1 Efficiency as deviation from a frontier

Efficiency, namely, the utilization of resources, is one of the most important topics of economic theory. Efficiency is the relationship between what an organization (producer, production unit, or any decision-making unit) produces and what it could feasibly produce, under the assumption of full utilization of the resources available (Hoyo et al., 2004). Within this conceptual framework, as stated by Kumbhakar and Lovell (2000, p. 15):

“efficiency represents the degree of success which producers achieve in allocating the available inputs and the outputs they produce, in order to achieve their goals ... namely ... to attain a high degree of efficiency in cost, revenue, or profit”.

As stated in del Hoyo et al. and Kumbhakar and Lovell (2000), efficiency is the ability of a decision – making unit to obtain the maximum output from a set of inputs (output orientation) or to produce an output using the lowest possible amount of inputs (input orientation). A production frontier refers to the maximum output attainable by given sets of inputs and existing production technologies. The production frontier defines the technical efficiency in terms of a minimum set of inputs in order to produce a given output or a maximum output produced by a given set of inputs. This approach involves selecting the mix of inputs which produces a given quantity of output at a minimum cost, namely the production frontier. If what a producer actually produces is less than what it could feasibly produce then it will lie below the frontier. The

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1I am gratefully indebted to the Hellenic State Scholarship Foundation for the scholarship granted
distance by which a firm lies below its production frontier is a measure of the firm’s inefficiency (Bera and Sharma, 1999).

Farrell (1957) was the first to empirically measure productive efficiency in terms of deviations from an ideal frontier. He also proposed a decomposition of economic efficiency into: a) technical efficiency ($TE$), which measures the ability of a firm to obtain the maximum output from given inputs, and b) allocative efficiency ($AE$), which measures the ability of a firm to use inputs in optimal proportions given their prices:

$$\text{Economic efficiency} = \text{Technical efficiency} + \text{allocative efficiency}$$

If the only information available are input and output quantities, and there is no information on input or output prices, then the type of efficiency that can be measured is technical efficiency. If price information on inputs and outputs is available, in addition to input and output quantities, then the type of efficiency that can be measured is allocative efficiency. Profit maximisation requires a firm to be both technically efficient (by producing the maximum output given the level of inputs employed), as well as allocative efficient (by using the right mix of inputs, or producing the right mix of outputs given their relative prices, respectively (Kumbhakar and Lovell, 2000). Nevertheless, in real economic life, producers are hardly fully productive efficient. The difference can be explained in terms of technical and allocative inefficiencies, as well as a range of unforeseen exogenous shocks, making it unlike all (or even any) producers, firms or, even, economies operate at the full efficiency frontier (Reifschneider and Stevenson, 1991). However, one of the main related questions is whether inefficiency occurs randomly, or whether some economic agents (producers, firms or economies) have predictably higher levels of inefficiency than others. That is the reason why estimating efficiency is one of the core tools of economic analysis. Firstly, efficiency estimation provides an indication of the percentage by which potential output could be increased, or potential cost could be decreased, in relation to the corresponding production frontier. The further below the frontier a producer lies, the more inefficient it is.

Regarding that the production frontier cannot be observed directly, several techniques have been developed in order to estimate efficiency. As broadly described in del Hoyo et al (2004) and Kortelainen (2008), the main methods of production frontiers and efficiency estimation may be classified into two core groups:

a) non – parametric models, regarding Data Envelopment Analysis, developed by Farrell (1957) and Charnes et al (1978), and

b) parametric models, regarding Deterministic Frontier Analysis and Stochastic Frontier Analysis, developed by Aigner et al, (1977) and Meeusen and van den Broeck (1977).

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2The type of efficiency that can be measured using a production possibility frontier is technical efficiency.
2 Deterministic production frontier models and technical efficiency

Aigner and Chu (1968) were the first researchers to estimate a deterministic frontier production function using Cobb-Douglas production function. They argued that, within a given industry, firms might differ from each other in their production processes, due to certain technical parameters in the industry, due to differences in scales of operation or due to organizational structures. Under this assumption, they considered a Cobb-Douglas production function, with an empirical frontier production model such as:

\[ q_{it} \leq f(x_{it}) \]  \hspace{1cm} (1)

This equation defines a production relationship between inputs, \( x \), and output \( q_{it} \), in which for any given \( x \), the observed value of \( q_{it} \) must be less or equal to \( f(x_{it}) \). Since the theoretical production function is an ideal (the frontier of efficient production), any non-zero disturbance is considered to be the result of inefficiency, which must have a negative effect on production function:

\[ q_{it} = f(x_{it}) - u_{it}, i = 1, 2, 3, 4, ..., I, t = 1, ..., N \]  \hspace{1cm} (2)

Taking natural logarithms, the model becomes:

\[ \ln q_{it} = \beta_0 + \ln x_{it} \beta - u_{it} \]  \hspace{1cm} (3)

where:

1. \( \ln q_{it} \) is the natural logarithm of the output of the \( i^{th} \) firm;
2. \( \ln x_{it} \) is the natural logarithms of inputs;
3. \( \beta \) is a column vector of the unknown parameters to be estimated;
4. \( u_{it} \) is a non-negative random variable associated with technical inefficiency, representing the shortfall of actual output from its maximum possible value.

Technical efficiency for the \( i^{th} \) firm is defined as the ratio of the observed output for the \( i^{th} \) firm relative to the potential output (frontier function):

\[ TE_{it} = \frac{\text{observed output}}{\text{potential maximum output}} = \frac{q_{it}}{\exp(x_{it} \beta)} = \frac{\exp(x_{it} \beta - u_{it})}{\exp(x_{it} \beta)} = \exp(-u_{it}), 0 \leq TE_{it} \leq 1 \]  \hspace{1cm} (4)

\[ \exp(-u_{it}) \hspace{1cm} (5) \]
and

\[ u_{it} = \ln(TE_{it}) \]  

(6)

Technical efficiency measure takes a value between zero and one:

1. \( TE_{it} = 1 \) shows that the producer is fully productive efficient and, correspondingly, the observed output \( q_i \) reaches its maximum obtainable value,

2. \( TE_{it} < 1 \) provides a measure of the shortfall of the observed output from maximum feasible output.

Letting:

\[ TE_{it} = \exp(-u_{it}) \times 0 \leq TE_{it} \leq 1 \]  

(7)

will ensure that the observed output lies below the frontier, that is:

\[ q_{it} \leq f(x_{it}\beta) \]  

(8)

Nevertheless, in this case, the model is deterministic, and all deviations form the frontier are assumed to be the result of technical inefficiency and no account is taken of any measurement errors (i.e. errors associated with the choice of functional form) or any statistical noise (i.e. omission of relevant variables from the vector \( x_{it} \)). This approach is dealt by the Stochastic Production Frontier models.

3 Stochastic production frontier models and technical efficiency

In the decade of 1970, deterministic production frontier model was extended by Afriat (1972), and more systematically by Aigner et al. (1977) and Meesuen and van den Broeck (1977). Based on the literature commencing with theoretical work by Debreu (1951) and Farrell (1957), Aigner et al. (1977) and Meesuen and van den Broeck (1977) extended the deterministic frontier approach in order to account not only for technical inefficiency, but also for any measurement errors or any statistical noise\(^3\). They developed a statistically and theoretically sound method for measuring efficiency, different is the sense that it allows random events to contribute to variations in producer output. Aigner et al. (1977) and Meesuen and van den Broeck (1977) proposed, almost simultaneously, but

\(^3\)Since the introduction of stochastic frontier analysis, it has been widely accepted that frontier models provide a number of advantages over non-frontier models (see, e.g., Forsund et al., 1980 and Bravo-Ureta and Pinherio, 1993). The economic literature on efficiency and stochastic frontier analysis has been rather extensive with numerous studies. To name just a few, there are influential research papers by Forsund et al. (1980) and Greene (1993, 1997), Bauer (1990), Battese (1992), Schmidt (1985), Cornwell and Schmidt (1996), Kalirajan and Shand (1999), and Murillo-Zamorano (2004), as well as book-length approaches, including Coelli et al. (1995), Coelli et al. (1998), Kumbhakar and Lovell (2000) and Fried et al (2008).
independently, a formulation within which observed deviations from the production function could arise from two sources: a) productive inefficiency, that would necessarily be negative, and b) effects specific to the firm, that could be of either sign. The model is intended to incorporate this feature, there is need to introduce another random variable representing any statistical noise or measurement errors. In order to capture this, the stochastic model includes a composite error term that sums a two-sided error term, measuring all effects outside the firm’s control, and a one-sided, non-negative error term, measuring technical inefficiency. The resulting frontier is presented in terms of a general production function, known as a ‘stochastic production frontier’:

\[ \ln q_{it} = x_{it}\beta + v_{it} - u_{it} \]  

(9)

where the observed response \( q_{it} \) is a scalar output, \( x_{it} \) is a vector of \( m \) inputs, \( \beta \) is a vector of the unknown technology parameters, \( f(x_{it}\beta) \) is the production frontier. As described in Coelli et al. (2005), in this case, a Cobb – Douglas stochastic frontier model takes the form:

\[ \ln q_{it} = \beta_0 + \beta_1 \ln x_{it} + v_{it} - u_{it} \]  

(10)

or

\[ q_{it} = \exp(\beta_0 + \beta_1 \ln x_{it} + v_{it} - u_{it}) \]  

(11)

or

\[ q_{it} = \exp(\beta_0 + \beta_1 \ln x_{it}) \times \exp(v_{it}) \times \exp(-u_{it}) \]  

(12)

where,

1. \( \exp(\beta_0 + \beta_1 \ln x_{it}) \): deterministic component
2. \( \exp(v_{it}) \): noise
3. \( \exp(-u_{it}) \): inefficiency

The model equation can be rewritten as:

\[ q_{it} = f(x_{it}\beta) \times \exp(v_{it} - u_{it}), u_i \geq 0 \]  

(13)

where \( u_i \) represents the shortfall of output from the frontier. The composite error structure is:

\[ \epsilon_{it} = v_{it} - u_{it} \]  

(14)

The stochastic econometric approach enables to attempt to distinguish the effects of noise and inefficiency, thereby providing the basis for statistical inference. The model is such that the possible production \( q_{it} \) is limited above by
the stochastic quantity $f(x_{it}, \beta) \times \exp(v_{it})$. The noise component $v_{it}$ is assumed to be independently and identically distributed (i.i.d.), symmetric, and distributed independently of $u_{it}$. The combined error term $\epsilon_{it} = v_{it} - u_{it}$ is therefore asymmetric since $u_{it} \geq 0$. Providing estimates of producer-specific technical efficiency, which is the ultimate objective of the estimation process in addition to obtaining estimates of the production technology parameters $\beta$ in $f(x_{it}, \beta)$, requires an extraction of separate estimates of statistical noise $v_{i}$ and technical inefficiency $u_{it}$ form the estimates of $\epsilon_{it}$ for each producer. Therefore, the distributional assumptions of the inefficiency term are required to estimate the technical inefficiency of each producer.

In order to define technical efficiency within the stochastic frontier framework, let us consider the above production function:

$$q_{it} = f(x_{it} \beta) + \epsilon_{it}$$

(15)

Under the assumption for the error term and ensuring that observed output lies below the stochastic frontier, the production function becomes:

$$q_{it} \leq f(x_{it} \beta) \times \exp(v_{it})$$

(16)

Consequently, we have:

$$TE_{it} = \frac{\text{observed output}}{\text{potential maximum output}} = \frac{f(x_{it} \beta) \times \exp(v_{it}) \times \exp(-u_{it})}{\exp(x_{it} \beta)} = \exp(-u_{it}) \times 0 \leq TE_{it} \leq 1$$

(17)

which will ensure that the observed output lies below the frontier. As stated above, following the inclusion of the second random error, the stochastic frontier model asserts that the composite error term of the function is made up of two independent components: a) of a two-sided random term, $v_{it}$, and b) by a one-sided positive error term $u_{it}$. The component $v_{it}$ represents factors that cannot be controlled by production units, measurement errors, and left-out explanatory variables. On the other hand, the component $u_{it}$ represents the shortfall from the production frontier due to inefficiency, which may be the result of cultural factors, such as attitude toward work; climatic factors, such as summers, or traditions, such as religious holidays.

Aigner et al. (1977) assumed that the stochastic error terms $v_{it}$ are independent and identically distributed (i.i.d.) normal random variables with mean zero and constant variance $\sigma_{v}^2$:

$$(v_{it}) \sim iidN(0, \sigma_{v}^2)$$

(18)
which denotes that the errors \( v_{it} \) are independently and identically distributed normal random variables with zero means and variances \( \sigma_v^2 \), and

\[
(u_{it}) \sim iidN(0, \sigma_u^2)
\]

which denotes that the errors \( u_{it} \) are independently and identically distributed normal random variables with zero means and variances \( \sigma_u^2 \).

The model assumes that each \( v_{it} \) is distributed independently of each \( u_{it} \) and that both errors are uncorrected with the explanatory variables in \( x_i \). In addition, it is assumed that:

1. \( E(v_{it}) = 0 \) (zero mean)
2. \( E(v_{it}^2) = \sigma_v^2 \) (homoskedastic)
3. \( E(v_{it}v_{jt}) = 0 \), for all \( i \neq j \) (uncorrelated)
4. \( E(u_{it}^2) = \text{constant} \) (homoskedastic)
5. \( E(u_{it}u_{jt}) = 0 \), for all \( i \neq j \) (uncorrelated)

For simplicity reasons, we restrict attention to firms which produce only one output \( q_{it} \) using only one input \( x_{it} \). Figure (2) shows the inputs and outputs of two firms \( A \) and \( B \). The deterministic component of the frontier model has been drawn to reflect the existence of diminishing returns to scale. Values to the input are measured along the horizontal axis and outputs are measured on the vertical axis. Firm \( A \) uses the input level \( x_A \) to produce the output \( q_A \), while Firm \( B \) uses the input level \( x_B \) to produce the output \( q_B \). If there were no inefficiency effects (that is, if \( u_A = 0 \) and \( u_B = 0 \)), then the so-called frontier outputs for firms \( A \) and \( B \) would be:

\[
q_A^* = \exp(\beta_0 + \beta_1 \ln x_A + v_A) \quad (20)
\]

and

\[
q_B^* = \exp(\beta_0 + \beta_1 \ln x_B + v_B) \quad (21)
\]

It is clear that the frontier output for firm \( A \) lies above the deterministic part of the production frontier only because the noise effect is positive (\( v_A > 0 \)), while the frontier output for firm \( B \) lies below the deterministic part of the frontier because the noise effect is negative (\( v_B < 0 \)). It can also been seen that the observed output of firm \( A \) lies below the deterministic part of the frontier because the sum of the noise and inefficiency effects is negative (\( v_A - u_A < 0 \)).

In this case, the prediction of technical efficiency is based on the conditional expectation

\[
E(\exp(-u_{it}) \mid \epsilon_{it}), \epsilon_{it} = v_{it} - u_{it} = y_{it} - f(x_{it}) \quad (22)
\]
where \( \epsilon_{it} = v_{it} - u_{it} \) is the combined error term.

The first step in predicting the technical efficiency \( TE_i \), is to estimate the parameters of the stochastic production frontier model. Even though, the entire term \( (v_{it} - u_{it}) \) is easily estimated for each observation, but a major problem is how to separate it into its two components. Estimation and hypothesis testing procedures in the case of stochastic frontiers is more complicated due to the fact that the right-hand side of the model includes two random terms – a symmetric error, \( v_i \) and a non-negative random variable \( u_i \). Trying to solve this problem, the relationship between \( q_i \) and \( v_i \) could be also expressed as

\[
q_i \sim iidN(x_i \beta, \sigma^2),
\]

where \( q_i \) denotes the \( i \)th observation on the dependent variable; \( x_i \) is a vector containing the explanatory variables; \( \beta \) is the associated vector of unknown parameters. The assumption of a certain inefficiency distribution as well as a normal noise distribution suggests the use of maximum likelihood estimation method (Behr and Tente, 2008), one of the commonly used methods of estimating the parameters of a stochastic frontier.

4 Stochastic Frontier model estimation - The Maximum Likelihood Estimation method

Since early, Aigner et al. (1977) first estimated the unknown parameters of the stochastic frontier model using the method of maximum likelihood (M.L.) method followed also widely in later decades by Greene (1982) and Coelli (1995), among others. Maximum likelihood (M.L.) estimation is a popular statistical method used for fitting a mathematical model to real world data. The concept of maximum likelihood (M.L.) estimation is based on the idea that a particular sample of observations is more likely to have been generated from some distributions than from others. Consequently, the maximum likelihood estimate of an unknown parameter is defined to be the value of the parameter that maximizes the probability (or likelihood) of randomly drawing a particular sample of observations.

In order to use the maximum likelihood principle to estimate the parameters of the production frontier function model, we make the assumption that the errors are normally distributed. This assumption is combined with the assumptions expressed above4:

1. \( \mathbb{E}(v_i) = 0 \) (zero mean)
2. \( \mathbb{E}(v_i^2) = \sigma^2 \) (homoskedastic)
3. \( \mathbb{E}(v_i v_j) = 0 \), for all \( i \neq j \) (uncorrelated).

Aigner et al. (1977) focused on the implicit assumption that the likelihood of inefficient behavior monotonically decreases for increasing levels of inefficiency. They parameterized the log-likelihood function for the half-normal model in terms of the variance parameters:

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4For a detailed analysis, see Coelli et al. (2005)
\[ \sigma^2 = \sigma_v^2 + \sigma_u^2 \]  \hspace{1cm} (23)

where \( \sigma^2 \) is a measure of the total variance of the combined error term

\[ \epsilon_{it} = v_{it} - u_{it} \]  \hspace{1cm} (24)

and

\[ \lambda^2 = \frac{\sigma_v^2}{\sigma_u^2} \geq 0 \]  \hspace{1cm} (25)

If \( \lambda = 0 \), there are no technical inefficiency effects and all deviations from the frontier are due to noise. Using this parameterization, the log–likelihood function is:

\[ \ln L(\mathbf{y} \mid \mathbf{\beta}, \mathbf{\sigma}, \mathbf{\lambda}) = -\frac{1}{2} \ln \left( \frac{\pi \sigma^2}{2} \right) + \sum_{i=1}^{I} \ln \Phi \left( \frac{\epsilon_{it} \lambda}{\sigma} \right) - \frac{1}{2 \sigma^2} \sum_{i=1}^{I} \epsilon_{it}^2 \]  \hspace{1cm} (26)

where \( \mathbf{y} \) is a vector of log–outputs, \( \epsilon_{it} = v_{it} - u_{it} \) is the composite error term and \( \Phi(\mathbf{x}) \) is the cumulative distribution function of the standard normal random variable evaluated at \( \mathbf{x} \).

Maximizing a log–likelihood function usually involves taking first–derivatives with respect to the unknown parameters and setting them to zero. However, since these first–order conditions are highly nonlinear and cannot be solved analytically for \( \mathbf{\beta}, \mathbf{\sigma}, \) and \( \mathbf{\lambda} \), we maximize the likelihood function using an iterative optimization procedure. This involves selecting starting values for the unknown parameters and systematically updating them until the values that maximize the log–likelihood function are found. In this case, the stochastic model is given by the equation:

\[ \ln q_{it} = x_{it} \mathbf{\beta} + v_{it} - u_{it}, \quad i = 1, 2, 3, ..., I \]  \hspace{1cm} (27)

along with \( (v_{it}) \sim iidN(0, \sigma^2_v) \) and \( (u_{it}) \sim iidN(0, \sigma^2_u) \). The parameters of the model take the form of \( x_{it} \) and \( \mathbf{\beta} \), with:

\[ x_i = \begin{bmatrix} 1 \\ t_i \\ \ln x_{1i} \\ \ln x_{2i} \\ \ln x_{3i} \\ 0.5(\ln x_{1i})^2 \\ \ln x_{1i} \ln x_{2i} \\ \ln x_{1i} \ln x_{3i} \\ 0.5(\ln x_{2i})^2 \\ \ln x_{2i} \ln x_{3i} \\ 0.5(\ln x_{3i})^2 \end{bmatrix} \]

\[ \mathbf{\beta} = \begin{bmatrix} \beta_0 \\ \theta \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_{11} \\ \beta_{12} \\ \beta_{13} \\ \beta_{22} \\ \beta_{23} \\ \beta_{33} \end{bmatrix} \]
where \( t_i \) is a time trend included to account for technological change. Technological advances often cause production functions to change over time, reflecting the industry-specific knowledge of technological developments and how they affect economic behavior within the industry.

In the case of the half-normal and exponential models, the null hypothesis is a single restriction involving a single parameter. If the model has been estimated using the method of M.L., we can test such an hypothesis using a z-test (because unconstrained M.L. estimators are asymptotically normally distributed). In the half-normal model, the null and alternative hypotheses are \( H_0: \sigma_u^2 = 0 \) and \( H_1: \sigma_u^2 > 0 \). In the case of half-normal model hypothesis, if the test statistic \( z \) exceeds the critical value (regarding the specified level of significance), so we reject the null hypothesis that there are no inefficient effects (at the specified (%) level of significance). According to the parameterization of Aigner, Lovell and Schmidt (1977), the hypotheses become \( H_0: \lambda = 0 \) and \( H_1: \lambda > 0 \). In this parameterization, the test statistic is:

\[
 z = \frac{\hat{\lambda}}{se(\hat{\lambda})} \sim N(0, 1) \tag{28}
\]

where \( \hat{\lambda} \) is the maximum likelihood estimator of \( \lambda \) and \( se(\hat{\lambda}) \) is the estimator for its standard error.

However, in the literature, apart from the half-normal distribution (Aigner et al., 1977), there are several variations of the model allowing for different distributions of the terms \( v \) and \( u \), such as truncated distributions, exponential distributions, or two-parameters gamma distributions (Kalirajan and Shand, 1999). Therefore, it is not uncommon to replace the half-normality assumption \((u_{it}) \sim iidN(0, \sigma_u^2)\) with one of the following assumptions:

1. model where the inefficiency disturbance is specified as a truncated (at 0) normal distribution: \((u_{it}) \sim iidN(\mu, \sigma_u^2)\), as described by Stevenson (1980).

2. model where the inefficiency disturbance is specified as an exponential distribution with mean \( \lambda \): \((u_{it}) \sim iidG(\lambda, \theta)\), as described by Meuens and van den Broeck (1977).

3. model where the inefficiency disturbance is specified as a gamma distribution with mean \( \lambda \) and degrees of freedom \( m \): \((u_{it}) \sim iidG(\lambda, m)\), as described by Greene (1990).

Normal, half-normal and exponential distributions are arbitrary choices due to lack of a priori justification for selecting a particular distributions form for the technical inefficiency effects. The choice of distributions specification is sometimes a matter of computational convenience, since estimation of some frontier models is automated in some software packages, rather than others. To give a general idea regarding these extensions,
Nevertheless, even though the original stochastic frontier production function has been extended, the vast majority of applied papers involve the estimation of a single equation half-normal stochastic frontier, in which the model is expressed as the output of a firm as a function of its inputs plus a compound error (inefficiency and random terms) to test the null hypothesis that there is no technical inefficiency in the industry (Coelli et al., 2005).

5 Case-study: Public capital efficiency

In modern economic world, economic growth rate varies enormously among countries. The explanation of these differences in economic performance may be one important contribution to public and private policies towards efficiency and growth enhancement. The first step towards this is to decompose growth into its main components. Economic growth can be decomposed into two main components: increases in factor inputs (capital accumulation) and improvements in productivity. The first component attributes growth differences to differences in physical resources, physical capital, and labour. Notwithstanding, reducing differences in factor inputs is not sufficient to guarantee a proportional reduction in economic performance differences. The main reason is that productivity differences, the second component, may also play a determinant role in economic growth. Increases in productivity may be achieved through technical change (shifts on the production frontier) and through reductions in production inefficiency (movements towards the frontier).

In order to estimate the inefficiency effect, we apply a stochastic frontier approach to estimate technical efficiency using a Cobb-Douglas production function, incorporating inputs in terms of labor, private capital and public capital, in a case-study with real economic data.

Economic literature regarding estimation of public capital productive efficiency was initiated by Aschauer (1989)\(^6\). More specifically, Aschauer (1989) was the first economist who clearly expanded the general production function to include public capital as an additional input, using a Cobb-Douglas production function form. The central point of the analysis is a production function which incorporates the stock of public capital a time \(t\), \(G_t\), as an input in the production process. The production function becomes:

\[
q_t = A_t f [K_t, G_t, L_t] 
\]

or

\[
q_t = A_t K_t^\alpha G_t^\beta L_t^\gamma, \quad \alpha, \beta, \gamma > 0
\]

Taking natural logarithms on both sizes of , we get a linear function:

\(^6\)Before Aschauer (1989), there were also researchers who included public capital as a factor of production, along with labor and private capital, such as Mera (1973), Ratner (1983) and Biehl (1986), even though in a broad way.
\[
\ln q_t = \ln A_t + \alpha \ln K_t + \beta \ln G_t + \gamma \ln L_t
\]  
(31)

where:

1. \( q_t \) is the real aggregate output within some area (region or country)
2. \( A_t \) is an index of economy-wide productivity, representing the level of technology (Hicks-neutral technological progress)
3. \( K_t \) denotes the stock of (non-residential) private fixed capital
4. \( L_t \) denotes employment (measured by total hours worked, or numbers of employees)
5. \( \alpha = \frac{d \ln q_t}{d \ln K_t} \) is the elasticity of output with respect to private capital
6. \( \beta = \frac{d \ln q_t}{d \ln G_t} \) is the elasticity of output with respect to public capital
7. \( \gamma = \frac{d \ln q_t}{d \ln L_t} \) is the elasticity of output with respect to labor

In the analysis of public capital efficiency, Aschauer (1989) has been the first to include public capital into the production function, as one of the inputs, along with private capital and employment. Since then, in general terms, the same approach has been used in the majority of public capital efficiency analysis research. Munell (1990) and more recent approaches, such as, Mamatzakis (2003) and Ligthart and Suarez (2005) followed the same method, considering public capital as an input in a neoclassical production model.

Aschauer (1989) concluded that more infrastructure can improve the productivity and also attract new establishments in a specific market or industry. This argument was also extended by Biehl (1986) and Seitz and Licht (1995), who investigated the influence of public capital in the formation of private investments, finding a significantly positive effect. Furthermore, researchers on economic growth such as Krugman (1991), Fujita et al. (1999) and Venables (1999) relate public capital investments to market access, transportation costs, technological externalities and agglomeration economies, all sources of accelerating growth. Their main argument is that the positive and scale externalities related to public capital investments are generated by inter-industry links and stimulated by the improvements in the access to consumers market, reduction of the cost of transactions, facilitation of the access to specialized services, availability of infrastructure, like telecommunication and transport, and spillovers of knowledge.

According to Aschauer (1989), public investment refers to expanding and improving the stock of infrastructure in roads, airports, water and sewage facilities, public transport and other utilities. Aschauer (1989) considered public capital to include transport infrastructure, electrical and gas facilities, water systems and sewers and any other public investments. This investment increases the productivity of private capital, making private investment more profitable and
accelerating economic growth\textsuperscript{7}. In this study, due to certain data availability and for simplicity reasons, we will estimate a time-series stochastic frontier model considering the public capital efficiency in Greek economy.

6 Methodology

As broadly analysed above, stochastic frontier analysis examines the relationship between output and input levels, using two error terms. One error term is the traditional normal error term in which the mean is zero and the variance is constant. The other error term represents technical inefficiency and may be expressed as a half-normal, truncated normal, exponential, or two-parameter gamma distribution. Technical efficiency is subsequently estimated via maximum likelihood estimation of the production function subject to the two error terms. Within this framework, we attempt to implement this method and estimate the impact of private and public capital on economic growth at the national level using time-series data. To estimate the parameters of the production function and the parameters in the equation of the expected inefficiency, we use a time-series single-stage model to investigate the inefficiency effects in stochastic production frontiers, applying the Maximum Likelihood method proposed by Kumbhakar et al. (1991), Reifschneider and Stevenson (1991) and Battese and Coelli (1995). The study follows the general research idea that stochastic production frontier model allows: a) technical inefficiency and input elasticities to vary over time in order to detect changes in the production structure, and b) inefficiency effects to be a function of a set of explanatory variables the parameters of which are estimated simultaneously with the stochastic frontier.

Efficiency is measured by separating the efficiency component from the overall error term. Economy may be off frontier because it is inefficient or because of random shocks or measurement errors. The model uses real GDP as the output and total employment, private capital and public capital as inputs. The model allows inefficiency to vary over time, and inefficiency effects to be a function of the level and composition of investment capital, private and public. As in Puig–Junoy (2001), we consider the sum of all individual production units as a single production unit and we assume away differences between firms within each national industry.

We assume a translog Cobb–Douglas production frontier function for Greek economy, covering years 1960–2001, with a data set of 42 annual observations. The data set is a time-series and the distribution chosen for the inefficiency component is the half-normal production. The estimation of the stochastic frontier is applied using the Maximum Likelihood (M.L.) method. With M.L.

\textsuperscript{7}The same approach is followed by recent research. See for example, Benos and Karagiannis (2008), who specified public capital as the tangible capital stock owned by the public sector, excluding military structures and equipment. More specifically, they also considered public capital to include investments in roads, railways, airports, and utilities such as sewerage and water facilities, hospitals, educational buildings, and the rest of public investment.
estimation, we choose parameters so as to maximize probability that observed sample of data is generated by a hypothesized process. Finally, we test the inefficiency hypothesis.

The analysis presented below is carried out using LIMDEP (Econometric Software). LIMDEP computes parameter estimates for the single equation variants of the stochastic frontier model. The log-likelihood functions for these models must be maximized using iterative optimization routines.

7 Data description

The application implements a stochastic frontier analysis based on real data from the Greek economy as case-study. The public sector comprises the general government and non-financial public corporations, such as public administration and defense services, compulsory social security services, public administration, educational services, and health care and social assistance, provided by the government (no private provision included) plus investment in infrastructure provided by public organizations.

The data were extracted by the OECD Analytical database and the National Statistical Service of Greece. Monetary values are evaluated at billions of national currency at 1995 prices. Total employment is evaluated at number of employees. Variables considered are expressed in the logarithmic form.

8 Model Application

According to methodology described above, we consider a Cobb-Douglas stochastic frontier production function, in the form:

\[ q_t = A_t \int f [K_t, G_t, L_t] \times \exp(v_{it}) \times \exp(-u_{it}) \]  \hspace{1cm} (32)

As in the case study by Coelli et al. (2005), for simplicity reasons, we will also use three inputs (private capital stock, public capital stock, labor). Apart from these three inputs, we additionally use a constant term, as well as a time variable, in order to include any technological change effects on production process. We transform the model variables, set as the natural logarithms (ln) of the initial variables and we estimate the model, under the assumption it is a half-normal frontier. In the following step, via LIMDEP software program, we completed a number of iterations in order to estimate the half-normal frontier model through Maximum Likelihood estimation.

As far as the hypothesis testing is concerned, the usual test in the analysis of stochastic frontiers is testing for the absence of inefficiency effects. As stated before, in the case of the half-normal models, the null hypothesis is a single restriction involving a single parameter. Since our model has been estimated using the method of Maximum Likelihood, we can test such an hypothesis using the simple z-test.

\footnotetext[8]{The same definition regarding public sector capital is also followed by Kamps (2004).}
In the half-normal model, the null and alternative hypotheses are:

1. Null Hypothesis: $H_0: \sigma_u^2 = 0$ (meaning that there are in inefficiency effects), and

2. Alternative Hypothesis: $H_1: \sigma_u^2 > 0$ (meaning that there are inefficiency effects).

Following the parameterization of Aigner, Lovell and Schmidt (1977), the hypotheses of the model become $H_0: \lambda = 0$ and $H_1: \lambda > 0$, respectively.

The values $\hat{\lambda}$ is estimated by the model to be 4.102 (the maximum likelihood estimator of $\lambda$) and the value $se(\hat{\lambda})$ is estimated by the model to be 4.290 (estimator for the standard error of the maximum likelihood estimator of $\lambda$). In this model parameterization, the test statistic $z = \frac{\hat{\lambda}}{se(\hat{\lambda})} \sim N(0, 1)$ becomes $z = \frac{4.102}{4.290} = 0.956$.

In the case of half-normal model hypothesis testing, the test statistic $z = 0.956$ is less than the critical value of $z_{0.95} = 1.645$ (level of significance 95%), so we cannot reject the null hypothesis that there are no inefficient effects. Therefore, we assume that all the efficiency deviations from the stochastic frontier are due to measurement errors and effects beyond the control of the producers (in this case - study, the economy). The following step is to divide public capital stock into different spending priorities, in order to estimate the particular efficiency of each one of these in public capital efficiency level.

9 Conclusion

The study was primarily motivated by the idea that deviations from the production frontier may not be entirely under the control of the production unit itself. As Kumbhakar and Lovell (2000:72) indicated, “the great virtue of stochastic production frontier models is that the impact on output of shocks due to variation in labor and machinery performance, vagaries of the weather, and just plain luck can at least in principle be separated from the contribution of variation in technical efficiency”.

Within this framework, measuring efficiency and productivity is a quite important task in economic analysis. First only by measuring efficiency and productivity, and by separating their effects from those of the general economic environment, can we explore hypotheses concerning the sources of efficiency or productivity differentials, as well as effectiveness of private practices and public policies designed to improve productive performance. Furthermore, efficiency and productivity measures are success indicators, by which producers are evaluated, so for the most efficient measurers to be taken, since productivity growth leads to improved economic and financial performance. Moreover, macro performance depends on micro performance and so the same reasoning applies to the study of the growth of nations (Lewis, 2004).
References


