right),
\]

where \( \mathcal{N}(\cdot) \) is the cumulative distribution function of a standard normal distribution. By \( F_v(v, \tau; v_t, t) = \text{Prob}\{v_r \leq v|v_t\} \) we denote the corresponding cumulative distribution function of \( v_t \) conditional on observing \( v_t \) at time \( t \). We observe that \( s_t = s(v_t) \) is a decreasing function of \( v_t \) implicitly defined by equation (13). From the latter equation we also find that \( s^{-1}(s) = \gamma_b(1-s) - \gamma_a \ln(s) \). The cumulative distribution function of consumption share \( s_t \) at time \( t \) conditional on observing \( s_t \) at time \( t \) can then be found as follows:

\[
 F(x, \tau; s_t, t) = \text{Prob}\{s_t \leq x|s_t\} = \text{Prob}\{s(v_t) \leq x|s_t\}
 = 1 - \text{Prob}\{v_r \leq s^{-1}(x)|v_t\}
 = 1 - \text{Prob}\{v_r \leq \gamma_b \ln(1-x) - \gamma_a \ln(x)|v_t\}
 = 1 - F_v(\gamma_b \ln(1-x) - \gamma_a \ln(x), \tau; v_t, t).
\]

Substituting \( v_t = \gamma_b \ln(1-s_t) - \gamma_a \ln(s_t) \) into (A79), differentiating CDF \( F(x, \tau; s_t, t) \) with respect to \( x \) and setting \( x = s \), we find that the transition PDF for \( s \) is given by:

\[
 f(s, \tau; s_t, t) = \left( \frac{\gamma_a}{s} + \frac{\gamma_b}{1-s} \right) f_v(\gamma_b \ln(1-s) - \gamma_a \ln(s), \tau; \gamma_b \ln(1-s_t) - \gamma_a \ln(s_t), t),
\]

where transition density \( f_v(v, \tau; v_t, t) \) is given by equation (A80).

The stationary distribution of variable \( v \), calculated in Veestraeten (2004), is given by:

\[
 f_v(v) = \frac{2\tilde{\mu}_v}{\sigma_v} \frac{\exp\left( (2\tilde{\mu}_v/\sigma_v^2)\bar{v} \right)}{\exp\left( (2\tilde{\mu}_v/\sigma_v^2)\bar{v} \right) - \exp\left( (2\tilde{\mu}_v/\sigma_v^2)\bar{v} \right)}.
\]

Proceeding in the same way as for the derivation of transition PDF (A80), we obtain stationary PDF (51) for consumption share \( s_t \). □
Proof of Corollary 2. The proof easily follows by substituting boundary conditions (34) into the equation (46) for volatility $\sigma_t$ at the boundary values $\underline{v}$ and $\overline{v}$. ■

Proof of Proposition 6. Consider Lagrangian (A6) for the dynamic optimization of investor $i$. Differentiating this Lagrangian with respect to $l_i$ and $c_{it}$, we obtain:

$$\frac{\partial V_i(W_{it}, v_i; l_i)}{\partial l_i} = \eta_t D_t \Delta t + e^{-\rho \Delta t} E_t^{\prime} \left[ \frac{\partial V_i(W_{it+\Delta t}, v_{it+\Delta t}; l_i)}{\partial l_i} \right],$$

(A82)

$$u'(c_{it}^*) = \eta_t.$$  
(A83)

By the envelope theorem (e.g., Back (2010, p.162)):

$$\frac{\partial V_i(W_{it}, v_i; l_i)}{\partial W} = u'(c_{it}^*),$$ (A84)

$$\frac{\partial V_i(W_{it+\Delta t}, v_{it+\Delta t}; l_i)}{\partial W} = u'(c_{it+\Delta t}^*).$$ (A85)

Substituting (54), (A83), (A84), and (A85) into equation (A82), and simplifying, we find:

$$\hat{S}_{it} = D_t \Delta t + E_t^{\prime} \left[ e^{-\rho \Delta t} u'(c_{it+\Delta t}^*) \frac{u'(c_{it}^*)}{u'(c_{it}^*)} \hat{S}_{i,t+\Delta t} \right].$$ (A86)

From equation (28), we recall that the SPD of investor $A$ is given by

$$\xi_{A,t+\Delta t} = e^{-\rho \Delta t + \Delta U_t} \frac{(c_{A,t+\Delta t}^*)^{-\gamma_A}}{(c_{A,t}^*)^{-\gamma_A}} D_{t+\Delta t},$$ (A87)

where $\Delta U_t = \max\{0; \mu_v \Delta t + \sigma_v \Delta w_t + J_t \Delta j_t - \overline{v}\}$. Rewriting equation (A86) for investor $A$ in terms of SPD (A87), we obtain:

$$\hat{S}_{At} = D_t \Delta t + E_t^{\prime} \left[ e^{-\Delta U_t} \frac{\xi_{A,t+\Delta t}}{\xi_{At}} \hat{S}_{A,t+\Delta t} \right].$$ (A88)

Following the same steps as in the proof of Lemma 2, we find that $\hat{S}_{At} = \hat{\Psi}_i(v_t; -\gamma_A)s(v_t)^{\gamma_A} D_t$, where $\hat{\Psi}_i(v; \theta)$ satisfies differential-difference equation (33) with boundary conditions (57).

Iterating equation (16) for stock and equation (A88) for shadow prices, we obtain:

$$S_t + (1 - l_A - l_B) D_t \Delta t = \frac{1}{\xi_t} E_t^{\prime} \left[ \sum_{\tau=t}^{\infty} \xi_{\tau} (1 - l_A - l_B) D_{\tau} \Delta t \right],$$ (A89)

$$\hat{S}_{At} = \frac{1}{\xi_t} E_t^{\prime} \left[ \sum_{\tau=t}^{\infty} e^{-U_{\tau}} \xi_{\tau} D_{\tau} \Delta t \right].$$ (A90)

Inequality $(S_t + (1 - l_A - l_B) D_t \Delta t) / (1 - l_A - l_B) > \hat{S}_{At}$ follows from the fact that $U_t = \sum_{\tau=t}^{\infty} \Delta U_{\tau}$ is a non-decreasing processes. In the continuous-time limit, we obtain that $S_t / (1 - l_A - l_B) > \hat{S}_{At}$. Hence, the liquidity premium $\Lambda_{At}$ is positive. The derivation of the shadow price of investor $B$ is analogous and available upon request.
Appendix B: Existence of boundaries $\underline{v}$ and $\overline{v}$.

**Proposition B.1.** Let $\Psi_i$ be the price-dividend ratio in the economy populated only by investor $i = A, B$. If ratios $\Psi_i$, given by equations (B11)-(B12) in the Appendix, are positive and finite, then there exist boundaries $\underline{v}$ and $\overline{v}$ that satisfy equations (38).

**Proof of Proposition B.1.** Let $\tilde{l}_A = l_A + k_A (1 - l_A - l_B)$ and $\tilde{l}_B = l_B + k_B (1 - l_A - l_B)$. Because $k_A + k_B < 1$, it is easy to observe that $1 - \tilde{l}_B > \tilde{l}_A$, and hence, $\gamma_B \ln(\tilde{l}_B) - \gamma_A \ln(1 - \tilde{l}_B) < \gamma_B \ln(1 - \tilde{l}_A) - \gamma_A \ln(\tilde{l}_A)$. Equations (38) can be rewritten as follows:

$$\frac{\hat{\Psi}(\overline{v}; 1 - \gamma_A)}{\hat{\Psi}(\overline{v}; -\gamma_A)} = \tilde{l}_A, \quad \frac{\hat{\Psi}(\underline{v}; 1 - \gamma_A)}{\hat{\Psi}(\underline{v}; -\gamma_A)} = 1 - \tilde{l}_B. \tag{B1}$$

Define

$$L_B(\underline{v}, \overline{v}) = \frac{\hat{\Psi}(\underline{v}; 1 - \gamma_A)}{\hat{\Psi}(\underline{v}; -\gamma_A)}. \tag{B2}$$

Substituting $\hat{\Psi}(\nu; \theta)$ from (40) into equation (B2), after some algebra, we obtain:

$$L_B(\underline{v}, \overline{v}) = \frac{\int_{\underline{v}}^{\overline{v}} [\hat{\psi}(\nu - y) - \hat{\psi}'(\nu - y)] s(y)^{-\gamma_A} \cdot s(y) dy}{\int_{\underline{v}}^{\overline{v}} [\hat{\psi}(\nu - y) - \hat{\psi}'(\nu - y)] s(y)^{-\gamma_A} dy}. \tag{B3}$$

$L_B(\underline{v}, \overline{v})$ is a weighted average of a decreasing function $s(y)$ from $\underline{v}$ to $\overline{v}$. By (B16) in Lemma B.1 below, function $[\hat{\psi}(\nu - y) - \hat{\psi}'(\nu - y)] s(y)^{-\gamma_A}$ is positive. Consequently $L_B(\underline{v}, \overline{v}) < s(\underline{v})$ and the function is decreasing in its first argument because

$$\frac{\partial}{\partial \underline{v}} L_B(\underline{v}, \overline{v}) = \frac{\int_{\underline{v}}^{\overline{v}} [\hat{\psi}(\nu - y) - \hat{\psi}'(\nu - y)] s(y)^{-\gamma_A} \cdot [L_B(\underline{v}, \overline{v}) - s(\underline{v})]}{\int_{\underline{v}}^{\overline{v}} [\hat{\psi}(\nu - y) - \hat{\psi}'(\nu - y)] s(y)^{-\gamma_A} dy} < 0. \tag{B4}$$

Consequently, for any $\nu \geq \gamma_B \ln(1 - \tilde{l}_A) - \gamma_A \ln(\tilde{l}_A)$,

$$L_B(\gamma_B \ln(\tilde{l}_B) - \gamma_A \ln(1 - \tilde{l}_B), \overline{v}) < s(\gamma_B \ln(\tilde{l}_B) - \gamma_A \ln(1 - \tilde{l}_B)) = 1 - \tilde{l}_B. \tag{B5}$$

Below, we prove that there exists a $\underline{v} < 0$ such that $L_B(\underline{v}, \overline{v}) > 1 - \tilde{l}_B$ for any $\overline{v} \geq \gamma_B \ln(1 - \tilde{l}_A) - \gamma_A \ln(\tilde{l}_A)$. Then, by the intermediate value theorem, equation (B2) has a solution $\underline{v}$ for any fixed $\overline{v}$. 

46
Using inequalities (B17) and (B18) from Lemma B.1 and inequality (B32) from Lemma B.2 below, we derive the following inequality:

\[
1 - L_B(V, v) = \frac{\int_V \left[ \hat{\psi}(\overline{v} - y) - \hat{\psi}'(\overline{v} - y) \right] s(y)^{-\gamma_A}(1 - s(y)) dy}{\int_V \left[ \hat{\psi}(\overline{v} - y) - \hat{\psi}'(\overline{v} - y) \right] s(y)^{-\gamma_A} dy} < \frac{\int_V \left[ -e^{z^+(\overline{v} - y)} \hat{\psi}'(0) \right] (2^{\gamma_B+1}e^y + 2^{\gamma_A}e^{\frac{1}{\gamma_B} - z^+}) dy}{\int_V \left[ -e^{z^+(\overline{v} - y - 1)}(z^+ - 1) \hat{\psi}(1) \right] s(\overline{v})^{-\gamma_A} dy} = \frac{\hat{\psi}'(0)e^{z^+} s(\gamma_B \ln(1 - \tilde{l}_B) - \gamma_A \ln(\tilde{l}_A))^\gamma_A}{(z^+ - 1)\hat{\psi}(1)} \cdot \frac{\int_V 2^{\gamma_B+1}e^{(1-z^+)y} + 2^{\gamma_A}e^{\frac{1}{\gamma_B} - z^+} dy}{\int_V e^{-z^+ y} dy}.
\]

(B6)

As \( y \) decreases, the denominator term \( e^{-z^+ y} \) increases exponentially faster than any term on the numerator. Consequently, the right-hand side of the above inequality converges to 0 as \( V \to -\infty \), which can be formally verified by L’Hôpital’s rule. Therefore, there exists a \( V < 0 \) not dependent on \( v \) such that \( 1 - L_B(V, v) < \tilde{l}_B \), or, equivalently,

\[
L_B(V, v) > 1 - \tilde{l}_B. \tag{B7}
\]

For a given \( \overline{v} \), \( L_B(\overline{v}, v) \) is an continuously decreasing function of \( v \) that takes different signs at the endpoints of the interval \([\overline{v}, \gamma_B \ln(\tilde{l}_B) - \gamma_A \ln(1 - \tilde{l}_B)]\). Therefore, by the intermediate value theorem, there exists unique \( \nu \in \left[ \overline{v}, \gamma_B \ln(\tilde{l}_B) - \gamma_A \ln(1 - \tilde{l}_B) \right] \) such that \( L_B(\nu, \overline{v}) = 1 - \tilde{l}_B \), and this defines a mapping \( \nu = m_B(\overline{v}) \). Since \( L_B \) has non-zero partial derivative with respect to \( \nu \), \( m_B(\cdot) \) is continuous by the implicit function theorem.

Similar to (B2), we define

\[
L_A(\nu, \overline{v}) = \frac{\hat{\Psi}(\overline{v}; 1 - \gamma_A)}{\hat{\Psi}(\overline{v}; -\gamma_A)}. \tag{B8}
\]
Substituting \( \hat{\Phi}(v, \theta) \) from (40) into (B8), after some algebra, we obtain:

\[
L_A(v, \tau) = \frac{\int_{\bar{v}}^{v} \left[q'(\bar{v} - y)\hat{\psi}(\tau - y) - q(\tau - y)\hat{\psi}'(\tau - y)\right] s(y)^{-\gamma_A} \cdot s(y) dy}{\int_{\bar{v}}^{v} \left[q'(\bar{v} - y)\hat{\psi}(\tau - y) - q(\tau - y)\hat{\psi}'(\tau - y)\right] s(y)^{-\gamma_A} dy}.
\]  

(B9)

Proceeding the same way as above, for any \( v \) less than or equal to \( \gamma_B \ln(\tilde{l}_b) - \gamma_A \ln(1 - \tilde{l}_b) \), there exists a \( \bar{v} \in \left[\gamma_B \ln(1 - \tilde{l}_b) - \gamma_A \ln(\tilde{l}_b), \nabla\right] \) that satisfies \( L_A(v, \bar{v}) = \tilde{l}_A \), where \( \nabla \) does not depend on \( v \). This defines a continuous mapping \( v = m_A(v) \).

Consider the following system of two equations with two unknowns:

\[
\bar{v} = m_A(v), \quad \bar{v} = m_B(\bar{v}),
\]  

(B10)

where \( m_A(\cdot) \) maps \( v \in (-\infty, \gamma_B \ln(\tilde{l}_b) - \gamma_A \ln(1 - \tilde{l}_b)) \) to \( \bar{v} \in \left[\gamma_B \ln(1 - \tilde{l}_b) - \gamma_A \ln(\tilde{l}_b), \nabla\right] \), and \( m_B(\cdot) \) maps \( \bar{v} \in \left[\gamma_B \ln(1 - \tilde{l}_b) - \gamma_A \ln(\tilde{l}_b), \infty\right) \) to \( v \in \left[\nabla, \gamma_B \ln(\tilde{l}_b) - \gamma_A \ln(1 - \tilde{l}_b)\right] \). Consider now a composition function \( m(v) = m_A(m_B(v)) \). Function \( m(\cdot) \) maps \( \bar{v} \in \left[\gamma_B \ln(1 - \tilde{l}_b) - \gamma_A \ln(\tilde{l}_b), \nabla\right] \) into itself. Because \( m(v) \) is continuous, it has a fixed point \( \bar{v} \) by the intermediate value theorem. Then, \( \bar{v} \) and \( v = m_B(\bar{v}) \) satisfy equations (B10).

As demonstrated in Barro (2009), the price-dividend ratios in homogeneous-investor economies populated by investors \( A \) and \( B \), respectively, are given by:

\[
\Psi_A = \frac{1}{\rho + (1 - \gamma_A)\mu_D + \frac{(1 - \gamma_A)\gamma_A}{2}\sigma_D^2 - \lambda(1 + J_D)^{1 - \gamma_A}},
\]  

(B11)

\[
\Psi_B = \frac{1}{\rho + (1 - \gamma_B)(\mu_D + \sigma_D\theta) + \frac{(1 - \gamma_B)\gamma_B}{2}\sigma_D^2 - \lambda_B(1 + J_D)^{1 - \gamma_B}}.
\]  

(B12)

After simple algebra, it can be shown that \( \Psi_A = 1/(\hat{\rho} - \hat{\lambda}) \) and \( \Psi_A = 1/(\hat{\rho} - \hat{\mu} - 0.5\hat{\sigma}^2 - \hat{\lambda}e^{-J}) \). Therefore, assumption (B14) in Lemma B.1 is equivalent to conditions \( \Psi_A > 0 \) and \( \Psi_B > 0 \). The latter conditions also follow from condition (14) in Section 2 when time is continuous.

The investor’s value functions are bounded because

\[
\left| \mathbb{E}_t \left[ \int_t^{\infty} e^{-\rho(t-s)} \frac{c_{i,t}^{\gamma_i}}{1 - \gamma_i} d\tau \right] \right| = \left| \mathbb{E}_t \left[ \int_t^{\infty} e^{-\rho(t-s)} \frac{s(v, t)^{1 - \gamma_i} D_{i,t}^{1 - \gamma_i}}{1 - \gamma_i} d\tau \right] \right| 
\leq \max \left\{ \frac{s(v)^{1 - \gamma_i} s(\bar{v})^{1 - \gamma_i}}{1 - \gamma_i}, \frac{\Psi_i D_{i,t}^{1 - \gamma_i}}{1 - \gamma_i} \right\} < +\infty.
\]  

(B13)
Lemma B.1 (Inequalities for $\hat{\psi}(x)$ and $\hat{\psi}'(x)$). Suppose, the model parameters are such that the following two inequalities are satisfied:

$$\hat{\rho} - \hat{\lambda} > 0, \quad \hat{\rho} - \hat{\mu} - \hat{\sigma}^2 \frac{2}{\hat{\lambda}} - \hat{\lambda} e^{-J} > 0,$$

where $\hat{\rho}, \hat{\lambda}, \hat{\mu}, \hat{\sigma}$ and $\hat{J}$ are given by equations (A43). Let function $q(x)$ be given by

$$q(x) = 1 - (\hat{\rho} - \hat{\lambda}) \int_0^x \hat{\psi}(y)dy.$$  \hspace{1cm} (B15)

Then, for all $x > 0$ and $\tau > \nu$ the following inequalities are satisfied:

$$\hat{\psi}(x) < 0, \quad \hat{\psi}'(x) < 0, \quad \hat{\psi}(x) - \hat{\psi}'(x) > 0,$$

$$q'(\tau - \nu)\hat{\psi}(x) - q(\tau - \nu)\hat{\psi}'(x) > 0.$$ \hspace{1cm} (B16)

Furthermore, there exists $z^+ > 1$ such that

$$\hat{\psi}(x) - \hat{\psi}'(x) > -e^{z^+(x-1)}(z^+ - 1)\hat{\psi}(1), \text{ for } x \geq 1,$$

$$\hat{\psi}(x) - \hat{\psi}'(x) < -e^{z^+x}\hat{\psi}'(0), \text{ for } x > 0.$$ \hspace{1cm} (B17, B18)

Proof of Lemma B.1. From definition (A49), $\hat{\psi}(x)$ satisfies equation:

$$\left[\hat{\rho} - \hat{\mu} z - \hat{\sigma}^2 \frac{2}{\hat{\lambda}} z^2 - \hat{\lambda} e^{-J}z\right] \mathcal{L} \left[\hat{\psi}(x)\right] = 1.$$ \hspace{1cm} (B19)

Dividing the above equation by $z$, applying inverse Laplace transform and using the fact that $\hat{\psi}(0) = 0$, we find that $\hat{\psi}(x)$ satisfies the following integro-differential equation:

$$\frac{\hat{\sigma}^2}{2} \hat{\psi}'(x) = -1 - \hat{\mu}\hat{\psi}(x) + (\hat{\rho} - \hat{\lambda}) \int_0^x \hat{\psi}(y)dy + \hat{\lambda} \int_{\max\{x-J,0\}}^x \hat{\psi}(y)dy.$$ \hspace{1cm} (B20)

Letting $x = 0$ in equation (B20), we obtain $\hat{\psi}'(0) < 0$. Therefore, because $\hat{\psi}(0) = 0$, $\hat{\psi}(x) < 0$ in some neighborhood of 0. We first prove that $\hat{\psi}(x) < 0$ for all $x > 0$. Suppose, on the contrary, that there exists $x > 0$ such that $\hat{\psi}(x) \geq 0$. Let $\underline{x} = \inf\{x \in \mathbb{R}^+ : \hat{\psi}(x) \geq 0\}$. By the continuity of $\hat{\psi}(x)$, we have $\hat{\psi}(\underline{x}) = 0$ and $\hat{\psi}(x) < 0$ for $x \in (0, \underline{x})$. Evaluating equation (B20) at $\underline{x}$, we obtain:

$$\frac{\hat{\sigma}^2}{2} \hat{\psi}'(\underline{x}) = -1 - \hat{\mu}\hat{\psi}(\underline{x}) + (\hat{\rho} - \hat{\lambda}) \int_0^{\underline{x}} \hat{\psi}(y)dy + \hat{\lambda} \int_{\max\{\underline{x}-J,0\}}^{\underline{x}} \hat{\psi}(y)dy$$

$$< -1 - \hat{\mu} \cdot 0 + (\hat{\rho} - \hat{\lambda}) \int_0^{\underline{x}} 0 \cdot dy + \hat{\lambda} \int_{\max\{\underline{x}-J,0\}}^{\underline{x}} 0 \cdot dy = -1.$$ \hspace{1cm} (B21)

The inequality (B21) is satisfied because $\hat{\rho} - \hat{\lambda} > 0$ by assumption (B14). However, $\hat{\psi}'(\underline{x}) < 0$ is inconsistent with $\underline{x}$ being the smallest positive number such that $\hat{\psi}(\underline{x}) = 0$.
because $\tilde{\psi}(x)$ cannot be a decreasing function at $x$. Therefore, we arrive at a contradiction, and hence, $\tilde{\psi}(x) < 0$ for all $x > 0$.

Consider function $h(z) \equiv \tilde{\rho} - \tilde{\mu}z - \tilde{\sigma}^2 z^2 - \tilde{\lambda}e^{-Jz}$. By assumption (B14), $h(0) > 0$ and $h(1) > 0$. It can be easily observed that $h(-\infty) = h(+\infty) = -\infty$. Therefore, by the intermediate value theorem there exist two real roots $z^- < 0$ and $z^+ > 1$ that satisfy equation $h(z) = 0$. Furthermore, function $h(z)$ is concave because $h''(z) < 0$. The concavity of $h(z)$ implies that $h(z) \geq 0$ for all $z \in [z^-, z^+]$.

Let $\tilde{z}$ be any number such that $\tilde{z} \in [z^-, z^+]$, and let $\tilde{\alpha}(x) \equiv e^{-\tilde{z}x} \tilde{\psi}(x)$. Next, we establish that $\tilde{\alpha}'(x) < 0$ for all $x \geq 0$. Differentiating equation (B20), we obtain:

$$\tilde{\alpha}'(x) = (\rho - \tilde{\mu}z - \tilde{\sigma}^2 z^2 - \tilde{\lambda}e^{-Jz}) \tilde{\alpha}(x) + \tilde{\lambda}e^{-Jz} \tilde{\alpha}(x) + \tilde{\lambda}e^{-Jz} \tilde{\alpha}(x) - \tilde{\alpha}(x) - \tilde{\alpha}(x - \tilde{J}) 1_{x \geq \tilde{J}}.$$  \hspace{1cm} (B22)

Substituting $\tilde{\psi}(x) = e^{-\tilde{z}x} \tilde{\alpha}(x)$ into equation (B22), after some algebra, we find:

$$\tilde{\alpha}'(x) = -(\tilde{\rho} + \tilde{\sigma}^2 \tilde{z}) \tilde{\alpha}(x) + (\bar{\rho} - \tilde{\mu} \tilde{z} - \tilde{\sigma}^2 \tilde{z}^2 - \tilde{\lambda}e^{-J\tilde{z}}) \tilde{\alpha}(x) + \tilde{\lambda}e^{-J\tilde{z}} \left[ \tilde{\alpha}(x) - \tilde{\alpha}(x - \tilde{J}) 1_{x \geq \tilde{J}} \right]$$

$$= -(\bar{\rho} + \tilde{\sigma}^2 \tilde{z}) \tilde{\alpha}(x) + (\bar{\rho} - \tilde{\mu} \tilde{z} - \tilde{\sigma}^2 \tilde{z}^2 - \tilde{\lambda}e^{-J\tilde{z}}) \int_0^x \tilde{\alpha}'(y) dy + \tilde{\lambda}e^{-J\tilde{z}} \int_{\max\{x-\tilde{J},0\}}^x \tilde{\alpha}'(y) dy. \hspace{1cm} (B23)$$

We observe that $\tilde{\alpha}(0) = \tilde{\psi}(0) = 0$, $\tilde{\alpha}'(0) = -\tilde{\psi}(0) + \tilde{\psi}'(0) < 0$ because $\tilde{\psi}(0) = 0$ and $\tilde{\psi}'(0) < 0$. The rest of the proof for $\tilde{\alpha}'(x) < 0$ is similar to that of $\tilde{\psi}(x) < 0$. Consequently, differentiating $\tilde{\alpha}(x)$ and dividing $\tilde{\alpha}'(x) < 0$ by $e^{-\tilde{z}x}$, we obtain:

$$\tilde{\psi}(x) - \tilde{\psi}'(x) > 0, \text{ for any } \tilde{z} \in [z^-, z^+]. \hspace{1cm} (B24)$$

In particular for $\tilde{z} = 0$ we find $\tilde{\psi}'(x) < 0$, and for $\tilde{z} = 1$ we find $\tilde{\psi}(x) - \tilde{\psi}'(x) > 0$. Therefore, we have proven the first three inequalities in (B16).

Next, we prove (B17) and (B18). For $x > 1$, using inequality (B24) and the fact that $\tilde{\alpha}(x) = e^{-\tilde{z}x} \tilde{\psi}(x)$ is a decreasing function, we establish inequality (B17) as follows:

$$\tilde{\psi}(x) - \tilde{\psi}'(x) = (1 - z^+) \tilde{\psi}(x) + (z^+ \tilde{\psi}(x) - \tilde{\psi}'(x)) > -e^{z^+}(z^+ - 1) - e^{z^+} \tilde{\psi}(x)$$

$$> -e^{z^+}(z^+ - 1) - e^{z^+} \tilde{\psi}(1). \hspace{1cm} (B25)$$

To prove (B18), let $\tilde{\alpha}(x) = -e^{-z^+ x} \tilde{\psi}'(x)$. Differentiating equation (B22) and rewriting it in terms of $\tilde{\alpha}(x)$, we derive the following equation:

$$\tilde{\alpha}'(x) = -(\tilde{\rho} + \tilde{\sigma}^2 z^+) \tilde{\alpha}(x) + \tilde{\lambda}e^{-Jz^+} \tilde{\alpha}(x) - \tilde{\lambda}e^{-\min\{x,J\} z^+} \tilde{\alpha}(\max\{x - \tilde{J}, 0\})$$

$$= -(\tilde{\rho} + \tilde{\sigma}^2 z^+) \tilde{\alpha}(x) + \tilde{\lambda}e^{-Jz^+} \int_{\max\{x-\tilde{J},0\}}^x \tilde{\alpha}'(y) dy + \tilde{\lambda}e^{-Jz^+} - \tilde{\lambda}e^{-\min\{x,J\} z^+} \tilde{\alpha}(0). \hspace{1cm} (B26)$$
Letting $x = 0$ in (B22), we find that $\hat{\psi}''(0) = -2(\hat{\mu}/\hat{\sigma}^2)\hat{\psi}'(0)$. Consequently,

$$\hat{\alpha}(0) = -\hat{\psi}'(0) > 0 \quad \hat{\alpha}'(0) = -\hat{\psi}''(0) + z^+\hat{\psi}'(0) = \frac{2}{\hat{\sigma}^2}(\hat{\mu} + \frac{\hat{\sigma}^2}{2}z^+)\hat{\psi}'(0) < 0,$$  \hspace{1cm} (B27)

where the last inequality hold because $z^+ > 1$ and $z^+\hat{\mu} + 0.5\hat{\sigma}^2z^+ = \hat{\rho} - \hat{\lambda}e^{-Jz^+} > \hat{\rho} - \hat{\lambda} > 0$. Similar to the above, we show that $\hat{\alpha}'(x) < 0$. Hence, we derive (B18) as follows:

$$\hat{\psi}(x) - \hat{\psi}'(x) < -\hat{\psi}'(x) = e^{z^+x}\hat{\alpha}(x) < e^{z^+x}\hat{\alpha}(0) = -e^{z^+x}\hat{\psi}'(0).$$  \hspace{1cm} (B28)

Finally, we prove the last inequality in (B16). We define $\tilde{\beta}(x) = e^{-z^+q(x)}$ and next prove that $\tilde{\beta}(x) < 0$. Proceeding in the same way as above, we express equation (B20) first in terms of $q(x)$ and then in terms of $\tilde{\beta}(x)$:

$$\tilde{\sigma}^2 q''(x) = -\tilde{\mu}q'(x) + \tilde{\rho}q(x) - \tilde{\lambda}q(\max\{x - \tilde{J}, 0\}),$$  \hspace{1cm} (B29)

$$\tilde{\sigma}^2 \tilde{\beta}''(x) = -(\tilde{\mu} + \tilde{\sigma}^2 z^+)\tilde{\beta}'(x) + \tilde{\lambda}e^{-Jz^+}\tilde{\beta}(x) - \tilde{\lambda}e^{-\min\{x,J\}z^+}\tilde{\beta}(\max\{x - \tilde{J}, 0\})$$

$$= -(\tilde{\mu} + \tilde{\sigma}^2 z^+)\tilde{\beta}'(x) + \tilde{\lambda}e^{-Jz^+}\int_{\max\{x - \tilde{J}, 0\}}^{x} \tilde{\beta}'(y)\text{d}y + \left[\tilde{\lambda}e^{-Jz^+} - \tilde{\lambda}e^{-\min\{x,J\}z^+}\right]\tilde{\beta}(0).$$  \hspace{1cm} (B30)

For $x = 0$ we observe that $\tilde{\beta}(0) = q(0) = 1$, $\tilde{\beta}'(0) = -z^+q(0) + q'(0) = -z^+q(0) - (\tilde{\rho} - \tilde{\lambda})\hat{\psi}(0) = -z^+ < 0$. Moreover, it is easy to observe that $\left[\tilde{\lambda}e^{-Jz^+} - \tilde{\lambda}e^{-\min\{x,J\}z^+}\right]\tilde{\beta}(0) \leq 0$ for all $x$. Proceeding as above, we find that $\tilde{\beta}'(x) < 0$, and hence, $q'(x) < z^+q(x)$. Using the latter inequality and $\tilde{\psi}(x) < 0$, we prove the last inequality in (B16):

$$q'(\overline{v} - \underline{v})\tilde{\psi}(x) - q(\overline{v} - \underline{v})\tilde{\psi}'(x) \geq z^+q(\overline{v} - \underline{v})\tilde{\psi}(x) - q(\overline{v} - \underline{v})\tilde{\psi}'(x)$$

$$= q(\overline{v} - \underline{v})\left[z^+\tilde{\psi}(x) - \tilde{\psi}'(x)\right] > 0. \hspace{1cm} \blacksquare$$  \hspace{1cm} (B31)

**Lemma B.2 (Inequality for consumption shares).** Let $s(v_i)$ denote the consumption share of investor $A$. Then, for all $v \in \mathbb{R}$ the following inequality is satisfied:

$$s(v)^{-\gamma_A}(1 - s(v)) \leq 2^{\gamma_B+1}e^v + 2^{\gamma_A}e^{v/\gamma_B}. \hspace{1cm} (B32)$$

**Proof of Lemma B.2.** First, we rewrite equation (13) in the following equivalent form:

$$s(v)^{-\gamma_A}(1 - s(v))^{\gamma_B} = e^v.$$  \hspace{1cm} (B33)

When $\gamma_B \leq 1$, from the above equation we obtain the following inequality:

$$s(v)^{-\gamma_A}(1 - s(v)) \leq s(v)^{-\gamma_A}(1 - s(v))^{\gamma_B} = e^v.$$  \hspace{1cm} (B34)
For $\gamma_B > 1$ and $1 - s(v) \geq 1/2$, we find that:

$$s(v)^{-\gamma_A}(1 - s(v)) \leq 2^{\gamma_B - 1} s(v)^{-\gamma_A}(1 - s(v))^{\gamma_B} = 2^{\gamma_B - 1} e^v.$$ (B35)

Finally, for $\gamma_B > 1$ and $s(y) \geq 1/2$ we have the following inequality:

$$s(y)^{-\gamma_A}(1 - s(y)) \leq 2^{\gamma_A - \gamma_A/\gamma_B} s(y)^{-\gamma_A/\gamma_B}(1 - s(y)) < 2^{\gamma_A} e^{v/\gamma_B}.$$ (B36)

Combining all the inequalities (B34)-(B36), we obtain inequality (B32). □