Liquidity Problems and Growth Implications,
a reconsideration of pension schemes role

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Abstract. We present a Schumpeterian growth model with overlapping generations. Consumers face liquidity shocks, which affect savings decision and capital accumulation. After introducing endogenous technological progress, these constraints are transferred to R&D investments. The research question addressed in this work is whether the design of a pension scheme matters in presence of liquidity problems. We show that it does matter and that it has growth implications.

Keywords. Schumpeterian growth - Overlapping Generations - Liquidity Shocks - Social Security

JEL Classification. D90, E21, H55, O40

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1 Introduction

Social security is one of the main source of government expenditure and it plays an important societal and economic role. There are various justifications for the existence of public forms of social security: some of them are more paternalistic, e.g. individuals are short-sight and do not save enough for their retirement period, others are more technical, justifying government intervention sustaining the inability of decentralized markets to reach Pareto efficiency. Whatever are the possible reasons or justifications, the introduction or modification of a social security program has important effects on saving decision, capital accumulation and economic growth. The aim of this paper is to shed some light on social security implications in particular on long term potential growth, hopefully helping in designing better schemes.
The seminal paper Samuelson (1975) constitutes our starting point. He develops a two periods life-cycle model, where there are two different generations overlapping. The main results found are that pay-as-you-go social security may restore dynamic efficiency reducing capital accumulation, while a fully funded scheme seems not to have any effects. This simple model is extremely useful to perform some macroeconomic analysis, specially about social security, this is why in the next section we briefly remind the construction of this model. This should help in grasping the differences between this standard framework and our analysis.
The first step we make is to consider liquidity shocks when an individual is young. Instead of including possible market imperfections, which would have automatically left room for policy intervention, we introduce consumption/liquidity uncertainty. This part of the model echoes Diamond and Dybvig (1983), in particular an extension in continuous time developed in von Thadden (1998). Ex-ante it is not obvious that social security may interfere with this economy bringing new results with respect to Samuelson (1975). This is because individuals may form a bank that provides insurances against the liquidity shocks. If we suppose that the government, which provides also social security, guarantees the deposits, we rule out the possibility of bank runs. Thus, the optimal allocation of resources is restored even in presence of these shocks. Deriving the savings flow solving the problem from a bank perspective (offering a contract where an agent can withdraw a certain quantity of deposits and determining his consumption level) leads to the same dynamic inefficiencies conditions fund by Samuelson (1975). The novelty is that fully funded social security increases consumption and reduces capital stock, even if less than a pay-as-you-go system. The different impact of the two social security regimes implies interesting policy implication, specially in terms of growth implications. We show how the transaction from a pay-as-you-go to a fully funded social pension increases potential growth and possibly restore dynamic efficiency.
In our model an individual faces two different optimization problems, one that is subject to liquidity problems when young and one when he retires. Part of the problem he has to solve is computing the optimal amount of savings he needs when he retires, in order to have enough resources to consume. This approach leads to a two-stage optimal control problem. For this type of optimization, the optimality conditions have been studied and developed for the first time in Tomiyama (1985). Two-stage optimal control techniques allow us to model switching regimes. This is useful to model the young and old period of life, which have different budget constraints and maximization problems, but being nevertheless part of the same consumption-savings problem. To the limits of our knowledge this paper is one of the few making use of this optimization technique. Another example can be found in Boucekkine, Raouf and Saglam, Cagri and Vall Ee (2004).
In the second step we extend our considerations to an endogenous growth model. This is particularly interesting since, as underlined in Saint-Paul (1992), when we try to endogenize growth,

designing a social security scheme which restores Pareto optimum is not possible anymore. This happens because it lowers economic growth, reducing consumers welfare. A possible solution that he suggests is to use investment subsidies. Instead we show that once capital can be used also in an R&D sector, social security may be Pareto improving.
In order to obtain endogenous growth we start from a particular version of the Schumpeterian growth model developed in Aghion and Howitt (2007). In that paper the authors develop a model which is a meeting point between neoclassical growth model of capital accumulation and Schumpeterian one, e.g. Romer (1990), Aghion and Howitt (1992). Their purpose is to show different interpretations of the same accounting data. Instead this model is useful for our purposes since it stresses the importance of capital as growth component, giving us an ideal setting in which we can extend our social security analysis. In our model savings and capital dynamics come from the overlapping generation model with liquidity shocks. Capital can be used in the production sector but also in R&D, in this way the constraints from the individual liquidity problems affects innovation and economic growth. We show that there exists an optimal level of capital, and the it is possible to use social security to achieve this level when the *laissez-faire* output is inefficient.
Again the story is simple: social security changes savings decision and capital accumulation, and when capital plays a role in production and innovation we have also economic growth effects. These should not be underestimate, since social security may become a powerful instrument also to restore optimal growth.
In order not to over-complicate the model, three decisions has been taken: (i) despite the structure presented is general enough to be adapted to study different optimal deposit contracts, allowing also the possibility of bank runs, we transcend from these considerations; (ii) we assume that all the agents are forced to retire at a certain time $t = 1$, but the optimal retirement time may be endogeneized since the two-stage optimal control techniques allow this possibility; (iii) the Schumpeterian framework offers good connections also with Industrial Organization theory, but these are not exploited here. All these elements are certainly interesting, but we do not want to loose our focus on the macroeconomic effects of social security. But, they show the extreme flexibility of the model we propose in this paper, and the possible directions of future research.
Our analysis makes use of a logarithm utility function, mainly for simplicity of exposition. It is possible to generalize our results for isoelastic utility functions without affecting the conclusions. Furthermore, whenever the risk aversion parameter is greater than one, there are more similarities with the model of Diamond-Dybvig since the consumption offered by the bank will be affected, and individuals will consumer higher when they withdraw before the end of the period. For more of these considerations see Diamond and Dybvig (1983).
The paper is structured as follows: in Section 2 we present the standard overlapping generation model showing also the results of Samuelson (1975). In 3 we show how starting from the simple overlapping generation model we can introduce liquidity shocks, drawing some new results about pension effects. In Section 4 we extend our results to the Schumpeterian framework already mentioned, and finally in 5 some conclusions are drawn.

2 Framework

2.1 The standard model

In this basic economy time or generation τ is a discrete variable and goes from 0 to ∞, with $\tau \in \mathbb{N}$. The population of generation τ is composed by L_τ individuals. Each individual $i \in L_\tau$ lives for two periods. In their first period of life agents are young, they are endowed with one unit of labor l which they supply inelastically in a competitive labor market, earning a real wage ω_τ. In the second period they are old and they retire. The life-cycle utility function U of each τ-individual is assumed to be separable and of the form

$$U_t(c_{1,\tau}, c_{2,\tau}) = \ln(c_{1,\tau}) + e^{-\rho} \ln(c_{2,\tau}) \tag{1}$$

where $\rho > 0$ is the discount rate, $c_{1,\tau}$ and $c_{2,\tau}$ denote respectively the consumption of an individual born at τ when young and when old. Population grows at a constant rate $n \in (-1, \infty)$, i.e. $L_\tau = e^{n\tau} L_0$ and we normalize the size setting $L_0 = 1$.
Firms produce an homogeneous good in a competitive framework, employing labor L_τ and capital K_τ. In the Cobb-Douglas production function, with capital elasticity equal to α, the production technology, A, for the moment, does not change. For simplicity we assume that the capital stock fully depreciates after its use. Hence the total production of our economy is

$$Y_\tau = F(K_\tau, L_\tau) = A(K_\tau)^\alpha (L_\tau)^{1-\alpha} \tag{2}$$

Because of the constant return to scale, we can rewrite the production function as $F(K_\tau, L_\tau) = L_\tau F(K_\tau/L_\tau, 1) \equiv L_\tau f(k_\tau)$, where we define the per capita level of capital $K\tau/L_\tau \equiv k_\tau$ and the per capita production function $F(K_\tau/L_\tau, 1) \equiv f(k_\tau)$. Since markets are competitive, marginal returns equal marginal costs in equilibrium. We can write the gross return of capital at τ as $R_\tau = e^{r_\tau} R_0$, where $r_\tau > 0$ is the rate of return of capital at τ, we assume that $R_0 = 1$.

2.2 Consumption-saving problem

In each generation τ every young individual allocates his resources ω_τ between the two periods of life, deciding how much to consume and save. Write s_τ the quantity of resources saved by a τ-individual, these are rent as capital to firms, receiving as return $R_{\tau+1}$ the next period, after that production has taken place. The consumption-saving maximization problem is

$$\begin{aligned} \max_{c_{1,\tau}, c_{2,\tau}, s_\tau} \quad & \ln(c_{1,\tau}) + e^{-\rho} \ln(c_{2,\tau}) \\ \text{subject to} \quad & c_{1,\tau} + s_\tau \leq \omega_\tau \\ & c_{2,\tau} \leq R_{\tau+1} s_\tau \end{aligned} \tag{3}$$

From the first order conditions we obtain

$$u'(\omega_\tau - s_\tau) = e^{-\rho} R_{\tau+1} u'(R_{\tau+1} s_\tau) \tag{4}$$

concavity guarantees that the first order conditions are necessary and sufficient to find the optimum level of savings $s_\tau = \omega_\tau/(1 + e^{-\rho})$. Adding the wage equilibrium condition ($\omega_\tau = (1 - \alpha)k_\tau^\alpha$), from $K_{\tau+1} = L_\tau s_\tau$ we obtain the unique steady state of capital/labor ratio

$$k_{eq} = \frac{\omega_\tau}{e^n(1+e^{-\rho})} = \left[\frac{(1-\alpha)}{e^n(1+e^{-\rho})}\right]^{\frac{1}{1-\alpha}} \tag{5}$$

A competitive equilibrium of the overlapping generation economy presented is defined as follows

Definition. *A competitive equilibrium is a series of factor prices, $\{R_\tau, \omega_\tau\}_{\tau=0}^{\infty}$, aggregate variables, $\{K_\tau, L_\tau, Y_\tau\}_{\tau=0}^{\infty}$, and individual variables, $\{c_{1,\tau}, c_{2,\tau}, s_\tau\}_{\tau=0}^{\infty}$, such that optimality conditions, 4, 5, are satisfied.*

2.3 Welfare

To study the welfare properties of this overlapping generation model we compare its results with the choice of a social planner maximizing a weighted average of every generation's utility, under the resources constraints, $F(K_\tau, L_\tau) = K_{\tau+1} + c_{1,\tau}L_\tau + c_{2,\tau-1}L_{\tau-1}$

$$\begin{aligned} \max_{c_{1,\tau},c_{2,\tau}} \quad & \sum_{\tau=0}^{\tau=\infty} e^{-\rho_p \tau}\left(\ln(c_{1,\tau}) + e^{-\rho}\ln(c_{2,\tau})\right) \\ \text{subject to} \quad & f(k_\tau) = e^n k_{\tau+1} + c_{1,\tau} + e^{-n}c_{2,\tau-1} \end{aligned} \tag{6}$$

where ρ_p is the social planner discount rate. The first order conditions are as in 4, thus consumption shares are allocated in the same way. Although the social planner problem takes into account the effects of capital accumulation on future generations, and competitive equilibrium over-accumulates capital whenever $r < n$, being dynamically inefficient and Pareto sub-optimal.

2.4 Pensions

Suppose that the government taxes young of generation τ by an amount ψ_τ. In a fully-funded social security program government invests these funds in the productive asset of the economy, the capital stock, leaving the returns to the workers. Instead if the government sets up a pay-as-you-go scheme, considering the population growth, it will distribute to each old agent a benefit $e^n\psi_\tau$. The new individual maximization problem becomes

$$\begin{aligned} \max_{c_{1,\tau},c_{2,\tau},s_\tau} \quad & \ln(c_{1,\tau}) + e^{-\rho}\ln(c_{2,\tau}) \\ \text{subject to} \quad & c_{1,\tau} + s_\tau + \psi_\tau \le \omega_\tau \\ & c_{2,\tau} \le R_{\tau+1}(s_\tau + \psi_\tau) \text{ if FF pension scheme} \\ & c_{2,\tau} \le R_{\tau+1}s_\tau + e^n\psi_\tau \text{ if PAYG pension scheme} \end{aligned} \tag{7}$$

Samuelson (1975) shows that a fully funded social security program actually does not affect the dynamic in presence of perfect financial markets, since the total amount invested is not affected. Instead a pay-as-you-go system reduces capital stock, since s_τ is lower than the case without social security for any $\psi_\tau > 0$.

Theorem. ***Samuelson (1975)***
Provided that capital markets are perfect, the fully funded pension scheme do not affect capital accumulation, consumption and welfare. Instead, a pay-as-you-go scheme decreases capital stock, and it may increase consumption and welfare in presence of over-accumulation of capital.

3 Introducing liquidity problems

3.1 Set up

Let divide each of the two periods of life in an infinite number of sub-periods t. When an agent is young, he faces a liquidity shock, such that his demand for consumption is singular at a certain t in the first period, without knowing which one in advance. Write $F : (0,1] \to [0,1]$ the cumulative distribution function of these i.i.d. shocks, with $F'(t) \equiv f(t)$ and $F(1) > F(t)$ for any $t \in [0,1)$. In the second period individuals simply smooth consumption over the sub-periods.
Investment return is increasing in t with cost-less liquidation, and the gross return function $R_\tau(t) = e^{r_\tau t}$ still has $r_\tau > 0$ as rate of return at τ.
These changes from the previous framework lead to a two stage control problem, with two budget constraints, one for each period of life. Since we have sub-periods, we specify also the sub-period for each variable, i.e. $c_{1,\tau}(t)$ is the consumption of a young agent of generation τ at t. Therefore the new consumption-saving problem is

$$\begin{aligned}
\max_{c_{1,\tau}(t), c_{2,\tau}(t)} \quad & \int_0^1 \ln\left(c_{1,\tau}(t)\right) f(t) dt + \int_1^2 e^{-\rho t} \ln\left(c_{2,\tau}(t)\right) dt \\
\text{subject to} \quad & ds_\tau(t)/dt = r_\tau s_\tau(t) - c_{1,\tau}(t) f(t) \text{ for } t \in [0,1] \\
& ds_\tau(t)/dt = r_\tau s_\tau(t) - c_{2,\tau}(t) \text{ for } t \in [1,2] \\
& s_\tau(0) = \omega_\tau, \qquad s_\tau(2) = 0
\end{aligned} \tag{8}$$

For $t \in [0,1]$ the individual faces a liquidity shock. This structure is an extension of the famous Diamond-Dybvig model in continuous time, introduced in von Thadden (1998). Instead for $t \in [1,2]$ the agent retires and consumes his savings. From the different construction of the two periods, we have the two differential equations which are also the individual resource constraints. Each agent still receives a wage ω_τ at the beginning of the first period, and we impose a no waste condition, giving us the starting and terminal points.
To solve this two stage control problem, we divide it in three steps: (I) solve the old period taking $s_\tau(1)$ as given and knowing that $s_\tau(2) = 0$; (II) solve the young period with free $s_\tau(1)$ and $s_\tau(0) = \omega_\tau$; (III) compute the optimal savings stock when an individual retires $s_\tau(1)$.
(I) Solve for $t \in [1,2]$
The Problem 8 for $t \in [1,2]$, reduces to

$$\begin{aligned}
\max_{c_{2,\tau}(t)} \quad & \int_1^2 e^{-\rho t} \ln\left(c_{2,\tau}(t)\right) dt \\
\text{subject to} \quad & ds_\tau(t)/dt = r_\tau s_\tau(t) - c_{2,\tau}(t) \\
& s_\tau(1) \text{ given}, s(2) = 0
\end{aligned}$$

The corresponding Hamiltonian of this problem is $\mathcal{H} : -e^{-\rho t} \ln\left(c_{2,\tau}(t)\right) + \lambda_2(t)\left(r_\tau s_\tau(t) - c_{2,\tau}(t)\right)$, and the first order necessary (and sufficient) conditions are

$$ds_\tau(t)/dt = r_\tau s_\tau(t) - c_{2,\tau}(t)$$

$$d\lambda_2(t)/dt = -\lambda_2(t) r_\tau$$

$$\lambda_2(2) \leq 0, \qquad s_\tau(2)\lambda_2(2) = 0$$

From the integration of these conditions we obtain

$$s_\tau(t) = s_\tau(1)e^{r_\tau t - r_\tau 1} + \frac{e^{r_\tau t}a}{\rho}(e^{-\rho t} - e^{-\rho}) \tag{9}$$

$$c_{2,\tau}(t) = ae^{(r_\tau - \rho)t}$$

$$\lambda_2(t) = -\frac{1}{a}e^{-r_\tau t}$$

Where a is a constant, which we need to solve. From the terminal condition $s_\tau(2) = 0$

$$s_\tau(1)e^{2r_\tau - r_\tau} + \frac{e^{2r_\tau}a}{\rho}(e^{-2\rho} - e^{-\rho}) = 0$$

and hence

$$a = \frac{\rho s_\tau(1)}{e^{r_\tau}(e^{-\rho} - e^{-2\rho})}$$

We can compute the optimal consumption value when an individual is old as a function of the savings stock when he retires

$$U_2 * (s_\tau(1)) \equiv \int_1^2 e^{-\rho t} \ln\left(\frac{\rho s_\tau(1)}{e^{r_\tau}(e^{-\rho} - e^{-2\rho})}e^{(r_\tau - \rho)t}\right) dt$$

(II) Solve for $t \in [0, 1]$

The Problem 8 for $t \in [0, 1]$, reduces to

$$\begin{array}{ll} \max\limits_{c_{1,\tau}(t)} & \int_0^1 \ln(c_{1,\tau}(t)) f(t)dt + U_2* \\ \text{subject to} & ds_\tau(t)/dt = r_\tau s_\tau(t) - c_{1,\tau}(t)f(t) \\ & s_\tau(0) = \omega_\tau, s_\tau(1) = free \end{array} \tag{10}$$

The corresponding Hamiltonian of this problem is $\mathcal{H} : -\ln(c_{1,\tau}(t)) f(t) + \lambda_1(t)(r_\tau s_\tau(t) - c_{1,\tau}(t)f(t))$, and the first order necessary (and sufficient) conditions are

$$ds_\tau(t)/dt = r_\tau s_\tau(t) - c_{1,\tau}(t)f(t)$$

$$d\lambda_1(t)/dt = -\lambda_1(t)r_\tau$$

$$s_\tau(0) = \omega_\tau$$

If we try to integrate the first condition we obtain $s_\tau(t) = e^{r_\tau t}\left(s_\tau(0) - \int_0^t e^{-r_\tau z} \ln c_{1,\tau}(z)f(z)dz\right)$. The integral causes some computation problems, without allowing us to identify a well defined solution. Instead, following the approach of von Thadden (1998), define $x_\tau(t)$ the per capita share of

resources that can be withdrawn from a bank where each young agent deposits his savings. It leads to a consumption $c_{1,\tau}(t) = x_\tau(t)R_\tau(t)\omega_\tau$. The new equation describing the savings dynamic is

$$ds_\tau(t)/dt = r_\tau s_\tau(t) - x_\tau(t)R_\tau(t)f(t)\omega_\tau$$

Integrating this last expression

$$s_\tau(t) = e^{r_\tau t} s_\tau(0)\left(1 - \int_0^1 x_\tau(z)f(z)dz\right) \tag{11}$$

Problem 10 can be restated as

$$\begin{array}{ll} \max\limits_{c_{1,\tau}(t)} & \displaystyle\int_0^1 \ln\left(x_\tau(t)R_\tau(t)\right) f(t)dt + U_2* \\ \text{subject to} & x_\tau(t) \text{ integrable on } [0,1] \\ & \displaystyle\int_0^1 x_\tau(t)f(t)dt\omega_\tau = \frac{s_\tau(0)}{R_\tau(0)} - \frac{s_\tau(1)}{R_\tau(1)} = \omega_\tau - s_\tau(1)e^{-r_\tau} \end{array}$$

From von Thadden (1998) we know: (i) each agent deposits his savings whenever $x_\tau^*(t) > 1$ (alone each agent faces the liquidity shock, and his expected utility is $\int_0^1 \ln\left(R_\tau(t)\right) f(t)dt$ which is the case when $x_\tau(t) = 1$); (ii) the problem in this form has as solution $x_\tau^*(t) = b'$, where b' is a constant. Thus 11 becomes

$$s_\tau(t) = e^{r_\tau t} s_\tau(0)\left(1 - b'\int_0^1 f(z)dz\right) = e^{r_\tau t}\left(s_\tau(0) - bF(t)\right) \tag{12}$$

where $b = b's_\tau(0)$. Going back to the first order conditions of the Hamiltonian, we can also compute

$$c_{1,\tau}(t) = be^{r_\tau t} \tag{13}$$

$$\lambda_1(t) = -\frac{1}{b}e^{-r_\tau t}$$

From the matching condition (or continuity) we should have that $\lambda_1(1) = \lambda_2(1)$, which implies

$$b = \frac{\rho s_\tau(1)}{e^{r_\tau}\left(e^{-\rho} - e^{-2\rho}\right)} \tag{14}$$

Combining 14 and 13, we have the optimal consumption in the first period as a function of of the savings stock when he retires, so now we need to compute $s_\tau(1)$.

(III) Compute $s_\tau(1)$

From the previous step we can combine 12 and 14, at $t = 1$

$$s_\tau(1) = e^{r_\tau}\left(s_\tau(0) - \frac{\rho s_\tau(1)}{e^{r_\tau}\left(e^{-\rho} - e^{-2\rho}\right)}F(1)\right)$$

re-arranging and since $F(1) = 1$, we obtain

$$s_\tau(1) = \frac{e^{r_\tau}\left(e^{-\rho} - e^{-2\rho}\right)}{\rho + e^{-\rho} - e^{-2\rho}}s_\tau(0) \tag{15}$$

where $s_\tau(0) = \omega_\tau$ is given.
This gives the optimal consumption levels and the savings when young (depending on t) in a *laissez-faire* economy

$$c_{1,\tau}(t) = \frac{\rho e^{r_\tau t}}{\rho + e^{-\rho} - e^{-2\rho}}\omega_\tau \qquad c_{2,\tau}(t) = \frac{\rho e^{(r_\tau - \rho)t}}{\rho + e^{-\rho} - e^{-2\rho}}\omega_\tau \tag{16}$$

$$s_\tau(t) = e^{r_\tau t}\left(1 - \frac{\rho}{\rho + e^{-\rho} - e^{-2\rho}}F(t)\right)\omega_\tau \tag{17}$$

There are two generations overlapping at each t, L_τ agents old with savings $s_\tau(t)$ following 9, and $e^n L_\tau$ agents young with savings $s_{\tau+1}(t)$ expressed by 12. Since the old savings must be kept liquid to finance consumption, the law of motion of capita is simply given by $K_{\tau+1}(t) = L_\tau s_\tau(t)$. In per capita terms and with 17, we obtain

$$k_{\tau+1}(t) = e^{r_\tau t - n}\left(1 - \frac{\rho}{\rho + e^{-\rho} - e^{-2\rho}}F(t)\right)\omega_\tau$$

Because of the shocks during time t, we are looking for optimal conditions across generations. Markets are still competitive, so wages should be equal to marginal productivity of capital. Thus we can compute the steady state value across generations of the capital-labor ratio in a *laissez-faire* economy

$$k_{eq}(t) = \left[(1-\alpha)e^{rt-n}\left(1 - \frac{\rho}{\rho + e^{-\rho} - e^{-2\rho}}F(t)\right)\right]^{\frac{1}{1-\alpha}} \tag{18}$$

A competitive equilibrium is still defined as in the previous section.

3.2 Welfare

As before, to study the welfare properties we compare the competitive equilibrium results with the choice of a social planner maximizing a weighted average of the utilities of every generation, under the resources constraints.

$$\begin{aligned} &\max_{c_{1,\tau}(t), c_{2,\tau}(t)} && \sum_{\tau=0}^{\tau=\infty} e^{-\rho_p \tau}\left(\int_0^1 \ln\left(c_{1,\tau}(t)\right) f(t)dt + \int_1^2 e^{-\rho t}\ln\left(c_{2,\tau}(t)\right)dt\right) \\ &\text{subject to} && f(k_\tau) = e^n k_{\tau+1} + \int_0^1 c_{1,\tau}(t)f(t)dt + e^{-n}\int_1^2 c_{2,\tau-1}(t)dt \end{aligned} \tag{19}$$

The first order conditions of the social planner problems are the same as in the individual one, but again the problems are structured in a similar way, thus there is not certain that the competitive equilibrium is equal to the optimal one. Define the total consumption at τ as $C_\tau \equiv \int_0^1 c_{1,\tau}(t)f(t)dt + e^{-n}\int_1^2 c_{2,\tau-1}(t)dt$. From the budget constraint we have $C_\tau = f(k_\tau) - e^n k_{\tau+1}$, thus the equilibrium level of capital-labor ratio maximizing consumption is $f'(k) = e^n$. Again the golden rule of capital requires $r = n$, which may not be the case. Dynamic inefficiencies arise also in presence of liquidity shocks.

3.3 Pensions under liquidity problems

Whenever we introduce a social security this may affect savings decision and capital accumulation in two ways: reducing for every generation $s_\tau(0)$, since wages are taxed, and increasing $s_\tau(1)$. In both pension schemes government taxes young of generation τ of an amount ψ_τ, and we simply have $s_\tau(0) = \omega_\tau - \psi_\tau$. The differences are in $s_\tau(1)$ where the government gives back what has taxed with the returns if we are in a fully funded case, or where it gives a benefit $b_\tau = e^n\psi_\tau$ in a pay-as-you-go case.

Proposition 1. *Under liquidity problems:*

- *A fully funded social security is not neutral. It decreases capital stock and increases consumption independently from the value of r and n;*
- *A pay-as-you-go scheme decreases capital stock and increases consumption in presence of over-accumulation of capital $r < n$;*
- *A pay-as-you-go scheme decreases capital stock more than a fully-funded pension one.*

Proof. In order to prove the three points, let first compute $s_\tau^{ff}(1) = s_\tau(1) + e^{r_\tau}\psi_\tau$ and $s_\tau^{payg}(1) = s_\tau(1) + e^n\psi_\tau$.
In case of a fully funded pension scheme, from 15 we have $s_\tau^{ff}(1) = s_\tau(1)+e^{r_\tau}\psi_\tau = \frac{e^{r_\tau}(e^{-\rho}-e^{-2\rho})}{\rho+e^{-\rho}-e^{-2\rho}}(\omega_\tau - \psi_\tau) + e^{r_\tau}\psi_\tau$, or rearranging

$$s_\tau^{ff}(1) = s_\tau(1) + e^{r_\tau}\psi_\tau = \frac{e^{r_\tau}(e^{-\rho} - e^{-2\rho})\omega_\tau + \rho e^{r_\tau}\psi_\tau}{\rho + e^{-\rho} - e^{-2\rho}} \tag{20}$$

In case of a pay-as-you-go pension scheme we obtain $s_\tau^{payg}(1) = s_\tau(1)+e^n\psi_\tau = \frac{e^{r_\tau}(e^{-\rho}-e^{-2\rho})}{\rho+e^{-\rho}-e^{-2\rho}}(\omega_\tau - \psi_\tau) + e^n\psi_\tau$, or rearranging

$$s_\tau^{payg}(1) = s_\tau(1) + e^n\psi_\tau = \frac{e^{r_\tau}(e^{-\rho} - e^{-2\rho})\omega_\tau + \left(e^n(\rho + e^{-\rho} - e^{-2\rho}) - e^{r_\tau}(e^{-\rho} - e^{-2\rho})\right)\psi_\tau}{\rho + e^{-\rho} - e^{-2\rho}} \tag{21}$$

Let now analyze the impact on consumption of the two social security scheme. From the two-stage optimal control problem, we can derive

$$c_{1,\tau}(t) = \frac{\rho e^{r_\tau t}}{e^{r_\tau}(e^{-\rho} - e^{-2\rho})} s_\tau(1) \qquad c_{2,\tau}(t) = \frac{\rho e^{r_\tau t}}{e^{(r_\tau-\rho)t}(e^{-\rho} - e^{-2\rho})} s_\tau(1)$$

The introduction of pension changes $s_\tau(1)$ in $s_\tau^{ff}(1)$ or $s_\tau^{payg}(1)$ depending on the type of social security. Since that $s_\tau^{ff}(1) > s_\tau(1)$ and $s_\tau^{payg}(1) > s_\tau(1)$ in case of $n > r$, consumption increases in both periods after the introduction of social security.
About capital implications, again from the two-stage optimal control problem we have

$$s_\tau(t) = e^{r_\tau t}\left(\omega_\tau - \psi_\tau - \frac{\rho s_\tau(1)}{e^{r_\tau}(e^{-\rho} - e^{-2\rho})}F(t)\right)$$

As for consumption, the introduction of pension changes $s_\tau(1)$ in $s_\tau^{ff}(1)$ or $s_\tau^{payg}(1)$, obtaining

$$s_\tau^{ff}(t) = e^{r_\tau t}\left(1 - \frac{\rho}{\rho + e^{-\rho} - e^{-2\rho}}F(t)\right)\omega_\tau - e^{r_\tau t}\psi - \left(\frac{\rho}{e^{-\rho} - e^{-2\rho}} - \frac{\rho}{\rho + e^{-\rho} - e^{-2\rho}}\right)F(t)\psi_\tau$$

$$s_\tau^{payg}(t) = e^{r_\tau t}\left(1 - \frac{\rho}{\rho + e^{-\rho} - e^{-2\rho}}F(t)\right)\omega_\tau - e^{r_\tau t}\psi - \left(\frac{\rho e^{r_\tau - n}}{e^{-\rho} - e^{-2\rho}} - \frac{\rho}{\rho + e^{-\rho} - e^{-2\rho}}\right)F(t)\psi_\tau$$

where $s_\tau(t) > s_\tau^{payg}(t)$ and $s_\tau(t) > s_\tau^{ff}(t)$. It is clear that in a pay-as-you-go system capital stock decreases, $k_{\tau+1} = e^{-n}s_\tau^{payg}(t)$, but this is the case also with fully funded social security, $k_{\tau+1} = e^{-n}(s_\tau^{ff}(t) + e^{r_\tau t}\psi)$.
Finally, from this last reasoning it is straightforward that pay-as-you-go decreases capital more than a fully funded pension scheme.

□

The effects on consumption are different from the standard model because of the presence of liquidity uncertainty. What remains to do is to analyze the welfare implications.

Proposition 2. *Under liquidity problems, the introduction of a social security program may increase welfare and restore dynamic efficiency in case of over-accumulation of capital*

Proof. Starting from Equation 5 and imposing $r = n$, we obtain the golden capital labor ratio level k_g. Using the Second Welfare Theorem argument we know that this level of capital is Pareto Optimal. A fully funded social security decreases capital stock, thus in case of over-accumulation ($r < n$ or $k_{eq}(t) > k_g$), affecting savings, it restores $k_{eq}(t) = k_g$ and with it efficiency. A similar discussion can be done for pay-as-you-go pension. □

4 Schumpeterian Growth

4.1 New production function

Individual consumption-saving problem is as in the previous section. Production takes place in every period τ. There is a final good, used as numeraire, which can be stored as capital. Wages are paid in final good. There are also intermediate goods, i, whose number is normalized to 1. Define $y_{i,\tau}^\alpha$ the flow of intermediate input i used at τ. They are produced with capital, while the final good is produced using intermediate goods and labor, still supplied inelastically by each individual. The new production function is

$$Y_\tau = \int_0^1 A_{i,\tau} y_{i,\tau}^\alpha di L_\tau^{1-\alpha} \qquad 0 < \alpha < 1 \tag{22}$$

where $A_{i,\tau}$ is the productivity variable, a measure of the quality of the input.
Production of intermediate input counts the fact that improved technologies for the intermediate production should be increasingly capital-intensive. Write $k_{i,\tau}$ the capital-labor ratio input in sector i at τ, then the intermediate input production function is simply

$$y_{i,\tau} = \frac{k_{i,\tau}}{A_{i,\tau}} \tag{23}$$

Each intermediate sector i is monopolized by a firm, which borrows capital from the bank at the beginning of each period τ. The production process needs exactly one period, it means that the firm pays $R_\tau(1) = e^{r_\tau}$. The marginal cost faced by the monopolist is $R_\tau(1)A_{i,\tau}$, since $A_{i,\tau}$ measures the quantity of capital necessary to produce of unit of intermediate good. Taking the derivative of Equation 22, we obtain the demand curve

$$p_{i,\tau} = dY_\tau/dy_{i,\tau} = \alpha A_{i,\tau} y_{i,\tau}^{\alpha-1}$$

Therefore, the profits $\pi_{i,\tau}$ of a monopolist at τ are $p_{i,\tau}y_{i,\tau} - e^{r_\tau}k_{i,\tau} = \alpha A_{i,\tau}y_{i,\tau}^{\alpha} - e^{r_\tau}A_{i,\tau}y_{i,\tau}$. Maximizing this expression for $y_{i,\tau}$ we obtain:

$$R_\tau(1) = e^{r_\tau} = \alpha^2 y_{i,\tau}^{\alpha-1} \tag{24}$$

Notice that the maximization problem is independent from i, which implies that it is valid in all sectors. Thus, $y_{i,\tau} = y_\tau$, and from 23 we have $\frac{k_{i,\tau}}{A_{i,\tau}} = \frac{k_\tau}{A_\tau}$, where $A_\tau = \int_0^1 A_{i,\tau}di$, i.e. the average productivity across sectors. Define $\hat{k}_\tau = \frac{k_\tau}{A_\tau}$, the productivity adjusted capital-labor ratio. We can combine 24 with the profit of the monopolist obtaining

$$\pi_{i,\tau} = A_{i,\tau}\alpha(1-\alpha)\hat{k}_\tau^{\alpha} \tag{25}$$

And we can obtain also the equilibrium wages

$$\omega_\tau = dY_\tau/dL_\tau = \alpha A_\tau \hat{k}_\tau^{\alpha} \tag{26}$$

4.2 R&D sector

Innovation in sector i increases its productivity in the next period, from $A_{i,\tau}$ to $\gamma A_{i,\tau+1}$, with $\gamma > 1$. The probability of an innovation μ in sector i per unit of time, depends on the amount of resources $N_{i,\tau}$ dedicated to R&D in that sector i at τ, and the research productivity η where the productivity adjustment captures the fishing-out effect

$$\mu = \eta N_{i,\tau}/A_{i,\tau}$$

Define $V_{i,\tau}$ the value of an innovation at τ for an entrepreneur at τ. If he invests $N_{i,\tau}$, his expected flow of returns is $\mu(N_{i,\tau},\eta)V_{i,\tau}$. Assume free entry in the research sector, implying an expected zero profit flow, $\mu(N_{i,\tau},\eta)V_{i,\tau} - N_{i,\tau} = 0$, which can be simplified in

$$\eta V_{i,\tau}/A_{i,\tau} = 1 \tag{27}$$

The interest rate at which the entrepreneur discount his flow of profits, should be equal to the actual value of profits divided by the expected present value of future one, minus the probability that somebody else innovates, i.e. $r_\tau = \pi_{i,\tau}/V_{i,\tau} - \mu(N_{i,\tau},\eta)$. Rewriting this equation we get

$$V_{i,\tau} = \frac{\pi_{i,\tau}}{r_\tau + \mu(N_{i,\tau},\eta)}$$

From 25, the stream of profits, given an innovation, adjusted by actual productivity is $\pi_{i,\tau}\ A_{i,\tau} \equiv \gamma\alpha(1-\alpha)\hat{k}_\tau^{\alpha}$. This together with the previous equation and 27 gives

$$\eta \frac{\gamma\alpha(1-\alpha)\hat{k}_\tau^\alpha}{r_\tau + \mu(N_{i,\tau}, \eta)} = 1 \tag{28}$$

which is the typical research arbitrage equation in Schumpeterian growth model.

4.3 Growth

Define $N_\tau = \int_0^1 N_{i,\tau} di$, the growth rate of the economy at τ is given by

$$g_\tau = \frac{A_{\tau+1} - A_\tau}{A_\tau} = (\gamma - 1)\mu(N_\tau, \eta)$$

Using also the Research Arbitrage equation 28 we obtain

$$g_\tau = (\gamma - 1)[\gamma\alpha(1-\alpha)\hat{k}_\tau^\alpha - r_\tau]$$

Finally we can use 24 to obtain $r_\tau = \ln(\alpha^2 \hat{k}_\tau^{\alpha-1})$, in order to have growth as a function of $\hat{k}_\tau$:

$$g_\tau = (\gamma - 1)[\gamma\alpha(1-\alpha)\hat{k}_\tau^\alpha - \ln(\alpha^2 \hat{k}_\tau^{\alpha-1})] \tag{29}$$

Thus growth rate increases as the productivity adjusted capital-labor ratio increases. As remarked in Aghion and Howitt (2007) this happens because it acts through two channels: first, increasing capital means higher wages, more consumption and so more profits for the monopolist which pushes innovation; second, more capital means also lower rate of capital return stimulating innovations as well. The innovation here introduced is that savings and capital are derived from an overlapping generation model with liquidity shocks.

4.4 Welfare

Define the total consumption at τ as $C_\tau \equiv \int_0^1 c_{1,\tau}(t)f(t)dt + e^{-n}\int_1^2 c_{2,\tau-1}(t)dt$. Under this production function, resources can be used for consumption, production of intermediate goods or R&D investments $Y_\tau = C_\tau + y_\tau + N_\tau$, where $y_\tau \equiv \int y_{i,\tau} di$. And from the R&D sector we can derive $\dot{A}_\tau = (\gamma - 1)\mu(N_\tau, \eta)A_\tau = (\gamma - 1)\eta N_\tau$. Combining these two last expressions, we obtain $\dot{A}_\tau = (\gamma - 1)\eta(Y_\tau - C_\tau - y_\tau)$, or substituting the variables

$$\dot{A}_\tau = (\gamma - 1)\eta(\alpha^{\frac{\alpha}{1-\alpha}} A_\tau L - \alpha^{\frac{1}{1-\alpha}} A_\tau L - C_\tau)$$

If look inside the single period of life we obtain a problem identical to the one solved by the consumer, thus instead of focusing on the division of consumption inside period of life, the social planner seeks to maximize consumption across generations.

$$\begin{aligned} \max_{C_\tau} \quad & \int_{\tau=0}^{\tau=\infty} e^{-\rho_p \tau} \ln C_\tau d\tau \\ \text{subject to} \quad & \dot{A}_\tau = (\gamma - 1)\eta(\alpha^{\frac{1}{1-\alpha}}(\alpha^\alpha - 1)A_\tau L - C_\tau) \end{aligned} \tag{30}$$

In this problem A_τ is the state variable and C_τ the control one. The corresponding Hamiltonian of the problem is $\mathcal{H} : \ln C_\tau + \lambda(t)[(\gamma - 1)\eta(\alpha^{\frac{1}{1-\alpha}}(\alpha^\alpha - 1)A_\tau L - C_\tau)]$ The necessary (and sufficient) conditions are

$$d\mathcal{H}/dC_\tau = 0 : 1/c - \lambda(\gamma - 1)\eta = 0$$

$$d\mathcal{H}/dA_\tau = \rho_p\lambda(t) - \dot{\lambda}(t) : (\gamma - 1)\eta(\alpha^{\frac{1}{1-\alpha}}(\alpha^\alpha - 1) = \rho_p\lambda(t) - \dot{\lambda}(t)$$

From which we obtain the optimal growth rate maximizing consumption for young and old in a any period τ

$$\frac{\dot{C}_\tau}{C_\tau} = g^* = (\gamma - 1)\eta\alpha^{\frac{1}{1-\alpha}}(\alpha^\alpha - 1) - \rho_p \tag{31}$$

The competitive growth equals the optimal one when

$$\gamma\alpha(1-\alpha)\hat{k}_\tau^\alpha - \ln(\alpha^2\hat{k}_\tau^{\alpha-1}) = \eta\alpha^{\frac{1}{1-\alpha}}(\alpha^\alpha - 1) - \rho_p$$

This leads to a more complicated formulation than the simpler $r = n$ conditions. The optimal level of the adjusted capital-labor ratio is

$$\hat{k}_{opt} = \left(\frac{z_\alpha e^{z_\alpha}}{\alpha^2\gamma}\right)^{\frac{1}{\alpha}} \tag{32}$$

where $z_\alpha \equiv \alpha^2 e^{\eta\alpha^{\frac{1+\alpha-\alpha^2}{1-\alpha}} + \eta\alpha^{\frac{1}{1-\alpha}} + \rho_p}$. Notice that $z_\alpha e^{z_\alpha}$ is a product logarithm function, and since z_α is positive, it means that we are in the principal branch of this function, which ensures that the optimal level of capital is greater than zero.
As in the previous cases, the competitive level of adjusted capital-labor ratio (Equation 18 divided by A_τ) may not be equal to the expression above, leaving room for policy implications.

4.5 Pensions

In this Schumpeterian framework, the choice of a scheme or the other has a stronger effect on the growth dynamic of the economy.

Proposition 3. *In a model of Schumpeterian growth with liquidity problems, social security may bring to optimal growth if inefficiencies lead to a case of over-accumulation of capital*

Proof. The introduction of social security decreases capital stock as shown in the previous section and propositions. This decreases also the steady state growth rate and the equilibrium value of $\hat{k}$. To see this it is sufficient to remind that $\hat{k}_\tau = k_\tau/A_\tau$ and that k_τ in a *laissez-faire* economy is equal to 18. The discussion of the previous section is still valid, social security scheme may restore the optimal condition $\hat{k} = \hat{k}_{opt}$. Similarly in case of over-accumulation and pay-as-you-go pension schemes. □

Another way to see the implications of social security policies on growth is to draw

$$GG : g_\tau = (\gamma - 1)[\gamma\alpha(1-\alpha)\hat{k}_\tau^\alpha - \ln(\alpha^2\hat{k}_\tau^{\alpha-1})]$$

$$KK : \hat{k} = k_{eq}(t)/A_\tau = 1/A_\tau\left[(1-\alpha)e^{rt-n}\left(1 - \frac{\rho}{\rho + e^{-\rho} - e^{-2\rho}}F(t)\right)\right]^{\frac{1}{1-\alpha}}$$

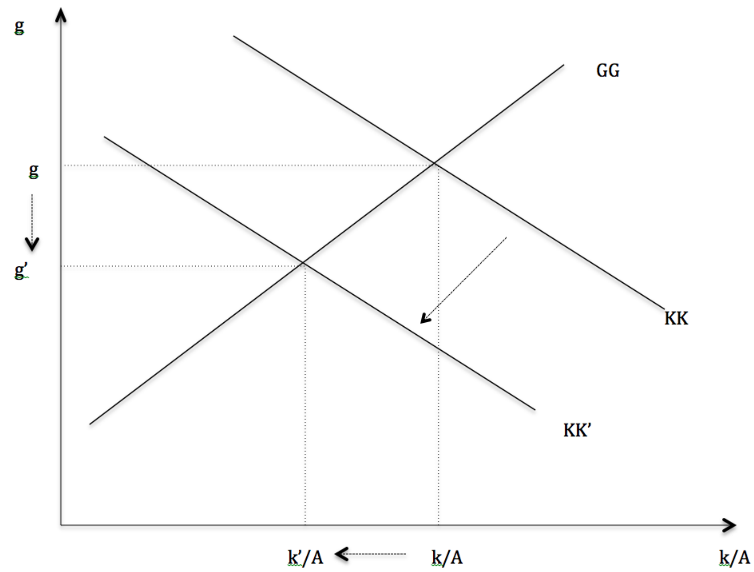

Figure 1: Growth consequences after the introduction of social security

The intersection of these two curves gives the optimal growth rate and adjusted capital-labor stock. The introduction of social security shifts the KK curve downwards since it reduces capital available, as shown in Figure 1.
Since the two pension systems decrease capital, but with different magnitudes, a switch from a pay-as-you-go system to a fully funded one may be beneficial specially in case of under-accumulation of capital, for example after a financial crises.

Proposition 4. *Moving from a pay-as-you-go system to a fully funded one increases growth, and may restore dynamic efficiency in case of under-accumulation of capital*

Proof. As argued in the previous section, a fully funded scheme reduces capital stock less than a pay-as-you-go one. Then, moving from a pay-as-you-go system to a fully-funded has the same effect of an increase of capital, which increases potential growth and it may also restore dynamic efficiency in case of under-accumulation of capital depending on the value of the parameters. □

A consequence of this proposition is that we should expect that economies with a social security closer to a fully funded type should exhibit higher economic growth with respect to countries with a pay-as-you-go system. Similarly to the previous proposition, we can draw also in this case the KK and GG curve. Moving from a pay-as-you-go system to a fully funded one shifts it upwards.

5 Conclusions

We have studied an overlapping generation model in a Schumpeterian framework, with both capital accumulation and endogenous technological progress. The idea was to develop a model to study the impact of social security in a richer setting. We have started from a simple overlapping generation model, reporting the results already existing in the literature, and showing how the introduction of liquidity shocks changes them. The Schumpeterian framework creates a useful model to study the role of social security in a setting where growth depends on capital accumulation and technological progress.

A policy maker may wonder whether he should accelerate or slowdown pension reforms after a financial crises. Our abstract settings offers a normative reply, pointing out that in a situation where the economy may be under-capitalized, a transition to a fully funded pension system may bring some benefits in terms of consumption and growth. Obviously in reality there also other considerations not to underestimate, like the transaction costs from one scheme to another. But this model still gives a sort of benchmark or a guideline for a policy maker, illustrating the dynamics that might be involved in case of liquidity problems.

To conclude let just remark that any consideration about social security must take into account not only inter-generational transfers but also the impact on capital dynamic. Focusing also on R&D the paper shows a potential link between alternative pension design and macroeconomic effects as well as the long term growth of the economy.

References

Aghion, P., and P. Howitt (1992): "A Model of Growth Through Creative Destruction," *Econometrica*, 60(2), 323–351.

——— (2007): "Capital, innovation, and growth accounting," *Oxford Review of Economic Policy*, 23(1), 79–93.

Boucekkine, Raouf and Saglam, Cagri and Vall Ee, T. (2004): "Technology Adoption Under Embodiment: A Two-Stage Optimal Control Approach," *Macroeconomic Dynamics*, 8(2), 250–271.

Diamond, D. W., and P. H. Dybvig (1983): "Bank Runs, Deposit Insurance, and Liquidity," *Journal of Political Economy*, 91(3), pp. 401–419.

Romer, P. M. (1990): "Endogenous Technlogical Change," *The Journal of Political Economy*, 98(5), S71–S102.

Saint-Paul, G. (1992): "Fiscal Policy in an Endogenous Growth Model," *The Quarterly Journal of Economics*, 107(4), 1243–1259.

Samuelson, P. (1975): "Optimum Social Security in a Life-Cycle Growth Model," *International Economic Review*, 16(3), 539–544.

Tomiyama, K. (1985): "Two-stage optimal control problems and optimality conditions," *Journal of Economic Dynamics and Control*, 9(3), 317–337.

von Thadden, E. (1998): "Intermediated versus Direct Investment : Optimal Liquidity Provision and Dynamic Incentive Compatibility," *journal of Financial Intermediation*, 7, 177–197.