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Investor Protection and Asset Prices∗

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Abstract

Empirical evidence suggests that investor protection has significant effects on ownership concentration and asset prices. We develop a dynamic asset pricing model to address the empirical regularities and uncover the underlying mechanisms at play. Our model features a controlling shareholder who endogenously accumulates control over a firm, and diverts a fraction of its output. In line with empirical evidence, better investor protection decreases stock holdings of controlling shareholders, increases stock mean-returns, and increases stock return volatilities when ownership concentration is sufficiently high. The model also predicts that better protection increases interest rates and decreases leverage.

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1. Introduction

In many countries corporations are dominated by controlling shareholders who can divert resources for their private benefit. Consequently, the protection of minority shareholders against expropriation by controlling shareholders is of much importance for understanding ownership concentration and asset prices. In line with this observation, the empirical literature documents significant effects of investor protection on asset prices and their dynamics. However, an overarching theory that sheds light simultaneously on the effects of investor protection on the firm ownership concentration, stock returns, interest rates, leverage, risk sharing, and extent of expropriation is missing in the literature. In this paper, we develop such a theory in a parsimonious general equilibrium setting and use it to address the empirical evidence on asset holdings of controlling shareholders and stock returns, and to provide new predictions.

The dynamic accumulation of control and the ability of shareholders to trade and share risks are key features of our analysis which play a leading role in determining the effects of investor protection on asset holdings and returns. Due to these key features, our analysis generates rich dynamics of asset holdings and returns, and reveals that the excess ownership concentration (relative to the full protection benchmark) is the main channel through which investor protection influences stock mean-returns, volatilities, interest rates, and leverage. These results help us address the related empirical evidence. In particular, in line with evidence (discussed below), we find that the stock holdings of controlling shareholders change in response to economic shocks and decrease with better investor protection, stock mean-returns increase with better protection, and stock return volatilities also increase with better protection when ownership concentration is sufficiently high. Our analysis also uncovers new predictions. In particular, we show that interest rates are higher with better protection, the effect of protection on volatility and controlling shareholders’ leverage is non-monotonic and depends on ownership concentration, and moreover, this leverage is lower in economies with full protection. We discuss these results in greater detail below.

We consider a dynamic general equilibrium economy with a representative competitive firm that produces an exogenous stream of output. The firm’s stock is owned by two types of shareholders with heterogeneous constant relative risk aversion (CRRA) preferences, a minority shareholder and a controlling shareholder who can divert a fraction of the firm’s output for himself. The diverted fraction satisfies an investor protection constraint, which limits the scope of available diversion strategies. This constraint becomes tighter with better protection and looser with higher stock holdings that increase the controlling shareholder’s
power over the firm. The diversion of output is further tempered by pecuniary costs of stealing. We provide tractable expressions for the equilibrium processes that admit intuitive comparative statics and are explicit up to the controlling shareholder’s stock holding, which is found numerically.

We find that the controlling shareholder’s stock holding is larger in economies with imperfect investor protection than in economies with full protection, consistent with the empirical evidence (La Porta, Lopez-de-Silanes, and Shleifer, 1999). Intuitively, poor protection expands the set of diversion strategies and increases the potential gains from higher control power over the firm, which induces buying more shares. However, the relationship between investor protection and optimal stock holdings is non-monotonic and depends on whether the investor protection constraint binds or not. This is because investor protection has two opposing effects on the controlling shareholder’s optimal portfolio decision. An increase in investor protection reduces the marginal benefit of control and hence reduces the incentive to acquire more shares, while on the other hand, makes the investor protection constraint relevant for a wider range of stock holdings thereby providing an incentive to acquire more shares to relax the constraint. The relative importance of these effects depends on the consumption share of the minority shareholder, which acts as an endogenous state variable in the model. We also find that the controlling shareholder’s stock holding changes in response to economic shocks, consistent with empirical evidence (e.g., Denis and Sarin, 1999; Holderness, Kroszner, and Sheehan, 1999). In particular, the controlling shareholder sells stocks to finance consumption in bad times, when the firm output is hit by negative shocks, and buys them back in good times.

The acquisition of the controlling shareholder’s shares is financed by leverage, and hence the leverage is higher in economies with imperfect protection. We establish a tight link between the leverage and stock holdings of the controlling shareholder. In particular, the leverage-stock price ratio in our model is simply given by the controlling shareholder’s excess ownership concentration, defined as the stock holding over and above his holding in the full protection economy. We also show that the fraction of diverted output is a hump-shaped function of the shares held by the controlling shareholder. In our model, a higher stock holding relaxes the investor protection constraint by increasing the control power over the firm, and the controlling shareholder diverts more as his holding increases. On the other hand, the higher stock holding decreases his incentive to divert due to pecuniary costs of stealing. After some point, the investor protection constraint no longer binds and the equilibrium amount of diverted output decreases with the stock holding.

We demonstrate that the stock mean-return decreases with poor investor protection in
equilibrium. This is consistent with the empirical evidence on the relation between the realized stock returns and the degree of corporate governance, entrenchment and managerial perks (e.g., Gompers, Ishii and Metrick, 2003; Yermack, 2006; Bebchuk, Cohen and Ferrell, 2009; Daines, Gow and Larcker, 2010, among others), although there is an ongoing discussion of the robustness of this relation (Core, Guay and Rusticus, 2006; Giroud and Mueller, 2011; Bebchuk, Cohen and Wang, 2013). We contribute to this discussion by providing a theoretical argument in favor of the positive relation between the stock mean-return and investor protection. Our intuition is that, in contrast to the minority shareholder, the controlling shareholder is compensated for holding risky assets not only by the risk premium but also by the fraction of the diverted output. Therefore, the controlling shareholder hoards shares even if the realized risk premium is low, which drives down the stock mean-return in equilibrium. Following this intuition, we show that the asset holdings of the controlling shareholder are determined by a previously unexplored quantity, which we refer to as the effective risk premium. We decompose the effective risk premium into a conventional risk premium implied by stock price dynamics and an additional term capturing the diverted output per share. In our decomposition the diverted output per share can be interpreted as an adjustment to the dividend that is received by the controlling shareholder.

We also show that the equilibrium stock return volatility is higher with imperfect protection than with full protection and exceeds the volatility of the aggregate output. Another novel prediction of our model is that across economies with varying imperfect protection, the relation between volatility and investor protection is non-monotone, and when the stock holding of the controlling shareholder is sufficiently high the volatility is higher in economies with better protection, consistent with the empirical evidence (e.g., Morck, Yeung and Yu, 2000; Jin and Myers, 2006; Bartram, Brown and Stulz, 2012). We find that the excess volatility relative to the full protection benchmark economy is proportional to the excess leverage-stock price ratio relative to the benchmark, and to the best of our knowledge, such a simple characterization of volatility in terms of leverage is new to the literature. Intuitively, leverage finances the acquisition of shares by the controlling shareholder when protection is low, and hence increases the sensitivity of the controlling shareholder’s wealth to economic shocks, which translates into higher stock return volatility via the state variable that tracks wealth inequality. The non-monotonicity of volatilities with respect to investor protection is explained by the non-monotonicity of the stock holdings, which determine leverage, as discussed earlier.

Furthermore, we find that the risk-free interest rates decrease with lower protection due to the following two effects in equilibrium. First, because of low risk premium and high
volatility, the minority shareholder turns to bond markets and is more willing to provide cheap credit. Second, the acquisition of shares by the controlling shareholder is partially covered by the diverted output, which moderates his demand for credit. These two effects are partially offset by the surge of the controlling shareholder’s leverage under poor protection, which increases the demand for borrowing, but the net effect on the equilibrium interest rate is negative.

Finally, we study the equilibrium effects of the pecuniary costs of stealing such as bribes, fines payable if controlling shareholders are caught stealing, payments to lawyers and other expenses for arranging diversion schemes. We find that a higher pecuniary cost of stealing is associated with a higher stock gross return and interest rate, and a lower volatility and stock holding of the controlling shareholder. However, the effects of pecuniary costs on stock holdings and volatility only exist when the investor protection constraint binds. Thus, modeling investor protection as a constraint, a novel feature of our model, is key in generating links among investor protection, endogenous accumulation of control and asset price dynamics.

We also consider extensions of our analysis that feature the separation of control and cash flow rights, and non-pecuniary costs of diverting output. We find that all our results on the effects of investor protection remain qualitatively the same and also uncover additional insights with these extensions. In particular, we show that the controlling shareholder diverts more output as the difference between the control and cash flow rights increases, consistent with the empirical findings of Claessens, Djankov, Fan, and Lang (2002). We further extend the analysis to incorporate cross-firm differences in investor protection, albeit in a simple one-period economy. We show that our earlier results on stock holdings, mean-returns, and interest rates remain valid. We additionally find that cross-firm differences in investor protection give rise to a positive investor protection premium, defined as the spread between mean-returns of stocks with higher and lower protection. This result is particularly valuable because much of the empirical evidence on the positive relation between stock returns and investor protection (discussed above) is for cross-sections of firms with different levels of investor protection whereas our main analysis features a representative firm.

Our paper also makes a methodological contribution by integrating a model with corporate frictions such as investor protection and acquisition of control into a general equilibrium asset pricing framework, albeit in a pure-exchange setting.¹ Solving models with frictions such as constraints on certain choice variables is a daunting task. We achieve tractability by

¹Focusing on a pure-exchange setting allows us to incorporate trading between different groups of shareholders in a tractable way. As discussed below, a general equilibrium production economy with investor protection and a buy-and-hold controlling shareholder has been studied by Albuquerque and Wang (2008).
allowing investors to optimize two-period CRRA preferences repeated over time, similar to models with myopic investors and overlapping generations. This approach allows us to focus on the effects of investor protection and abstract away from hedging demands for stocks, which are more relevant for the portfolio choice literature. However, we demonstrate that the equilibrium processes in the full protection benchmark economy share some features of dynamic Lucas-type (1978) economies with heterogeneous investors. In particular, the Sharpe ratio in the latter economy and our model have the same structure. A notable feature of our model is that it is stationary in the sense that both shareholders survive in the long run and the distributions of their consumption shares are non-degenerate. The stationarity is achieved by endowing shareholders with non-financial labor incomes, since future incomes help investors gradually accumulate wealth and our preference specification keeps investors’ wealth positive, restricting financial losses.

1.1. Related Literature

The most related to our paper are the works that study asset pricing implications of investor protection against expropriation by controlling shareholders. Shleifer and Wolfenzon (2002) introduce a static model that explains why firms are larger and more valuable with better investor protection. Albuquerque and Wang (2008) develop a dynamic production economy with buy-and-hold controlling shareholders who extract more benefits in larger firms. They show that weaker investor protection implies over-investment which leads to higher stock risk premia, volatility, and interest rates. The stock mean-returns, volatility, and interest rates are constant due to the absence of trading between shareholders. Giannetti and Koskinen (2010) study a static model of two countries with different levels of investor protection, where investors make portfolio decisions at the initial date and do not rebalance their portfolios. They find that stock returns decrease with weaker protection, similar to our paper. We complement these works by focusing on different aspects of investor protection arising due to its effect on asset demands and accumulation of control. In particular, the controlling shareholder in our model extracts private benefits by dynamically increasing his stock holding in the firm whereas in these works the stock holding is fixed. Consequently, we uncover new economic forces that decrease stock mean-returns and interest rates when investor protection is low, leading also to excess stock return volatility, leverage, time-variation of all equilibrium processes and wealth transfers between different categories of shareholders.

Our paper is also related to the literature where the stock holdings of controlling shareholders are endogenous and influence managerial incentives and firm output in a dynamic
setting. DeMarzo and Urosevic (2006) and Gorton, He, and Huang (2014) present models with stock trading and risk sharing between a controlling shareholder and small investors. The controlling shareholder in these models trades dynamically and accounts for the relationship between her stock holding on one hand and effort level, firm output, and stock price on the other. In particular, the latter paper shows that stock mean-returns and volatility increase or decrease depending on investors’ risk aversions. Haddad (2015) studies a dynamic model with endogenous ownership concentration where active shareholders increase the output mean-growth rate and are compensated by the firm for being under-diversified. He shows that active capital amplifies stock-return volatility. Jung, Subramanian, and Zeng (2016) study a model where a large shareholder faces a tradeoff between diversification and incentive provision to firm managers determining the value of the firm. Our paper differs from these works in that we focus on the economic effects of investor protection in a setting where the controlling shareholder’s power over the firm and the ability to divert output depend on his stock holdings. As a result, we uncover new economic effects of investor protection on excess ownership concentration, mean-returns, volatility, interest rates, and leverage.

Also related to our paper are various works that share some of the key features of our analysis such as investor protection, expropriation of minority shareholders, and endogenous ownership concentration. Doidge, Karolyi, and Stulz (2004) consider a static model where controlling shareholders of firms with growth opportunities reduce their cost of capital by cross-listing firms in the U.S., which commits them to better investor protection. Doidge, Karolyi, and Stulz (2007) show, both theoretically and empirically, the importance of country characteristics for the quality of investor protection. Dow, Gorton, and Krishnamurthy (2005) study a model with the separation of ownership and control where firm managers expropriate shareholders by diverting cash flows to inefficient investments. Aslan and Kumar (2012) present a three-period model in which the endogenous choice of ownership concentration affects the cost of borrowing, probability of default, and the post-default restructuring. Li and Li (2018) show, both theoretically and empirically, that better corporate governance leads to higher stock returns during market upturns when firms have good growth options, but low stock returns in the downturns. In contrast to this literature, our paper studies the joint effects and interactions of investor protection, expropriation, and ownership concentration, which give rise to distinct economic predictions.
2. The Economy with Investor Protection

We consider a pure-exchange continuous-time infinite-horizon economy with a representative firm that produces one consumption good and is owned by two types of shareholders with heterogeneous control power over the firm. In this Section, we discuss the firm, the financial markets, and shareholder optimization problems.

2.1. Firms and Financial Markets

The uncertainty is represented by a filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)\), on which is defined a Brownian motion \(w\). The stochastic processes are adapted to the filtration \(\{\mathcal{F}_t, t \in [0, \infty]\}\), generated by \(w\). There is one representative firm in the economy which stands for a large number of identical firms. The firm produces an exogenous stream of output \(\hat{D}_t\), which follows a geometric Brownian motion (GBM) process:

\[
d\hat{D}_t = \hat{D}_t [\mu_D dt + \sigma_D dw_t],
\]

where the output mean-growth rate \(\mu_D\) and volatility \(\sigma_D\) are constants.

There are two types of shareholders, a controlling \(C\) and a representative minority \(M\) shareholders. The representative minority shareholder stands for a group of identical shareholders. Fractions \(l_C\) and \(l_M\) of the aggregate output are paid to the controlling and minority shareholders as their labor incomes, respectively. The shareholders trade continuously in two securities, a riskless bond in zero net supply with an instantaneous interest rate \(r_t\) and a stock in positive net supply, normalized to one unit. The stock is a claim to the stream of dividends, which are paid each date \(t\) out of the net output after paying labor incomes, given by \(D_t = (1 - l_C - l_M)\hat{D}_t\). The dividend payout is determined by the controlling shareholder, as discussed below. We focus on Markovian equilibria in which bond price, \(B_t\), and stock price, \(S_t\), follow processes

\[
\begin{align*}
\text{d}B_t &= B_t r_t dt, \\
\text{d}S_t &= S_t [\mu_t dt + \sigma_t dw_t],
\end{align*}
\]

where the interest rate \(r_t\), stock mean-return \(\mu_t\), and volatility \(\sigma_t\) are endogenously determined in equilibrium, and the bond price at time 0 is normalized so that \(B_0 = 1\).
2.2. Investor Protection and Shareholder Objectives

The minority shareholder does not have control power, and cannot influence the dividend payout policy. The controlling shareholder can divert a fraction \( x_t \) of the firm’s output for himself. The remaining non-diverted output \((1 - x_t)D_t\) is paid as a time-\( t \) dividend. The diverted fraction \( x_t \) is constrained by investor protection in the economy, so that \( x_t \leq (1 - p)q(n_t) \), where \( p \in [0,1] \) is interpreted as the protection of minority shareholders, with higher \( p \) indicating better investor protection, and \( q(n) \in [0,1] \) indicating the controlling shareholder’s power over the firm. Consequently, the above investor protection constraint on fraction \( x_t \) is determined jointly by investor protection \( p \) and the controlling shareholder’s power \( q(n) \) over the firm. To simplify the analysis, we assume that \( q(n) = n \), so that the power over the firm is linearly increasing in the the number of shares, and the controlling shareholder has the same control and cash flow rights in the firm. In Section 5.1 we consider a generalization of the model with a more elaborate control power \( q(n) = an \), which allows to separate between control power \( q(n) \) and cash flow rights \( n \) by varying parameter \( a \), and show that the economic implications of the model remain the same.

The investor protection constraint captures the fact that better protection restricts the set of available expropriation strategies. Controlling shareholders may divert output by employing a wide range of complex strategies rather than outright theft. For example, cash flows can be tunneled through intra-group activities which can be economically large (Bertrand, Mehta, and Mullainathan, 2002; Cheung, Rau, and Stouraitis, 2006; Khanna and Yafeh, 2007; Jiang, Lee and Yue, 2010). Companies can give each other, or to controlling shareholders directly, high (or low) interest loans (Bertrand, Mehta, and Mullainathan, 2002), pay special dividends to controlling shareholders (DeAngelo and DeAngelo, 2000), engage in abnormal sales (Jian and Wong, 2010; Lo, Wong, and Firth, 2010), sell assets below or above their market values (Cheung, Rau, and Stouraitis, 2006) and provide loan guarantees (Berkman, Cole and Fu, 2009). One could also include jobs given to relatives, large bonuses, perquisites etc. that are enjoyed by controlling shareholders as a form of wealth transfer from minority shareholders.

The investor protection constraint also demonstrates that more power \( q(n) \) over the firm makes it easier to orchestrate wealth transfers for private benefits by expanding the set of available diversion strategies through various channels. First such channel is that a higher stake in the firm gives the controlling shareholder better control over the board and its decisions which allows implementing diversion strategies that would otherwise not be possible. For instance, Anderson and Reeb (2004) analyze board composition of family
controlled firms and find that families often seek to minimize the presence of independent directors. The literature provides both empirical and anecdotal evidence of how controlling shareholders divert firm assets to their private use without substantive interference from the board (e.g., DeAngelo and DeAngelo, 2000; Anderson and Reeb, 2004; Enriques and Volpin, 2007). Second channel is that a higher stake thwarts possible takeover attempts by third parties, widely considered as a market disciplining mechanism (Grossman and Hart, 1980) in which poorly performing firms become targets for acquisition by third parties (perhaps as a result of extraction of private benefits). Consistent with this argument Stulz (1988) shows that an increase in the fraction of voting rights controlled by management decreases the probability of a successful tender offer, mitigating the disciplining role played by outside takeover threats. Anti-takeover clauses are often used as part of governance indices (Gompers, Ishii and Metrick, 2003). Third possible channel is that a larger stake of the controlling shareholder deters a formation of large blocks by other shareholders who might oppose the extraction of private benefits through better monitoring.

We further assume that the controlling shareholder incurs a pecuniary cost $f(x, D)$ from diverting output, in line with the related literature (e.g., Shleifer and Wolfenson, 2002; Albuquerque and Wang, 2008). The cost function $f(x, D)$ is an increasing function of the diverted fraction $x_t$ and net output $D_t$. Throughout the paper, we assume for tractability that the cost function $f(x_t, D_t)$ is quadratic in $x_t$ and given by

$$f(x_t, D_t) = \frac{k x_t^2 D_t}{2},$$

where the parameter $k$ captures the magnitude of the cost. The pecuniary costs of diverting output may include bribes, fines payable if controlling shareholders are caught stealing, payments to lawyers and other expenses for arranging diversion schemes. The pecuniary cost does not disappear from the economy but is paid to the minority shareholder either via a government transfer or by the controlling shareholder.

In reality, there could also be non-pecuniary costs of diverting output such as disutility from stealing due to social norms that promote fairness, honesty and morality (Kahneman, Knetsch and Thaler, 1986), and loss of reputation. Such non-pecuniary costs reduce the controlling shareholder’s utility without affecting the budget constraint. In Section 5.2 we discuss an alternative formulation with non-pecuniary costs and find that all our main economic mechanisms at play and results remain equally valid for this alternative.

The shareholder protection constraint $x_t \leq (1 - p)n_t$ and the cost function $f(x_t, D_t)$ capture different barriers to expropriation in the economy, and hence, lead to distinct economic implications in equilibrium, as highlighted in Section 4.3 below. The former constraint prox-
ies for legal protection of minority shareholders that limits wealth transfer strategies. In contrast, cost function $f(x_t, D_t)$ quantifies pecuniary costs of stealing.

All shareholders have standard constant relative risk aversion (CRRA) preferences

$$u_i(c) = \frac{c^{1-\gamma_i} - 1}{1 - \gamma_i}, \quad i = C, M,$$

(5)

with the risk aversion parameters $\gamma_M \geq \gamma_C > 0$. The controlling shareholder being less risk averse is natural in our setting since a typical endogenous occupation choice model would predict that less risk averse people self-select into entrepreneurial activities and become controlling shareholders (Kihlstrom and Laffont, 1979). For tractability, we assume that investors are guided by myopic preferences over current consumption $c$ and wealth $W$, given by:

$$V_i(c_t, W_t, W_{t+dt}) = \rho u_i(c_t)dt + (1 - \rho dt)\mathbb{E}_t[u_i(W_{t+dt})],$$

(6)

where utility function $u_i(\cdot)$ is given by (5), $i = \{C, M\}$, and $\rho > 0$ is a time-preference parameter. Myopic preferences admit considerable tractability yielding closed-form solution for the optimal portfolios (Section 3.1), and are widely employed in various contexts (e.g., DeLong, Shleifer, Summers and Waldmann, 1990; Pastor, 2000; Acharya and Pedersen, 2004). They also help us focus on the effects of investor protection and abstract away from hedging demands, which are more relevant for the portfolio choice literature.

The preferences (6) have the additional benefit that, along with labor income, they make our economy stationary in that both shareholders continue to have significant economic impact in the long run (Section 3.2). The reason is that these preferences require the shareholders to hold positive next-period financial wealth $W_{t+dt}$ at all times, capturing effects similar to those of collateral constraints in economies where investors have limited liability and receive non-pledgeable incomes that cannot be used to back risky positions. Income non-pledgeability may arise in our setting due to the difficulties of borrowing against future stolen output or due to the inability to confiscate the labor income of a defaulted counterparty. The literature demonstrates that the constraint $W_{t+dt} \geq 0$ naturally emerges in such

\footnote{We thank an anonymous referee for pointing this out. Consistently, Hvide and Panos (2014) provide empirical evidence that risk tolerant individuals are more likely to become entrepreneurs.}

\footnote{Such preferences may also naturally arise in an OLG-type framework. Furthermore, in many models logarithmic preferences give rise to investor myopia, similar to that in the objective function (6) (e.g., Detemple and Murthy, 1997; Basak and Croitoru, 2000). In particular, it can be shown that the value function of a dynamic infinite-horizon consumption choice problem with logarithmic preferences has the following structure: $J(W_t, z_t) = (1/\rho) \ln(W_t) + J(z_t, t)$, where $z_t$ is a certain state variable. Then, from the Hamilton-Jacobi-Bellman equation for the dynamic problem it is immediate to observe that solving the dynamic problem is equivalent to solving a myopic problem with an objective function $\rho \ln(c_t)dt + (1 - \rho dt)\mathbb{E}_t[\ln(W_{t+dt})]$. The objective function (6) retains the latter structure of problems with logarithmic preferences but has the additional benefit of accounting for risk aversion $\gamma_i$.}
economies to prevent defaults by requiring investors to collateralize their risky positions with pledgeable financial securities, even when investors have an infinite planning horizon (e.g., Chien and Lustig, 2010; Chabakauri and Han, 2018). The non-pledgeability of future income also significantly reduces its (shadow) value to investors (Chabakauri and Han, 2018), diminishing its impact on portfolio choice and consumption. Our economy then becomes stationary because keeping the wealth \( W_{t+dt} \) positive restricts financial losses while future incomes help investors gradually accumulate wealth even when their current wealth is close to zero.

### 2.3. Shareholders’ budget constraints and optimizations

Each shareholder chooses consumption \( c_t \), the number of shares \( n_t \), and the controlling shareholder additionally chooses the fraction \( x_t \) of diverted output for private consumption. Each shareholder’s wealth at time \( t \) is given by \( W_t \equiv b_t B_t + n_t S_t \), where \( b_t \) is the number of units of bonds in the shareholder’s portfolio. The shareholders’ dynamic self-financing budget constraints are as follows:

\[
\begin{align*}
    dW_t &= \left( W_t r_t + n_t (S_t (\mu_t - r_t) + (1 - x_t) D_t) - c_t + l_t \widehat{D}_t \right) dt \\
    &\quad + \left( 1_{\{i=C\}} (x_t D_t - f(x_t, D_t)) + 1_{\{i=M\}} f(x_t, D_t) \right) dt + n_t S_t \sigma_t dw_t,
\end{align*}
\]

where \( S_t (\mu_t - r_t) + (1 - x_t) D_t \) is the gross dollar excess return on the stock in absolute terms, \((1 - x_t) D_t \) is the dividend per share, \( l_t \widehat{D}_t \) is shareholder \( i \)'s labor income, \( x_t D_t \) is the diverted output, \( f(x_t, D_t) \) is the pecuniary cost of diverting output, and \( 1_{\{i\}} \) denotes the indicator function.

The minority shareholder maximizes the following objective function over current consumption \( c \) and next period’s wealth \( W \):

\[
    \max_{n_t, c_t} V_M(c_t, W_t, W_{t+dt}),
\]

where the function \( V_M(\cdot) \) is given by equation (6) for \( i = M \), subject to self-financing budget constraint (7). The controlling shareholder maximizes his objective function

\[
    \max_{x_t, n_t, c_t} V_C(c_t, W_t, W_{t+dt}),
\]

where the function \( V_C(\cdot) \) is as given in (6) for \( i = C \), subject to the budget constraint (7), the investor protection constraint \( x_t \leq (1 - p) n_t \) and the maximum share constraint \( n_t \leq 1 \). An
important feature of our model is that we allow the controlling shareholder to dynamically rebalance the portfolio of assets, consistent with empirical evidence on the dynamics of firm ownership by controlling shareholders such as managers and board members (e.g., Denis and Sarin, 1999; Holderness, Kroszner, and Sheehan, 1999). Finally, we note that the controlling shareholders in our setting act as price takers and do not manipulate their firm’s stock price.5

3. Equilibrium with Investor Protection

In this Section, we first solve for investors’ optimal strategies in a partial equilibrium setting, in which asset price dynamics are taken as given. Then, by substituting the optimal strategies into the market clearing conditions, we obtain the dynamics of asset prices in equilibrium.

3.1. Shareholders’ Optimal Strategies

We now solve for the optimal consumptions and stock holdings of controlling and minority shareholders. We first note that the maximization of objective functions (8) and (9) turns out to be equivalent to separate optimization problems for consumption $c_t$ and stock holding $n_t$.6

In particular, investor $i$’s optimal consumption maximizes the following objective function:

$$\max_{c_t} \rho c_t^{1-\gamma_i} - \gamma_i^{-1} W_t^{-\gamma_i} c_t,$$

whereas the optimal stock holding $n^{*}_{Ct}$ of the controlling shareholder and diverted fraction $x_t^*$ solve a quadratic optimization problem

$$\max_{n_t, x_t} J_C(n_t; x_t) = \frac{n_t S_t}{W_{Ct}} (\mu_t - r_t) + (1 - x_t) \frac{D_t}{W_{Ct}} + x_t \frac{D_t}{W_{Ct}} - x_t^2 \frac{k}{2} \frac{D_t}{W_{Ct}} - \gamma_C \frac{\sigma_t}{2}$$

5This is due to the following reasons. First, their trading, consumption and stealing decisions do not affect Sharpe ratios of their own stocks because all firms are small and have identical outputs driven by the same Brownian motion $w_t$. In such an economy, any deviation of a firm’s Sharpe ratio from that of the market portfolio leads to an arbitrage opportunity (e.g., Basak and Croitoru, 2000), which can be easily eliminated by minority shareholders who do not face any trading frictions. Second, we posit that the minority shareholders cannot observe the stock demands of controlling shareholders and believe that firms are identical and the controlling shareholders do not manipulate prices. Consequently, they are indifferent to which stocks to hold. If they observe that the stock price of a particular firm $k$ deviates from the prices of other firms they assume that too many minority shareholders bought or sold stocks of firm $k$ and trade in the direction of eliminating the mispricing. As a result, the controlling shareholders do not attempt to alter stock prices knowing that the mispricing will be eliminated, which confirms the beliefs of the minority shareholders that all firms are identical and the controlling shareholders do not manipulate prices.

6To demonstrate this, we rewrite the second term in equation (6) for investor preferences as $\mathbb{E}_t[u_i(W_{it+\delta t})] = u_i(W_{it}) + \mathbb{E}_t[du_i(W_{it})]$, apply Itô’s Lemma to $u_i(W_{it})$, where $u_i(\cdot)$ is given by (7), and then, after some algebra, we find that the optimal consumption and the number of shares solve two separate optimization problems.
subject to constraints $x_t \leq (1-p)n_t$ and $n_t \leq 1$, and the optimal stock holding $n^*_{ct}$ of the minority shareholder solves the optimization problem

$$\max_{n_t} J_{ct}(n_t) = \frac{n_t S_t}{W_{st}} \left( (\mu_t - r_t) + (1 - x_t) \frac{D_t}{S_t} \right) - \frac{\gamma_m (n_t S_t \sigma_t)}{2}. \quad (12)$$

Solving for shareholders’ optimal consumptions, we find that consumptions $c^*_t$ are given by:

$$c^*_t = \rho^{1/\gamma} W_t. \quad (13)$$

The shareholders’ consumption-wealth ratios are constant, similar to frictionless models with logarithmic preferences.\(^7\)

Solving the portfolio choice problem of the controlling shareholder is complicated by the presence of constraints on stock holding $n$ and diverted fraction of output $x$, which renders the value function (11) non-concave function of stock holding $n$. We proceed in two steps. We first maximize the objective function $J_C(n; x)$ in equation (11) with respect to $x$, taking the stock holding $n$ as given, and find the optimal fraction of diverted output $x^*(n)$. Then, we substitute the fraction $x^*(n)$ back into equation (11) and find the optimal stock holding $n^*_{ct}$ by maximizing the objective function $J_C(n; x^*(n))$. Proposition 1 below summarizes our results in partial equilibrium.

**Proposition 1 (Partial equilibrium).** The fraction of diverted output $x^*_t$ and optimal stock holdings $n^*_{ct}$ are given by:

$$x^*(n_t) = \min \left( (1-n_t)/k; (1-p)n_t \right); \quad (14)$$

$$n^*_{ct} = \begin{cases} 
\frac{\mu_t - r_t + (2-p) \frac{D_t}{S_t}}{\gamma_m \sigma_t^2 S_t W_{ct}} + \left( 2(1-p) + (1-p)^2 \frac{k}{S_t} \right) \frac{D_t}{S_t}, & \text{if } J_C^* = J_C(n^*_{ct,1}; x^*_t) \text{ (region 1)}, \\
\frac{\mu_t - r_t + \frac{k-1}{k} \frac{D_t}{S_t}}{\gamma_m \sigma_t^2 S_t W_{ct} - \frac{k}{S_t} \frac{D_t}{S_t}}, & \text{if } J_C^* = J_C(n^*_{ct,2}; x^*_t) \text{ (region 2)}, (15) \\
\frac{1}{1 + (1-p)k^2}, & \text{if } J_C^* = J_C(n^*_{ct,3}; x^*_t) \text{ (region 3)}, \\
\frac{\mu_t - r_t + (1-x_t) \frac{D_t}{S_t}}{\gamma_m \sigma_t^2 S_t W_{st}}, & \text{if } J_C^* = J_C(n^*_{ct,4}; x^*_t) \text{ (region 4)};
\end{cases}$$

\(^7\)We note that in settings with intertemporal preferences, the portfolio and consumption decisions of investors in markets with frictionless trading are guided by the total wealth given by $W_t = W_t^* + \mathbb{E}_t[\int_t^\infty \xi \epsilon r d\tau]/\xi_t$, where $\xi_t$ is the state price density and $\epsilon$ is the income stream. However, in reality it might be difficult to borrow against the diverted output, and moreover the income might not be pledgeable, which reduces the value of the income stream. As discussed in Subsection 2.2, the non-pledgeability of income is captured in a simple form by the shareholder preferences (6).
where $J_2^* = \max \left( J_c(n_{ct,1}^*; x_1^*), J_c(n_{ct,2}^*; x_1^*), J_c(n_{ct,3}^*; x_1^*), J_c(n_{ct,4}^*; x_1^*) \right)$ and $J_c(n; x)$ is as in (11). Moreover, the shareholders’ optimal consumptions are given by Equation (13).

The optimal fraction of diverted output $x^*(n_t)$ is a hump-shaped function of the number of shares $n_t$, and is depicted on Figure 1. On one hand, the endogenous accumulation of control power by the controlling shareholder relaxes the investor protection constraint and allows him to divert more as his stock holdings increase. On the other hand, marginal benefit of stealing decreases due to larger cash flow rights. Initially, the equilibrium level of stealing increases as the controlling shareholders’ stock holdings increase. However, after the kink in Figure 1, the investor protection constraint no longer binds and the equilibrium amount of diverted output decreases with the controlling shareholder’s stock holdings.

The hump-shaped relation between the diverted output and the number of shares is a notable feature of our model. Previous literature focuses only on cases where the fraction of diverted output is exogenous or is decreasing due to larger cash flow rights as in our region 2 (Shleifer and Wolfenzon, 2002; Albuquerque and Wang, 2008). Region 1 where the diverted output increases with the stock holding $n$ is in line with the evidence that firms without controlling shareholders have higher valuations than firms with controlling shareholders (Laeven and Levine, 2008). Therefore, when a firm transitions from being widely held (when $n \approx 0$) to being controlled by a large shareholder its value would go down. On the other hand, Claessens, Djankov, Fan and Lang (2002) show that the firm value increases with increases in the stock holding of the controlling shareholder, consistent with a decreasing function $x^*(n)$. However, we note that the latter work uses a cutoff level of $n = 10\%$ to identify a firm as being effectively controlled by a shareholder, and hence truncates the region where $x^*(n)$ might be an increasing function of the stock holding $n$.

The controlling shareholder’s optimal stock holding $n_{ct}^*$ in Proposition 1 captures the tradeoff between the benefits and costs of diverting the output. The expression for stock holding $n_{ct}^*$ in Equation (15) differs across four regions in the space of the state variables. Region 1 is such that the fraction of the diverted output is given by $x^*(n_{ct}^*) = (1 - p)n_{ct}^*$, that is, the investor chooses to divert the maximum possible fraction of output. In this region, laws and regulations that protect minority shareholder rights is the binding constraint on stealing. This is because marginal benefit from stealing is high with low ownership rights, the constraint on stealing is tight because controlling shareholder’s power is low and the pecuniary cost of stealing is low at low levels of stealing.

Region 2 is such that $x^*(n_{ct}^*) = (1 - n_{ct}^*)/k$, that is, the cost of diverting output kicks in. In this region, the controlling shareholder has higher power over the firm which makes...
Figure 1

**Optimal Fraction of Diverted Output** $x^*(n)$

This Figure shows the tent-shaped optimal fraction of diverted output $x^*$ as a function of the controlling shareholder’s stake $n$ in the firm.

the constraint imposed by investor protection relatively relaxed. On the other hand, high stake in the firm reduces incentive to expropriate. Consequently, in this region, the cost of stealing rather than investor protection determines the optimal amount of stealing. We observe that, after simple algebra, the stock holding $n^*_c$ in region 2 can be rewritten in the following equivalent way:

$$n^*_c = \frac{\mu_t - r_t + (1 - x^*_t) \frac{D_t}{S_t}}{\gamma_c \sigma_t^2 \frac{S_t}{W_{ct}}},$$

where $x^*(n^*_c) = (1 - n^*_c)/k$. Therefore, both types of shareholders hold a simple mean-variance portfolio when the economy is in region 2, where the investor protection constraint is not binding.\(^8\)

Finally, we discuss regions 3 and 4. Region 3 is a point where $(1 - p)n^*_ct = (1 - n^*_ct)/k$, and hence the two forces of diverting the maximum and the cost of stealing equate. Region

---

\(^8\)This result follows from the envelope condition and is due to the fact that the derivative of objective function $J_c(n; x_t)$ in (11) with respect to $x_t$ is zero in region 2 because fraction of diverted output $x_t$ is chosen precisely to satisfy the first order condition with respect to $x_t$, that is, $\partial J_c(n; x_t)/\partial x_t = 0$. Therefore, the first order condition with respect to stock holding $n_c$ is given by $\partial J_c(n; x_t)/\partial n_t = 0$, and hence, the optimal stock holdings of controlling and minority shareholders are similar.
4 is where \( n_{ct}^* = 1 \), and hence the controlling shareholder has full control over the firm.

### 3.2. Asset Price Dynamics in Equilibrium

In this subsection, we derive the equilibrium mean-return \( \mu_t \), volatility \( \sigma_t \), riskless rate \( r_t \) and shareholder stock holdings \( n_{it}^* \). The definition of equilibrium in our pure-exchange economy is standard: the equilibrium is a set of processes \( r_t, \mu_t \) and \( \sigma_t \), optimal stock and bond holdings, \( n_{it}^* \) and \( b_{it}^* \), and consumptions \( c_{it}^* \) that satisfy the market clearing conditions

\[
\begin{align*}
n_{ct}^* + n_{Mt}^* &= 1, \quad (18) \\
 b_{ct}^* + b_{Mt}^* &= 0, \quad (19) \\
 c_{ct}^* + c_{Mt}^* &= \hat{D}_t. \quad (20)
\end{align*}
\]

All equilibrium processes are derived as functions of minority shareholder’s share in the aggregate consumption, defined as \( y_t = c_{Mt}^*/\hat{D}_t \). Similarly to the literature on equilibrium with heterogeneous investors (e.g., Chabakauri, 2013) the consumption share \( y_t \) of one of the investors emerges as a crucial state variable that determines the dynamics of asset prices in the economy. Following the literature, we conjecture and then verify that the consumption share follows a Markovian process

\[
dy_t = \mu_{yt} dt + \sigma_{yt} dw_t, \quad (21)
\]

where drift \( \mu_{yt} \) and volatility \( \sigma_{yt} \) are determined in equilibrium as functions of \( y_t \).

To facilitate the intuition, we provide first the equilibrium in the benchmark economy with full investment protection \( p = 1 \), and then compare it with the equilibrium with imperfect protection. In the benchmark economy, the difference in risk aversions \( \gamma_C \) and \( \gamma_M \) is the only source of heterogeneity between investors \( C \) and \( M \).

**Proposition 2 (Benchmark equilibrium with full protection \( p = 1 \)).** In the economy with full investor protection \( p = 1 \) the shareholders’ optimal consumptions are given by (13). The stock mean-return, Sharpe ratio, volatility, and the interest rate are given by:

\[
\begin{align*}
\mu_{t, bmk} &= r_{t, bmk} + \kappa_{t, bmk} \sigma_{t, bmk} - \frac{1 - l_M - l_C}{y_t \rho^{-1/\gamma_M} + (1 - y_t) \rho^{-1/\gamma_C}}, \quad (22) \\
\kappa_{t, bmk} &= \Gamma_t \sigma_D, \quad (23) \\
\sigma_{t, bmk} &= \sigma_D + \frac{y_t (1 - y_t) \left( \rho^{-1/\gamma_M} - \rho^{-1/\gamma_C} \right) \left( \frac{1}{\gamma_M} - \frac{1}{\gamma_C} \right) \Gamma_t \sigma_D}{y_t \rho^{-1/\gamma_M} + (1 - y_t) \rho^{-1/\gamma_C}}, \quad (24) \\
r_{t, bmk} &= \mu_D + y_t \rho^{1/\gamma_M} + (1 - y_t) \rho^{1/\gamma_C} - \Gamma_t \sigma_D^2 - l_M \rho^{1/\gamma_M} - l_C \rho^{1/\gamma_C}, \quad (25)
\end{align*}
\]
respectively, where $\Gamma_t$ is the risk aversion of the representative investor, given by:

$$\Gamma_t = \frac{1}{y_t/\gamma_M + (1 - y_t)/\gamma_C}. \quad (26)$$

The minority shareholder’s consumption share volatility and drift are

$$\sigma_{yt}^{bnk} = y_t(1 - y_t)\Gamma_t \sigma_D \left( \frac{1}{\gamma_M} - \frac{1}{\gamma_C} \right), \quad (27)$$

$$\mu_{yt}^{bnk} = y_t(1 - y_t) \left( \rho_{1/\gamma_C} - \rho_{1/\gamma_M} + \left( \frac{1}{\gamma_M} - \frac{1}{\gamma_C} \right) \Gamma_t (\Gamma_t - 1) \sigma_D^2 \right)$$

$$+ (1 - y_t) l_M \rho_{1/\gamma_M} - y_t l_C \rho_{1/\gamma_C}. \quad (28)$$

The shareholders’ optimal stock holdings are given by:

$$n_{ct}^{bnk} = \frac{1 - y_t}{\gamma_M \rho^{-1/\gamma_M}} + \frac{1 - y_t}{\gamma_C \rho^{-1/\gamma_C}}, \quad n_{mt}^{bnk} = \frac{y_t}{\gamma_M \rho^{-1/\gamma_M}} + \frac{y_t}{\gamma_C \rho^{-1/\gamma_C}}. \quad (29)$$

The distribution of consumption share $y$ is stationary and its probability density function (pdf) is given in closed form by:

$$\psi(y) = \exp \left\{ a_0 + a_1 y - \frac{a_C}{y} - \frac{a_M}{1 - y} \right\} \frac{y^{bc} (1 - y)^{-b_M}}{\Gamma^2}, \quad (30)$$

where the constants $a_0, a_1, a_i > 0$, and $b_i, i = M, C$, are provided in the Appendix.

Proposition 2 provides in closed-form all the relevant equilibrium processes in the benchmark economy with full investor protection. Similar to models with non-myopic investors, the expressions for the interest rate and Sharpe ratio depend on the risk aversion $\Gamma$ of the representative unconstrained investor. Moreover, the expression for the Sharpe ratio (23) is as in the economy with two non-myopic CRRA investors with risk aversions $\gamma_C$ and $\gamma_M$.

The expressions for the volatility $\sigma_y$ and drift $\mu_y$ of the state variable process indicate the stationarity of its distribution. In particular from equations (27) and (28) we observe that the volatility $\sigma_y$ is zero at the boundaries $y = 0$ and $y = 1$, while the drift is $\mu_y = l_M \rho^{1/\gamma_M} > 0$ at $y = 0$ and $\mu_y = -l_C \rho^{1/\gamma_C} < 0$ at $y = 1$. Therefore, the boundaries $y = 0$ and $y = 1$ are repulsive. When the consumption share $y$ approaches the boundary, its volatility decreases and the drift pushes it back into the internal region. This behavior at the boundaries indicates that none of the investors becomes extinct in the long run. The probability density function (pdf) of the state variable $y$ is obtained in closed form given by (30). As discussed in Section 2, the stationarity arises because the preferences require the shareholders to maintain positive financial wealth and non-financial income allows them to accumulate wealth over time.
Proposition 3 (Equilibrium with imperfect protection). In the equilibrium with imperfect protection $p < 1$ the shareholders’ optimal consumptions are given by (13), and the stock mean-return, Sharpe ratio, volatility, and the interest rate are given by:

\[
\begin{align*}
\mu_t &= r_t + \sigma_t \kappa_t - \left(1 - x^*_t\right) \frac{1 - l_M - l_C}{yt \rho^{-1/\gamma_M} + (1 - yt) \rho^{-1/\gamma_C}}, \\
\kappa_t &= \kappa_t^{bnk} - \left(n^*_t - n_{ct}^{bnk}\right) \Gamma_t \sigma_D \rho^{1/\gamma_M} (1 - n^*_t) + \rho^{1/\gamma_C} n^*_t n_{mt}^{bnk} \sigma_D, \\
\sigma_t &= \sigma_t^{bnk} + \sigma_t^{bnk} \left(n^*_t - n_{ct}^{bnk}\right) \left(\rho^{1/\gamma_M} - \rho^{1/\gamma_C}\right) \rho^{1/\gamma_M} (1 - n^*_t) + \rho^{1/\gamma_C} n^*_t n_{mt}^{bnk} \sigma_D, \\
r_t &= \gamma_M \left(\frac{n^*_t}{n_{ct}^{bnk}} - \frac{n_{ct}^{bnk}}{n_{ct}]^{bnk}}\right) + (1 - l_M - l_C) x^*_t \rho^{1/\gamma_C} - (\rho^{1/\gamma_M} - \rho^{1/\gamma_C}) \left(1 - l_M - l_C\right) \frac{k(x^*_t)^2}{2}
\end{align*}
\]

\[
\frac{(n^*_t - n_{ct}^{bnk}) \Gamma_t \sigma_D \rho^{1/\gamma_M} (1 - n^*_t) + \rho^{1/\gamma_C} n^*_t n_{mt}^{bnk} \sigma_D}{\rho^{1/\gamma_M} (1 - n^*_t) + \rho^{1/\gamma_C} n^*_t n_{mt}^{bnk}}
\]

respectively, and the minority shareholder’s consumption share $y$ volatility and drift are

\[
\begin{align*}
\sigma_{yt} &= \sigma_{yt}^{bnk} - \left(n^*_t - n_{ct}^{bnk}\right) yt \Gamma_t \sigma_D \rho^{1/\gamma_M} (1 - n^*_t) + \rho^{1/\gamma_C} n^*_t n_{mt}^{bnk} \sigma_D, \\
\mu_{yt} &= yt \left(\frac{k_y^2}{\gamma_M} - \rho^{1/\gamma_M} - \mu_D\right) - \sigma_{yt} \sigma_D + l_M \rho^{1/\gamma_M} + (1 - l_M - l_C) \frac{k(x^*_t)^2}{2} \rho^{1/\gamma_M},
\end{align*}
\]

where $\kappa^{bnk}, \sigma^{bnk}, \gamma^{bnk}, \sigma^{bnk}$ and $n^{bnk}$ are the corresponding equilibrium processes in the full protection economy given by (23)–(27), and (29), respectively. The controlling shareholder’s optimal stock holding $n^*_t$ solves the fixed-point equation

\[
n^*_t = \text{argmax}_{n_t, n_t \leq 1} \left\{ \frac{n_t S_t}{W_C} \left( \mu_t - r_t + (1 - x^*(n_t)) \frac{D_t}{S_t} \right) + x^*(n_t) \frac{D_t}{W_C} - \frac{2}{2} \left( \frac{n_t S_t}{W_C} \sigma_t \right)^2 \right\},
\]

where $x^*(n)$ is given by equation (14), $\mu_t$, $r_t$ and $\sigma_t$ are given by equations (31)–(33) and themselves depend on $n^*_t$ in equilibrium, and ratios $D_t/W_C$ and $S_t/W_C$ are given by:

\[
\begin{align*}
\frac{D_t}{S_t} &= \frac{1 - l_M - l_C}{yt \rho^{-1/\gamma_M} + (1 - yt) \rho^{-1/\gamma_C}}, \\
\frac{D_t}{W_C} &= \rho^{1/\gamma_C} \frac{(1 - l_M - l_C)}{1 - yt}, \\
\frac{S_t}{W_C} &= \frac{yt \rho^{-1/\gamma_M} + (1 - yt) \rho^{-1/\gamma_C}}{(1 - yt) \rho^{-1/\gamma_C}}.
\end{align*}
\]

Proposition 2 derives the equilibrium processes (31)–(34) as functions of minority shareholder’s consumption share $y_t$ and demonstrates that these processes are given by their
counterparts (22)–(25) in the benchmark economy, plus additional adjustment terms arising due to imperfect protection. These adjustment terms depend on the fraction of diverted output $x_t$ and the excess ownership concentration relative to the full protection benchmark, given by $n_{ct}^* - n_{ct}^{bmk}$. In contrast to the full protection benchmark economy, simple closed-form expressions for stock holdings $n^*_t$ are no longer available. Imperfect investor protection gives rise to complex dynamics of equilibrium processes via the effects of protection on the stock holding of the controlling shareholder $n_{ct}^*$. In particular, stock holding $n_{ct}^*$ now solves a fixed-point problem in equation (37) in which the equilibrium processes on the right-hand side of this equation are functions of $n_{ct}^*$ itself.

The structure of the equilibrium processes in Proposition 3 is sufficiently tractable admitting several comparative statics. In particular, from equations (32) and (33) we observe that the effects of investor protection on the stock Sharpe ratio $\kappa$ and volatility $\sigma$ depend on the excess ownership concentration $n_{ct}^* - n_{ct}^{bmk}$. Intuitively, one would expect that controlling shareholders hold more shares under weak investor protection, and hence, $n_{ct}^* - n_{ct}^{bmk} \geq 0$, as we confirm in Section 4 after computing the equilibrium. Therefore, the expressions for the volatility (33) and the Sharpe ratio (32) suggest that weaker investor protection increases volatility and decreases the Sharpe ratio relative to the full protection benchmark.

From equation (35) we also observe that the state variable volatility $\sigma_y$ is negative because $\sigma_y^{bmk}$ in (27) is negative and $n_{ct}^* - n_{ct}^{bmk} \geq 0$. Hence, the state variable $y$ is negatively correlated with shocks to output $dw$. As a result, this variable is countercyclical in the sense that it tends to be higher (lower) during periods of negative (positive) output shocks $dw$ which we label as bad (good) times. The countercyclicality of $y$ arises because the controlling shareholder is under-diversified and less risk averse than the minority shareholder. Therefore, negative output shocks hurt the controlling shareholder and benefit the minority shareholder so that the minority shareholder’s consumption share $y$ increases in bad times.

The economy with weak investor protection remains stationary. The reason for the stationarity is that at the boundaries $y = 0$ and $y = 1$ the equilibrium processes coincide with those in the full protection benchmark, and hence the boundaries are repulsive as in the benchmark. This is due to the fact that at both boundaries there is no stealing in equilibrium because at $y = 0$ the economy is dominated by minority shareholders and at $y = 1$ it is dominated by controlling shareholders who do not steal from themselves when they control the entire economy, as discussed in Section 2. However, the pdf of the consumption share $y_t$ is no longer possible in closed form, and we compute it by Monte-Carlo simulation in Section 4. We also observe that the consumption share $y_t$ is not a martingale because its drift $\mu_y$ is non-zero and is reverted back into the interior of $(0, 1)$ interval when $y_t$ is close to
the boundary. Consequently, the consumption share $y_t$ and all equilibrium processes which are functions of $y_t$ are predictable over short time intervals.

We note that the endogenous accumulation of control and the participation of the controlling shareholder in asset markets are key ingredients for explaining certain empirical regularities. In particular, endogenizing the stock holdings allows us to explain the higher ownership concentration in countries with low protection and shed light on the portfolio choice between the risky asset, which provides higher control rights, and the riskless asset. Furthermore, dynamic asset holdings generate endogenous wealth transfers between controlling and minority shareholders which give rise to the stochastic time-variation in asset returns and excess volatility. Finally, these asset holdings also give rise to leverage and help us shed light on the role of leverage in the accumulation of control and its effect on the stock return volatility.

4. Economic Implications of Investor Protection

In this Section, we present our results with plots as depicted in Figures 2-6 for a plausible set of baseline parameters as functions of the consumption share $y$ of the minority shareholders in the economy.\(^9\) The comparative statics results are reported holding the consumption share $y$ fixed. The state variable $y$ is countercyclical, as discussed in Section 3.2, and hence the dependence on this variable provides information on how the equilibrium processes change over good and bad times in the economy.

Figure 2 presents the controlling shareholder’s equilibrium stock holding $n_C^*$, the fraction of diverted output $x^*$, and the controlling shareholder’s leverage-stock price ratio for different values of investor protection $p$ in the economy, holding the stealing cost parameter $k$ fixed. Then, we use the results in Figure 2 for the analysis of equilibrium expected gross returns, interest rates, and volatilities as depicted in Figure 3 for the same protection $p$ and stealing cost $k$ parameters. Figure 4 presents the stationary distributions of the consumption share $y$ for the same sets of parameter values as in previous figures. Figure 5 explores the effects of the stealing cost by varying parameter $k$ and holding investor protection $p$ fixed. Finally,

\(^9\)We set $\mu_D = 1.5\%$ and $\sigma_D = 13\%$, matching the mean-growth rate and volatility of dividends (e.g., Brennan and Xia, 2001; Dumas, Kurshev, and Uppal, 2009), and set $\gamma_C = 3$, $\gamma_M = 3.5$, $\rho = 0.01$, and $k = 3$. We set the labor income shares to $l_C = 0.1$ and $l_M = 0.5$. The labor income shares capture the fact that although individual controlling shareholders may have large incomes they are fewer than minority shareholders in reality. Hence, their combined income is still lower than the income of minority shareholders. We recall that the shareholders in our model are representative, that is, each shareholder stands for a group of identical shareholders.
Figure 2
The effect of investor protection on controlling shareholder's stock holding, fraction of diverted output and leverage

This Figure shows the equilibrium stock holdings $n^*_C$, fraction of diverted output $x^*$ and leverage-stock price ratio $L/S$ as functions of consumption share $y$ for fixed stealing cost parameter $k = 3$ and different levels of investor protection $p$, for baseline parameter values.

Figure 6 shows the expected stock gross returns and interest rates in an economy without the investor protection constraint for a range of cost parameters $k$. The numerical approach for deriving the equilibrium processes is explained in the Appendix.

4.1. Stock Holdings and Diverted Output

We start our analysis with Figure 2 presenting the controlling shareholder’s stock holding, fraction of diverted output and leverage. Panel (a) of Figure 2 demonstrates that lower protection tends to increase the controlling shareholder’s stock holding $n^*_C$ relative to the full
protection benchmark, i.e., \( n^*_ct \geq n_{ct}^{bmk} \), in the region where the investor protection constraint binds. This is because when investor protection is imperfect the controlling shareholder can divert a larger fraction of output when he owns more shares. This gives the controlling shareholder an incentive to acquire more shares in equilibrium when his consumption share is low (i.e., \( y \) is high). However, when the controlling shareholder’s consumption share is high (i.e., \( y \) is low), the stock holding is the same as in the benchmark economy, i.e., \( n^*_ct = n_{ct}^{bmk} \). In this case, the controlling shareholder holds a large number of shares, which reduces the benefits of diverting the output. Therefore, the economy is in region 2 (equation (15) of Proposition 1), where the investor protection constraint does not bind and the diversion of output is tempered only by the cost of stealing. The cost parameter \( k \), however, does not depend on the stock holding. Consequently, buying shares beyond the benchmark level \( n_{ct}^{bmk} \) does not help divert more output and harms the controlling shareholder by reducing portfolio diversification benefits. Hence, \( n^*_ct = n_{ct}^{bmk} \) in region 2.

We note that the controlling shareholder’s stock holding \( n^*_ct \) is non-monotone in protection \( p \). In particular, in panel (a) we observe that whether stock holding \( n^*_ct \) is higher in the economy with \( p = 0.9 \) or \( p = 0.6 \) critically depends not only on the protection \( p \) but also on the consumption share \( y \). This is because a decrease in \( p \) has two opposing effects on stock holding \( n^*_ct \). On one hand, stock holding \( n^*_ct \) increases because the controlling shareholder can earn extra return by diverting the output. On the other hand, it expands region 2 in equation (15) for stock holding \( n^*_ct \), in which the cost of stealing comes into play. The latter effect can be seen in panel (b) of Figure 2, from which we observe that the area in which stealing is a decreasing function of the consumption share \( y \) becomes larger as protection \( p \) goes down.

Panel (b) of Figure 2 shows the fraction of diverted output \( x^* \) and how it is affected by investor protection. As would be expected, the fraction of diverted output is considerably reduced in economies with better protection. The diverted output \( x^*(n^*_ct) \) has a kink at the separation point of regions 1 and 2 in equation (15) for the stock holding \( n^*_ct \), which correspond to situations when the investor protection constraint \( x_t \leq (1 - p)n_t \) is binding or not, respectively. The latter constraint is loose when \( n \) is sufficiently large, so that the controlling shareholder does not want to steal from himself. Therefore, because the stock holding \( n^*_ct \) is a decreasing function of consumption share \( y \) (panel (c) of Figure 2), the economy is in region 1 (region 2) when \( y \) lies to the right-hand (left-hand) side of the kink. We remark that all the equilibrium processes have kinks, which arise via the dependence of the latter processes on the fraction of diverted output \( x^*(n^*_ct) \).

The economy transits from region 1 to region 2 following a sequence of positive shocks
because the controlling shareholder accumulates more wealth and stock. The economy moves from region 2 to region 1 if it is hit by bad shocks \( dw < 0 \) in which case the controlling shareholder decreases his stock holding to finance consumption in bad times. These transitions between the regions and the time-variation of the state variable \( y \) make the diverted output \( x^* (n^*_{ct}) \) and all other equilibrium processes time-varying in our model.

Panel (c) of Figure 2 shows the controlling shareholder’s leverage-stock price ratio. The leverage is given by the wealth invested in stocks in excess of total wealth. The personal leverage in the full protection and imperfect protection economies is given by \( L^b_{ct} = n^b_{ct} S^b_{ct} - W^b_{ct} \) and \( L_t = n^*_{ct} S_t - W_{ct} \), respectively. Then, taking into account that the ratios \( S^b_{ct} / W^b_{ct} \) and \( S_t / W_{ct} \) are the same, conditional on the consumption share \( y \), and given in equation (38), we obtain the leverage as:

\[
\frac{L_t}{S_t} = \frac{L^b_{ct}}{S^b_{ct}} + n^*_{ct} - n^b_{ct}.
\]

Equation (39) reveals that the excess ownership concentration \( n^*_{ct} - n^b_{ct} \) is exactly equal to the change in leverage \( L_t / S_t - L^b_{ct} / S^b_{ct} \). Therefore, the acquisition of additional shares in economies with poor protection is financed by leverage because, as shown below, the borrowing is cheap in such economies. Hence, the leverage increases, as shown on panel (c) of Figure 2. We note that the leverage here represents the personal leverage of the controlling shareholder while the firm is unlevered.

4.2. Stock Return, Volatility, and Interest Rate

Panel (a) of Figure 3 shows that a higher investor protection \( p \) leads to a higher gross stock return \( \mu + (1 - x^*) D / S \), where \( \mu \) is the mean capital gain and \( (1 - x^*) D / S \) is dividend yield. This finding is consistent with the empirical literature documenting a positive relationship between corporate governance and realized returns. Future returns are positively correlated with a governance index of shareholder rights (Gompers, Ishii and Metrick, 2003), lower managerial perks (Yermack, 2006), a lower entrenchment index (Bebchuk, Cohen and Ferrell, 2009), and a governance index (AGR) from Audit Integrity (Daines, Gow and Larcker, 2010). We note that these empirical findings are for cross-sections of firms with different investor protections whereas our main analysis features a representative firm. However, our main insights and intuition remain valid for a cross-section of firms, as we demonstrate in Section 5.3 below, albeit in a simple one-period setting.

Similar relations are documented in other countries and cross-country studies. Firms with higher governance scores in Germany (Drobetz, Schillhofer, and Zimmermann, 2004),
The effect of investor protection on stock gross returns, volatilities and interest rates

This Figure shows the equilibrium gross stock returns $\mu + (1 - x^*)D/S\%$, stock return volatilities $\sigma$ and interest rates $r$ as functions of consumption share $y$ for fixed stealing cost parameter $k = 3$ for different levels of investor protection $p$, for baseline parameter values.

firms that do not engage in tunneling using inter-corporate loans in China (Jiang, Lee and Yue, 2010) and countries with better legal institutions (Lombardo and Pagano, 2006) have higher returns. Despite the supporting empirical evidence, it is not immediately clear why such a relationship between expected returns and investor protection exists in equilibrium. For example, taking away a constant fraction of dividends reduces the value of the firm but does not affect the expected return in equilibrium. There is also an ongoing discussion about whether the empirical relationship is robust (Core, Guay and Rusticus, 2006; Giroud and Mueller, 2011; Bebchuk, Cohen and Wang, 2013). Therefore, further guidance from theory, as a contribution to this debate, would be helpful.
To understand the intuition, we first consider the benchmark economy with full protection \( p = 1 \). In this economy, gross stock returns are determined by investors’ risk aversions and are sufficiently high to compensate investors for risk taking. Lower investor protection \( p < 1 \) opens up an opportunity to divert firm cash flows to benefit the controlling shareholders. Therefore, the controlling shareholder is compensated for excessive risk taking not only via risk premia but also via stealing. Consequently, the compensation for risk of the controlling shareholder is determined by a new previously unexplored quantity which we refer to as the effective risk premium, and which appears to be higher than the risk premium implied by the stock price dynamics. More formally, from the budget constraint (7) of the controlling shareholder \( C \) it is immediate to observe that his effective risk premium for holding stocks is given by \( \mu - r + (1 - x^*)D/S + x^*D/(S n^*_C) \), and hence, is higher than the risk premium \( \mu - r + (1 - x^*)D/S \) for the minority shareholder by the diverted output yield per share \( x^*D/(S n^*_C) \). Therefore, a low risk premium \( \mu - r + (1 - x^*)D/S \) implied by asset prices is indeed consistent with our equilibrium because the stock market clears due to high demand for stocks by the controlling shareholders. Consistent with our intuition, it has been documented that the demand by controlling shareholders for voting shares increases with poor investor protection, which then affects prices.\(^{10}\)

Panel (b) depicts the stock return volatility \( \sigma \) and demonstrates that in our calibration of the model it is higher than in the benchmark economy with full protection, i.e., \( \sigma \geq \sigma_{bmk} \), and hence, the stock price is more volatile than output. This result is consistent with the comparative statics in Section 3.2. The volatility increases because with imperfect protection \( p < 1 \) the controlling shareholder holds more shares than in the full protection benchmark, and hence is under-diversified. Therefore, his wealth and consumption are more volatile, which then translates into the stock market volatility via the market clearing conditions.

We note an important link between the volatility and leverage. In particular, combining equations (33) and (39) for the volatility and leverage, after simple algebra, we obtain the following expression for the percentage change in volatility in terms of change in leverage:

\[
\frac{\sigma_t - \sigma_{t, bmk}}{\sigma_{t, bmk}} = \left( \frac{L_t}{S_t} - \frac{L_{t, bmk}}{S_{t, bmk}} \right) \frac{\rho^{1/\gamma_M} - \rho^{1/\gamma_C}}{\rho^{1/\gamma_M}(1 - n^*_t) + \rho^{1/\gamma_C} n^*_t}.
\]

Equation (40) explains the close resemblance of volatility \( \sigma \) and leverage-stock price ratio \( L/S \). Moreover, in our calibration, the latter equation implies a positive relationship between the change in volatility and the change in leverage. When the investor protection constraint

\(^{10}\)This is consistent with the evidence that the value of control is negatively correlated with variation in investor protection across countries (Nenova, 2003; Dyck and Zingales, 2004) and over time (Albuquerque and Schroth, 2010).
does not bind, that is, the economy is in region 2, the equilibrium stock holding $n^*_C$, is the same as in the benchmark economy, as elaborated in Section 4.1. Consequently, equation (39) implies that there is no excess leverage in the economy in region 2, and hence equation (40) implies that the volatility is the same as in the full protection benchmark.

Our analysis further uncovers previously unknown effects of investor protection on stock return volatilities. First, in the region where the investor protection constraint is binding, stock return volatility $\sigma$ is a concave function of minority shareholder’s consumption share $y$, consistent with the empirical evidence provided by Gul, Kim and Qui (2010). Second, similar to the controlling shareholder’s stock holding $n^*_C$, we observe that volatility $\sigma$ is non-monotone in protection $p$. This is because the interaction between volatility $\sigma$ and protection $p$ depends on the minority shareholder’s consumption share $y$. Clearly, the income inequality or distribution of wealth could have a direct effect on portfolio holdings of controlling versus minority shareholders and our model implies that this would interact with the effect of investor protection on equilibrium volatility.

Empirical evidence indicates that the idiosyncratic and total volatilities are higher in more developed countries such as U.S. (Morck, Yeung and Yu, 2000; Bartram, Brown and Stulz, 2012), where investor protection is also relatively higher. Indeed, minority investor protection, property rights protection and opaqueness are suggested as likely culprits (Morck, Yeung and Yu, 2000; Jin and Myers, 2006; Bartram, Brown and Stulz, 2012). Our model can potentially shed light on this empirical relation between investor protection and volatility. For example, in economies where the stock holding of the controlling shareholder is sufficiently high, and hence the corresponding consumption share $y$ is sufficiently low, stock price volatility can be higher in an economy with high level of investor protection (e.g., point B in panel (b) of Figure 3) than in an economy with low level of protection (e.g., point A in panel (b) of Figure 3).

Panel (c) of Figure 3 shows that the interest rate $r$ is lower in the economy with poor protection. Because the risk premium faced by the minority shareholders is low, investment in stocks is less attractive for them than in the economy with full protection. Therefore, the minority shareholders run to the bond market, and hence, are willing to provide cheap credit, which decreases the interest rates. Furthermore, the purchases of stocks can be partially covered by the diverted output, which contributes to the decreases in interest rates. The negative effect of stealing on interest rates is captured by the second and third terms equation (34) for the interest rate $r$. We also observe that the gross stock return and the interest rate have similar shapes, and their comparison shows that the interest rates account for a significant fraction of the variation in gross returns. In particular, the ratio
Figure 4
Stationary probability density functions of consumption share $y$
This Figure shows the probability density function of minority shareholders’ consumption share $y$ for different levels of investor protection, for the cost parameter $k = 3$ and the baseline parameter values.

$r/(\mu + (1 - x^*)D/S)$ ranges from 0.32 (when $y \approx 0$) to 0.54 (when $y \approx 1$).

From the results on Figures 2 and 3, we observe that the effects of investor protection are more conspicuous when both investors have significant stock holdings, i.e, the economy is away from the boundaries $y = 0$ and $y = 1$. Intuitively, when $y \approx 1$ the controlling shareholder accounts for a tiny fraction of aggregate wealth and consumption, and hence, the effect of stealing is small. Furthermore, when $y \approx 0$ the economy is dominated by the controlling shareholder. As a result, the controlling shareholder holds almost all shares, i.e., $n^*_{ct} \approx 1$, and hence, the diverted fraction $x^*_t = \min((1 - n^*_{ct})/k; (1 - p)n^*_{ct})$ is small because, given the cost of diverting the output, the controlling shareholder finds it sub-optimal to divert the output of a firm in which he is entitled to almost 100% of cash flows. Therefore, there is no diversion of output when $y = 0$ and the effect of investor protection vanishes.

Figure 4 shows the stationary probability density functions for the consumption share of minority shareholders $y$. The distribution of the variable $y$ is stationary due to the presence of labor incomes and the shareholders’ wealth being positive at all times, as elaborated in Section 3. The distributions are not available in closed form, and so we compute them using Monte-Carlo simulations of the consumption share $y$ dynamics given in equation (21). These distributions are shifted towards the boundary $y = 1$ because the minority shareholders as a group receive higher labor income than the controlling shareholder. The distributions have supports concentrated over a small interval $[0.7, 0.85]$. Consequently, the unconditional
effects of investor protection are similar to the conditional effects when the consumption share $y$ lies in that interval.

We also note that consumption share $y$ takes a broader range of values before its distribution converges to the stationary one. Therefore, the comparison of the effects for different levels of protection $p$ conditional on the variable $y$ should be performed for the entire interval $(0, 1)$. This conditional comparison has two economic interpretations. First, it reveals the effects of an unanticipated change in investor protection $p$ when the equilibrium processes change in response to $p$ but variable $y$ has not yet adjusted to the new level. Second, such a comparison reveals the effects of investor protection across economies that have different levels of protection $p$ but similar distribution of consumption.

4.3. Effect of Stealing Costs

Figure 5 shows the effect of a change in the cost function parameter $k$ on equilibrium processes when the level of investor protection is fixed. It demonstrates that higher cost of stealing (i.e., high cost parameter $k$) is associated with higher stock gross return and interest rate, and lower volatility and stock holding of the controlling shareholder. Overall, the effect of higher cost of stealing on equilibrium is consistent with the effect of better protection $p$.

A comparison of the results in Figures 3 and 5 reveals an important difference between the economic effects of the protection $p$ and cost $k$ parameters on stock gross returns and interest rates. In particular, the change in protection $p$ has stronger effects when consumption share $y$ is high (i.e., the economy is in region 1), while the change in stealing costs $k$ has stronger effects when consumption share $y$ is relatively low (i.e., the economy is in region 2). Intuitively, as can be formally seen from expression (14) for diverted fraction of output $x^*$, this is due to the fact that lowering the cost of stealing $k$ increases the diverted output $x^*$ only when the investor protection constraint does not bind, that is, when the economy is in region 2 where the variable $y$ is low.

The effect of the cost parameter $k$ on the stock volatility and holdings is present only when consumption share $y$ is large, in contrast to the effect of $k$ on gross stock return and interest rate. The reason for such an asymmetry is as follows. When consumption share $y$ is low, the economy is dominated by the controlling shareholder, and hence, the stock holding of the controlling shareholder $n^*_C$ is high. Therefore, the investor protection constraint $x_t \leq (1 - p)n_t$ is not binding because the controlling shareholder does not have an incentive to steal from himself. Hence, the economy is in region 2 where the diversion of output is only tempered by the cost of stealing and the stock holding is the same as in...
The effect of stealing costs on stock gross returns, volatilities, interest rates, and stock holdings

This Figure shows the equilibrium gross returns $\mu + (1 - x^*)D/S$, stock return volatilities $\sigma$, interest rates $r$, and stock holdings $n^*_{C_t}$ as functions of consumption share $y$ for fixed investor protection $p = 0.6$ and different levels of cost parameter $k$, for baseline parameter values.

the benchmark economy. Consequently, the stock volatility equation (33) implies that the volatility is the same as in the benchmark economy.

Furthermore, the cost parameter $k$ has a significant effect on stock return $\mu_t$ and interest rate $r_t$ in region 2 via the terms that depend on the fraction of diverted output $x^*_t = (1 - n^*_{C_t})/k$ in equations (31) and (34) when consumption share $y$ is low. The intuition is that despite the fact that $n^*_{C_t} = n^*_{C_t}^{bmk}$ in region 2, the wealth and consumption of the controlling shareholder continue to increase at a higher rate than the wealth and consumption of the minority shareholder due to the diverted cash flow $x^*_t D_t$ in the budget constraint (7) of the controlling shareholder. Therefore, the controlling shareholder extracts higher effective
returns from holding stocks, which affects the valuation of stocks and their returns.

Finally, to shed further light on the distinction between the effects of investor protection constraint and the cost function we consider an economy without the investor protection constraint so that stealing is only moderated by the pecuniary cost function. Figure 6 presents the gross stock returns and interest rates in such an economy for different values of the cost parameter $k$. This figure shows that lowering the cost of stealing decreases asset returns, consistent with the results in Figure 5 and the effects of weaker investor protection in Section 4.1.

With the investor protection constraint absent, the volatility of stock returns and the stockholdings of the controlling shareholder are the same as in the benchmark economy for the same values of the variable $y$, and hence are not shown for brevity. This is because region 2 is now the entire interval $(0, 1)$ and the controlling shareholder’s portfolio in that region is the same as in the benchmark economy, as discussed in Section 4.1. However, this result does not imply that there is no effect of the stealing cost on volatility $\sigma_t$ and stock holding $n_{Ct}^*$. The cost parameter $k$ still affects all equilibrium processes via the distribution of the minority shareholder’s consumption share $y$. This consumption share will be smaller in economies with low stealing costs where the controlling shareholder has more consumption and wealth. As a result, the stockholding $n_{Ct}^*$ will be larger in such economies. The effect of
the cost parameter $k$ on volatility $\sigma$ is ambiguous because $\sigma$ is a non-monotone function of the variable $y$ (panel (b) of Figure 3).

5. Extensions

In this Section, we consider extensions of our analysis that feature the separation of control and cash flow rights, non-pecuniary costs of diverting output, and cross-firm differences in investor protection. We find that the underlying mechanisms at play and our results on the effects of investor protection remain qualitatively the same as in Section 4, and also uncover additional insights due to these extended formulations.

5.1. Separation of Cash flow and Control Rights

We now incorporate into our analysis situations when controlling shareholders’ control power is disproportionately larger or smaller than their cash flow rights. Such situations commonly arise when firms are indirectly controlled via other firms through pyramidal structures or some of their stocks do not have voting rights. A simple way to capture the separation of control and cash flow rights within our framework is to consider a more general control function $q(n) = an$, where $n$ is the stock holding representing the cash flow rights and $a$ is the control parameter representing the control power per unit of cash flow right. A control parameter $a < 1$ arises when the controlling shareholder has less control rights than cash flow rights because, for instance, some of his stock holdings do not have voting rights. A control parameter $a > 1$ captures the situations when the control rights exceed cash flow rights, for example, due to pyramidal ownership when the firm is indirectly controlled via another firm in which the controlling shareholder has significant power.

Following the same steps as in our main analysis of investor protection with $q(n) = n$, we find that the equilibrium Sharpe ratio, volatility, interest rate, and drift and volatility of consumption share $y$ are given by equations (32)–(36) of Proposition 3, and the stock holding $n^*_c$ solves the fixed point problem (37), as in the main model, but the diverted output $x^*_t$ in the latter equations now depends on the control parameter $a$ and is given by $x^*(n; a) = \min((1 - n)/k; (1 - p)an)$. Therefore, the diverted output $x^*(n; a)$ is the key channel through which the differences in control and cash flow rights affect other equilibrium processes. Consequently, we find that the economic effects of investor protection remain the same as in Section 4, although we do not report the equilibrium processes for brevity.

Figure 7 depicts the fraction of diverted output $x^*(n; a)$ and highlights two additional
economic insights. First, the controlling shareholder diverts a larger fraction of the output as the difference between the control and cash flow rights increases (i.e., parameter $a$ increases), consistent with the empirical findings of Claessens, Djankov, Fan, and Lang (2002). Second, a higher control parameter $a$ shrinks region 1, where the investor protection constraint binds (region to the right of the kinks in Figure 7). The intuition is that with higher $a$ the controlling shareholder can divert a larger fraction of output even with low cash flow rights $n$. Consequently, the cost of stealing becomes large even for low $n$, which induces the controlling shareholder to steal less than what is allowed by the investor protection constraint. The control parameter $a$ also affects all other equilibrium processes through the fraction of diverted output $x^*(n; a)$. In particular, we find that that larger control $a$ reduces gross stock mean-returns.

5.2. Non-pecuniary Costs of Diverting Output

We here extend our analysis to feature non-pecuniary costs of diverting output, instead of pecuniary costs in our main analysis. The non-pecuniary costs capture the possibility that misuse of control power is more costly in terms of the required effort and reputation, or agents feel disutility from stealing due to social norms that promote fairness, honesty and morality. In particular, Kahneman, Knetsch and Thaler (1986) argue that people have a preference for fairness and provide examples when economic agents commonly allocate resources fairly to others even when they are free to do otherwise. Consequently, the controlling shareholder
may not only be limited by laws and regulations that protect minority shareholders but also by social norms that promote fairness. These norms could be self enforced by personal emotions or by others through disapproval, ridicule or ostracism (Posner, 1997). We capture the effect of norms in the controlling shareholder’s decision to steal simply by incorporating a disutility from stealing.

The non-pecuniary cost function \( f(x, D) \) is the same as the pecuniary cost given by (4). However, this cost is now interpreted as the certainty equivalent of the disutility of stealing and directly affects the controlling shareholder’s preferences:

\[
V_C(c_t, W_t, W_{t+dt}) = \rho u_C(c_t)dt + (1 - \rho dt)E_t[u_C(W_{t+dt} - f(x_t, D_t))dt],
\]

(41)

where \( u_C(c) \) is the utility function in (5). The controlling shareholder maximizes the objective function (41) subject to the investor protection constraint \( x_t \leq (1 - p)q(n_t) \), maximum share constraint \( n_t \leq 1 \), and the budget constraint. The utility of the minority shareholder is as before, and the shareholders’ budget constraints are as follows:

\[
dW_{it} = (W_{it}r_t + n_{it}(S_t(\mu_t - r_t) + (1 - x_t)D_t) - c_{it} + l_iD_t) dt + 1_{i=c}x_tD_t + n_{it}S_t\sigma_t dw_t,
\]

(42)

and no longer include the cost function \( f(x, D) \).

The optimal portfolio \( n_C^* \) of the controlling shareholder in the partial equilibrium can be obtained following the same steps as in Proposition 1 and turns out to be the same as for the main model with pecuniary costs in equation (15). This is because the non-pecuniary costs are subtracted from wealth \( W_{t+dt} \), and hence their effect is similar to that of pecuniary costs, which reduce the wealth through the budget constraint (7). However, the dynamics of wealths in the economies with pecuniary and non-pecuniary costs, given by (7) and (42), are different.

The equilibrium can be derived along the same lines as in Proposition 3. The equilibrium processes are again given by equations as in (32)–(36) but without the quadratic costs \( k(x_t^*)^2/2 \). All economic effects of investor protection on the equilibrium processes remain qualitatively the same as in the main analysis of Section 4. This is because the pecuniary costs directly affect only the interest rate \( r \) and drift \( \mu_y \), and moreover, have small effects on these processes. In particular, the term \( (\rho^{1/\gamma_M} - \rho^{1/\gamma_C})(1 - l_M - l_C)k(x_t^*)^2/2 \) through which the costs affect the interest rate (34) has a magnitude smaller than 0.1% for our baseline parameter values.
5.3. Investor Protection and Cross-Section of Asset Returns

The empirical literature on the effects of investor protection on firm stock returns is largely cross-sectional. Hence, it would be valuable to extend our main analysis to feature an additional firm with a different level of investor protection. However, such an extension turns out to be intractable in a dynamic continuous-time setting due to cumbersome expressions for equilibrium processes, an extra state variable, complex differential equations and fixed-point problems for optimal stock holdings.\(^{11}\) Therefore, we adopt a simpler, one-period economy. This economy lends considerable tractability to the analysis of stock holdings and mean-returns but does not allow us to study stock return volatilities because they are equal to exogenous dividend volatilities. For further tractability, we exclude pecuniary costs of stealing to avoid non-convexity of the controlling shareholder’s optimization (discussed in Section 3.1).\(^{12}\)

We consider an economy with two dates, 0 and 1, and three equally likely states \(\omega_1, \omega_2, \text{ and } \omega_3\) at date \(t = 1\). There are two representative firms that produce equal outputs \(D\) at the initial date and state-dependent outputs \(\tilde{D}_1\) and \(\tilde{D}_2\) at the final date \(t = 1\), given by

\[
\tilde{D}_j = D[1 + \mu_D + \sigma_D \Delta w_j], \quad j = 1, 2, \tag{43}
\]

where the output mean-growth rate \(\mu_D\) and volatility \(\sigma_D\) are constants, and \(\Delta w_j\) are economic shocks. The shocks take values \(\Delta w_1 \in \{\sqrt{3}/2, 0, -\sqrt{3}/2\}\) and \(\Delta w_2 \in \{\sqrt{3}/2, -\sqrt{3}/2, 0\}\) in states \(\omega_1, \omega_2, \omega_3\), respectively. Their mean and variance are given by \(\mathbb{E}[\Delta w_j] = 0\) and \(\text{var}[\Delta w_j] = 1\), respectively, and the correlation is 0.5. The outputs (43) are deliberately chosen to be ex-ante identical to exclude confounding sources of firm heterogeneity.

The economy is populated by representative controlling and minority shareholders. The shareholders trade in three securities, a riskless bond in zero net supply with interest rate \(r\) and two stocks with prices \(S_1\) and \(S_2\), which represent claims to dividends paid by firms 1 and 2 at the final date, respectively. Hence, the underlying financial market is complete.

Firm 1 has imperfect investor protection. The controlling shareholder \(C\) of firm 1 can divert a fraction \(x\) of this firm’s output, subject to the investor protection constraint \(x \leq \)

\(^{11}\)The analysis of stock prices and volatilities is complicated even in frictionless economies with heterogeneous investors, where these processes are given by complex expressions in terms of special functions and integrals (e.g., Chabakauri, 2013). Moreover, a dynamic two-firm setting requires an additional state variable capturing the dynamics of the output share of one of the firms, which further complicates the analysis.

\(^{12}\)We justify removing pecuniary costs as follows. First, these costs are introduced in our main analysis for realism, and the economic effects are primarily driven by the investor protection constraint. Second, they have similar effects on asset returns as investor protection constraint (Section 4.3). Third, firms are more likely to differ in investor protection implied by corporate culture rather than in pecuniary costs such as bribes, fines, and similar expenses that are likely to be determined at an economy level.
similar to our main analysis, where \( n_{C1} \) is this shareholder’s holding of stock 1. Firm 2 has full protection. This firm may also have a controlling shareholder who manages it. However, this shareholder cannot divert output, and hence faces the same portfolio and consumption choice as a minority shareholder. Therefore, we subsume this shareholder and all minority shareholders within the representative minority shareholder \( M \) of firm 1. In an economy with full protection for both firms the stock prices are the same, \( S_1 = S_2 \), because the outputs (43) are equivalent from the viewpoint of date 0 before shocks \( \Delta w_j \) are realized. Hence, the difference in investor protections is the only source of cross-sectional variation in stock prices and returns in this setting.

The shareholders have preferences over initial and final consumptions \( c \) and \( \tilde{c} \), given by:

\[
u_i(c) + \beta \mathbb{E}[\nu_i(\tilde{c})], \quad i = C, M, \tag{44}\]

where \( \nu_i(\cdot) \) are CRRA utilities given by (5) and \( \beta > 0 \) is a time-preference parameter. The shareholders’ self-financing budget constraints are given by

\[
\tilde{c}_i = (W_i - c_i)(1 + r) + n_{i1}((1 - x)\tilde{D}_1 - S_1) + n_{i2}(\tilde{D}_2 - S_2) + 1_{i = C}x\tilde{D}_1, \quad i = C, M, \tag{45}\]

where \( W_i \) denotes initial wealth.

The minority shareholder \( M \) maximizes the objective function (44) with respect to consumption \( c \) and stock holdings \( n_1 \) and \( n_2 \) subject to the budget constraint (45). The controlling shareholder \( C \) maximizes the same objective with respect to \( c, n_1, n_2, \) and diverted output \( x \) subject to the budget constraint (45), the investor protection constraint \( x \leq (1 - p)n_{C1} \), and a non-negative date \( t = 1 \) financial wealth constraint

\[
\tilde{W}_C = (W_C - c_C)(1 + r) + n_{C1}((1 - x)\tilde{D}_1 - S_1) + n_{C2}(\tilde{D}_2 - S_2) \geq 0. \tag{46}\]

The latter constraint rules out unrealistic risky positions backed by future stolen output, consistent with our main analysis where a similar constraint emerges due to shareholder preferences over next-period financial wealth (Section 2.2).

We derive the equilibrium processes as functions of the consumption share \( y \) of the minority shareholder at date \( t = 0 \). Finding equilibrium reduces to solving a system of four non-linear equations for the controlling shareholder’s consumptions in states \( \omega_1, \omega_2, \omega_3, \) and fraction \( x \). These equations are significantly more complex than in the full protection economy where the system breaks down into separate one-dimensional equations for consumptions in states \( \omega \). We solve the model numerically, as detailed in the Appendix.

The effects of investor protection on stock returns, interest rates, and stock holdings are in line with our main analysis. For brevity, we here only present the stock mean-returns,
Figure 8
The effect of firm 1 investor protection on stock 1 mean-returns, investor protection premiums, and interest rates
This Figure shows the equilibrium stock 1 mean-return $\mu_1$, investor protection premium $\mu_2 - \mu_1$ and interest rates $r$ as functions of the consumption share $y$ for different levels of firm 1 investor protection $p$ and baseline parameter values.

defined as $\mu_1 = \mathbb{E}[(1-x)\hat{D}_1]/S_1 - 1$ and $\mu_2 = \mathbb{E}[\hat{D}_2]/S_2 - 1$, which are the focus of the related empirical literature, and also the interest rate $r$ for comparison. Figure 8 reports the mean-return $\mu_1$, spread $\mu_2 - \mu_1$, and interest rate $r$ for different values of firm 1 investor protection $p$ and baseline parameter values.\(^{13}\) Panels (a) and (c) demonstrate that mean-return $\mu_1$ and rate $r$ are lower with weaker investor protection, as in our main analysis.

Panel (b) depicts the investor protection premium, defined as the spread $\mu_2 - \mu_1$ between the mean-returns of stocks with full and imperfect protection, respectively. This premium is zero when both firms have full protection because the firm outputs are ex-ante identical.

\(^{13}\)We set $\mu_D = 1.5\%$, $\sigma_D = 13\%$, $\gamma_C = 3$, $\gamma_M = 3.5$, as in our main analysis, and $\beta = 0.95.$
Our key result is that this premium is positive when firm 1 has imperfect protection $p < 1$, in line with the empirical evidence discussed in Section 4.2. The intuition is that if the premium remains zero for $p < 1$ (i.e., $\mu_2 = \mu_1$) the controlling shareholder would be willing to invest more in stock 1 to receive extra compensation in the form of diverted output $x\tilde{D}_1$ thereby reducing the mean-return $\mu_1$. Hence the positive premium in equilibrium.

6. Conclusion

We develop a dynamic asset pricing model where a controlling shareholder can divert a firm’s output but is constrained by investor protection and pecuniary costs of stealing. By incorporating endogenous accumulation of control, we show that controlling shareholder’s excess ownership concentration interacts with investor protection to determine equilibrium level of stock mean-return, volatility and interest rates. We demonstrate that in equilibrium the controlling shareholder’s asset concentration in the firm is larger with imperfect investor protection. We also find that better investor protection increases stock mean-returns, and increases stock return volatilities when ownership concentration is sufficiently high. Our findings provide support for the empirical evidence on asset prices.

Our main analysis employs a preference specification that exhibits myopia. It would be of interest to extend our analysis to the familiar case of dynamic preferences of Epstein and Zin (1989) with heterogeneous risk aversions and unit intertemporal elasticities of substitution, in line with the literature on economies with heterogeneous investors (e.g., Găretean and Panageas, 2015). However, such an analysis appears rather difficult because solving the model is further complicated by additional hedging demands in shareholder portfolios, non-linear differential equations for the value functions, and the non-convexity of the controlling shareholder’s portfolio optimization.
Appendix: Proofs

Proof of Proposition 1. We observe that the controlling shareholder’s objective function (11) is a quadratic function of the share of diverted output $x_t$. Maximizing this function with respect to $x_t$ subject to the constraint $x_t \leq (1 - p)n_t$, we obtain the optimal fraction of diverted output as $x^*(n_t) = \min \left((1 - n_t)/k; (1 - p)n_t\right)$. Substituting $x^*(n_t)$ back into the objective function (11), we find that the objective function $J_c(n_t)$ is given by:

$$J_c(n_t) = \begin{cases} J_{c1}(n_t), & n_t \leq \frac{1}{1+(1-p)k}; \\ J_{c2}(n_t), & n_t > \frac{1}{1+(1-p)k}; \end{cases}$$

(A.1)

where $J_{c1}(n_t)$ and $J_{c2}(n_t)$ are quadratic functions of $n_t$ defined as follows:

$$J_{c1}(n_t) = n_t S_t \left( \frac{\mu_t - r_t}{W_{ct}} + \frac{1 - (1 - p)n_t}{k} \right) + \frac{(1 - p)n_t D_t}{W_{ct}} - \frac{k (1 - p)^2 n_t^2 D_t}{2 W_{ct}} - \frac{\gamma_c}{2} \left( \frac{n_t S_t}{W_{ct}} \sigma_t \right)^2;$$

(A.2)

$$J_{c2}(n_t) = \frac{n_t S_t}{W_{ct}} \left( \frac{\mu_t - r_t}{k} + \frac{1}{k} \right) + \frac{1 - n_t}{k} \frac{D_t}{W_{ct}} - \frac{\gamma_c}{2} \left( \frac{n_t S_t}{W_{ct}} \sigma_t \right)^2.$$

(A.3)

It is immediate to observe that the function $J_{c1}(n_t)$ is a concave function of $n_t$ and achieves a unique global maximum given by $n_{c1,1}^*$ in equation (15). In contrast, the function $J_{c2}(n_t)$ can be either convex or concave, depending on the cost parameter $k$. It achieves a global maximum or minimum (depending on $k$) at point $n_{c1,2}^*$ in equation (15). Two other potential points of global maximum are $n_{c1,3}^* = 1/(1 + (1 - p)k)$, where $J_{c1}(n_t) = J_{c2}(n_t)$, and $n_{c1,4}^* = 1$ at which the constraint $n_t \leq 1$ becomes binding. We then determine the global maximum by direct search over points $n_{c1,1}^*$, $n_{c1,2}^*$, $n_{c1,3}^*$, $n_{c1,4}^*$ to find the point at which the function achieves global maximum, which gives rise to equation (15). The minority shareholder’s optimal portfolio (16) is easily obtained by maximizing the quadratic concave objective function (12).

Proof of Proposition 2. We consider the benchmark economy with full protection. Because there is no stealing in this economy, and hence $x_t = 0$, both investors have the same objective function (12) and their portfolios are given by

$$n_{ct}^{bmk} = \frac{\mu_t - r_t + \frac{D_t}{S_t}}{\gamma_c \sigma_t^2 \frac{S_t}{W_{ct}}}; \quad n_{mt}^{bmk} = \frac{\mu_t - r_t + \frac{D_t}{S_t}}{\gamma_m \sigma_t^2 \frac{S_t}{W_{mt}}}.$$

(A.4)
In what follows we omit the superscript "\(bmk\)" over the benchmark equilibrium processes for convenience. Substituting the portfolios (A.4) and optimal consumptions \(c_i^* = \rho^{1/\gamma_i}W_{it}\) into the shareholders’ self-financing budget constraints (7) with \(x_t = 0\), we obtain that their wealths under the optimal strategies follow the dynamics:

\[
dW_{it} = W_{it} \left( r_t + \frac{\kappa_t^2}{\gamma_t} - \rho^{1/\gamma_i} \right) dt + l_t \hat{D}_t dt + \frac{W_{it} \kappa_t}{\gamma_t} dw_t, \tag{A.5}
\]

where \(\kappa \equiv (\mu_t - r_t + D_t/S_t)/\sigma_t\) is the Sharpe ratio. Substituting the shareholders’ optimal consumptions \(c_i^* = \rho^{1/\gamma_i}W_i\) into the consumption clearing condition (20), we obtain the equation \(\rho^{1/\gamma_c}W_{ct} + \rho^{1/\gamma_M}W_{mt} = \hat{D}_t\). Applying Itô’s Lemma to both sides of this equation, matching the \(dt\) and \(dw\) terms and then dividing both sides of the resulting equations by the output \(\hat{D}_t\), we obtain the following system of equations for \(r\) and \(\kappa\):

\[
(1 - y_t) \left( r_t + \frac{\kappa_t^2}{\gamma_c} - \rho^{1/\gamma_c} \right) + y_t \left( r_t + \frac{\kappa_t^2}{\gamma_M} - \rho^{1/\gamma_M} \right) = \mu_d - \rho^{1/\gamma_M} l_M - \rho^{1/\gamma_c} l_c, \tag{A.6}
\]

\[
(1 - y_t) \frac{\kappa_t}{\gamma_c} + y_t \frac{\kappa_t}{\gamma_M} = \sigma_d. \tag{A.7}
\]

Solving equations (A.6) and (A.7), we obtain the interest rate (25) and Sharpe ratio (23).

Next, we obtain the drift \(\mu_{yt}\) and volatility \(\sigma_{yt}\) of the consumption share \(y_t = c_{mt}/\hat{D}_t\). Because \(c_{mt} = \rho^{1/\gamma_M}W_{mt}\), the consumption share can be rewritten as \(y_t = \rho^{1/\gamma_M}W_{mt}/\hat{D}_t\), where \(W_{mt}\) follows the process (A.5). Applying Itô’s Lemma to both sides of this equation for the consumption share \(y_t\), and matching \(dt\) and \(dw\) terms, we obtain equations (27) and (28) for \(\sigma_y\) and \(\mu_y\), respectively.

Towards obtaining the volatility \(\sigma_t\) and the trading strategies, we first derive the ratios \(\hat{D}_t/S_t, W_{ct}/S_t,\) and \(W_{mt}/S_t\). The ratio \(D_t/S_t\) is found using the market clearing condition \(W_{ct} + W_{mt} = S_t\) and optimal consumptions \(c_i^* = \rho_i W_{it}\) as follows:

\[
\frac{\hat{D}_t}{S_t} = \frac{\hat{D}_t}{W_{ct} + W_{mt}} = \frac{\hat{D}_t}{\frac{c_{ct}\rho^{-1/\gamma_c} + c_{mt}\rho^{-1/\gamma_M}}{1 - y_t\rho^{-1/\gamma_c} + y_t\rho^{-1/\gamma_M}}}, \tag{A.8}
\]

The ratios \(W_{ct}/S_t\) and \(W_{mt}/S_t\) are determined analogously to \(D_t/S_t\) as follows:

\[
\frac{W_{ct}}{S_t} = \frac{W_{ct}}{W_{ct} + W_{mt}} = \frac{c_{ct}\rho^{-1/\gamma_c}}{c_{ct}\rho^{-1/\gamma_c} + c_{mt}\rho^{-1/\gamma_M}} = \frac{1}{(1 - y_t)\rho^{-1/\gamma_c} + y_t\rho^{-1/\gamma_M}}, \tag{A.9}
\]

\[
\frac{W_{mt}}{S_t} = \frac{W_{mt}}{W_{ct} + W_{mt}} = \frac{c_{mt}\rho^{-1/\gamma_M}}{c_{ct}\rho^{-1/\gamma_c} + c_{mt}\rho^{-1/\gamma_M}} = \frac{1}{(1 - y_t)\rho^{-1/\gamma_c} + y_t\rho^{-1/\gamma_M}}.
\]
\[
\frac{W_{Mt}}{S_t} = 1 - \frac{W_{ct}}{S_t} = \frac{yt\rho^{-1/\gamma_M}}{(1 - yt)\rho^{-1/\gamma_C} + yt\rho^{-1/\gamma_M}}.
\] (A.10)

The market clearing in the stock and bond markets implies that the aggregate wealth equals the value of the stock market, that is, \(W_{ct} + W_{Mt} = S_t\). From the latter equation and the expressions for the optimal consumption \(c_i^* = \rho^{1/\gamma_i}W_{it}\) we obtain the stock price:

\[
S_t = c_{ct}^*\rho^{-1/\gamma_C} + c_{Mt}^*\rho^{-1/\gamma_M} = ((1 - yt)\rho^{-1/\gamma_C} + yt\rho^{-1/\gamma_M})\hat{D}_t.
\] (A.11)

Applying Itô’s Lemma to equation (A.11) we find that

\[
\sigma_t = \sigma_D + (\rho^{-1/\gamma_M} - \rho^{-1/\gamma_C})\sigma_{yt}\hat{D}_t/S_t.
\]

Substituting \(\sigma_y\) from (27) and \(\hat{D}_t/S_t\) from (A.8) into the latter equation for \(\sigma_t\) we obtain the volatility \(\sigma_t\) in equation (24). The stock return is then given by

\[
\mu_t = \kappa_t\sigma_t + rt - (1 - l_M - l_C)\hat{D}_t/S_t,
\]

which yields equation (22).

Finally, we obtain the optimal portfolios (29) by substituting the equilibrium processes into equations (A.4). The processes \(\mu_t, rt, \sigma_t\) are substituted from (22)–(24), the ratio \(D_t/S_t \equiv (1 - l_M - l_C)\hat{D}_t/S_t\) is substituted from (A.8), and \(W_{it}/S_t\) from (A.9) and (A.10).

The stationary pdf is given by (Karlin and Taylor, 1981, p. 221):

\[
\psi(y) = \frac{\exp\left(a_0 + \int_{0.5}^{y} \frac{2\mu_y(z)}{\sigma_y(z)^2} dz\right)}{\sigma_y(y)^2},
\] (A.12)

where \(a_0\) is a normalizing constant that makes the pdf integrate to 1. After some tedious algebra, we find that:

\[
\frac{2\mu_y}{\sigma_y^2} = a_1 + \frac{a_c}{y^2} - \frac{a_M}{(1 - y)^2} + \frac{b_c}{y} + \frac{b_M}{1 - y}.
\] (A.13)

where the constants are given by:

\[
a_1 = -\frac{2\rho^{1/\gamma_C} - \rho^{1/\gamma_M}}{\sigma_D^2},
\]

\[
b_i = \frac{2}{\sigma_D^2(1/\gamma_M - 1/\gamma_C)^2} \left(\frac{1}{\gamma_i} \left(\frac{1}{\gamma_M} - \frac{1}{\gamma_C}\right) \frac{\gamma_i - 1}{\gamma_i} + \frac{\rho^{1/\gamma_M}l_M - \rho^{1/\gamma_C}l_C}{\gamma_i^2} \frac{1}{\gamma_M} \left(\frac{1}{\gamma_M} - \frac{1}{\gamma_C}\right)\right),
\]

\[
a_c = \frac{\rho^{1/\gamma_M}l_M}{\gamma_C^2}, \quad a_M = \frac{\rho^{1/\gamma_C}l_C}{\gamma_M^2}.
\] (A.14)
Substituting (A.13) into (A.12) and integrating, we obtain the pdf (30). Because $a_M > 0$ and $a_C > 0$, the pdf (30) converges to 0 exponentially fast at the boundaries. Consequently, \( \lim_{y \to 0} \psi(y) = \lim_{y \to 1} \psi(y) = 0 \), and \( \int_0^1 \psi(y) < +\infty \), and hence, the pdf is well-defined.

**Proof of Proposition 2.** Equation (31) for the mean-stock return $\mu_t$ follows readily from the definition of the Sharpe ratio $\kappa_t = (\mu_t - r_t + (1 - x_t^*)D_t/S_t)/\sigma_t$ and the expression (A.8) for the ratio $\hat{D}_t/S_t$. Next, we derive the interest rate $r_t$. The wealths of the controlling and minority shareholders satisfy the budget constraints (7). Applying Itô’s Lemma to both sides of the consumption clearing condition $\rho^{1/\gamma_C}W_{ct} + \rho^{1/\gamma_M}W_m = \hat{D}_t$, matching $dt$ and $dw$ terms, and dividing both sides by $\hat{D}_t$, we obtain the equations:

\[
(1 - y_t) \left( r_t - \rho^{1/\gamma_C} + \frac{n^*_c S_t \rho \kappa_t}{W_{ct}} \right) + (1 - l_M - l_C) \left( x_t^* - 0.5k(x_t^*)^2 \right) \rho^{1/\gamma_C} + l_C \rho^{1/\gamma_C} \\
\quad + y_t \left( r_t - \rho^{1/\gamma_M} + \frac{n^*_M S_t \sigma_t \kappa_t}{W_{mt}} \right) + 0.5k(x_t^*)^2 \rho^{1/\gamma_M} + l_M \rho^{1/\gamma_M} = \mu_D, \quad (A.15)
\]

\[
(1 - y_t) \frac{n^*_c S_t \sigma_t}{W_{ct}} + y_t \frac{n^*_M S_t \sigma_t}{W_{mt}} = \sigma_D. \quad (A.16)
\]

From equations (A.15)–(A.16) and equations (A.9)–(A.10) for the ratios $W_{C}/S$ and $W_{M}/S$ (which remain the same for the model with imperfect protection), we obtain:

\[
r_t = \mu_D + y_t \rho^{1/\gamma_M} + (1 - y_t) \rho^{1/\gamma_C} - \kappa_t \sigma_D - l_M \rho^{1/\gamma_M} - l_C \rho^{1/\gamma_C} - (1 - l_M - l_C)x_t^* \rho^{1/\gamma_C} - (\rho^{1/\gamma_M} - \rho^{1/\gamma_C})(1 - l_M - l_C) \frac{k(x_t^*)^2}{2}, \quad (A.17)
\]

\[
\sigma_t = \frac{\sigma_D}{(\rho^{1/\gamma_M} (1 - n^*_c) + \rho^{1/\gamma_C} n^*_c)(y_t \rho^{-1/\gamma_M} + (1 - y_t) \rho^{-1/\gamma_C})}. \quad (A.18)
\]

From equation (16) for $n^*_M$ and the fact that $n^*_M = 1 - n^*_C$ we find that $\kappa_t = \gamma_M \sigma_t (1 - n^*_c)/(W_{mt}/S_t)$. Substituting the ratio $W_{M}/S$ from (A.10) and $\sigma_t$ from (A.18) into the latter equation, we obtain:

\[
\kappa_t = \frac{\gamma_M \sigma_D (1 - n^*_c) \rho^{1/\gamma_M} 1}{\rho^{1/\gamma_M} (1 - n^*_c) + \rho^{1/\gamma_C} n^*_c y_t}. \quad (A.19)
\]

Using exactly the same steps as above, we obtain the following expressions for $\sigma_{bmk}$ and $\kappa_{bmk}$ in the benchmark economy:

\[
\sigma_{bmk} = \frac{\sigma_D}{(\rho^{1/\gamma_M} (1 - n^*_c) + \rho^{1/\gamma_C} n^*_c)/(y_t \rho^{-1/\gamma_M} + (1 - y_t) \rho^{-1/\gamma_C})}. \quad (A.20)
\]

\[
\kappa_{bmk} = \frac{\gamma_M \sigma_D (1 - n^*_c) \rho^{1/\gamma_M} 1}{\rho^{1/\gamma_M} (1 - n^*_c) + \rho^{1/\gamma_C} n^*_c y_t}. \quad (A.21)
\]
We next rewrite \( \kappa_t \) as \( \kappa_t = \kappa_t^{bmk} + (\kappa_t - \kappa_t^{bmk}) \), where \( \kappa_t^{bmk} \) is given by equation (A.21). Substituting (A.19) into the right-hand side of the latter equation, after some algebra, we obtain equation (32) for the Sharpe ratio. Similarly, we rewrite the volatility as \( \sigma_t = \sigma_t^{bmk} + (\sigma_t - \sigma_t^{bmk}) \), where \( \sigma_t^{bmk} \) is given by equation (A.20). Substituting (A.18) into the right-hand side of the latter equation, after some algebra, we obtain equation (33) for \( \sigma_t \). Substituting \( \kappa_t \) from (32) into equation (A.17) for \( r_t \), after some algebra, we obtain equation (34) for \( r_t \).

Next, we find the volatility \( \sigma_t \) and drift \( \mu_t \) of the consumption share \( y_t = c_t^*/\hat{D}_t \). Noting that \( c_t^* = \rho_1^*/\gamma_t M_t \), we apply Itô’s Lemma to \( y_t = \rho_1^*/\gamma_t M_t/\hat{D}_t \) and obtain \( \sigma_t \) and \( \mu_t \) by matching the \( dw \) and \( dt \) terms on both sides of the equation, as in the benchmark economy.

The controlling shareholder’s optimization problem (11) implies the fixed-point equation (37) for the stock holding \( n_{st}^* \), in which the equilibrium processes depend on \( n_{st}^* \) itself. The ratios \( D_t/S_t \) and \( S_t/W_{ct} \) in (38) are derived in the same way as in equations (A.8)–(A.10). The ratio \( D_t/W_{ct} \) in (38) is found by multiplying ratios \( D_t/S_t \) and \( S_t/W_{ct} \).

**Lemma A.1 (Equilibrium stock holding).** The optimal stock holding \( n_{ct,t}^* \) of the controlling shareholder in equilibrium has the following representation:

\[
\begin{align*}
  n_{ct,t}^* &= \begin{cases}
    n_{ct,1}^* + n_{ct,2}^* \left( 1 - \frac{n_{ct,1}^*}{n_{ct,3}^*} \right) \left( \rho^{1/\gamma_t} n_{ct,1}^* + \rho^{1/\gamma_t} (1 - n_{ct,1}^*) \right)^2 \\
    \times \frac{(1 - p)(1 - l_t - l_c)y_t \rho^{-1/\gamma_t}}{\gamma_t \sigma_D^2}, & \text{(region 1),} \\
    \frac{1 - y_t}{\rho^{1/\gamma_t} c_t} + \frac{y_t}{\rho^{1/\gamma_t} M_t}, & \text{(region 2),} \\
    \frac{1}{1 + (1 - p)k}, & \text{(region 3),} \\
    1 & \text{(region 4),}
  \end{cases}
\end{align*}
\]  

(A.22)

Furthermore, for all consumption shares \( y \in [0, 1] \) there always exists \( n_{ct,1}^* \in [0, 1] \), which solves a third degree polynomial equation.

**Proof of Lemma A.1.** First, we find the stock holding \( n_{ct,2}^* \) in region 2. The constraint \( x_t \leq (1 - p)q(n) \) is not binding by the definition of region 2. Hence, \( n_{ct,2}^* \) satisfies the
following first order condition:

\[
\frac{S_t}{W_{ct}} \left( \mu_t - r_t + (1 - x^*_t) \frac{D_t}{S_t} \right) - \gamma_c n_t \left( \frac{S_t \sigma_t}{W_{ct}} \right)^2 + \left( \frac{S_t}{W_{ct}} \left( \mu_t - r_t + (1 - x^*_t) \frac{D_t}{S_t} \right) + \frac{x^*_t - \frac{k(x^*_t)^2}{2}}{x} \right) (x^*(n))^' = 0.
\]  

(A.23)

The second line in (A.23) is zero from the first order condition for \(x^*_t\). Hence, we obtain:

\[
n^*_c = \frac{\mu_t - r_t + (1 - x^*_t) \frac{D_t}{S_t}}{\gamma_c \sigma_t^2 \frac{S_t}{W_{ct}}}.
\]  

(A.24)

Substituting \(\kappa_t\) from (A.19), \(\sigma_t\) from (A.18), \(D_t/W_{ct}\) and \(S_t/W_{ct}\) from (38) into the expression for \(n^*_{ct,2}\) in (A.24), after some algebra, we obtain:

\[
n^*_{ct,2} = \frac{1 - y_t}{\rho^{1/\gamma_c} \gamma_c} + \frac{y_t}{\rho^{1/\gamma_M} \gamma_M}.
\]  

(A.25)

We note that \(n^*_{ct,3}\) and \(n^*_{ct,4}\) remain the same as in (15).

To obtain the stock holding \(n^*_{ct,1}\) in region 1, we observe that by the definition of region 1, \(n^*_{ct,1}\) satisfies the following first order condition for the objective function \(J_{c1}(n_t)\) in (A.2):

\[
n^*_{ct,1} = \frac{\kappa_t}{\gamma_c (S_t/W_{ct} \sigma_t)} + \frac{(1 - p) D_t/W_{ct} (1 - n^*_{ct,1} - k(1 - p)n^*_{ct,1})}{\gamma_c (S_t/W_{ct} \sigma_t)^2}.
\]  

(A.26)

Substituting \(\kappa_t\) from (A.19) and \(\sigma_t\) from (A.18), \(D_t/W_{ct}\) and \(S_t/W_{ct}\) from (38) into equation (A.26), after straightforward algebra, we obtain the first line of (A.22). It can be easily verified that the left-hand side of the equation in the first line of (A.22) is lower than its right-hand side for \(n^*_{ct,1} = 0\), and vice versa for \(n^*_{ct,1} = 1\). Therefore, by the intermediate value theorem, there exists \(n^*_{ct,1} \in [0, 1]\).  

Numerical method for the main analysis in Section 4. First, we derive the optimal stock holding \(n^*_{ct}\) as a function of the consumption share \(y\) in each of the regions 1, 2, 3, and 4 in equation (15). Lemma A.1 above demonstrates the existence of the stock holding \(n^*_{ct}\) and derives it as an implicit function of the consumption share \(y_t\), which can easily be computed by solving a third-degree polynomial equation.\(^{14}\) Then, for each value of the

\(^{14}\)As demonstrated in Lemma A.1, this polynomial always has a solution in the interval \([0, 1]\). Although in general this solution may not be unique, we only obtain unique solutions in our calibration of the model. Furthermore, because the objective function (A.1) may have regions of non-concavity, the optimal portfolio holding \(n^*_{ct}\) may appear to be a discontinuous function of \(y\). However, such a discontinuity requires extreme values of exogenous model parameters and does not occur in our calibration of the model.
consumption share $y$, we substitute $n^{*}_{ct,1}$, $n^{*}_{ct,2}$, $n^{*}_{ct,3}$, and $n^{*}_{ct,4}$ in turn into the objective function on the right-hand side of (37) and find the element that maximizes the objective function. This way, we obtain the optimal stock holding $n^{*}_{ct}$. Substituting $n^{*}_{ct}$ into the expressions for equilibrium processes in Proposition 2, we obtain all equilibrium processes as functions of the consumption share $y$.

**Numerical method for the model with cross-section of firms in Section 5.3.** For tractability, we solve the optimization of shareholder $C$ subject to the financial wealth constraint (46) using the method of penalty functions. In particular, we incorporate the latter constraint into the utility function by adding a penalty term $\tilde{\beta}\mathbb{E}[u_c(\tilde{W})]$, where $\tilde{\beta} = 10^{-7}$ is a small penalty parameter. Economically, this term captures the disutility of violating constraint (46). We then maximize utility function $u_c(c) + \beta\mathbb{E}[u_c(\tilde{c})] + \tilde{\beta}\mathbb{E}[u_c(\tilde{W})]$, subject to the budget constraint (45) and investor protection constraint $x \leq (1 - p)n$. The latter constraint always binds because there are no pecuniary costs of stealing, and hence $x = (1 - p)n$. We also use the fact that $\tilde{W} = \tilde{c} - XD_1$.

The FOCs with respect to $c$ and $\tilde{c}$ for both shareholders, stacked together, give equations:

$$
\frac{1}{1 + r} = \beta \mathbb{E} \left[ \frac{u'_c(c^*_c)}{u''_c(c^*_c)} \right] + \tilde{\beta} \mathbb{E} \left[ \frac{u'_c(c^*_c - x^*D_1)}{u''_c(c^*_c)} \right] = \beta \mathbb{E} \left[ \frac{u'_M(c^*_M)}{u''_M(c^*_M)} \right], \quad (A.27)
$$

$$
S_1 = \beta \mathbb{E} \left[ (2 - p - 2x^*)D_1 \frac{u'_c(c^*_c)}{u''_c(c^*_c)} \right] + \tilde{\beta} \mathbb{E} \left[ (1 - 2x^*)D_1 \frac{u'_c(c^*_c - x^*D_1)}{u''_c(c^*_c)} \right] = \beta \mathbb{E} \left[ D_1 \frac{u'_M(c^*_M)}{u''_M(c^*_M)} \right], \quad (A.28)
$$

$$
S_2 = \beta \mathbb{E} \left[ D_2 \frac{u'_c(c^*_c)}{u''_c(c^*_c)} \right] + \tilde{\beta} \mathbb{E} \left[ D_2 \frac{u'_c(c^*_c - x^*D_1)}{u''_c(c^*_c)} \right] = \beta \mathbb{E} \left[ D_2 \frac{u'_M(c^*_M)}{u''_M(c^*_M)} \right]. \quad (A.29)
$$

Multiplying the budget constraint (45) for shareholder $C$ in turn by $\Delta w_1$ and then by $\Delta w_2$ and taking expectations, we obtain two equations:

$$
\mathbb{E}[\tilde{c}_C \Delta w_1] = (n^{*}_{1C}(1 - x^*) + x^*)D\sigma_D + 0.5n^{*}_{2C}D\sigma_D, \quad (A.30)
$$

$$
\mathbb{E}[\tilde{c}_C \Delta w_2] = 0.5(n^{*}_{1C}(1 - x^*) + x^*)D\sigma_D + n^{*}_{2C}D\sigma_D. \quad (A.31)
$$

Noting that $n^{*}_{1C} = x^*/(1 - p)$ and solving (A.30)–(A.31) we obtain:

$$
\left( x^* + \frac{x^*(1 - x^*)}{1 - p} \right) D\sigma_D = (4/3)\mathbb{E} [\tilde{c}_C(\Delta w_1 - 0.5\Delta w_2)]. \quad (A.32)
$$
Let $c^* = 2Dy$, and define date $t = 1$ consumption share $\tilde{y} = \tilde{c}^*_m/(\tilde{D}_1 + \tilde{D}_2)$ in state $\omega$. Then, using the market clearing conditions for consumption, we find $c_c^* = 2D(1 - y)$, $\tilde{c}_c = (\tilde{D}_1 + \tilde{D}_2)(1 - \tilde{y})$. Substituting consumptions in terms of the variables $y$ and $\tilde{y}$ into equations (A.27)–(A.29), we find the interest rate $r$.

Equations (A.27)–(A.29) and (A.32), and also solving equations (A.27)–(A.29) for marginal utility ratios $u'_m(c^*_m)/u'_m(c^*_m)$, we obtain the following system of equations for $\tilde{y}$ and $x$:

\[
\left(\frac{\tilde{y}\tilde{D}_1 + \tilde{D}_2}{y\tilde{D}}\right)^{-\gamma M} = \left(\frac{1 - \tilde{y}\tilde{D}_1 + \tilde{D}_2}{1 - y\tilde{D}}\right)^{-\gamma C} + \frac{\beta}{\beta}\left(\frac{1 - \tilde{y}\tilde{D}_1 + \tilde{D}_2}{1 - y\tilde{D}} - \frac{x^* \tilde{D}_1}{1 - y\tilde{D} - 1}\right)^{-\gamma C} + \delta \tilde{v}, \tag{A.33}
\]

where variable $\tilde{v}$ takes values $v_k$ in states $\omega_k$, given by:

\[
\delta = \frac{(1 - p)E \left[ \frac{\tilde{D}_1}{D}\left(\frac{1 - \tilde{y}\tilde{D}_1 + \tilde{D}_2}{1 - y\tilde{D}}\right)^{-\gamma C} - x^* E \left[ \frac{\tilde{D}_1}{D}\left(\frac{\tilde{y}\tilde{D}_1 + \tilde{D}_2}{y\tilde{D}}\right)^{-\gamma M} \right] \right]}{1 - x^*(1 + E[\tilde{D}_1/D])}, \tag{A.34}
\]

\[
\left( x^* + \frac{x^*(1 - x^*)}{1 - p} \right) \sigma_D = (4/3)E \left[ \frac{\tilde{D}_1 + \tilde{D}_2}{D}(1 - \tilde{y})\Delta w_1 - 0.5\Delta w_2 \right], \tag{A.35}
\]

We then solve equations (A.33)–(A.35) numerically along the following steps:

(i) For each value of $y$, consider equation (A.33) for fixed $\delta$. For each state $\omega$, this is an equation for one variable $\tilde{y}(\omega)$, which we simply denote as $\tilde{y}$. Solving it by the method of bisections, we find $\tilde{y}(\delta; x^*)$.

(ii) Substituting $\tilde{y}(\delta; x^*)$ into (A.34), we solve the latter equation using the method of bisections and find $\delta(x^*)$.

(iii) Substituting $\tilde{y}(\delta(x^*); x^*)$ into (A.35), by the same method we find $x^*$.

(iv) Hence, we find date $t = 1$ consumption shares $\tilde{y}$ in all states $\omega$.

(v) Then, we find $c^*_m = 2Dy$ and $\tilde{c}^*_m = (\tilde{D}_1 + \tilde{D}_1)\tilde{y}$. Substituting these optimal consumptions into (A.27)–(A.29), we find the interest rate $r$, and stock prices $S_1$ and $S_2$. 

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References


