The Analytic Theory of a Monetary Shock*

Fernando Alvarez
University of Chicago and NBER

Francesco Lippi
EIEF and LUISS

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Abstract

We propose a new method to analyze the propagation of a once and for all shock in a large class of sticky price models. The method is based on the eigenvalue-eigenfunction representation of the cross-sectional process for price adjustments and provides a thorough characterization of the entire impulse response function for any moment of interest. We use the method to discuss several substantive applications, such as (i) a general analytic characterization of "selection effects", (ii) parsimonious representation of the impulse response function and (iii) volatility shocks in monetary models. We conclude by showing the method can also be applied to models featuring asymmetric return functions and asymmetric law of motion for the state.

JEL Classification Numbers: E3, E5

Key Words: Menu costs, Impulse response, Dominant Eigenvalue, Selection.

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1 Introduction

Economists are often faced with the analysis of dynamic high dimensional objects, such as cross sectional distributions of incomes, prices, or other economic variables of interest. This is the case for instance when studying impulse response functions, resulting from the dynamics of selected moments computed on the distribution of interest. We present a powerful method for such analyses, which typically require solving the partial differential equation that characterizes the evolution of the distribution of interest. The method is the eigenvalue-eigenfunction decomposition that allows for a neat separation of the time-dimension from the state-dimension of the problem, providing a tractable solution to a non-trivial problem. For instance Krieger (2002a,b) proposes to use it to represent states in macroeconomic models. Few recent papers in economics have applied such method to study the transition of the cross-section distribution of incomes, Gabaix et al. (2016), and to analyze asset pricing for long term risk, Hansen and Scheinkman (2009).

We apply this method to analytically characterize the entire impulse response function to a once and for all monetary shock in a large class of sticky price models including versions of Taylor (1980), Calvo (1983), Golosov and Lucas (2007), a version of the "CalvoPlus" model by Nakamura and Steinsson (2010), as well as the multi-product models of Midrigan (2011), Bhattarai and Schoenle (2014) and Alvarez and Lippi (2014), and the model with "price-plans" as in Eichenbaum, Jaimovich, and Rebelo (2011) and Alvarez and Lippi (2018). The key features of these models are that firms, subject to idiosyncratic shocks, face a price setting problem featuring (possibly random) menu costs, as well as "price plans" (i.e. the possibility of choosing 2 prices instead of a single one upon resetting). As in most of the GE literature on the topic we abstract from strategic complementarities to retain tractability. These problems are typically computationally intensive and numerical solutions may hinder a clear understanding of the mechanism at work. The approach we propose greatly facilitates the solution of such models, which we show in many cases has a simple-to-derive analytic form, while at the same time unveiling the key forces and deep parameters, behind the results. Our results provide straightforward characterizations for several features of interest of a large class of models, such as impulse responses, duration analysis, the dynamics of some moments of interest (such as the mean or the variance) after an aggregate shock. Unlike several previous analytic investigations, focusing mostly on approximations for small monetary shocks, the analysis applies to shocks of any size as well as to shock sto higher moments, such as uncertainty shocks. We illustrate the power of the method by discussing three substantive economic applications.

We begin by providing an analytic characterization of the "selection effect", which is one of the main reason why different sticky price models yield different real effects. The selection effect, first discussed by Golosov and Lucas, refers to the fact that the prices that adjust following a monetary shock are those of a selected group of firms. For instance, following a monetary expansion, it is more likely to observe price increases (price changes by firms with a low markup) than price decreases. This contrasts with models where adjusting firms are not systematically selected, such as models of rational inattentiveness, or models where the times of price adjustment are exogenously given such as the Calvo model. We present an analytic result showing how the selection effect creates a wedge between the duration of price changes and that of output. The two durations coincide when there is no selection. We show that such wedge is visible in the magnitude of the eigenvalues that control, respectively, the dynamics of the survival function of prices and the dynamics of output.

Second we discuss the possibility to obtain a parsimonious approximate characterization of the impulse response function by using selected eigenvalues. The question arises since in Hansen and Scheinkman (2009) and Gabaix et al. (2016) the dominant eigenvalue is a convenient and accurate description of an otherwise complicated infinitely dimensional object. It is thus natural to ask whether a single eigenvalue might be found to represent an approximate impulse response function. We present several results. We show that the dominant eigenvalue, which characterizes the tail behaviour of the income distribution in Gabaix et al. (2016), gives the asymptotic hazard rate of price changes. Yet the impulse response of output, including its tail (or asymptotic behavior), is unrelated to the dominant eigenvalue. This is because the output's IRF depends on the difference between price increases and decreases, as opposed to the hazard rate of any price change, i.e. the hazard rate of either increases or decreases. Indeed, as we consider models with "less selection" on the price changes –say going from Golosov and Lucas menu cost model towards Calvo pricing, or increasing the number of products in the multi product model–, the dominant and second eigenvalues get closer to each other and the behavior of the tail of the output IRF, but not the whole function, is better characterized by the dominant eigenvalue. We show that in general it is not possible to summarize the impulse response function, or even its tail behavior, with a single eigenvalue-eigenfunction pair. As a concrete example of a case where no single eigenvalue can be used to characterize the IRF we discuss a class of sticky price models that gives rise to a hump-shaped impulse response function.[1] We provide analytic conditions for the hump-shaped impulse response to arise and show that in such cases no single eigenvalue can summarize the impulse response. Similar results are established for multiproduct models. We show however that an interesting special case exists for the canonical menu cost model with 1 good, no price plans can be effectively summarized by a single eigenfunction.

The third application studies how the propagation of monetary shocks is affected by the volatility of shocks faced by firms. Bloom (2009) analyzed the macroeconomic impact of uncertainty shocks, and others scholars have shows that recessions are times in which the volatility of shocks is higher. We use our method to analyze whether monetary policy is more or less powerful in recessions, interpreted as times in which volatility is high. The question relates to recent quantitative analysis by Vavra (2014). The analysis shows that the answer crucially relies on the time elapsed since the volatility shock occurred. We show that the propagation of a monetary shock that occurs together with the volatility shock differs substantively from the propagation of a shock that occurs a long time after the new volatility is in place. The reason is that in the first case, which we label the "short run", the new

[1] We show that the hump-shaped behavior may emerge in economies where the price setting technology features "price plans" and some randomness in the cost of price changes (weak selection).

volatility immediately affects the firm's decision rule, but the cross sectional distribution of firms is still the old invariant distribution. For instance, an increase in uncertainty widens the inaction region, so that no firm is initially located close to the boundary and the number of adjustments is lower (so that monetary policy is more powerful). This effect reverses in the long run: as the new invariant distribution gets settled, the higher uncertainty will trigger more adjustments and thus a less powerful monetary policy. The model also allows us to quantify the time that it takes for the long-run effect to settle in.

Finally, we discuss how to apply the method to setups that, unlike they baseline sticky-price model described in this paper, allow for an asymmetric return function as well as an asymmetric law of motion for the state (e.g. high inflation), or both. We show that, although analytical results are now harder to obtain, the analysis involved problems remains simple and retains tractability. Such a framework is tractable and seems a promising avenue to study, among others, the role of higher-order shocks of the kind discussed by Fernandez-Villaverde et al. (2011) and Fernandez-Villaverde et al. (2015) in context where decisions rules and the state of the economy feature various kinds of asymmetries.

The paper is organized as follows. The next section describes the setup: the main objects we analyze as well as the baseline sticky price model uses in our applications. Section 3 gives the main result of the paper: the analytic representation of the impulse response function. Some applications of our method are presented in the next three sections: Section 4 uses the setup to discuss "selection effects" in monetary models; Section 5 explores the possibility to approximate the impulse response by using only a few "important" eigenvalues; finally Section 6 explores how uncertainty shocks affect the propagation of monetary shocks. Section 7 consider the n dimensional case of the multiproduct model. A generalization of the method to problems featuring various types of asymmetries is given in Section 8. We conclude by briefly discussing avenues for future work in Section 9.

2 Set up

This section introduces the main objects of our analysis. First we set up a standard mathematical definition of the impulse response. Second, we present a simple baseline sticky-price model that is used to illustrate several applications of interest.

The standard set up is made by the following objects: the law of motion of the Markov process $\{x(t)\}$ for each individual firm, the function of interest $f(x)$, the cross-sectional initial distribution of x, denoted by $P(x;0)$. At this general level the set-up and definition of an impulse response is closely related to the one in Borovicka, Hansen, and Scheinkman (2014). The law of motion for the process $f(x)$, with $x \in X \equiv [\underline{x}, \bar{x}]$, is also Markov and is described using

$$\mathcal{H}(f)(x,t) = \mathbb{E}\left[f\left(x\left(t\right)\right)|x(0) = x\right] \tag{1}$$

where the operator $\mathcal{H}$ computes the t period ahead expected value of the function $f : X \to \mathbb{R}$ conditional on the state $x = x(0)$. Next we describe the initial distribution of x, which we denote by $P(\cdot;0) : X \to \mathbb{R}$. This represents the measure of firms that start with value smaller or equal than x at time $t = 0$, each of them following the stochastic process described in $\mathcal{H}(f)$, with independent realizations. We allow the distribution $P(x;0)$ to have mass points. In particular P has a piecewise continuous derivative (density) which we extend to the entire domain, so that $p(\cdot,0) : [\underline{x}, \bar{x}] \to \mathbb{R}$, where P can have countably many jump discontinuities (mass points), denoting the difference between the right and left limits by $p_m(\cdot;0) : \{x_k\}_{k=1}^{\infty} \to \mathbb{R}$, so that x_k is the location of the mass points.[2] While the possibility of handling a distribution with mass point is of theoretical interest, most of the economic applications that we discuss have no mass points and hence the density p will suffice. Summarizing, we assume that the "shocks" are idiosyncratic, and that the initial condition is

[2]Given our assumption on P we can write the expectations of any function g as:

$$\int_{\underline{x}}^{\bar{x}} g(x)\, dP(x;0) = \int_{\underline{x}}^{\bar{x}} g(x)\, p(x;0)dx + \sum_{k=1}^{\infty} \; g(x)\, p_m(x_k;0)\,.$$

given by a cross sectional distribution $P(\cdot, 0)$.

We are interested in the standard impulse response function H defined for each $t > 0$ as:

$$H\left(t; f, P - \bar{P}\right) = \int_{\underline{x}}^{\bar{x}} \mathcal{H}(f)(x,t) \, \left[dP(x;0) - d\bar{P}(x)\right] \tag{2}$$

where $\bar{P}$ is the invariant distribution of x, the distribution that x will converge to in the long run. The ergodicity of $\{x\}$ implies that we can also write

$$H\left(t; f, P - \bar{P}\right) = \int_{\underline{x}}^{\bar{x}} \mathcal{H}(f)(x,t) \, dP(x;0) \, - \int_{\underline{x}}^{\bar{x}} f(x) d\bar{P}(x) \, .$$

The interpretation of $H(t)$ is the expected value of the cross-sectional distribution of f *in deviation from its steady state value*, where each $x(t)$ has followed the Markov process associated with $\mathcal{H}(\cdot)$ and whose cross sectional distribution at time zero is given by $P(\cdot; 0)$. In other words, for ergodic processes we are forcing the impulse response to go to zero as t diverges. Since we evaluate H only for the difference between two measures, i.e. only for signed measures, when it is convenient we introduce the notation $\hat{P} \equiv P - \bar{P}$ and likewise for the densities $\hat{p} = p - \bar{p}$. Thus

$$\hat{P}(x,t) \equiv P(x,t) - \bar{P}(x) \text{ for all } x \in [\underline{x}, \bar{x}] \text{ and for all } t \geq 0 \, .$$

We also define the discounted cumulative response, given by the (discounted) area under the impulse response function $H(t; f, \hat{P})$:

$$\mathbf{H}(r; f, \hat{P}) = \int_0^\infty e^{-rt} \, H(t; f, \hat{P}) \, dt \, . \tag{3}$$

For many problems, such as the sufficient statistics for monetary policy discussed in Alvarez, Le Bihan, and Lippi (2016), the cumulative response $\mathbf{H}$ is a convenient summary for the effects of a shock, but of course such statistics is not informative about *the shape* of the impulse response function.

We give an alternative definition of the impulse response, which uses a stopping time τ, and a modified expectation operator $\mathcal{G}$ defined as:

$$\mathcal{G}(f)(x,t) = \mathbb{E}\left[1_{\{t\leq\tau\}} f\left(x\left(t\right)\right)|x(0)=x\right] \tag{4}$$

The operator $\mathcal{G}$ computes the t period ahead expected value of the function $f : X \to \mathbb{R}$ starting from the value of the state $x = x(0)$, conditional on x surviving. The indicator function $1_{\{t\leq\tau\}}$ becomes zero when the first adjustment following the shock occurs at the stopping time τ.

In the context of the price setting models with sS rules we refer to the operator $\mathcal{H}$ as the one for the problem with "reinjection", i.e. one in which the operator keeps following the firm until an adjustment occurs (at time τ). In contrast, we refer to the operator $\mathcal{G}$ as one for the problem without "reinjection", i.e. not tracking the firm after the first adjustment. Note that $\mathcal{G}$ is losing measure with time, i.e. $\mathcal{G}(1)(x,t) \leq 1$, and the inequality can be strict for most x's. To be concrete, in the sticky price models discussed below the stopping time τ will be given by the occurrence of a price adjustment.

We define the impulse response function G for each $t > 0$ as:

$$G(t; f, P) = \int_{\underline{x}}^{\bar{x}} \mathcal{G}(f)(x,t)\, dP(x;0)\,. \tag{5}$$

The interpretation of $G(t)$ is the expected value of the cross-sectional distribution of f, conditional on surviving and where each $x(t)$ has followed the Markov process, and whose cross sectional distribution at time zero is given by $P(\cdot;0)$. Note the difference with $\mathcal{H}$, which was defined for measures relative to their steady state value. In the case without "reinjection" we don't need to subtract the invariant, since as t diverges the measure of surviving units converges to zero, and so does the impulse response.

As done above for H, we define the cumulative effect as the area under the impulse response produced by G, formally:

$$\mathbf{G}(r; f, P) = \int_0^\infty e^{-rt}\, G(t; f, P)\, dt\,. \tag{6}$$

While H is the impulse response as commonly defined, it turns out that G is simpler to characterize and that in a large class of problems an equivalence holds so that the cumulative impulse response $\mathbf{G}$ coincides with the cumulative response $\mathbf{H}$ (see Proposition 1 below). Moreover, Proposition 2 below will show that under slightly more stringent conditions the impulse response $G(t; f, P)$ will also coincide with $H(t; f, \hat{P})$ for all t. Given the simpler nature of G we first develop our main results for setups where the $G = H$ equivalence holds, and present more general results for the cases where it does not hold in Section 8.

The initial condition. The primal impulse in the setup is encoded in the initial condition, $P(x, 0)$, which denotes the distribution of the state variable $x \in (\underline{x}, \bar{x})$ at time zero. In particular $P(\cdot; 0)$ describes the cross-sectional distribution of the state immediately after the shock. As time elapses the initial distribution will converge to the invariant distribution $\bar{P}(x)$, tracing out the impulse response for the function of interest $f(x)$. We discussed above that our method allows the initial distribution to have mass points. This can be useful, for instance, if the initial shock is large enough to displace a non-negligible mass of agents onto the return point x^*. Also notice that our formulation allows us to handle a variety of shocks. Several papers have focussed on a small uniform displacement $\delta > 0$ of the whole distribution relative to the invariant $\bar{P}(x)$, what Borovicka, Hansen, and Scheinkman (2014) label the "marginal response function". In this case the initial condition is $P(x, 0) = \bar{P}(x+\delta)$ and it is straighforward that we can rewrite the signed measure $\hat{P}(x, 0) \equiv P(x, 0) - \bar{P}(x)$ using a first order expansion as $\hat{P}(x, 0) = \delta \bar{P}'(x)$. We will sometime focus on such marginal shocks for convenience and to relate to the literature. We stress, however, that our method can handle any type of initial condition, such as one triggered by a large shock, or the one triggered by a higher order shock.

2.1 A baseline price setting problem

This section lays out the price setting problem solved by a firm in the Calvo plus setup, where the firm is allowed to change prices either by paying the menu cost or by receiving a random free adjustment opportunity. This setup nests several models of interest, from the canonical menu cost problem to the pure Calvo model. Analogous results to the ones described in this section can be derived for models with Price-plans as well as for the multiproduct price-setting problem with slight changes in the math of the problem.

Firm's problem in the Calvo plus model. We describe the price setting problem for a firm in steady state. The firm cost follows a Brownian motion with variance σ^2 and drift μ, where the latter is due to inflation. The firm can change its price at any time paying a fixed cost $\psi > 0$. At exogenously given times, which occur with a Poisson rate ζ, the firm faces a zero menu cost. The price gap x is defined as the price currently charged by the firm relative to the price that will maximize current profits, which is proportional to the firm cost. The optimal policy is to change prices either when the price gap x reaches either of two barriers, $\underline{x} < \bar{x}$, or when the menu cost is zero. In either case, at the time of a price change the firm sets a new price which determines a price gap x^*.

The flow cost of the firm is given by $R(x)$. An example is $R(x) = Bx^2$, where the coefficient B measures the curvature of the profit function measured around the static profit maximizing point. The firm maximizes expected discounted profits, using a discount rate $r > 0$. Thus, the optimal policy can be describe by three numbers: $\underline{x} < x^* < \bar{x}$, whose values can be found by solving the value function for the cost in the inaction region:

$$(r + \zeta)v(x) = R(x) + \mu v'(x) + \frac{\sigma^2}{2}v''(x) + \zeta v(x^*) \text{ for } x \in [\underline{x}, \bar{x}] \tag{7}$$

and imposing the relevant boundary conditions: value matching, smooth pasting, and optimal return point:

$$v(\bar{x}) = v(\underline{x}) = v(x^*) + \psi, \text{ and } v'(\underline{x}) = v'(\bar{x}) = v'(x^*) = 0 \quad . \tag{8}$$

The density $\bar{p}$ of the invariant distribution for price gaps generated by the policy $\{\underline{x}, x^*, \bar{x}\}$ and the parameters $\{\mu, \sigma^2\}$ is the solution to the Kolmogorov forward equation:

$$\zeta \bar{p}(x) = -\mu \bar{p}'(x) + \frac{\sigma^2}{2} \bar{p}''(x) \text{ for } x \in [\underline{x}, \bar{x}], x \neq x^* \tag{9}$$

with boundary conditions at the exit points, unit mass, and continuity requirements:

$$\bar{p}(\underline{x}) = \bar{p}(\bar{x}) = 0, \int_{\underline{x}}^{\bar{x}} \bar{p}(x)dx = 1, \text{ and } \bar{p} \text{ continuous at } x = x^* \, .$$

The boundary conditions at the exit points are immediate in sS models with fixed costs since no mass can ever accumulate at the boundary of the inaction region as long as $\sigma > 0$.

2.2 Cumulative impulse response

We now discuss a useful result concerning the cumulated impulse response function that holds in all models described in the previous section. The result states that if the function of interest is $f(x) = R'(x)$, then the cumulated impulse response can be readily computed using the derivative of the value function v, given any arbitrary initial distribution. The function $R'(x)$ is of interest because in several problems the derivative of the firm's function is proportional to the firm's contribution to aggregate output. Recall that the initial condition for the problem is the signed measure $\hat{P}(\cdot, 0) \equiv P(\cdot, 0) - \bar{P}(\cdot)$ where $\bar{P}$ is the invariant distribution. We have the following result (see Appendix A for all proofs in the paper):

Proposition 1. Consider the problem described by $\mathbb{P} \equiv \{\mu, \sigma, r, \psi, \zeta, R(\cdot)\}$. Let $\{\underline{x}, x^*, \bar{x}\}$ and $v(\cdot)$ be the optimal policy and value functions solving the problem $\mathbb{P}$, so the thresholds and value function solve equation (7) and equation (8). Let R' be the derivative of the return function R and $P(\cdot, 0)$ be the distribution of x right after the shock. Then:

$$\mathbf{H}(r, R', \hat{P}) = \int_{\underline{x}}^{\bar{x}} v'(x) d\hat{P}(x; 0) = \mathbf{G}(r, R', \hat{P}) \quad .$$

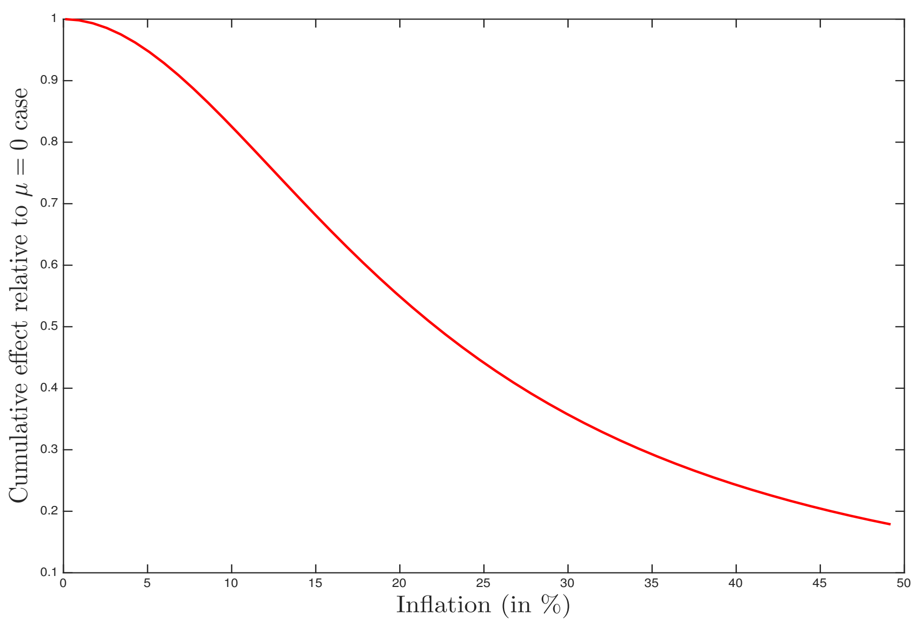
The result is surprising to us. The problem features asymmetric sS bands and the state has a drift, yet we can compute the cumulative IRF as in the symmetric case with no drift. Moreover, when computed this way the value of the optimal return point x^* is irrelevant –literally it does not enter in the computations.[3] We make several comments to this result. First, notice that the proposition holds for $f = R'$, not for an arbitrary function f. The first equality says that if one has solved for the value function in equation (7)-(8), then it has the cumulative IRF for the function $f = R'$. Note that for $R(x) = Bx^2$, i.e. for a second order approximation to the profit function, the derivative $R'(x) = 2Bx$, and hence it is proportional to the firm's contribution of the IRF of output. In particular, the cumulative IRF of output after a monetary shock is obtained by setting $p(x, 0) = \bar{p}(x + \delta)$ and $f(x) = -R'(x)/(2B) = -x$. Second, we stress that the proposition holds for any initial condition $\hat{P}$. This allows us to study either small shocks, such as an infinitesimal displacement of the invariant distribution $\bar{P}$, as well as large shocks that give rise to any type of initial signed mass $\hat{P} = P(0) - \bar{P}$.

Proposition 1 yields a straightforward analysis of an otherwise computationally intensive question. The last equality states that we can obtain the cumulative IRF by simply keeping track of each firm until the time it makes the first adjustment following the shock. This is convenient both for analytical computation as well as for simulations. Figure 1 illustrates this point by exploring how the the cumulated output response, following a small monetary shock, changes with the inflation rate. To this end we use a simple Golosov-Lucas menu cost model calibrated to produce 1 price adjustment per year at zero inflation. We then vary the

[3] An identical result holds for the model with price plans. We relegate this result to an appendix since the math is more cumbersome in that case.

inflation rate and compute the cumulated area as $\mathbf{H}(r, R', \hat{P}) = \int_{\underline{x}}^{\bar{x}} v'(x)\bar{p}'(x;0)dx$ where the value function $v(x)$ solves equation (7) and equation (8), and the density function $\bar{p}(x)$ solves equation (9). The small shock assumption is used by postulating that the distribution right after the shock is equal to a small displacement of the invariant, namely that $p(x,0) = \bar{p}(x+\delta)$, so that $\hat{p}(x) = p(x,0) - \bar{p}(x) \approx \bar{p}'(x)\delta$, where δ is the aggregate monetary shock. The figure shows that the cumulated output effect is not responsive to inflation when inflation is low (it is easy to prove that the function has a zero derivative at $\pi = 0$). As inflation increases however the real (cumulated) effect of policy vanishes fast. At an inflation rate equal to 50% per year the effect is about 1/5th of the effect at low inflation.

Figure 1: Cumulative output response at different inflation rates



Note: the figure plots uses Proposition 1 to compute the cumulated output response, following a small monetary shock, at different inflation rates. The underlying menu cost model is calibrated to produce 1 price adjustment per year at zero inflation.

2.3 Impulse Response Functions for symmetric sS problems

We define a problem and its associated sS rule to be symmetric if the state variable has no drift, so that $\mu = 0$, and if the return point x^* is equidistant from the upper and lower barriers: $x^* - \underline{x} = \bar{x} - x^*$, so that $x^* = (\underline{x} + \bar{x})/2$. Indeed, when $\mu = 0$ and $R(x)$ is symmetric the *optimal* decision rule is symmetric. Let $\{x(t)\}$ be the value of the state for a firm following the optimal policy. Let $g(x; t, x^*)$ be the density of distribution of $x(t)$, conditional on $x(0) = x^*$. This distribution is symmetric with respect to x around x^* for all $t > 0$, i.e. its density is given by $g(z + x^*; t, x^*) = g(-z + x^*; t, x^*)$ for all $z \in [0, \bar{x} - x^*]$. This symmetry comes from the combination of the symmetry of the distribution of a BM without drift, and the symmetry of the boundaries relative to the optimal return point.

Next we present a proposition, for the symmetric case, under which the standard IRF $H(t)$ coincides with $G(t)$, i.e. the simple IRF computed stopping after the first adjustment. The conditions for this to happen are that either the function of interest f, or that the initial distribution $P(\cdot, 0)$ are antisymmetric, where a function $\nu(x)$ is antisymmetric about x^* if it satisfies $\nu(x - x^*) = -\nu(x^* - x)$ for all $x \in [\underline{x}, \bar{x}]$.

Proposition 2. Assume the problem is symmetric. Then if either

(i) the function of interest $f : [\underline{x}, \bar{x}] \to \mathbb{R}$ is antisymmetric and $P(\cdot, 0)$ is arbitrary

(ii) the signed measure (initial condition) $p(\cdot, 0) - \bar{p}(\cdot) : [\underline{x}, \bar{x}] \to \mathbb{R}$ as well as its mass points $p_m(\cdot, 0) - \bar{p}_m(\cdot) : \{x_k\}_{k=1}^{\infty} \to \mathbb{R}$ are antisymmetric and $f(\cdot)$ is arbitrary

we have that $G(t; f, P) = H\left(t; f, P - \bar{P}\right)$ for all t.

See Appendix A for all proofs in the paper. The proposition's requirement that either the function of interest f, or the initial distribution $P - \bar{P}$, is anti-symmetric is not that restrictive for our applications. The main function of interest for the paper, used to compute the IRF for output, is anti-symmetric in the class of models we analyse. For example $f(x) = -x$ in the Calvo$^+$ model. Also, our benchmark case in this class of models is that the density $p - \bar{p}$ is anti-symmetric. This is because in our setup what matters is $p - \bar{p}$, namely the deviation from a symmetric steady after a monetary shock, which can be shown to be antisymmetric –we will state and prove this later. We believe that the strongest assumption is the symmetry on the sS decision rules, which is appropriate for monetary models in the neighborhood of price stability, but it may be inappropriate for other set-ups. An extension to analyse problems with asymmetries is studied in Section 8.

3 Analytic Impulse Response Functions

This section presents a fundamental decomposition that allows us to develop an analytic solution for the operator $\mathcal{G}(f)(x,t)$ defined in equation (4). The main assumption of this section is that the problem is symmetric, i.e. that x has no drift ($\mu = 0$), and that the stopping barriers $\underline{x}, \bar{x}$ are symmetric around the optimal return point x^* (see Section 2.3). As shown in Proposition 2 we know that in this case the impulse response is equivalent to the one of a problem without reinjections and is given by $G(t, f, P)$. We see the symmetric setup as a convenient starting point for the analysis and one which is relevant to analyze most state of the art sticky price models. In Section 8 and Appendix B we show how to extend the analysis presented here to problems with asymmetries and drift, which will involve reinjections. While such problems are in principle more involved we show that the method retains tractability and remains useful.

Assume the process for the firm's price gap is given by a drift-less brownian motion, with instantaneous variance per unit of time σ^2. The stopping time, i.e. the rule at which prices are change, is given by the first time at which $x(t)$ hits either $\underline{x}$ or $\bar{x}$, or that a Poisson counter, with instantaneous rate ζ, changes its value. The definition of $\mathcal{G}(f)(x,t)$ as an expected value implies that this function must satisfy the following partial differential equation:

$$\partial_t \mathcal{G}(f)(x,t) = \frac{\sigma^2}{2} \partial_{xx} \mathcal{G}(f)(x,t) - \zeta \mathcal{G}(f)(x,t) \text{ for all } x \in [\underline{x}, \bar{x}] \text{ and } t > 0 \tag{10}$$

with boundary conditions

$$\mathcal{G}(f)(\underline{x},t) = \mathcal{G}(f)(\bar{x},t) = 0 \text{ for all } t > 0 \text{ and} \tag{11}$$

$$\mathcal{G}(f)(x,0) = f(x) \text{ for all } x \in [\underline{x}, \bar{x}] \tag{12}$$

where boundary conditions for $t > 0$ are an implication from $\bar{x}$ and $\underline{x}$ being exit points, and hence close to them the survival rate goes to zero. The boundary condition at $t = 0$ follows directly from the definition of $\mathcal{G}(f)$.

To solve for $\mathcal{G}(f)$ we consider a class of functions, with typical member φ_j, so that the solution for $\mathcal{G}(\varphi_j)$ is multiplicatively separable in (x, t) as in

$$\mathcal{G}(\varphi)(x,t) = e^{\lambda_j t}\,\varphi_j(x)$$

for some constant λ_j. Thus the partial differential equation in (10) becomes the following ordinary differential equation:

$$\lambda_j\varphi_j(x) = \varphi_j''(x)\frac{\sigma^2}{2} - \zeta\varphi_j(x) \text{ for all } x \in [\underline{x}, \bar{x}] \text{ with boundary conditions } \varphi_j(\bar{x}) = \varphi_j(\underline{x}) = 0\ .$$

Note that the boundary condition for $\mathcal{G}(\varphi_j)$ at $t = 0$, in equation (12), holds by construction. We will denote the φ_j as eigenfunctions and the corresponding λ_j as eigenvalues. The eigenvalues and eigenfunctions are:

$$\lambda_j = -\left[\zeta + \frac{\sigma^2}{2}\left(\frac{j\,\pi}{\bar{x} - \underline{x}}\right)^2\right] \quad \text{and} \quad \varphi_j(x) = \frac{1}{\sqrt{(\bar{x} - \underline{x})/2}}\sin\left(\frac{[x - \underline{x}]}{[\bar{x} - \underline{x}]}\,j\,\pi\right) \tag{13}$$

This is the solution to a well known problem which has been studied extensively.[4] A key property of the set of eigenfunctions $\{\varphi_j\}$ is that they form an orthonormal base for the functions $f : [\underline{x}, \bar{x}] \to \mathbb{R}$ and for which $\int_{\underline{x}}^{\bar{x}}[f(x)]^2 dx < \infty$ so that we can write:

$$\hat{f}(x) \equiv \sum_{j=1}^{\infty} b_j[f]\varphi_j(x) \text{ for all } x \in [\underline{x}, \bar{x}]$$

[4]That this is the solution of the o.d.e. follows since $\sin(0) = \sin(j\pi) = 0$ for all $j \geq 1$ and also because $\sin''(x) = -\sin(x)$ for all x. Matching coefficients in the o.d.e. we obtain the eigenvalues.

where $\int_{\underline{x}}^{\bar{x}}(\hat{f}(x)-f(x))^2dx = 0$ and where the coefficients $b_j[f]$ are:

$$b_j[f] = \int_{\underline{x}}^{\bar{x}} f(x)\varphi_j(x)dx \text{ for all } j \geq 1\,,$$

which uses that $\{\varphi_j\}$ is an orthonormal base. Next we present a key Lemma for the representation of the projections (expectations) of the function of interest f:

LEMMA 1. Assuming that $f : [\underline{x}\,,\, \bar{x}] \to \mathbb{R}$ is piece-wise differentiable, with countably many discontinuities, and $\int_{\underline{x}}^{\bar{x}} [f(x)]^2\, dx < \infty$ then:

$$\mathcal{G}(f)(x,t) = \sum_{j=1}^{\infty} e^{\lambda_j\, t}\, b_j[f]\, \varphi_j(x) \;\text{ for all } x \in [\underline{x}, \bar{x}] \tag{14}$$

Now we turn to the impulse response. We define the projection coefficients for the initial distribution $P(\cdot,0)$ as follows:

$$b_j[P(\cdot,0)] = \int_{\underline{x}}^{\bar{x}} \varphi_j(x)dP(x;0) \equiv \int_{\underline{x}}^{\bar{x}} \varphi_j(x)p(x;0)dx + \sum_{k=1}^{\infty} \varphi_j(x)p_m(x_k;0)$$

so that if P has no mass points, it coincides with the definition for a function f. Using the definition of G, the result in Lemma 1, and the definition of $\{b_j[P]\}$ we can write the impulse response function as follows:

PROPOSITION 3. Assume that $f : [\underline{x}\,,\, \bar{x}] \to \mathbb{R}$ is piece-wise differentiable, with countably many discontinuities, and $\int_{\underline{x}}^{\bar{x}} [f(x)]^2\, dx < \infty$. Furthermore we assume that P has a piecewise continuous density and at most countably many mass points, then:

$$G(t;f,P) = \sum_{j=1}^{\infty} e^{\lambda_j t} b_j[f] b_j[P(\cdot,0)]$$

For the cumulative impulse response defined in equation (6) we then have:

$$\mathbf{G} \equiv \int_0^\infty G(t)dt = \sum_{j=1}^{\infty} \frac{1}{-\lambda_j} b_j[f]\, b_j\left[P(\cdot, 0)\right] \tag{15}$$

For instance, the impulse response of output can be obtained using $f(x) = -x$, since the contribution of output of each firm is proportional to their price gap. It is also straightforward to analyze the slope of the impulse response at $t = 0$, a model feature we discuss later in the application to the calvo$^+$ model.

A simple example. We illustrate a concrete application of the above results using the canonical menu-cost model, obtained by setting $\mu = \zeta = 0$ in the problem of Section 2.1, which yields the symmetric inaction region $\underline{x} = -\bar{x}$ with optimal return $x^* = 0$. To compute the impulse response of output we use $f(x) = -x$ since the contribution of a firm to the deviation of output (relative to steady state) is inversely proportional to its price gap. Integrating $f(x)$ against $\varphi_j(x)$ we find the Fourier coefficients $b_j[f]$:

$$b_j[f] = \frac{4\bar{x}^{3/2}}{j\,\pi} \text{ for } j = 2, 4, 6, \ldots, \text{ and } b_j[f] = 0 \text{ otherwise.} \tag{16}$$

In this example we consider a *small* monetary shock so that the initial condition is given by $p(x, 0) - \bar{p}(x) = \delta\bar{p}'(x)$, as discussed above. The invariant distribution for this model $\bar{p}(x)$ is triangular, so that $\bar{p}'(x)$ is a step function equal to $1/(\bar{x}^2)$ for $x \in [-\bar{x}, 0)$ and equal to $-1/(\bar{x})$ for $x \in (0, \bar{x}, 0]$. We thus construct the Fourier series for $b[\bar{p}']$ by integrating $\bar{p}'(x)$ against $\varphi_j(x)$, this gives the coefficients

$$b_j[\bar{p}'] = \frac{8}{j\,\pi\bar{x}^{3/2}} \text{ if } j = 2 + i4 \text{ for } i = 0, 1, 2, \ldots, \text{ and } b_j[\bar{p}'] = 0 \text{ otherwise.} \tag{17}$$

Thus the impulse response $b_j[\bar{p}']b_j[f]$ coefficients are:

$$b_j[\bar{p}']b_j[f] = \frac{32}{(j\,\pi)^2} \text{ if } j = 2 + i4 \text{ for } i = 0, 1, 2, \ldots, \text{ and } b_j[\bar{p}']b_j[f] = 0 \text{ otherwise.}$$

and the output impulse response, which we denote by $Y(t) = G(t)/\delta$ is:

$$Y(t) = \sum_{i=0}^{\infty} \frac{32}{((2+4i)\,\pi)^2} e^{-N \frac{((2+4i)\pi))^2}{8} t}$$

where $N = \sigma^2/\bar{x}^2$ is the average number of price changes per period, the only parameter in this expression.

4 Characterization of the selection effect

This section uses the results of Section 3 to provide an analytic illustration of why different models display different degrees of "selection". To ensure the results are applicable we focus on a symmetric problem (symmetric return function and law of motion). The term selection, coined by Golosov and Lucas, refers to the fact that the prices that adjust following a monetary shock are those of a selected group of firms. For instance, following a monetary expansion, it is more likely to observe price increases (price changes by firms with a low markup) than price decreases. This contrasts with models where adjusting firms are not systematically selected, such as models of rational inattentiveness, or models where the times of price adjustment are exogenously given such as the Calvo model. It is known that different amounts of selection critically affect the propagation of monetary shocks. Our application casts light on the mechanism behind this result.

We present an analytic result showing how the selection effect creates a wedge between the duration of price changes and that of output. The two durations coincide when there is no selection. In this case the frequency of price changes is a sufficient statistic for the output effect of monetary policy. We show that such wedge is visible in the magnitude of the eigenvalues that control, respectively, the dynamics of the survival function of prices and the dynamics of output. Next we illustrate this result using the Calvo-plus model, a model that nests several special cases featuring different degrees of selection, from Golosov-Lucas to the pure Calvo model. The result also holds in several other models, featuring multiproduct firms or price plans.

Application to the Calvo$^+$ model. Next we use the decision problem defined in Section 2.1, and assume zero inflation ($\mu = 0$) and a quadratic profit function $R = x^2$. It is straightforward that $\bar{x} = -\underline{x} > 0$ and that the optimal return is $x^* = 0$. Given the policy parameters $\{-\bar{x}, \bar{x}\}$ and the law of motion of the state $dx = \sigma dW$ it is immediate that the eigenvalues-eigenfunctions of the problem are those computed in equation (13). Since the eigenvalues depend on the speed at which prices are changed, we find it convenient to rewrite them in terms of the average number of price changes per unit of time, N and ϕ.

To this end we compute the expected number of adjustments per unit of time, the reciprocal of the expected time until an adjustment,

$$N = \frac{\zeta}{1 - \text{sech}(\sqrt{2\phi})} \quad \text{where} \quad \phi \equiv \frac{\zeta \bar{x}^2}{\sigma^2} \quad .$$

Note that as $\bar{x} \to \infty$ then $N \to \zeta$, which is the Calvo model where all adjustment occur after an exogenous poisson shock. As $\zeta \to 0$ then $N \to \sigma^2/\bar{x}^2$ so that the model is Golosov and Lucas. This single parameter $\phi \in (0, \infty)$ controls the degree to which the model varies between Golosov-Lucas and Calvo. Note that with this parameterization we can distinguish between N and the importance of the randomness in the menu cost ζ vs the width of the barriers, $\bar{x}^2/\sigma^2$. Indeed ζ/N, the share of adjustment due to random free-adjustments, depends only on ϕ. We let:

$$\frac{\zeta}{N} = \ell(\phi) \text{ where this function is defined as } \ell(\phi) = 1 - \text{sech}\left(\sqrt{2\phi}\right) \quad .$$

The function $\ell(\cdot)$ is increasing in ϕ, and ranges from 0 to 1 as ϕ goes from 0 to ∞. Using the formula for N and equation (13) we have:

$$\lambda_j = -\zeta - \frac{\sigma^2}{\bar{x}^2} \frac{(j\pi)^2}{8} = -\zeta \left[1 + \frac{(j\pi)^2}{8\phi}\right] = -N\,\ell(\phi) \left[1 + \frac{(j\pi)^2}{8\phi}\right] \quad . \tag{18}$$

Interpretation of the Dominant Eigenvalue. The dominant eigenvalue has the interpretation of the asymptotic hazard rate of price changes. In particular, let $h(t)$ be the hazard rate of price spells as a function of the duration of the price spell t. Let τ be the stopping time for prices, i.e. τ is the first time at which $\sigma W(t)$, which started at $W(0) = 0$, either hits $\bar{x}$ or $\underline{x} = -\bar{x}$, or that the Poisson process changes. Let $S(t)$ be the survival function, i.e.: $S(t) = \Pr\{\tau \geq t\}$. Notice that the function of interest to compute the survival function is the indicator $f(x) = 1$ for all $t < \tau$. The hazard rate is defined as $h(t) = -S'(t)/S(t)$. We have:

Proposition 4. The Survival function $S(t)$ depends only on the odd-indexed eigenvalues-eigenfunctions, i.e. (λ_i, φ_i) for $i = 1, 3, 5, \ldots$. Let $h(t)$ be the hazard rate of price changes. Then, the dominant eigenvalue λ_1 is equal to the asymptotic hazard rate, i.e.

$$S(t) = \sum_{j=1}^{\infty} e^{\lambda_{2j-1} t}\, b_{2j-1} \quad \text{and} \quad -\lambda_1 = \lim_{t \to \infty} h(t)$$

where $b_j \equiv b_j[1]\, b_j[\delta_0]$ where δ_0 is the Dirac delta function.

Average survival function. Next we derive a result, closely related to the previous one, showing that the even-index eigenvalue-eigenfunctions ($j = 2, 4, \ldots$) are irrelevant (literally they do not appear) in the equation for the average number of price changes following a monetary shock.

Define the average survival function $Q(t)$ to be the number of firms that survive up to period t among those that started at time zero with a distribution of x given by $p(x; 0)$. In terms of our notation this corresponds to $f(x) = 1$, as in the survival function described above. The difference with the survival function $S(\cdot)$ defined above, is that the initial condition is $p(\cdot; 0)$ instead of the dirac function. Specifically, $S(t)$ is the probability of a firm surviving up to time t conditional on starting with $x = 0$ at time zero. Instead $Q(t)$ is the fraction of firms surviving up to t for a cohort that starts at zero with distribution $p(\cdot; 0)$. The function $Q(\cdot)$ is important because our objective is to characterize the impulse response after an aggregate shock, which is modeled as a particular initial distribution $p(\cdot; 0)$. Indeed, together with the output impulse response, it allows us to decompose the effect of a monetary shock into the effect on the probabilities of price changes vs the effect on the average price level at each horizon.

The definition above gives us:

$$Q(t) = \int_{\underline{x}}^{\bar{x}} q(x,t)\, p(x;0)\, dx \quad \text{where} \quad q(x,t) = \mathcal{G}(1)(x,t)$$

so $q(x,t)$ is the probability that a firm with x at time zero will survive until time t. Before setting the next result we define as $\bar{Q}(\cdot)$ the average survival function at steady state, i.e. Q when $p(x,0) = \bar{p}(x)$, where $\bar{p}$ is the density of the invariant distribution.

Proposition 5. The average survival function $Q(t)$ for the Calvo$^+$ model after a small monetary shock is equal to its steady state value, i.e. $Q(t) = \bar{Q}(t)$ for all $t \geq 0$. The function $\bar{Q}$, depends only on the odd-indexed eigenfunctions-eigenvalues ($j = 1, 3, ...$) and $-\lambda_1$ is the asymptotic hazard rate of Q, i.e.

$$Q(t) = \sum_{j=1}^{\infty} e^{\lambda_{2j-1} t}\, b_{2j-1} \quad \text{and} \quad -\lambda_1 = \lim_{t \to \infty} h(t)$$

where $b_j \equiv b_j[1]\, b_j[\bar{p}]$.

Proposition 5 means one should not expected to detect difference in the average hazard rates for price changes before and after a (small) monetary shock. This is because what is important for the output IRF is the impact of the monetary shock on the average price level. For example, an equal number of increases and decreases on prices do not contribute to output IRF.

Irrelevance of dominant eigenvalue for output IRF. Next we show that the dominant eigenvalue λ_1, as well as all other odd-indexed eigenvalue-eigenfunction pairs, play no role in the output impulse response. Consider the output coefficients in the impulse response, given by equation (16). It is apparent that the coefficients $b_j[f]$ for all the odd-indexed eigenvalues-eigenfunctions ($j = 1, 3, \ldots$) are zero, i.e. the loading of these terms are zero. This implies that the coefficient corresponding to the dominant eigenvalue λ_1 is zero. The first non-zero term, which we call the "leading" eigenvalue, involves λ_2. This is because $\varphi_j(\cdot)$ is symmetric around $x = 0$ for j odd, and antisymmetric for j even. Thus:

$$\int_{\underline{x}}^{\bar{x}} \varphi_j(x) f(x) dx = 0 \implies b_j[f] = 0 \text{ for } j = 1, 3, \ldots$$

This happens since all the odd-indexed eigenfunctions φ_j ($j = 1, 3, ...$) are symmetric functions, and thus the projection onto them of an asymmetric function, such as $f(x) = -x$, yields a zero b_j coefficient. We summarize this result in the next proposition.

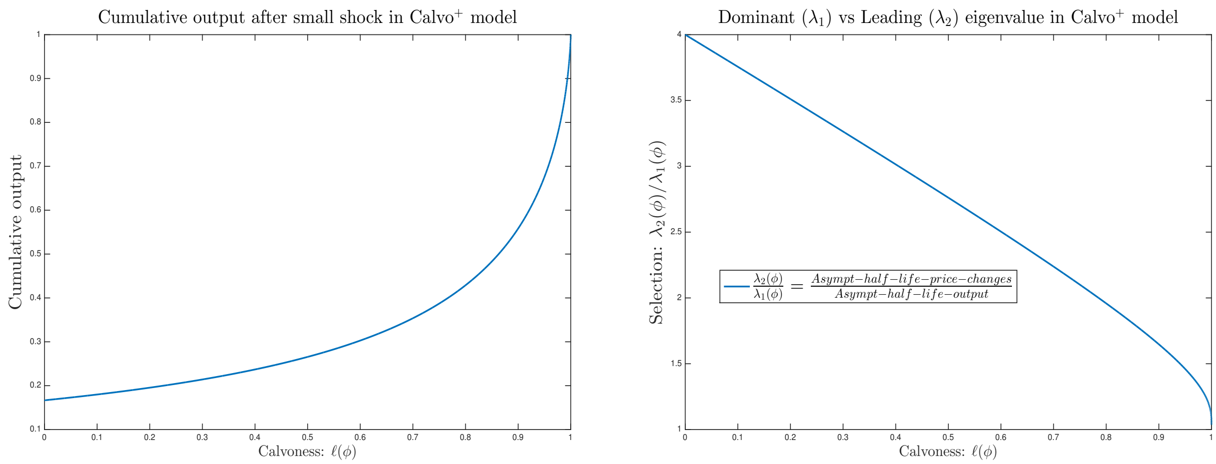
PROPOSITION 6. The output impulse response function for the Calvo$^+$ model depends only on the even-indexed eigenvalue-eigenfunctions (λ_j, φ_j), and has zero loadings on the odd-indexed ones, such as the dominant eigenvalue. Thus the first leading term corresponds to the second eigenvalue:

$$G(t) = \sum_{j=1}^{\infty} e^{\lambda_{2j} t} \, b_{2j} \quad \text{and} \quad \lambda_2 = \lim_{t \to \infty} \frac{\log Y(t)}{t}$$

where $b_j \equiv b_j[f] \, b_j[p(\cdot, 0)]$.

The proposition states that only half of the eigenvalues (those with an even index) show

Figure 2: Selection effect in Calvo$^+$ model



up in the output impulse response function. The largest eigenvalue is λ_2, which we call the "leading" eigenvalue of the output response function. It is interesting to notice that the dominant eigenvalue λ_1 does not appear in the impulse response for output. Notice the difference with the survival function where the only eigenvalues that appear are those with an odd-index. The right panel of Figure 2 plots the ratio between the leading eigenvalue for output λ_2 and the dominant eigenvalue λ_1. It is straightforward to see that the ratio is $\frac{\lambda_2}{\lambda_1} = \frac{8\phi+4\pi^2}{8\phi+\pi^2}$ depends only on ϕ, so that it can be immediately mapped into the "Calvoness" of the problem $\ell(\phi) \in (0,1)$. It appears that the ratio, which can also be interpreted as a ration between the asymptotic duration of price changes over the asymptotic duration of the output impulse response, is monotonically decreasing in ℓ, and converges to 1 as $\ell \to 1$. The economics of this result is that the shape of the impulse response of output depends on the differential impact of the aggregate shock on price *increases* and price *increases*. Instead the dominant eigenvalue controls the tail behavior of price *changes*, both increases and decreases. As $\ell \to 1$ selection disappears from the model and the two durations coincide. The left panel of the figure uses the particular case of a small monetary shock (developed in detail in the next subsection) to illustrate that as ℓ increases the cumulated output effect become larger due to a muted smaller selection effect.

4.1 How do monetary shocks affect the dispersion of prices?

We conclude this section with a discussion of a relevant topic in monetary models that concerns the welfare effects of shocks. In many monetary models the presence of price stickiness implies that welfare of the representative consumer depends on the dispersion of prices, or the dispersion of markups (i.e. prices relative to a flexible price benchmark where dispersion is nil). It is therefore of interest to analyze how the dispersion of markups behaves following a small monetary shock of size δ.[5] Let $\mathcal{D}(t,\delta) \equiv var(x,t;\delta)$ denote the cross sectional variance of markups t periods after the monetary shock δ hits an economy at the steady state. We have the following result:

PROPOSITION 7. Assume the initial condition $\hat{p}$, the signed mass right after the aggregate shock, is odd. Then a small monetary shock δ does not have a first order effect on the dispersion of markups, namely $\frac{\partial}{\partial\delta}\mathcal{D}(t,\delta)|_{\delta=0} = 0$, for all t. More generally a zero first order effect occurs on all even centered moments of the distribution of markups (such as Kurtosis). Instead, uneven moments, such as the the mean markup (proportional to total output) or the the skewness of the distribution display a non-zero first order effect.

5 Is there a parsimonious representation of the IRF?

This section discusses whether the leading eigenvalue gives a good summary description of the entire impulse response. To analyze this case we focus on a small monetary shock that causes a small displacement of the invariant distribution. We assume a symmetric problem and present results for the baseline Calvo$^+$ model as well as for a model with price plans.

Initial Condition $p(\cdot,0)$ for the impulse response to a monetary shock. The invariant density function $\bar{p}$ solves the Kolmogorov forward $\zeta\bar{p}(x) = \sigma^2/2\bar{p}''(x)$ in the support,

[5] We are thankful to Nobu Kiyotaki for posing this question to us.

except at $x = 0$, integrates to one and it is zero at $\pm\bar{x}$. This gives:

$$\bar{p}(x) = \begin{cases} -\frac{\theta\left[e^{-\theta x} - e^{2\theta\bar{x}}e^{\theta x}\right]}{2\left[1-2e^{\theta\bar{x}}+e^{2\theta\bar{x}}\right]} & \text{if } x \in [-\bar{x}, 0] \\ -\frac{\theta\left[e^{\theta x} - e^{2\theta\bar{x}}e^{-\theta x}\right]}{2\left[1-2e^{\theta\bar{x}}+e^{2\theta\bar{x}}\right]} & \text{if } x \in [0, \bar{x}] \end{cases} \quad \text{where} \quad \theta = \sqrt{2\zeta/\sigma^2} \quad . \tag{19}$$

The invariant distribution $\bar{p}(\cdot)$ is symmetric with $\bar{p}(x) = \bar{p}(-x)$ and $\bar{p}'(x) = -\bar{p}'(-x)$ for all $x \in [-\bar{x}, 0)$. The density of the distribution is continuous but non-differentiable at the injection point $x = 0$. For concreteness we focus below on the response to a small monetary shock δ, starting at the steady state. For a small shock we can disregard the fraction of firms that change prices on impact, i.e. this effect is of order δ^2. Thus the initial condition is

$$p(x, 0) = \bar{p}(x + \delta) = \bar{p}(x) + \bar{p}'(x)\delta + o(\delta) \text{ for all } x \in [-\bar{x}, 0) \text{ and } x \in (0, \bar{x}]$$

Notice that we can write:

$$G(t, \delta) = \frac{\partial}{\partial \delta} G(t, \delta)\big|_{\delta=0} \delta + o(\delta)$$

From now on for the Impulse response function of output we simply write

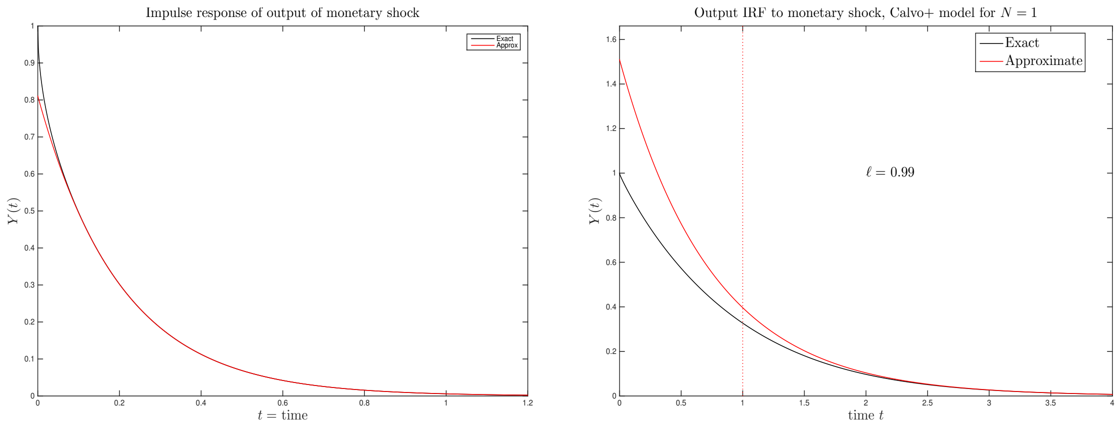
$$Y(t) \equiv \frac{\partial}{\partial \delta} G(t, \delta)\big|_{\delta=0} = \sum_{j=1}^{\infty} e^{\lambda_{2j} t} b_{2j}[f] b_{2j}[\bar{p}']$$

which is the output impulse response per unit of the monetary shock.

Proposition 8. The coefficients for the impulse response to a small monetary shock in

Figure 3: Exact vs approximate IRF

Golosov Lucas ($\ell = 0$) | Calvo$^+$ model



the Calvo $^+$ model are given by:

$$b_j(\phi) \equiv b_j[\bar{p}']b_j[f] = \begin{cases} 0 & \text{if } j \text{ is odd} \\ 2\left[\frac{1+\cosh\left(\sqrt{2\phi}\right)}{\cosh(\sqrt{2\phi})-1}\right]\left[\frac{1}{1+\frac{j^2\pi^2}{8\phi}}\right] & \text{if } j \text{ is even and } \frac{j}{2} \text{ is odd} \\ -2\left[\frac{1}{1+\frac{j^2\pi^2}{8\phi}}\right] & \text{if } j \text{ is even and } \frac{j}{2} \text{ is even} \end{cases}$$

Simple calculus gives the limit as $\phi \to 0$ and $\phi \to \infty$, while keeping N constant, which are respectively the Golosov-Lucas and the Calvo model. The solution for the first case, $\phi \to 0$, was given in equation (16) and equation (17). The second case, $\phi \to \infty$, is peculiar because in this limit the spectrum is no longer discrete. To see this notice that each of the eigenvalues $\lambda_j \to -\zeta$ and:

$$\lim_{\phi\to\infty} b_j(\phi) = \begin{cases} 0 & \text{if } j \text{ is odd} \\ 2 & \text{if } j \text{ is even and } \frac{j}{2} \text{ is odd} \\ -2 & \text{if } j \text{ is even and } \frac{j}{2} \text{ is even} \end{cases}$$

Note that in this case, using just the "leading" eigenvalue, i.e. the term with the first non-zero weight, gives an impulse response $G(t)$ that is twice as large than the true one, for each t. This is in stark contrast with the case when $\phi \to 0$ where the approximation is extremely accurate. The difference is less than 1.5%, to see this note that when $\phi \to 0$ then $\mathbf{G} \to 1/(6N) \approx 0.1677/N$. On the other hand, using only the term corresponding to the second eigenvalue we obtain $b_2/(-\lambda_2) = 16/(\pi^4)/N \approx 0.1643/N$.

The next proposition gives a characterization of the ratio between the true area under the impulse response and the approximate one, computed using only the leading eigenvalue:

Proposition 9. Define the ratio of the approximate cumulative impulse response based on the second eigenvalue to the area under the impulse response as

$$m_2(\phi) = \frac{b_2(\phi)/\lambda_2(\phi)}{\sum_{j=1}^{\infty} b_j(\phi)/\lambda_j(\phi)} = 2\frac{\left[1 + \cosh\left(\sqrt{2\phi}\right)\right]}{\left[\cosh(\sqrt{2\phi}) - 1 - \phi\right]\left[1 + \frac{\pi^2}{2\phi}\right]^2}$$

The function $m_2(\phi)$ goes from $m_2(0) = (16/\pi^4)6 \approx 0.9855$, $m_2(\phi)' > 0$ and $m_2(\phi) \to 2$ as $\phi \to \infty$.

We can also use the expression for the coefficients of the impulse response to show that the slope of Y at $t = 0$ is minus infinity. This is intuitive since, after the shock, there are firms that are just on the boundary of the inaction region where they will increase prices, but there are no firms at the boundary at which they want to decrease prices.

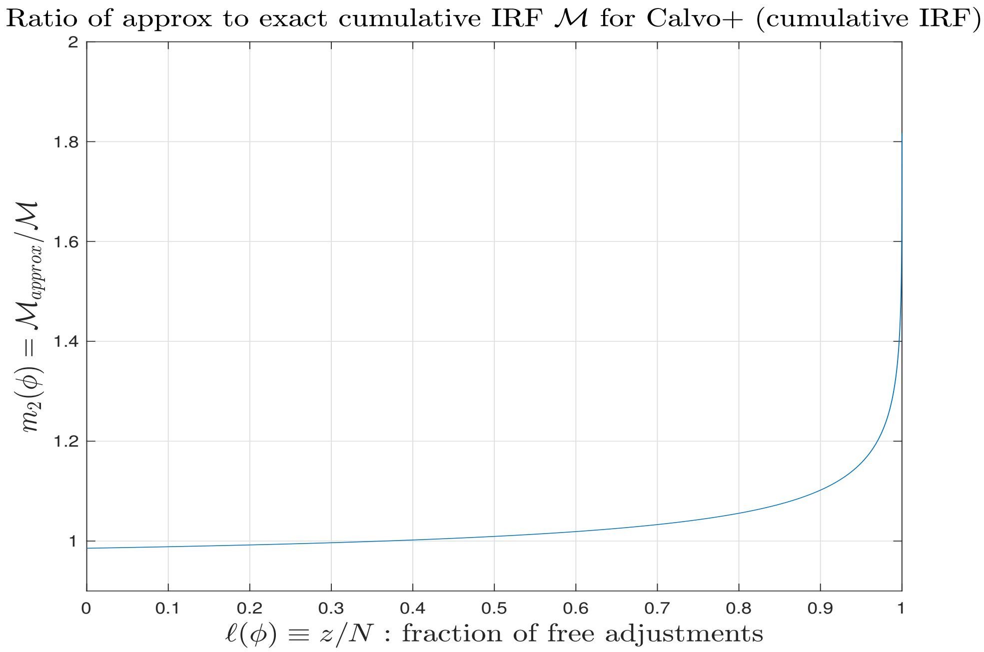
Proposition 10. The derivative of the IRF with respect to t at $t = 0$ is given by:

$$\frac{\partial}{\partial t} Y(t)|_{t=0} = -\infty$$

if $0 \leq \phi < \infty$.

Note that when $\phi \to \infty$, so we get the pure Calvo model, the impulse response because $Y(t) = \exp(-Nt)$, and thus $Y'(0)$ is finite. Indeed Appendix F analyzes the the output IRF

Figure 4: Calvo$^+$ model



for the pure Calvo model.

5.1 Price Plans and the hump-shaped Output IRF

The model with price plans assumes that upon paying the menu cost the firm can choose two, instead of one price. At any point in time the firm is free to charge either price within the current plan, but changing to a new plan (another pair of prices) is costly. The idea was first proposed by Eichenbaum, Jaimovich, and Rebelo (2011) to model the phenomenon of temporary price changes (prices that move from a reference value for a short period of time and then return to it). In Alvarez and Lippi (2018) we provide an analytic solution to this problem and characterize the determinants of $\bar{x}$, the threshold where a new plan is chosen, as well as the optimal prices within the plan, named $\tilde{x}$ and $-\tilde{x}$. When $x \in [-\bar{x}, 0]$ the firm charges $-\tilde{x}$ and when $x \in (0, \bar{x}]$ it charges $\tilde{x}$. The invariant density of price gaps is still given by equation (19). For a given threshold $\bar{x}$ the value of $\tilde{x}$ is given by:

$$\tilde{x} = \bar{x}\left[\frac{\exp\left(\sqrt{2\phi}\right) - \exp\left(-\sqrt{2\phi}\right) - 2\sqrt{2\phi}}{\sqrt{2\phi}\left(\exp\left(\sqrt{2\phi}\right) + \exp\left(\sqrt{-2\phi}\right) - 2\right)}\right] \equiv \bar{x}\,\rho(\phi) > 0$$

where $\phi = \bar{x}^2\zeta/\sigma^2$ and the function $\rho(\phi) \in (0, 1/3)$. The contribution to the output of a firm with price gap x is, instead of $f(x) = -x$, the following function:

$$\tilde{f}(x) = \begin{cases} -x - \tilde{x} & \text{if } x \in [-\bar{x}, 0) \\ -x + \tilde{x} & \text{if } x \in (0, \bar{x}] \end{cases}$$

By the linearity of Fourier series we can add to the coefficients of the function $f(x) = -x$, shown in equation (16) the ones of the step function:

$$f_0(x) = \begin{cases} -\tilde{x} & \text{if } x \in [-\bar{x}, 0) \\ +\tilde{x} & \text{if } x \in (0, \bar{x}] \end{cases}$$

Importantly, we note the function $\tilde{f}(x) = f(x) + f_0(x)$ is still an asymmetric function. The function f_0 has Fourier sine coefficients equal to:

$$b_j[f_0] = -\frac{8\bar{x}^{3/2}\rho(\phi)}{j\,\pi} \text{ if } j = 2 + i4 \text{ for } i = 0, 1, 2, \ldots, \text{ and } b_j[f_0] = 0 \text{ otherwise}$$

From here we conclude that:

$$b_j^0(\phi) = b_j[f_0]b_j[\bar{p}'] = \begin{cases} 0 & \text{if } j \text{ is odd} \\ -4\,\rho(\phi)\left[\frac{1+\cosh\left(\sqrt{2\phi}\right)}{\cosh(\sqrt{2\phi})-1}\right]\left[\frac{1}{1+\frac{j^2\pi^2}{8\phi}}\right] & \text{if } j \text{ is even and } \frac{j}{2} \text{ is odd} \\ 0 & \text{if } j \text{ is even and } \frac{j}{2} \text{ is even} \end{cases}$$

Thus the impulse response is given by:

$$Y_{\text{Plan}}(t) = Y_{\text{Calvo}^+}(t) + \sum_{j=1}^{\infty} b_j^0(\phi) e^{\lambda_j(\phi)t}$$

While the impulse response is monotone decreasing in the Calvo$^+$ model, in the price plan model the impulse response can be hump shaped. Indeed, as the ϕ increases, the impulse response goes from decreasing to humped shaped. The reason for this difference is that in the price plan model there is a non-negligible impact effect, due to the firm that within the plan adjust from one price to the other. The difference across models, as we change ϕ is that invariant distribution where randomness is more important has more firms with price gaps close to zero, and hence more firms which can change from one price to the other within the plan. This is because the higher is the impact effect on prices, the smaller is the effect on output. The next proposition indeed shows that when $\phi = 1$, so that $\zeta = \sigma^2/\bar{x}^2$, i.e. the number of plan changes that correspond to a pure barrier model equals to one with a pure random plan changes are equal. For $\phi = 1$ the impulse response has a "hump", but it is infinitesimal.

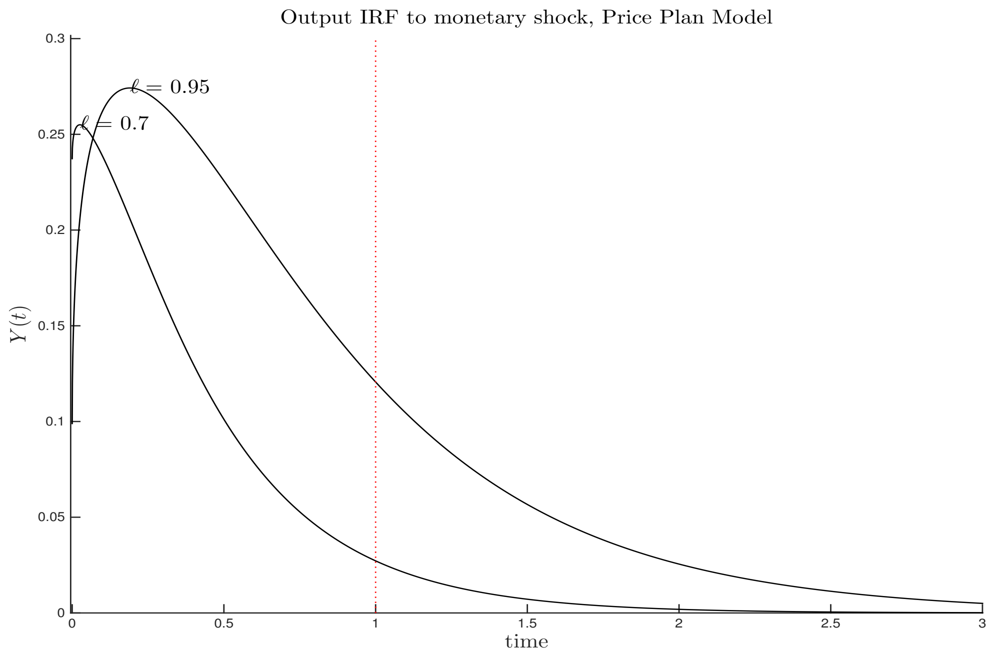
Proposition 11. For $0 \leq \phi \leq 1$, the impulse response $Y(t)$ is decreasing. For $\phi > 1$, the impulse response is hump shaped. The value for $t_0 N$ at which the maximum is reached, $Y'(t_0 N) = 0$, increases relative to expected time to adjustment $1/N$.

Given that the impulse response is hump-shaped any single eigenvalue cannot approximate the impulse response. Even for low values of ϕ, where the impulse response is monotone, using λ_2 gives a very bad approximation.

6 Monetary propagation with volatility shocks

We use the pure menu cost model ($\ell = 0$) to conduct two comparative statics exercise: the objective is to analyze how the propagation of a monetary shock is affected by an innova-

Figure 5: Price Plan model



tion of the "volatility shocks", namely a permanent changes in the common value for the idiosyncratic volatility σ.[6] The first comparative static (i), consists on comparing the effect of a monetary shock δ across two steady states, one with low idiosyncratic volatility σ, and the other one with high volatility $\tilde{\sigma} = \left(1 + \frac{d\sigma}{\sigma}\right)\sigma$. We refer to this as the *long run* effect of volatility on the output IRF after a monetary policy, since it is the effect of an unanticipated monetary shock once the distribution of price gaps have achieved its new invariant distribution. In this case the firm's decision rules will correspond to the new volatility $\tilde{\sigma}$, which will generates the new invariant distribution of price gaps. The second comparative static (ii), consist on starting with the steady state with volatility σ, then having a permanent change to $\tilde{\sigma} = \left(1 + \frac{d\sigma}{\sigma}\right)\sigma$, and measuring the effect of a monetary shock immediately after the shock to volatility. We refer to this as the *impact* effect of volatility on the output IRF after a monetary policy shock. As in the previous case, the forward looking firm's decision rules immediately reflect the new volatility $\tilde{\sigma}$. The difference with the first case is that the initial

[6] The object under investigation has the flavour of a cross-partial derivative: how the impact of a monetary shock δ on output (a partial at time t) is affected by a change in volatility σ.

distribution of price gaps correspond to the stationary distribution corresponding to the old decision rules, i.e the decision rules using volatility σ. In each case, we will trace the impulse response of output to a monetary shock, which we denote by $\tilde{Y}(t)$, to distinguish it from the one $Y(t)$ that is the impulse response when volatility stays constant.

Proposition 12. Let $Y(t)$ be the output impulse response to a monetary shock for the idiosyncratic volatility σ, and let the new volatility of shocks be $\tilde{\sigma} = \left(1 + \frac{d\sigma}{\sigma}\right)\sigma$. The *long run* effect of the volatility shock $\frac{d\sigma}{\sigma}$ on the impulse response of output to a monetary shock is:

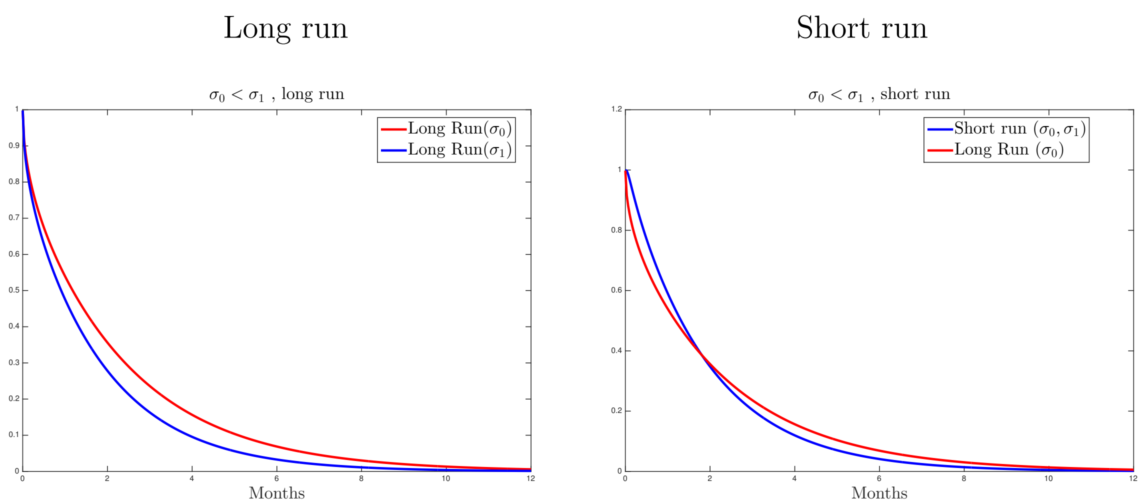
$$\tilde{Y}(t) = Y\left(t\left(1 + \frac{d\sigma}{\sigma}\right)\right) \quad \text{for all } t \geq 0 \ . \tag{20}$$

The *short run* effect of the volatility shock $\frac{d\sigma}{\sigma}$ on the impulse response of output to a monetary shock is:

$$\tilde{Y}(t) = \left(1 + \frac{d\sigma}{\sigma}\right) \ Y\left(t\left(1 + \frac{d\sigma}{\sigma}\right)\right) \ \text{for all } t \geq 0 \quad . \tag{21}$$

Few comments are in order. First, for this proposition we use the form of the decision rules for the threshold $\bar{x}$, which as the discount rate goes to zero is $\bar{x} = \left(6\frac{\psi}{B}\sigma^2\right)^{\frac{1}{4}}$ where ψ is the fixed cost –as fraction of the frictionless profit and B which is the curvature of the profit function around the frictionless profit. This implies that the elasticity of $\bar{x}$ to σ is $1/2$. This elasticity is the so called "option value" effect on the optimal decision rules. Second, the rescaling of time as in the old IRF, as in $Y\left(t\left(1 + \frac{d\sigma}{\sigma}\right)\right)$ in both expressions reflects the change in the eigenvalues, which depend on the value of N, the implied average number of price changes per unit of time, as $\lambda_j = -N\left(\pi j\right)^2/8$. Recall that $N = (\sigma/\bar{x})^2$, and hence all the eigenvalues change proportionally with σ. Third, for the case of the impact effect and in which $\tilde{\sigma} > \sigma$, the invariant distribution just before the monetary shock is narrow than the new range of inaction that corresponds to the new wider barriers. This explains the extra multiplicative term level $\left(1 + \frac{d\sigma}{\sigma}\right)$ in the impact effect in equation (21), since firms have price gaps that are discrete away from the inaction bands. Hence, prices react more slowly, generating the extra effect on output. The logic for the case where $\tilde{\sigma} < \sigma$ is similar. Fourth we have found that Y is differentiable with respect to $\tilde{\sigma}$, when evaluated at $\tilde{\sigma} < \sigma$. It is surprising that the right and left derivatives (corresponding to the cases of increases and decreases on σ) are the same, because in the case of a decrease in σ there is a positive mass of firms that change prices on impact. Nevertheless, this effect is of smaller order of magnitude than the change on σ. The proof works out both cases separately.

Figure 6: Uncertainty and the propagation of monetary shocks



7 Multiproduct firms

In Alvarez and Lippi (2014) we consider a model where a firm produces n different products, but it faces increasing returns in the price adjustment: if a pays a fixed cost it can adjust simultaneously the n prices. Variations on this model have been studied by Midrigan (2011), Bhattarai and Schoenle (2014). In Alvarez and Lippi (2014) we explain the sense in which such a model is realistic. We extend our results for impulse responses to this multidimemsional setup. We show that the characterization of the selection effect, as the difference between the survival function and the output IRF holds in this model, with the number of products n serving as the parameter that control selection. We also show that in this case a single eigenvalue gives a poor characterization of the output IRF.

In the multiproduct model the price gap is given by a vector of n price gaps, each of them given by an independently standard BM's $(p_1, p_2, \ldots p_n)$, driftless and with innovation variance σ^2. We are interested only on two functions of this vector, the sum of its squares and it sum:

$$y = \sum_{i=1}^{n} p_i^2 \text{ and } z = \sum_{i=1}^{n} p_i$$

It is interesting to notice that while the original state is n dimensional, (y, z) can be described as a two dimensional diffusion –see Alvarez and Lippi (2014) and Appendix C for details.

We are interested in the sum of its squares y because in Alvarez and Lippi (2014) under the assumption of symmetric demand the optimal decision rule is to adjust the firm time that y hits a critical value $\bar{y}$. We are interested in z, the sum of the price gaps, because this give the contribution of firm to the deviation of the price level relative to the steady state value, and hence $-z$ is proportional to its contribution to output. Note that the domain of (y, z) is $0 \leq y \leq \bar{y}$ and $-\sqrt{ny} \leq z \leq \sqrt{n\,y}$. In Alvarez and Lippi (2014) we show that the expected number of adjustments per unit of time is given by $N = \frac{n\sigma^2}{\bar{y}}$ and also give a characterization of $\bar{y}$ in terms of the parameters for the firm's problem. For the purpose in this paper we find it convenient to rewrite the state as (x, w) defined as

$$x = \sqrt{y} \text{ and } w = \frac{z}{\sqrt{ny}}.$$

In Lemma 2 in Appendix C we analyze the behavior of the $(x, w) \in [0, \bar{x}] \times [-1, 1]$ process with $\bar{x} \equiv \sqrt{\bar{y}}$. Clearly we can recover (y, z) from (x, w). For instance, $z = w\sqrt{nx}$. Yet with this change on variables, even though the original problem is n dimensional, we define a two dimensional process for which we can analytically find its associated eigenfunctions and eigenvalues for the operator:

$$\mathcal{G}(f)\left((x,w),t\right) = \mathbb{E}\left[f\left(x(t),w(t)\right)\, 1_{y\geq\bar{y}} \,\middle|\, (x(0),w(0)) = (x,w)\right]$$

where $f : [0,\bar{x}] \times [-1,1] \to \mathbb{R}$. The relevant p.d.e. is defined and its solution via eigenfunctions and eigenvalues, is characterized in Proposition 18 in Appendix C. Moreover the eigenfunctions and eigenvalues are indexed by a countably double infinity indices $\{m,k\}$.

Eigenfunctions. The eigenfunctions φ have a multiplicative nature, so $\varphi_{m,k}(x,w) = h_m(w)g_{m,k}(x)$ where for each number of products n then h_m and $g_{m,k}$ are known analytic functions indexed by k and by (k,m) respectively. Indeed h_m are scaled Gegenbauer polynomials, and $g_{m,k}$ are scaled Bessel functions –see Proposition C for the exact expressions and definition.[7]

Eigenvalues. For each n the eigenvalues can be also indexed by a countably double-infinity $\{\lambda_{m,k}\}$. As in the baseline case, the eigenvalues are proportional to N, the expected number of price changes per unit of time:

$$\lambda_{m,k} = -N\,\frac{\left(j_{\frac{n}{2}-1+m,k}\right)^2}{2\,n} \quad \text{for } m = 0,1,\ldots, \text{ and } k = 1,2,\ldots$$

$j_{\nu,k}$ denote the ordered zeros of the Bessel function of the first kind $J_\nu(\cdot)$ with index ν.

The second sub-index k in the root of the Bessel function denote their ordering, so $k=1$ is the smallest positive root. Also fixing k the roots $j_{m+\frac{n}{2}-1,k}$ are increasing in m. Thus, the *dominant* eigenvalue is given by $\lambda_{0,1}$. We will argue below that the smallest (in absolute value) eigenvalue that is featured in the (marginal) output IRF is $\lambda_{1,1}$. A very accurate approximation of the eigenvalues consists on using the first three leading terms in its expansion, as is given by: $j_{\nu,k} \approx \nu + \nu^{1/3}2^{-1/3}a_k + (3/20)(a_k)^2 2^{1/3}\nu^{-1/3}$ where a_k are the zeros of the Airy function. [8] Using this approximation into the expression for the eigenvalues, one can

[7]The Gegenbauer polynomials are orthogonal to each other, and so are the Bessel functions when using an appropriately weighted inner product, as defined in Appendix C.

[8]In our case, we are interested in $k=1$ which is about $a_1 = -2.33811$. See Figure 10 in the APP where

see that keeping fixed N, the absolute value both $\lambda_{0,1}$ and $\lambda_{1,1}$ go to infinity, and that the difference between the two decreases and converges to $N/2$. Figure 7 displays the difference between these two eigenvalues.

Impulse response. As before, we want to compute $G(t)$, the conditional expectation of $f : [0, \bar{x}] \times [-1, 1] \to \mathbb{R}$ for (x, w) following equation (37)-equation (36), integrated with respect to $p(w, x; 0)$. We are interested in functions $f : [0, \bar{x}] \times [-1, 1]$ that can be written as:

$$f(x, w) = \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} b_{m,k}[f] \varphi_{m,k}(x, w)$$

Using the same logic as in the one dimensional case:[9]

$$G(t) = \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} e^{\lambda_{m,k}\, t}\, b_{m,k}[f]\, b_{m,k}\left[p(\cdot, 0)/\omega\right]\, \langle \varphi_{m,k}, \varphi_{m,k} \rangle$$

where the term $\langle \varphi_{m,k}, \varphi_{m,k} \rangle$ appears because the have, as it is customary in this case, use an orthogonal, but not orthonormal base, and where $\omega(w, x)$ is a weighing function appropriately defined – see Appendix C. So that $b_{m,k}\left[p(\cdot, 0)/\omega\right]$ are the projections of the ratio of the functions $p(\cdot, 0)$ and ω.

Functions of interest. We analyze two important functions of interest f. The first one a constant, $f(w, x) = 1$ which is used to compute the measure of firms that have not adjusted, or the survival function $S(t)$. The second one is the one that gives the average price gap among the n product of the firm, i.e. $f(w, z) = -z/n = -wx/\sqrt{n}$. This is, as before, the negative of the average across the n products of the price gaps. This is the function f used for the impulse response of output to a monetary shock. An important property of the Gegenbauer polynomials is that the $m = 0$ equals a constant, for $m = 1$ is proportional to w, and in general for m odd are antisymmetric on w and symmetric for even m. Thus for $f = 1$ we can use just the Gegenbauer polynomial with $m = 0$ and all the Bessel functions

we plots both eigenvalues, as well as its approximation for several n.

[9] See Appendix C for a derivation

corresponding to $m = 0$ and $k \geq 1$. Instead for $f(w, x) = wx/sqrtn = z$ we can use just the Gegenbauer polynomial with $m = 1$ and all the Bessel functions corresponding to $m = 0$ and $k \geq 1$.

Initial shifted distribution for a small shock. We have derived the invariant distribution of (z, y) in Alvarez and Lippi (2014). Using the change in variables (y, z) to $y = x^2$ and $z = \sqrt{yn}\, w =$ we can define the steady state density as $\bar{p}(w, x) = \bar{h}(w)\bar{g}(x)$ – see Appendix C for the expressions. We perturb this density with a shock of size δ in each of the n price gaps. We want to subtract δ to each component of $(p_1, ..., p_n)$. This means that the density for each $x = ||p||$ just after the shock becomes the density of $x(\delta) = ||(p_1 + \delta, \ldots, p_n + \delta)||$ just before. Likewise the density corresponding to each w becomes th one for $w(\delta) = (z + n\delta)/(\sqrt{n}\, x(\delta))$. We consider the initial condition given by density $p_0(w, x; \delta) = \bar{h}(w(\delta))\bar{g}(x(\delta))$. We will use the first order terms, which are appropriate for the case of a small shock δ. The expressions can be found in Appendix C.

Interpretation of dominant eigenvalue, and irrelevance for the marginal IRF. We are now ready to generalize our interpretation of the dominant eigenvalue (as well as those corresponding to symmetric functions of z), as well as its irrelevance for the marginal output IRF.

Proposition 13. The coefficient of the marginal impulse response of output for a monetary shock are a function of the $\{\lambda_{1,k}, \varphi_{1,k}\}_{k=1}^{\infty}$ eigenvalue-eigenfunctions pairs, so that:

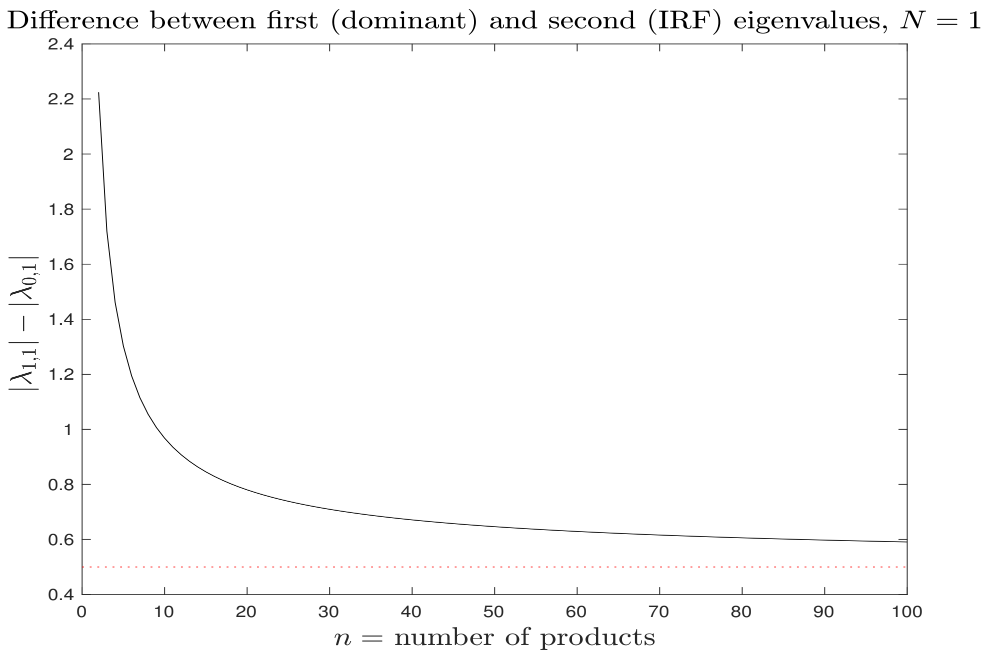
$$Y(t) = \sum_{k=1}^{\infty} b_{1,k}\, e^{\lambda_{1,k}\, t} \quad \text{and} \quad -\lambda_{1,1} = \lim_{t \to \infty} \frac{\log |Y(t)|}{t}$$

where $b_{1,k} = b_{1,k}\, [wx/\sqrt{n}]\; b_{1,k}\, [\bar{p}'(w, x)]$. In particular, the dominant eigenvalue $\lambda_{0,1}$ does not characterize the limiting behavior of the impulse response. Instead the survival function for price changes $S(t)$, can be written in terms of $\{\lambda_{0,k}, \varphi_{0,k}\}_{k=1}^{\infty}$, and hence the asymptotic hazard rate is equal to the dominant eigenvalue $\lambda_{0,1}$, i.e.

$$S(t) = \sum_{k=1}^{\infty} b_{0,k}\, e^{\lambda_{0,k}\, t} \quad \text{and} \quad -\lambda_{0,1} = \lim_{t\to\infty} \frac{\log S(t)}{t}$$

where $b_{0,k} = b_{0,k}\,[1]\; b_{0,k}\,[\delta_0]$ where δ_0 is the Dirac delta function for $(p_1, \ldots, p_n)$ transformed to the (x, w) coordinates. Recall that $0 > \lambda_{0,1} > \lambda_{1,1}$.

Figure 7: Difference of eigenvalues for multiproduct model



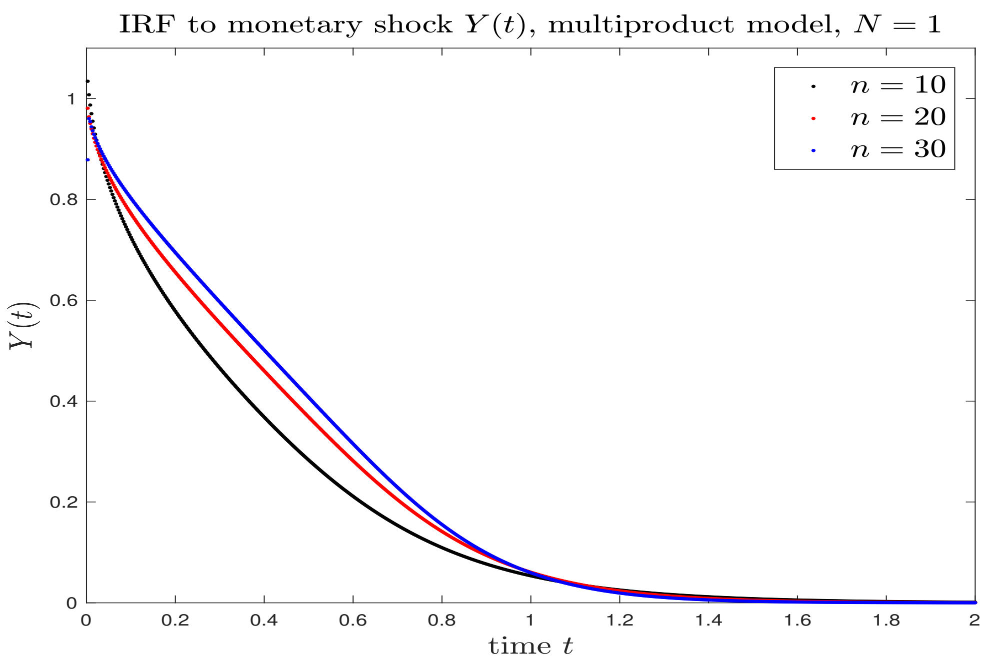
Keeping fixed $N = 1$ for all n

Given the importance of the difference between the eigenvalues $\lambda_{1,k}$ and $\lambda_{0,k}$ we show that for a fixed k they both increase with n, but it difference decreases to asymptote to $1/2$.

Proposition 14. Fixing $k \geq 1$, the k^{th} eigenvalue for the IRF $Y(\cdot)$ given by $\lambda_{1,k}$ and the k^{th} eigenvalue for the survival function $S(\cdot)$ given by $\lambda_{1,k}$ both increase with the number of products n, diverging towards $-\infty$ as $n \to \infty$. The difference $\lambda_{0,k} - \lambda_{1,k} > 0$ decreases with n, converging to $1/2$ as $n \to \infty$.

Figure 10 and Figure 7 illustrates Proposition 14 for the case of $k = 1$, i.e. the eigenvalue that dominates the long run behaviour of the survival and IRF functions. Proposition 14 extends the result for all k. Increasing the number of products n in the multi product model decreases the selection effect at the time of a price change. As n goes to infinity, the eigenvalues that control the duration of the price changes (S) and those that control the marginal output IRF (Y) converge. This result shows that the characterization of selection effect in terms of dynamics controlled by two different types of eigenvalues is present not only in the Calvo$^+$ model, but also in this setup.

In Appendix C we include Proposition 19 which gives an closed form solution for $\bar{p}'(w, x; 0)$ and for the coefficients for $b_{1,k}$ of the output impulse response function. All these expressions depends only of the number of products n. Instead we include a figure of the impulse responses for three values of n. It is clear both the output IRF and the survival function cannot be well described using one eigenfunction-eigenvalue for large n. For instance, as $n \to \infty$ the output's IRF Y becomes a linearly declining function until it hits zero at $t = 1/N$, and the survival function S is zero until it becomes infinite at $t = 1/N$.



8 Asymmetric problems: dealing with reinjection

In this section we study the impulse response function for problems where the symmetry assumptions of Proposition 2 do not hold. In such a case the computation of the impulse response function requires keeping track of firms after their first adjustment, so that the impulse response function $H(t)$ cannot be computed by means of the simpler operator $G(t)$. The solution to this problem is to compute the law of motion of the cross-sectional distribution using the Kolmogorov forward equation, keeping track of the reinjenctions that occur after the adjustments.

The nature of reinjection in our set up differs from the one in Gabaix et al. (2016) and hence we cannot use their results for the ergodic case. The added complexity of our case originates because the exit points (of our pricing problem) are not independent of where x is, as in the case of poisson adjustments. Rather, prices are changed when either barrier $\underline{x}$ or $\bar{x}$ is hit, and then the measure of products whose prices are changed are all reinjected at single value, the optimal return point x^*.

The set-up consists of an unregulated BM $dx = \mu dt + \sigma dB$, which returns to a single point x^* the first time that x hits either of the barriers $\underline{x}$ or $\bar{x}$ or that a Poisson counter with intensity ζ changes. As implied by Proposition 2 we cannot ignore the reinjections at x^* if either $x^* \neq (\bar{x} + \underline{x})/2$ or $\mu \neq 0$. For simplicity consider the case with no drift, so we set $\mu = 0$. We can use the method in Appendix B to modify the result accordingly. This set up can be used to study the problem of a firm with a non-symmetric period return function in an economy without inflation ($\mu = 0$). In this case the optimal decision rule implies $x^* \neq (\bar{x} + \underline{x})/2$, i.e. the reinjection point (after adjustment) is not located in the middle of the inaction region. Note that the number of price adjustment per unit of time is given by $N = \frac{\sigma^2}{(\bar{x}-\underline{x})^2\alpha(1-\alpha)}$.

Let $\hat{p}(x)$ denote the initial condition for the density of firms relative the invariant distribution $\bar{p}(x)$, i.e. $\hat{p}(x) = p(x) - \bar{p}(x)$ for some density p, where $\bar{p}$ is the asymmetric (steady state) tent map. Notice that to analyze small shocks δ, i.e. an initial condition $p(x) = \bar{p}(x + \delta)$ the signed measure is $\hat{p}(x) = \delta \bar{p}'(x)$ by a simple Taylor expansion and mass preservation requires that $\int_{\underline{x}}^{\bar{x}} \hat{p}(x) dx = 0$, so that

$$\hat{p}(x)/\delta = \begin{cases} \frac{2}{(\bar{x}-\underline{x})^2 \alpha} & \text{if } x \in [\underline{x}, x^*) \\ \frac{-2}{(\bar{x}-\underline{x})^2 (1-\alpha)} & \text{if } x \in (x^*, \bar{x}] \, . \end{cases} \tag{22}$$

We define the Kolmogorov forward operator $\mathcal{H}^*(\hat{p}) : [\underline{x}, \bar{x}] \times \mathbb{R}_+ \to \mathbb{R}$ for the process with reinjection, where $\mathcal{H}^*(\hat{p})(x,t)$ denotes the cross-sectional density of the firms t periods after the shock. In this case we have that for all $x \in [\underline{x}, x^*) \cup (x^*, \bar{x}]$ and for all $t > 0$:

$$\partial_t \mathcal{H}^*(\hat{p})(x,t) = \frac{\sigma^2}{2} \partial_{xx} \mathcal{H}^*(\hat{p})(x,t) - \zeta \mathcal{H}^*(\hat{p})(x,t) \tag{23}$$

with boundary conditions:

$$\mathcal{H}^*(\hat{p})(\underline{x},t) = \mathcal{H}^*(\hat{p})(\bar{x},t) = 0 \quad , \quad \lim_{x \uparrow x^*} \mathcal{H}^*(\hat{p})(x,t) = \lim_{x \downarrow x^*} \mathcal{H}^*(\hat{p})(x,t) \tag{24}$$

$$\partial_x^- \mathcal{H}^*(\hat{p})(\bar{x},t) - \partial_x^+ \mathcal{H}^*(\hat{p})(x^*,t) + \partial_x^- \mathcal{H}^*(\hat{p})(x^*,t) - \partial_x^+ \mathcal{H}^*(\hat{p})(\underline{x},t) = \frac{2\zeta}{\sigma^2} \tag{25}$$

$$\mathcal{H}^*(\hat{p})(x,0) = \hat{p}(x) \text{ for all } x \in [\underline{x}, \bar{x}] \tag{26}$$

The p.d.e. in equation (23) is standard, we just note that it does not need to hold at the reinjection point x^*. The boundary conditions in equation (24) are also standard, given that $\underline{x}$ and $\bar{x}$ are exit points, and that with $\sigma^2 > 0$, the density must be continuous everywhere. The condition in equation (25) ensures that the measure is preserved, or equivalently that there is no change in total mass across time: $\int_{\underline{x}}^{\bar{x}} \mathcal{H}^*(\hat{p})(x,t) dx = \int_{\underline{x}}^{\bar{x}} \hat{p}(x) dx$ for all t. This is a small extension of Proposition 1 in Caballero (1993).

We can use $\mathcal{H}^*$ to compute the Impulse response function defined above as follows:

$$H(t, f, \hat{p}) = \int_{\underline{x}}^{\bar{x}} f(x) \, \mathcal{H}^*(\hat{p})(x,t) \, dx \, . \tag{27}$$

If $\zeta > 0$ and $\bar{x} \to \infty$ as well as $\underline{x} = \infty$, we will have the pure Calvo case, and we can use the ideas in Gabaix et al. (2016), and thus the case without reinjection and with reinjection are quite similar. To highlight the difference, we consider the opposite case, and set $\zeta = 0$ and use an eigenvalue decomposition of $\mathcal{H}^*$.

Proposition 15. Assume that $\zeta = 0$ and that α is not a rational number. The orthonormal eigenfunctions of $\mathcal{H}^*$ are:

$$\varphi_j^m(x) = \sqrt{\frac{2}{(\bar{x} - \underline{x})}} \sin\left(\left[\frac{x - \underline{x}}{\bar{x} - \underline{x}}\right] 2\pi j\right) \text{ if } x \in [\underline{x}, \bar{x}] \tag{28}$$

$$\varphi_j^l(x) = \sqrt{\frac{2}{(x^* - \underline{x})}} \sin\left(\left[\frac{x - \underline{x}}{x^* - \underline{x}}\right] 2\pi j\right) \text{ if } x \in [\underline{x}, x^*] \text{ and 0 otherwise} \tag{29}$$

$$\varphi_j^h(x) = \sqrt{\frac{2}{(\bar{x} - x^*)}} \sin\left(\left[\frac{x - x^*}{\bar{x} - x^*}\right] 2\pi j\right) \text{ if } x \in [x^*, \bar{x}] \text{ and 0 otherwise} \ , \tag{30}$$

with corresponding eigenvalues:

$$\lambda_j^m = -\frac{\sigma^2}{2}\frac{(2\pi j)^2}{(\bar{x} - \underline{x})^2} \ , \ \lambda_j^l = -\frac{\sigma^2}{2}\frac{(2\pi j)^2}{(x^* - \underline{x})^2}, \text{ and } \lambda_j^h = -\frac{\sigma^2}{2}\frac{(2\pi j)^2}{(\bar{x} - x^*)^2} \ , \tag{31}$$

for all $j = 1, 2, \ldots$. The eigenfunctions in the set $\{\varphi_j^m\}_{j=1}^\infty$ are orthogonal to each other, and so are those in the set $\{\varphi_j^l, \varphi_j^h\}_{j=1}^\infty$. The eigenfunctions $\{\varphi_j^m, \varphi_j^l, \varphi_j^h\}_{j=1}^\infty$ span the set of functions $g : [\underline{x}, \bar{x}] \to \mathbb{R}$, piecewise differentiable, with countably many discontinuities, and with $\int_{\underline{x}}^{\bar{x}} g(x)dx = 0$.

By defining $\mathcal{H}^*$ for initial conditions given by the differences of a density relative to the density of the invariant distribution, we are excluding the zero eigenvalue and its corresponding eigenfunction, the invariant distribution $\bar{p}$ from the its representation. From the proposition we see what are the first two non-zero eigenvalues.

$$\lambda_1 = -\frac{\sigma^2}{2}\left(\frac{2\pi}{\bar{x} - \underline{x}}\right)^2 > \lambda_2 = -\frac{\sigma^2}{2}\left(\frac{2\pi}{\max\{(\bar{x} - x^*), (x^* - \underline{x})\}}\right)^2 \tag{32}$$

Notice that the difference between the first and the second eigenvalues depends on the asymmetry of the bands.

The proposition also proves that the eigenfunctions φ_j^k with $k = \{l, h, m\}$ form a base, so that projecting the initial condition onto them is possible. It is however more involved than in the symmetric case since the eigenfunctions are not all orthogonal with each other, e.g. $\langle \varphi_j^m, \varphi_j^h \rangle \neq 0$. Note however that $\{\varphi_j^m, \lambda_j^m\}$ coincide withe the antisymmetric eigenfunction and eigenvalues for the case without reinjection and $\mu = \zeta = 0$. Because of this, any piecewise differentiable function $g : [\underline{x}, \bar{x}] :\to \mathbb{R}$ that is antisymmetric around $(\underline{x} + \bar{x})/2$ can be represented, in a L^2 sense, as a Fourier series using $\{\varphi_j^m\}$.

In spite of the lack of orthogonality, the general logic for constructing the impulse response function is the same. Given the projection of the initial condition on the eigenfunctions

$$\hat{p}(x, 0) = \sum_k \sum_{j=1}^{\infty} a_j^k \varphi_j^k(x)$$

where $k = \{l, h, m\}$. We use the linearity of $\mathcal{H}^*$ to write the operator in equation (27) as

$$\mathcal{H}^*(\hat{p})(x, t) = \mathcal{H}^* \left(\sum_k \sum_{j=1}^{\infty} a_j^k \varphi_j^k \right)(x, t) = \sum_k \sum_{j=1}^{\infty} a_j^k \mathcal{H}^*(\varphi_j^k)(x, t) = \sum_k \sum_{j=1}^{\infty} a_j^k e^{\lambda_j^k t} \varphi_j^k(x)$$

where the last equality uses that the $\varphi_j^k(x)$ are eigenfunctions. Thus, given the a_j^k coefficients (whose computation is discussed below), we can write the impulse response in equation (27) as

$$H(t, f, \hat{p}) = \sum_{\{k=l,h,m\}} \sum_{j=1}^{\infty} e^{\lambda_j^k t} \, a_j^k \int_{\underline{x}}^{\bar{x}} f(x) \, \varphi_j^k(x) \, dx \, .$$

or, computing the inner products $b_j^k[f] = \int_{\underline{x}}^{\bar{x}} f(x) \, \varphi_j^k(x) \, dx$

$$H(t, f, \hat{p}) = \sum_{k=\{l,h,m\}} \sum_{j=1}^{\infty} e^{\lambda_j^k t} \, a_j^k \, b_j^k[f] \, . \tag{33}$$

A straightforward numerical approach to finding the projection coefficients a_j^k requires running a simple linear regression of $\hat{p}(x,0)$ on the basis $\{\varphi_j^h(x), \varphi_j^l(x), \varphi_j^m(x)\}_{j=1}^J$ (up to some order frequency J). For the output impulse response, given the function of interest $f(x) = -x$, the projection coefficients are also readily computed $b_j^k[f] = \int_{\underline{x}}^{\bar{x}} f(x)\varphi_j^k(x)dx$ for $k = \{m, l, h\}$, and $j = 1, 2, 3....$ which gives

$$b_j^m[f] = \frac{(\bar{x} - \underline{x})^{3/2}}{\sqrt{2}\pi j} \quad , \quad b_j^l[f] = \frac{(x^* - \underline{x})^{3/2}}{\sqrt{2}\pi j} \quad , \quad b_j^h[f] = \frac{(\bar{x} - x^*)^{3/2}}{\sqrt{2}\pi j} \quad . \tag{34}$$

Figure 8: Response to monetary shock for asymmetric problem

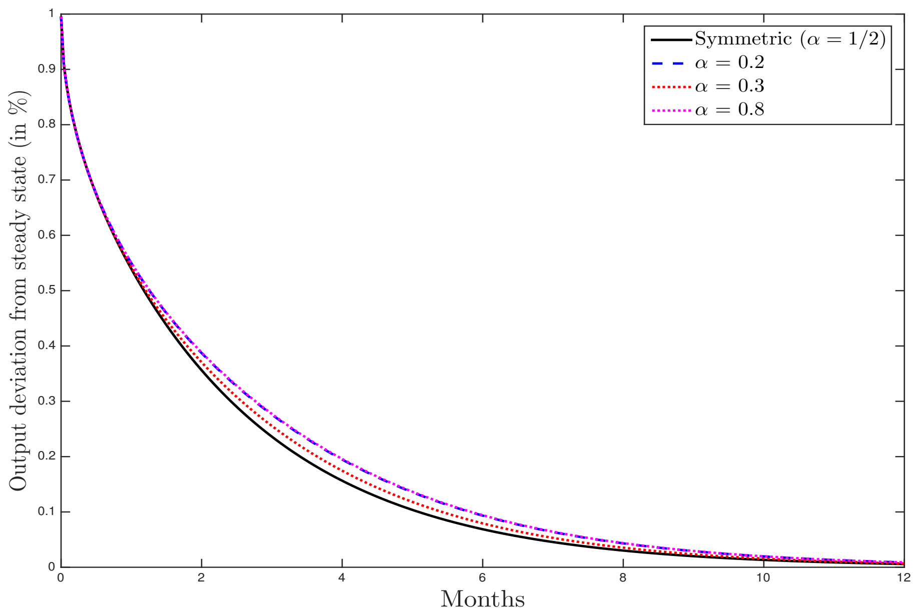


Figure 9 displays some impulse response generated by asymmetric problems where $\alpha \neq 1/2$ and contrasts them to the one produced by the symmetric problem where $\alpha = 1/2$. Two remarks are in order: first, modest degrees of asymmetry do not have a major effect on the impulse response: the impulse response function for $\alpha = 0.4$ would be barely distinguishable from the symmetric impulse response. Second, once quantitatively large asymmetries are considered, such as the small values of α considered in the figure, the impulse response becomes more persistent than the symmetric one. The presence of the asymmetry makes the convergence to the mean x value of the invariant distribution slower; this is intuitive since for symmetric problems the mean of the distribution is obtained right after the first adjustment, while this is not anymore true.

Irrelevance of the sign of the reinjection point. We discuss here the irrelevance of whether the optimal return point x^* is to the left of the interval's midpoint, as when $x^* < 0$, or to the right of it hence with $x^* > 0$. Formally we consider two problems: the first one has $\alpha = 1/2 - z$ where $z \in (0, 1/2)$ and the second problem has $\tilde{\alpha} = 1/2 + z$. We will show that, somewhat surprisingly to us, the sign of the optimal return point x^* is irrelevant for the impulse response which is the same one for the problem with α and for the one with $\tilde{\alpha}$. We have the following result

Proposition 16. Consider the inaction region for x defined by the interval $(-\bar{x}, \bar{x})$, let $z \in (0, 1/2)$ be a non-rational number. Consider a problem with reinjection point $\alpha = 1/2 - z$ and another problem with reinjection point $\tilde{\alpha} = 1/2 + z$. Then the impulse response function is the same for both problems.

Figure 9 illustrates the results of the proposition by showing that the impulse response for $\alpha = 0.2$ coincides with the one for $\alpha = 0.8$. An important implication of this property is that the derivative of the impulse response function with respect to α evaluated at $alpha = 1/2$ must be zero, which explains why small deviations from the symmetric benchmark produce results that are essentially almost indistinguishable from those produced by the symmetric case. Overall this result suggests that the symmetric benchmark is an accurate approximation of problems with modest degrees of asymmetry.

9 Conclusion and Future Work

(TBD)

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A Proofs

Proof. of Proposition 1. Using the definitions of $\mathbf{H}$, H and $\mathcal{H}$ note that for any f:

$$
\begin{aligned}
\mathbf{H}(r,f,p) &\equiv \int_0^\infty e^{-rt} H(t;f,p)dt \\
&= \int_0^\infty e^{-rt} \left[\int_{\underline{x}}^{\bar{x}} \mathcal{H}(f)(x,t)\, p(x,0)\, dx \right] dt \\
&= \int_0^\infty e^{-rt} \int_{\underline{x}}^{\bar{x}} \mathbb{E}\left[f(x(t)) \,|\, x(0)=x \right] \, p(x,0)\, dx\, dt \\
&= \int_{\underline{x}}^{\bar{x}} \int_0^\infty e^{-rt} \mathbb{E}\left[f(x(t)) \,|\, x(0)=x \right] \, p(x,0)\, dt\, dx \\
&= \int_{\underline{x}}^{\bar{x}} \mathbb{E}\left[\int_0^\infty e^{-rt} f(x(t)) dt \,|\, x(0)=x \right] \, p(x,0)\, dx
\end{aligned}
$$

where the first equality is the definition of $\mathbf{H}$, the second and third lines use the definition of H and $\mathcal{H}$, the fourth line exchanges the order of integration, the fifth line uses the linearity of the expectations operator (where p stands for a generic signed measure).

The rest of the proof consists of two steps. The first is to show that if $f(x) = R'(x)$, then

$$
v'(x) = \mathbb{E}\left[\int_0^\infty e^{-rt} f(x(t)) dt \,|\, x(0)=x \right] \text{ for all } x\,.
$$

This establishes the first equality in the proposition. The second step is to show that this expression implies that its expectation over x using the signed measure $p(\cdot,0)$ gives $\mathbf{G}(r,R',p)$, which establishes the second equality in the proposition.

For the first step we take the value function in inaction for the Calvo$^+$ problem and differentiate it with respect to x obtaining:

$$
(r+\zeta)v'(x) = R'(x) + \mu v''(x) + \frac{\sigma^2}{2} v'''(x) \text{ for } x \in [\underline{x}, \bar{x}]
$$

Moreover the boundary conditions of the Calvo$^+$ problem (smooth pasting and optimal return) give:

$$
v'(\underline{x}) = v'(\bar{x}) = v'(x^*) = 0
$$

Thus, given the ode stated above for v' and $v'(\underline{x}) = v'(\bar{x}) = v'(x^*)$, which act as value matching, then:

$$
v'(x) = \mathbb{E}\left[\int_0^\infty e^{-rt} R'(x(t)) dt \,|\, x(0)=x \right]
$$

which is the sequence problem corresponding to the Hamilton Jacobi Bellman equation written above.

For the second step we notice that $v'(x^*) = 0$ and hence:

$$
v'(x) = \mathbb{E}\left[\int_0^\tau e^{-rt} R'(x(t))\, dt + e^{-r\tau} v'(x(\tau)) \,|\, x(0)=x \right]
$$

where τ is the stopping time denoting the first time that x hits either $\underline{x}$ or $\bar{x}$ or that the free adjustment opportunity occurs. Since $x(\tau) = x^*$ and $v'(x^*) = 0$, then:

$$v'(x) = \mathbb{E}\left[\int_0^\tau e^{-rt} R'(x(t))\, dt |\, x(0) = x\right]$$

Thus to compute the cumulative impulse response it suffices to compute the stochastic integral up to the first price change. □

Proof. (of Proposition 2). Using the definitions of $\mathcal{H}$, $\mathcal{G}$ and τ we have the following recursion:

$$\mathcal{H}(f)(x,t) = \mathcal{G}(f)(x,t) + \mathbb{E}\left[1_{\{t>\tau\}}\mathcal{H}(f)(x^*, t-\tau)\middle|x\right] \text{ for all } x \in [\underline{x}, \bar{x}] \text{ and for all } t > 0.$$

We argue that $D(x,t) \equiv \mathbb{E}\left[1_{\{t>\tau\}}\mathcal{H}(f)(x^*, t-\tau)\middle|x\right]$ is symmetric in x around $x^* = (\underline{x}+\bar{x})/2$. This uses that since the law of motion is symmetric so $g(x,t)$ is symmetric. This in turn implies that the probability of hitting either barrier at time s, starting with $x(0) = x$, is symmetric in x, which directly implies the symmetry of $D(x,t)$.

We first consider case (i) for which we show that $D(x,t) = 0$ for all x, t. This follows since $\mathcal{H}(f)(x^*, s) = 0$ for all s. This in turn follows because f is antisymmetric, thus we have $\mathbb{E}\left[f(x(t))\,|x(\tau) = x^*\right] = 0$, which follows immediately by the symmetry of the distribution $g(x,t)$ and the antisymmetric property of f. It follows that $\mathbb{E}\left[1_{\{t\geq\tau\}}f(x(t))\,|x(0) = x\right] = 0$. Hence, since $\mathcal{H} = \mathcal{G}$, this implies that $G(t) = H(t)$ for any $p(\cdot, t)$.

Now we turn to case (ii) we use that $D(x,t)$ is symmetric and that

$$H(t,f,p) - G(t,f,p) = \int_{\underline{x}}^{\bar{x}} D(x,t)\left(p(x,0) - \bar{p}(x)\right)\, dx \quad .$$

Since $D(x,t)$ is symmetric and $p(x,0) - \bar{p}(x)$ is antisymmetric we have that the right hand side is zero so that $H(t) = G(t)$. □

Proof. of Lemma 1. The proof uses the linearity of $\mathcal{G}$ to write $\mathcal{G}(\hat{f} + f - \hat{f}) = \mathcal{G}(\hat{f}) + \mathcal{G}(f - \hat{f})$. The projection $\hat{f}$ converges pointwise to f at any point at which f is differentiable. Additionally, by hypothesis, f is not differentiable at most at countably many points. Finally, we have defined $\mathcal{G}(f)(x,t) = \mathbb{E}\left[1_{\{t<\tau\}}f(x(t))|x(0) = x\right]$. We note that this expected value is given by the integral that uses a continuous density, i.e. the density of BM starting at $x(0) = x$ and reaching $x(t) = y$ at t, which is continuous on y for $t > 0$. Hence the function $f - \hat{f}$ is non-zero only at countably many points, and thus its integral with respect to continuous density is zero, i.e. $\mathcal{G}(f-\hat{f})(x,t)=0$ for all x and $t>0$. Then we have

$$\begin{aligned}\mathcal{G}(f)(x,t) &= \mathcal{G}\left(\hat{f}\right)(x,t)+\mathcal{G}(f-\hat{f})(x,t)=\mathcal{G}\left(\hat{f}\right)(x,t)\\ &=\mathcal{G}\left(\sum_{j=1}^{\infty} b_j[f]\varphi_j\right)(x,t)=\sum_{j=1}^{\infty} b_j[f]\mathcal{G}\left(\varphi_j\right)(x,t)\\ &=\sum_{j=1}^{\infty} b_j[f]e^{\lambda_j t}\varphi_j(x)\end{aligned}$$

where we have used the linearity of $\mathcal{G}$, the definition of $\hat{f}$, and the form of the solution for $\mathcal{G}\left(\varphi_j\right)(x,t)$. $\square$

Proof. (of Proposition 8) Given the expression for $\bar{p}$ we get

$$\bar{p}'(x)=\begin{cases}-\frac{\theta^2\left[-e^{-\theta x}-e^{2\theta\bar{x}}e^{\theta x}\right]}{2\left[1-2e^{\theta\bar{x}}+e^{2\theta\bar{x}}\right]} & \text{if } x\in[-\bar{x},0]\\ -\frac{\theta^2\left[e^{\theta x}+e^{2\theta\bar{x}}e^{-\theta x}\right]}{2\left[1-2e^{\theta\bar{x}}+e^{2\theta\bar{x}}\right]} & \text{if } x\in[0,\bar{x}]\end{cases}$$

We can write them as:

$$b_j[\bar{p}']=\begin{cases}0 & \text{if } j \text{ is odd}\\ \left[\frac{1+\cosh(\bar{x}\theta)}{\cosh(\bar{x}\theta)-1}\right]\left[\frac{2\theta^2 j\pi}{4\theta^2\bar{x}^2+j^2\pi^2}\right] & \text{if } j \text{ is even and } \frac{j}{2} \text{ is odd}\\ -\left[\frac{2\theta^2 j\pi}{4\theta^2\bar{x}^2+j^2\pi^2}\right] & \text{if } j \text{ is even and } \frac{j}{2} \text{ is even}\end{cases}$$

A first step is to compute:

$$\int_{-\bar{x}}^{\bar{x}}\bar{p}'(x)\varphi_j(x)dx=2\int_{-\bar{x}}^{0}\bar{p}'(x)\varphi_j(x)dx \tag{35}$$

for $j=2,4,6,\ldots$. The function $\bar{p}'$ is antisymmetric and φ_j is antisymmetric for j even, with respect to $x=0$. For j odd, this integral is zero, since φ_j is symmetric.

Thus for $j=2,4\ldots$ we have:

$$\begin{aligned}&\int_{-\bar{x}}^{\bar{x}}\bar{p}'(x)\varphi_j(x)dx=\frac{\theta^2}{\left[1-2e^{\theta\bar{x}}+e^{2\theta\bar{x}}\right]}\int_{-\bar{x}}^{0}\left[e^{-\theta x}+e^{2\bar{\theta}x}e^{\theta x}\right]\sin\left(\frac{(x+\bar{x})}{2\bar{x}}j\pi\right)dx\\ &=\frac{e^{\bar{x}\theta}4\theta^2\bar{x}}{\left[1-2e^{\theta\bar{x}}+e^{2\theta\bar{x}}\right]}\frac{\left[j\pi\left(1-\cosh\left(\bar{x}\theta\right)(-1)^{j/2}\right)\right]}{4\theta^2\bar{x}^2+\pi^2 j^2}\end{aligned}$$

Thus the coefficients for j even are

$$\begin{aligned} b_j[\bar{p}'] &= \frac{1}{\bar{x}} \int_{-\bar{x}}^{\bar{x}} \bar{p}'(x) \varphi_j(x) dx \\ &= \frac{e^{\bar{x}\theta} 4\theta^2}{\left[1 - 2e^{\theta \bar{x}} + e^{2\theta \bar{x}}\right]} \frac{\left[j\pi \left(1 - \cosh\left(\bar{x}\theta\right) (-1)^{j/2}\right)\right]}{4\theta^2 \bar{x}^2 + \pi^2 j^2} \\ &= \left[\frac{1 - \cosh\left(\bar{x}\theta\right)(-1)^{j/2}}{\cosh(\bar{x}\theta) - 1}\right] \left[\frac{2\theta^2 j\pi}{4\theta^2 \bar{x}^2 + j^2 \pi^2}\right] \end{aligned}$$

Combining this with the expression for $b[f]$ we have the desired result. $\square$

Proof. (of Proposition 9) Rewriting the expression for m_2:

$$\begin{aligned} m_2(\phi) &= \frac{b_2(\phi)/\lambda_2(\phi)}{\sum_{j=1}^{\infty} b_j(\phi)/\lambda_j(\phi)} = \frac{b_2(\phi)/\lambda_2(\phi)}{Kurt(\phi)/(6\,N)} \\ &= \frac{\left[\frac{1+\cosh\left(\sqrt{2\phi}\right)}{\cosh(\sqrt{2\phi})-1}\right] \left[\frac{8(2\phi)}{4(2\phi)+4\pi^2}\right]}{N\,\ell\left(\sqrt{2\phi}\right)\left[1+\frac{\pi^2}{2\phi}\right]} \frac{N\left(\exp(\sqrt{2\phi}) + \exp(-\sqrt{2\phi}) - 2\right)^2}{(\exp(\sqrt{2\phi}) + \exp(-\sqrt{2\phi}))(\exp(\sqrt{2\phi}) + \exp(-\sqrt{2\phi}) - 2 - 2\phi)} \\ &= 2 \frac{\left[1 + \cosh\left(\sqrt{2\phi}\right)\right]}{\left[\cosh(\sqrt{2\phi}) - 1 - \phi\right]\left[1 + \frac{\pi^2}{2\phi}\right]^2} \end{aligned}$$

where the first line follows from the definition, and the first equality from the sufficient statistic result in Alvarez, Le Bihan, and Lippi (2016). The second line uses the expression for b_2, λ_2 derived above, as well as the expression for the Kurtosis derived in Alvarez, Le Bihan, and Lippi (2016). The third line uses the expression for ℓ. The remaining lines are simplifications. $\square$

Proof. (of Proposition 10)

$$\begin{aligned} \frac{\partial}{\partial t} Y(t)|_{t=0} &= -+\infty \lim_{M\to\infty} \sum_{j=1}^{M} b_j(\phi)\lambda_j(\phi) = \lim_{M\to\infty} \sum_{i=0}^{M} \left[b_{2+4i}\lambda_{2+4i} + b_{4+4i}\lambda_{4+4i}\right] \\ &= -N\ell(\phi) \lim_{M\to\infty} \sum_{i=0}^{M} 2 \left(\left[\frac{1+\cosh\left(\sqrt{2\phi}\right)}{\cosh(\sqrt{2\phi}) - 1}\right] - 1\right) \\ &= -2N\ell(\phi) \lim_{M\to\infty} M \left(\left[\frac{1+\cosh\left(\sqrt{2\phi}\right)}{\cosh(\sqrt{2\phi}) - 1}\right] - 1\right) \end{aligned}$$

which diverges towards minus infinity for any $0 \le \phi < \infty$. $\square$

Proof. (of Proposition 7) Use the definition $\mathcal{D}(t,\delta) \equiv var(x,t;\delta)$ to write

$$\frac{\partial}{\partial \delta} \mathcal{D}(t,\delta)\Big|_{\delta=0} = \frac{\partial}{\partial \delta} \mathbb{E}\left[x(t)^2;\delta\right]\Big|_{\delta=0} - \frac{\partial}{\partial \delta} \left(\mathbb{E}\left[x(t);\delta\right]\right)^2\Big|_{\delta=0}$$

which gives

$$\frac{\partial}{\partial \delta}\mathcal{D}(t,\delta)\Big|_{\delta=0} = \frac{\partial}{\partial \delta}\mathbb{E}\left[x(t)^2;\delta\right]\Big|_{\delta=0} - 2\left(\mathbb{E}\left[x(t);0\right]\right)\frac{\partial}{\partial \delta}\left(\mathbb{E}\left[x(t);\delta\right]\right)\Big|_{\delta=0}$$

Next we argue that the right hand side is zero. First notice that $\frac{\partial}{\partial\delta}\mathbb{E}\left[x(t)^2;\delta\right]\Big|_{\delta=0} = 0$ since this is the impulse response of a symmetric function of interest $f = x^2$ its projection on all antisymmetric eigenfunctions is zero, i.e. the projection coefficients $b_{2j-1}[f] = 0$ for $j = 1,2,....$ Likewise, since $\hat{p}$ is antisymmetric, then its projection on all symmetric eigenfunctions is zero i.e. the projection coefficients $b_{2j}[\hat{p}] = 0$ for $j = 1,2,....$ Hence the impulse response of $x(t)^2$ to an aggregate shock is zero at all t. Second, notice that $\mathbb{E}\left[x(t);0\right] = 0$ by definition since x is symmetrically distributed around zero. The extension to other even moments of x is straightforward. $\square$

Proof. (of Proposition 12) First we consider case (i), i.e. the *long run* effect of a volatility shock $\frac{d\sigma}{\sigma}$, so that $\tilde{\sigma} = \left(1+\frac{d\sigma}{\sigma}\right)\sigma$ on the impulse response of output to a monetary shock. We note that the expression for $Y(t)$ derived above does not feature $\bar{x}$, which is a function of σ. Indeed the only place where σ enters in the expression for $Y(t)$ is on the eigenvalues. Since, $\bar{x} = \left(6\frac{\psi}{B}\sigma^2\right)^{\frac{1}{4}}$, then $d\log\bar{x} = 1/2 d\log\sigma$ and $d\log N = 2\left(d\log\bar{x} - d\log\sigma\right)$, hence $d\log N = d\log\sigma$. Substituting this into the eigenvalue $\lambda_j = -\tilde{N}(j\pi)^2/8 = -(1+\frac{d\sigma}{\sigma})N(j\pi)^2/8$ were N is the average number of price changes before the volatility shock. Using the expression for the impulse response in terms of the post-shock objects we have:

$$\tilde{Y}(t) = \sum_{j=1}^{\infty} b_j[f]b_j[\bar{p}']e^{-\left(1+\frac{d\sigma}{\sigma}\right)N\frac{(j\pi)^2}{8}t} = \sum_{j=1}^{\infty} b_j[f]b_j[\bar{p}']e^{-N\frac{(j\pi)^2}{8}t\left(1+\frac{d\sigma}{\sigma}\right)} = Y\left(t\left(1+\frac{d\sigma}{\sigma}\right)\right)$$

and we obtain the desired result.

Now we consider case (ii), i.e. the *impact* effect of a volatility shock $\frac{d\sigma}{\sigma}$, so that $\tilde{\sigma} = \left(1+\frac{d\sigma}{\sigma}\right)\sigma$ on the impulse response of output to a monetary shock. As in the previous case the eigenvalues can be written as functions of the shock and the old value of the expected number of price changes. Also as the previous case we have $f(x) = -x$. The difference is on the initial distribution $p(x,0)$. The initial condition is given by $p(x,0) = \bar{p}(x+\delta;\bar{x}(\sigma))$ where we write $\bar{x}(\sigma))$ to indicate that the distribution depends on σ. Indeed, since we are using the expression for $\tilde{Y}(t)$ in terms of the value of $\bar{x}$ that correspond to the post-shock value of σ, we need to consider the effect of $\bar{x}$ of a decrease of σ in the proportion $d\sigma/\sigma$. We will take a second order expansion of $p(x,0) = \bar{p}(x+\delta; \bar{x}(\sigma))$ in with respect to δ and σ.

$$
\begin{aligned}
p(x;0) = \bar{p}(x+\delta;\bar{x}(\sigma)) &= \bar{p}(x) + \frac{\partial}{\partial x}\bar{p}(x+\delta;\bar{x}(\sigma))\big|_{\delta=0}\delta - \frac{\partial}{\partial \bar{x}}\bar{p}(x+\delta;\bar{x}(\sigma))\big|_{\delta=0}\frac{\partial \bar{x}(\sigma)}{\partial \sigma}d\sigma \\
&+ \frac{1}{2}\frac{\partial^2}{\partial x^2}\bar{p}(x+\delta;\bar{x}(\sigma))\big|_{\delta=0}\delta^2 \\
&+ \frac{1}{2}\frac{\partial^2}{\partial^2 \bar{x}}\bar{p}(x+\delta;\bar{x}(\sigma))\big|_{\delta=0}\left(\frac{\partial \bar{x}(\sigma)}{\partial \sigma}\right)^2 d\sigma^2 + \frac{1}{2}\frac{\partial}{\partial \bar{x}}\bar{p}(x+\delta;\bar{x}(\sigma))\big|_{\delta=0}\frac{\partial^2 \bar{x}(\sigma)}{\partial \sigma^2}d\sigma^2 \\
&- \frac{\partial^2}{\partial \bar{x}\partial x}\bar{p}(x+\delta;\bar{x}(\sigma))\big|_{\delta=0}\frac{\partial \bar{x}(\sigma)}{\partial \sigma}d\sigma\delta + o\left(||(\delta, d\sigma)||^2\right)
\end{aligned}
$$

for $x \in [\underline{x}, \bar{x}]$ and $x \neq 0$. Recall that:

$$
\bar{p}(\delta + x; \bar{x}) = \begin{cases} +\frac{x+\delta}{\bar{x}^2} & \text{if } x \in [-\bar{x}, 0) \\ -\frac{x+\delta}{\bar{x}^2} & \text{if } x \in (0, \bar{x}] \end{cases}
$$

Using this functional form we have:

$$
\frac{\partial}{\partial x}\bar{p}(\delta + x; \bar{x}) = \begin{cases} +\frac{1}{\bar{x}^2} & \text{if } x \in [-\bar{x}, 0) \\ -\frac{1}{\bar{x}^2} & \text{if } x \in (0, \bar{x}] \end{cases}
$$

$$
\frac{\partial}{\partial \bar{x}}\bar{p}(\delta + x; \bar{x})\big|_{\delta=0} = \begin{cases} -\frac{x}{\bar{x}^2}\frac{2}{\bar{x}} & \text{if } x \in [-\bar{x}, 0) \\ +\frac{x}{\bar{x}^2}\frac{2}{\bar{x}} & \text{if } x \in (0, \bar{x}] \end{cases}
$$

a symmetric function of x.

$$
\frac{\partial^2}{\partial x \partial \bar{x}}\bar{p}(x+\delta; \bar{x}) = \begin{cases} -\frac{1}{\bar{x}^2}\frac{2}{\bar{x}} & \text{if } x \in [-\bar{x}, 0) \\ +\frac{1}{\bar{x}^2}\frac{2}{\bar{x}} & \text{if } x \in (0, \bar{x}] \end{cases}
$$

$$
\frac{\partial^2}{\partial \bar{x}^2}\bar{p}(\delta + x; \bar{x}) = \begin{cases} +\frac{x}{\bar{x}^2}\frac{6}{\bar{x}^2} & \text{if } x \in [-\bar{x}, 0) \\ -\frac{x}{\bar{x}^2}\frac{6}{\bar{x}^2} & \text{if } x \in (0, \bar{x}] \end{cases}
$$

also a symmetric function of x. Finally we have $\frac{\partial^2}{\partial x^2}\bar{p}(x+\delta;\bar{x}) = 0$.

Now we compute the terms $b_j[f]b_j[p(\cdot, 0)]$. The first order term for $d\sigma$ is zero because f is antisymmetric and the first derivative with respect to $\bar{x}$ are symmetric. The second order terms for $d\sigma^2$ are zero since f is antisymmetric and the first and second derivative with respect to $\bar{x}$ are symmetric. The second order term δ^2 is zero because the second derivative with respect to δ is zero. This leave us with two non-zero term. The first order term on δ, which is the term for the IRF with respect to a monetary shock, and the second order term corresponding to the cross-derivative. For this term we note that, using that $\bar{x}$ has elasticity

1/2 with respect to σ,

$$\begin{aligned}
\frac{\partial}{\partial \delta}\bar{p}(\delta + x; \bar{x})\delta &= -\frac{\partial^2}{\partial \delta \partial \bar{x}}\bar{p}(x+\delta;\bar{x})\frac{\partial \bar{x}(\sigma)}{\partial \sigma}d\sigma\delta \\
&= -\frac{\partial^2}{\partial \delta \partial \bar{x}}\bar{p}(x+\delta;\bar{x})\bar{x}(\sigma)\left[\frac{\partial \bar{x}(\sigma)}{\partial \sigma}\frac{\sigma}{\bar{x}(\sigma)}\right]\frac{d\sigma}{\sigma}\delta \\
&= -\frac{2}{\bar{x}(\sigma)}\frac{\partial}{\partial \delta}\bar{p}(\delta + x;\bar{x})\bar{x}(\sigma)\frac{1}{2}\frac{d\sigma}{\sigma}\delta \\
&= -\frac{\partial}{\partial \delta}\bar{p}(\delta + x;\bar{x})\frac{d\sigma}{\sigma}\delta
\end{aligned}$$

Thus we have that each j term is given by the sum of the terms corresponding to the first order term on δ and the second-order term corresponding to the cross-derivative:

$$b_j[f]b_j\left[\bar{p}'(\cdot)\right]\delta + b_j[f]b_j\left[\bar{p}'(\cdot)\right]\delta\frac{d\sigma}{\sigma} = b_j[f]b_j\left[\bar{p}'(\cdot)\right]\delta\left(1+\frac{d\sigma}{\sigma}\right)$$

This gives the second adjustment for $\tilde{Y}(t)$ in terms of the expression for $Y(t)$. $\square$

Proof. (of Proposition 11) The slope of the impulse response function at $t=0$ is given by:

$$\begin{aligned}
\frac{\partial}{\partial t}Y(t)|_{t=0} &= \lim_{M\to\infty}\sum_{j=1}^{M} b_j(\phi)\lambda_j(\phi) = \lim_{M\to\infty}\sum_{i=0}^{M}\left[b_{2+4i}\lambda_{2+4i} + b_{4+4i}\lambda_{4+4i}\right] \\
&= -N\ell(\phi)\lim_{M\to\infty}\sum_{i=0}^{M} 2\left(\left[\frac{1+\cosh\left(\sqrt{2\phi}\right)}{\cosh(\sqrt{2\phi})-1}\right](1-2\rho(\phi))-1\right) \\
&= -N\ell(\phi)\lim_{M\to\infty} 2M\left(\left[\frac{1+\cosh\left(\sqrt{2\phi}\right)}{\cosh(\sqrt{2\phi})-1}\right](1-2\rho(\phi))-1\right)
\end{aligned}$$

Replacing the value of $\rho(\phi)$ we have hat

$$\begin{aligned}
&\left[\frac{1+\cosh\left(\sqrt{2\phi}\right)}{\cosh(\sqrt{2\phi})-1}\right]\left[1-2\left[\frac{\exp\left(\sqrt{2\phi}\right)-\exp\left(-\sqrt{2\phi}\right)-2\sqrt{2\phi}}{\sqrt{2\phi}\left(\exp\left(\sqrt{2\phi}\right)+\exp\left(\sqrt{-2\phi}\right)-2\right)}\right]\right]-1 \\
&= \left[\frac{1+\cosh\left(\sqrt{2\phi}\right)}{\cosh(\sqrt{2\phi})-1}\right]\left[\frac{\sqrt{2\phi}\cosh\left(\sqrt{2\phi}\right)-2\sinh\left(\sqrt{2\phi}\right)+\sqrt{2\phi}}{\sqrt{2\phi}\left(\cosh\left(\sqrt{2\phi}\right)-1\right)}\right]-1
\end{aligned}$$

Thus, this expression is equal to zero at $\bar{\phi}$ solving:

$$\begin{aligned}
&\sqrt{2\phi}\left(\cosh\left(\sqrt{2\phi}\right)-1\right)^2 \\
&= \left[1+\cosh\left(\sqrt{2\phi}\right)\right]\left[\sqrt{2\phi}\cosh\left(\sqrt{2\phi}\right)-2\sinh\left(\sqrt{2\phi}\right)+\sqrt{2\phi}\right]
\end{aligned}$$

Analysis of this functions shows that $\bar{\phi} = 1$. We can also check that at $\phi = 1$ we have:

$$\frac{\partial^2}{\partial t^2} Y(t)|_{t=0} = \lim_{M\to\infty} \sum_{j=1}^{M} b_j(\phi)\lambda_j(\phi) = \lim_{M\to\infty} \sum_{i=0}^{M} [b_{2+4i}\lambda_{2+4i}\lambda_{2+4i} + b_{4+4i}\lambda_{4+4i}\lambda_{4+4i}]$$
$$= N\ell(\phi) \lim_{M\to\infty} \sum_{i=0}^{M} 2 \left(\left[\frac{1+\cosh\left(\sqrt{2\phi}\right)}{\cosh(\sqrt{2\phi})-1} \right] (1-2\rho(\phi))\,|\lambda_{2+4i}| - |\lambda_{4+4i}| \right) = -\infty$$

since $|\lambda_{2+4i}| > |\lambda_{4+4i}$ and since:

$$\left[\frac{1+\cosh\left(\sqrt{2\phi}\right)}{\cosh(\sqrt{2\phi})-1} \right] (1-2\rho(\phi)) - 1 = 0 \quad \text{at} \quad \phi = 1\,.$$

From here we conclude two things. At $\phi = 1$ the slope of the impulse response at $t = 0$ is zero, it is a maximum. But the first derivative changes radically for strictly positive t. Thus, the "hump" is extremely short for ϕ just above one. □

Proof. (of Proposition 15) The proof consists on checking that the functions described are the only ones that satisfy the sufficient conditions equation (23), equation (24), equation (25) and equation (26) for $\mathcal{H}^*(\varphi_j^k(x)e^{\lambda_j^k t})$ for $k \in l, m, j$ and $j = 1, 2, \ldots$.

First, let's consider the case of eigenvalues $\lambda \neq 0$. In this case the only non-constant real function that satisfy the o.d.e.: $\lambda\varphi(x) = \partial_{xx}\varphi(x)\sigma^2/2$ for $\lambda < 0$ is $\varphi(x) = \sin(\phi + \omega x)$ for some ϕ and for $\lambda = -\frac{\sigma^2}{2}\omega^2$. The cases below use this characterization when the function is not constant, to determine the values of ϕ and ω.

Second, consider the case of functions that are differentiable in the entire domain $[\underline{x}, \bar{x}]$. The continuity at $x = x^*$ is satisfied immediately. In this case, the o.d.e. : $\lambda\varphi(x) = \partial_{xx}\varphi(x)\sigma^2/2$, with boundaries equation (24) and equation (25) is satisfied only by $\varphi(x) = \varphi_j^m(x)$ for all $j = 1, 2, ...$ This gives the particular value of ϕ and ω, for $j = 1, 2, \ldots$, and hence no other differentiable function different from zero satisfy all the conditions.

Third, consider the case of functions $\varphi(x)$ which are constant in an interval of strictly positive length included in $[\underline{x}, x^*]$. Then $\varphi(x) = 0$, to satisfy the boundary condition equation (24) at $x = \underline{x}$. Then $\varphi(x) = 0$ for all $x \in [\underline{x}, x^*)$, since φ can only be non-differentiable at $x = x^*$. Then, for $x \in (x^*, \bar{x}]$ it has to be differentiable, non-identically equal to zero, satisfy $\varphi(x^*) = 0$ so that it is continuous at $x = x^*$, and also $\varphi(\bar{x}) = 0$, to satisfy the boundary condition equation (24) at $x = \bar{x}$. Finally, to satisfy the measure preserving condition equation (25), it has to be of the form of $\varphi_j^l(x)$ for $j = 1, 2, \ldots$.

Fourth, consider the case of functions $\varphi(x)$ which are constant in an interval of strictly positive length included in $[x^*, \bar{x}]$. Following the same steps as in the previous case, we obtain that $\varphi(x) = \varphi_j^h(x)$ for $j = 1, 2, \ldots$ for this case.

For the fifth and remaining case, we consider the case of functions $\varphi(x)$ which are non-constant for all intervals included in $[\underline{x}, \bar{]}$, and that $\varphi(x)$ is not differentiable at $x = x^*$. To satisfy the o.d.e. in each segment $[\underline{x}, x^*)$ and $(x^*, \bar{x}]$ then we must have $\varphi(x) = \sin\left(\underline{\phi} + \underline{\omega} x\right)$ and $\varphi(x) = \sin\left(\bar{\phi} + \bar{\omega} x\right)$ in each of the respective segments. Since the eigenvalue has to be the same for all segments, then we have that $\underline{\omega} = \bar{\omega} \equiv \omega$. The eigenfunction φ must be measure preserving, so that

$$0 = \cos\left(\underline{\phi} + \omega x^*\right) - \cos\left(\underline{\phi} + \omega \underline{x}\right) + \cos\left(\bar{\phi} + \omega \bar{x}\right) - \cos\left(\bar{\phi} + \omega x^*\right)$$

To satisfy the boundary conditions equation (24) we require $\sin\left(\underline{\phi} + \omega \underline{x}\right) = \sin\left(\bar{\phi} + \omega \bar{x}\right) = 0$. Thus $\cos\left(\underline{\phi} + \omega \underline{x}\right) = \pm 1$ and $\cos\left(\bar{\phi} + \omega \bar{x}\right) = \pm 1$. Hence, we have that:

$$\text{either } 0 = \cos\left(\underline{\phi} + \omega x^*\right) - \cos\left(\bar{\phi} + \omega x^*\right) \text{ or } \pm 2 = \cos\left(\underline{\phi} + \omega x^*\right) - \cos\left(\bar{\phi} + \omega x^*\right)$$

In the first case we have:

$$0 = \cos\left(\underline{\phi} + \omega x^*\right) - \cos\left(\bar{\phi} + \omega x^*\right) \text{ and } 0 = \sin\left(\underline{\phi} + \omega x^*\right) - \sin\left(\bar{\phi} + \omega x^*\right)$$

so the function is differentiable at $x = x^*$, which is a contradiction. So we must have the second case, and because eigenfunctions are defined up to sign, must have:

$$2 = \cos\left(\underline{\phi} + \omega x^*\right) - \cos\left(\bar{\phi} + \omega x^*\right) \text{ and } 0 = \sin\left(\underline{\phi} + \omega x^*\right) - \sin\left(\bar{\phi} + \omega x^*\right)$$

Using the properties of cos it must be the case that $\sin\left(\underline{\phi} + \omega x^*\right) = \sin\left(\bar{\phi} + \omega x^*\right) = 0$. Then, φ must be zero in the extremes of each of the two following segment $[\underline{x}, x^*]$ and $[x^*, \bar{x}]$. This requires that $[\underline{x}, x^*]$ and $[x^*, \bar{x}]$ be an in an multiple integer of each other, since in each of the segments φ is a sine function with the same frequency ω which is zero at the two extremes. But this violate that $[\underline{x}, x^*]/[x^*, \bar{x}]$ is not rational.

Now we show that the eigenfunctions span the densities for the signed measures. It suffices to show that if $g : [\underline{x}, \bar{x}] \to \mathbb{R}$ is in the domain of $\mathcal{H}^*$, and $\langle g, \varphi_j^m \rangle = \langle g, \varphi_j^l \rangle = \langle g, \varphi_j^h \rangle = 0$ for all $j = 1, 2, \ldots$, then it must be that $g = 0$.

As a way of contradiction, suppose we have a function $g \neq 0$ that g is orthogonal to all the eigenfunctions. Given that the eigenfunctions can span antisymmetric functions defined in different domains as explained above, it must be that g is a symmetric function as defined in $[\underline{x}, \bar{x}]$, so that it is orthogonal to $\{\varphi_j^m\}_{j=1}^{\infty}$, and also a symmetric in the following restricted domains $[\underline{x}, x^*]$ and $[x^*, \bar{x}]$, so that g is also orthogonal each of eigenfunctions $\{\varphi_j^l\}_{j=1}^{\infty}$ and $\{\varphi_j^h\}_{j=1}^{\infty}$ when defined in the restricted domains.

Now, without loss of generality, assume that $x^* < (\underline{x} + \bar{x})/2$. Below we sketch a proof that for a function g to be even (or symmetric) in these three domains, it must be the case that $[\bar{x} - x^*]$ is an integer multiple of $[x^* - \underline{x}]$, which contradicts the assumption that $[x^* - \underline{x}]/[\bar{x} - x^*]$ is not a rational number.

Let $L = x^* - \underline{x}$. To arrive to this conclusion we first notice that since g must be symmetric in the entire domain $[\underline{x}, \bar{x}]$, then it must be the case that g has identical symmetric shape in the segment $[\underline{x}, \underline{x} + L]$ than in the segment $[\bar{x} - L, \bar{x}]$. Then using that g is symmetric in the restricted domain $[x^*, \bar{x}]$, it must be that it also has the same symmetric shape in the interval $[x^*, x^* + L]$ than in both intervals $[\underline{x}, \underline{x} + L]$ and $[\bar{x} - L, \bar{x}]$. If it is the case that $x^* + L = \bar{x} - L$, then $[\bar{x} - x^*]$ is an integer multiple of $[x^* - \underline{x}]$, and find a contradiction. If this is not the case, i.e. if $x^* + L < \bar{x} - L$, we use the g is symmetric in the entire domain, to say that again g must take the same symmetric shape in the interval $[\bar{x} - 2L, \bar{x} - L]$. Now either $x * + L = \bar{x} - 2L$, which gives a contradiction, or we continue using the symmetry of g in either the entire domain $[\underline{x}, \bar{x}]$ or in the restricted domain $[x^*, \bar{x}]$ until we get that $[\bar{x} - x^*]$

is an integer multiple of $[x^* - \underline{x}]$, which is a contradiction with our assumption. Formally, this can be set up as an induction step, but it requires to develop enough notation, which we skip to shorten. $\square$

Proof. of Proposition 16. Consider the first problem with $\alpha < 1/2$. Normalize (WLOG) the interval width to $2\bar{x} = 1$ and rewrite the initial condition as $\hat{p} = \hat{p}^s + \hat{p}^a$, respectively the symmetric and antisymmetric component as

$$\hat{p}^s(x) = \begin{cases} \frac{1-2\alpha}{\alpha(1-\alpha)} \\ \frac{-1}{(1-\alpha)} \end{cases} \quad , \quad \hat{p}^a(x) = \begin{cases} \frac{1}{\alpha(1-\alpha)} & \text{for} \quad x \in (-\bar{x}, -z) \cup (z, \bar{x}) \\ 0 & \text{for } x \in (-z, z) \end{cases}$$

Notice that for $\alpha = 1/2 - z$ the slope of the antisymmetric part is either zero or $\frac{1}{1/4-z^2}$. The same obtains for $\tilde{\alpha} = 1/2 + z$. Thus the asymmetric component of the initial condition $\hat{p}^a(x)$ is the same for α and for $\tilde{\alpha}$. The symmetric component $\hat{p}^s(x)$ is as follows

$$\hat{p}^s(x, \alpha) = \begin{cases} \frac{2z}{1/4-z^2} \\ \frac{-1}{1/2+z} \end{cases} \quad , \quad \hat{p}^s(x, \tilde{\alpha}) = \begin{cases} \frac{-2z}{1/4-z^2} & \text{for} \quad x \in (-\bar{x}, -z) \cup (z, \bar{x}) \\ \frac{1}{1/2+z} & \text{for } x \in (-z, z) \end{cases}$$

which reveals that the symmetric component of the initial condition for the problem with $\tilde{\alpha}$ is given by -1 times the symmetric component of the initial condition for the problem with α.

Now let's consider the consequences for the output impulse response as defined in equation (27). For the problem with α we use the decomposition $\hat{p} = \hat{p}^s + \hat{p}^a$ and the linearity of $\mathcal{H}^*$ to write the IRF as

$$H_\alpha(t, f, \hat{p}(\alpha)) = H_\alpha(t, f, \hat{p}^a(\alpha)) + H_\alpha(t, f, \hat{p}^s(\alpha))$$

where we use the subscript to emphasize that this is the impulse response for the problem with reinjection point α. Using the properties for the initial condition associated to the problem with $\tilde{\alpha}$ discussed above we can the write its impulse response as

$$H_{\tilde{\alpha}}(t, f, \hat{p}(\tilde{\alpha})) = H_{\tilde{\alpha}}(t, f, \hat{p}^a(\alpha)) + H_{\tilde{\alpha}}(t, f, -\hat{p}^s(\alpha))$$

where we used that $\hat{p}^a(\alpha) = \hat{p}^a(\tilde{\alpha})$ and that $\hat{p}^s(\alpha) = -\hat{p}^s(\tilde{\alpha})$.

It is immediate to see that $H_\alpha(t, f, \hat{p}^a(\alpha)) = H_{\tilde{\alpha}}(t, f, \hat{p}^a(\alpha))$, i.e. that the IRF component triggered by the asymmetric part of the initial condition, is the same in both problems. This follows since $\hat{p}^a(\alpha) = \hat{p}^a(\tilde{\alpha})$ and because both problems share the same identical base for asymmetric functions, given by the eigenfunctions φ_j^m.

Finally, we argue that $H_\alpha(t, f, \hat{p}^s(\alpha)) = H_{\tilde{\alpha}}(t, f, -\hat{p}^s(\alpha))$. To see this notice that the symmetric part of the impulse response function is obtained by projecting the initial condition on the orthogonalized symmetric eigenfunctions v_j^k, where $k = \{l, h\}$, produced by e.g. the Gram-Schmidt algorithm. The key is to notice that the symmetrized eigenfunction for the problem with $\tilde{\alpha}$, equals -1 times the eigenfunctions for the problem with α, formally $v_j^k(\alpha) = -v_j^k(\tilde{\alpha})$. Inspection if the eigenfunctions φ_j^h and φ_j^l reveals that, for all $x \in (-\bar{x}, \bar{x})$ they obey $\varphi_1^h(x; x^* = -z) = -\varphi_1^l(-x; x^* = z)$. It therefore follows that $H_\alpha(t, f, \hat{p}(\alpha)) = H_{\tilde{\alpha}}(t, f, \hat{p}(\tilde{\alpha}))$.

$\square$

B Symmetric problem with drift (inflation)

In this section we introduce a non-zero drift μ to the process for x and solve for the eigenfunctions and eigenvalues for $\mathcal{G}(f)(x,t)$, i.e. for the process without reinjection. To lighten the notation we use $g(x,t) = \mathcal{G}(f)(x,t)$.

Proposition 17. Assume that the process has a drift μ, and variance σ^2, so the Kolmogorov backward equation is:

$$\partial_t \mathcal{G}(f)(x,t) = \partial_x \mathcal{G}(f)(x,t)\mu + \partial_{xx}\mathcal{G}(f)(x,t)\frac{\sigma^2}{2} \text{ for all } x \in [\underline{x}, \bar{x}]$$

with boundary conditions $\mathcal{G}(f)(\underline{x},t) = \mathcal{G}(f)(\bar{x},t) = 0$ for all $t > 0$ and $\mathcal{G}(f)(x,0) = f(x)$ for all x. The eigenvalues, eigenfunctions, projections, and inner product are given by:

$$\begin{aligned}
\lambda_j &= -\left[\frac{1}{2}\frac{\mu^2}{\sigma^2} + \frac{\sigma^2}{2}\left(\frac{j\,\pi}{\bar{x}-\underline{x}}\right)^2\right] \text{ for all } j = 1, 2, \ldots \\
\varphi_j(x) &= \sqrt{\frac{2}{\bar{x}-\underline{x}}}\,\sin\left(\left[\frac{x-\underline{x}}{\bar{x}-\underline{x}}\right] j\pi\right) e^{-\frac{\mu}{\sigma^2}x} \text{ for all } x \in [\underline{x}, \bar{x}] \text{ where} \\
b_j[f] &= \frac{\langle f, \varphi_j\rangle}{\langle \varphi_j, \varphi_j\rangle} \text{ where } \langle a, b\rangle \equiv \int_{\underline{x}}^{\bar{x}} a(x)b(x)e^{2\frac{\mu}{\sigma^2}x}dx
\end{aligned}$$

Thus the solution for $\mathcal{G}(f)$ is:

$$\mathcal{G}(f)(x,t) = \sum_{j=1}^{\infty} e^{\lambda_j t} b_j[f]\,\varphi_j(x) \text{ for all } t \geq 0 \text{ and } x \in [\underline{x}, \bar{x}].$$

Proof. (of Proposition 17) To lighten the notation, denote $g(x,t) = \mathcal{G}(f)(x,t)$. We start by rewriting g as

$$g(x,t) = h(x,t)s(x)$$

for some function s, so that the Kolmogorov backward equation is:

$$s(x)\partial_t h(x,t) = [s'(x)h(x,t) + s(x)\partial_x h(x,t)]\,\mu + [s''(x)h(x,t) + 2s'(x)\partial_x h(x,t) + s(x)\partial_{xx}h(x,t)]\,\frac{\sigma^2}{2}$$

We will take $s(x) = \exp(ax)$ for some constant a to be determined. Replacing this function and its derivatives, and cancelling we get

$$\begin{aligned}
\partial_t h(x,t) &= [ah(x,t) + \partial_x h(x,t)]\,\mu + \left[a^2 h(x,t) + 2a\partial_x h(x,t) + \partial_{xx}h(x,t)\right]\frac{\sigma^2}{2} \\
&= h(x,t)\left[a\mu + a^2\frac{\sigma^2}{2}\right] + \partial_x h(x,t)\left[\mu + a\sigma^2\right] + \partial_{xx}h(x,t)\frac{\sigma^2}{2}
\end{aligned}$$

Setting $a = -\frac{\mu}{\sigma^2}$ into the p.d.e. for h we have:

$$\begin{aligned}
\partial_t h(x,t) &= h(x,t)\left[-\frac{\mu^2}{\sigma^2} + \frac{\mu^2}{\sigma^4}\frac{\sigma^2}{2}\right] + \partial_{xx}h(x,t)\frac{\sigma^2}{2} \\
&= h(x,t)\left[-\frac{\mu^2}{\sigma^2} + \frac{1}{2}\frac{\mu^2}{\sigma^2}\right] + \partial_{xx}h(x,t)\frac{\sigma^2}{2} \\
&= -h(x,t)\frac{1}{2}\frac{\mu^2}{\sigma^2} + \partial_{xx}h(x,t)\frac{\sigma^2}{2}
\end{aligned}$$

Since $s(x) \neq 0$, the boundary conditions for h are the same as for g, namely $h(\underline{x},t) = h(\bar{x},t) = 0$. The equation for the eigenvalues-eigenfunctions for h is the same as in the Calvo+ model

$$\left(\lambda_j + \frac{1}{2}\frac{\mu^2}{\sigma^2}\right)\phi_j(x) = \partial_{xx}\phi_j(x)\frac{\sigma^2}{2}$$

so that we have the same expression for the eigenvalues-eigenfuctions:

$$\begin{aligned}
\lambda_j &= -\left[\frac{1}{2}\frac{\mu^2}{\sigma^2} + \frac{\sigma^2}{2}\left(\frac{j\,\pi}{\bar{x}-\underline{x}}\right)^2\right] \text{ for all } j = 1,2,\ldots \\
\phi_j(x) &= \sin\left(\left[\frac{x-\underline{x}}{\bar{x}-\underline{x}}\right] j\pi\right) \text{ for all } x \in [\underline{x},\bar{x}]
\end{aligned}$$

Thus we can write the solution for h as:

$$h(x,t) = \sum_{j=1}^{\infty} e^{\lambda_j t}\,\frac{\int_{\underline{x}}^{\bar{x}} h(x',0)\phi_j(x')dx'}{\int_{\underline{x}}^{\bar{x}}(\phi_j(x'))^2 dx'}\,\phi_j(x)$$

Multiplying both sides by $s(x)$:

$$h(x,t)s(x) = \sum_{j=1}^{\infty} e^{\lambda_j t}\,\frac{\int_{\underline{x}}^{\bar{x}} h(x',0)\phi_j(x')dx'}{\int_{\underline{x}}^{\bar{x}}(\phi_j(x'))^2 dx'}\,\phi_j(x)s(x)$$

Thus we define

$$\begin{aligned}
\langle a,b\rangle &= \int_{\underline{x}}^{\bar{x}} a(x)b(x)\frac{1}{s(x)^2}dx \text{ and} \\
\varphi_j(x) &= \phi_j(x)s(x) \text{ so that} \\
\langle \varphi_j,\varphi_i\rangle &= 0 \text{ if } i \neq j \text{ since } \int_{\underline{x}}^{\bar{x}} \phi_j(x)\phi_i(x)dx = 0 \text{ for } i \neq i
\end{aligned}$$

Note that we can write the solution for g as follows:

$$\frac{\langle g(x,0),\varphi_j\rangle}{\langle\varphi_j,\varphi_j\rangle} = \frac{\int_{\underline{x}}^{\bar{x}} g(x,0)\varphi_j(x)\frac{dx}{s(x)^2}}{\int_{\underline{x}}^{\bar{x}}(\varphi_j(x))^2\frac{dx}{s(x)^2}} = \frac{\int_{\underline{x}}^{\bar{x}} \frac{g(x,0)}{s(x)}\phi_j(x)dx}{\int_{\underline{x}}^{\bar{x}}(\phi_j(x))^2 dx} = \frac{\int_{\underline{x}}^{\bar{x}} h(x,0)\phi_j(x)dx}{\int_{\underline{x}}^{\bar{x}}(\phi_j(x))^2 dx}$$

where we use that $g(x,0) = s(x)h(x,0)$. $\square$

C Details of the multiproduct model

Law of motion for y, z.

$$dy = \sigma^2 n\, dt + 2\sigma \sqrt{y}\, dW^a$$
$$dz = \sigma\sqrt{n}\left[\frac{z}{\sqrt{ny}}\, dW^a + \sqrt{1-\left(\frac{z}{\sqrt{ny}}\right)^2}\, dW^b\right]$$

where W^a, W^b are independent standard BM's.

Lemma 2. Define

$$x = \sqrt{y} \;\text{ and } w = \frac{z}{\sqrt{ny}}$$

so that the domain is $0 \le x \le \bar{x} \equiv \sqrt{\bar{y}}$ and $-1 \le w \le 1$. They satisfy:

$$dx = \sigma^2 \frac{n-1}{2\,x} dt + \sigma dW^a \tag{36}$$
$$dw = \frac{w}{x^2}\left(\frac{1-n}{2}\right) dt + \frac{\sqrt{1-w^2}}{x}\, dW^b \tag{37}$$

We look for a solution to the eigenvalue-eigenfunction problem (λ, φ) given by equation (36) and equation (37). They must satisfy

$$\lambda\varphi(w,x) = \varphi_x(w,x)\sigma^2\left(\frac{n-1}{2x}\right) + \varphi_w(w,x)\frac{w}{x^2}\left(\frac{1-n}{2}\right) + \frac{1}{2}\varphi_{ww}(w,x)\frac{(1-w^2)}{x^2} + \frac{1}{2}\sigma^2\varphi_{xx}(w,x)$$

for all $(x,w) \in [0,\bar{x}] \times [-1,1]$, with $\varphi(\bar{x}, w) = 0$, all w and φ^2 integrable.

Proposition 18. The eigenfunctions-eigenvalues of (w,x) satisfying equation (36)-equation (37) denoted by $\{\varphi_{m,k}(\cdot), \lambda_{m,k}\}$ for $k = 1, 2, \ldots$ and $m = 0, 1, \ldots$ are given by:

$$\varphi_{m,k}(x,w) = h_m(w)\, g_{m,k}(x) \;\text{ where}$$
$$h_m(w) = C_m^{\frac{n}{2}-1}(w) \;\text{ for } m = 0,1,2,\ldots \text{ and}$$
$$g_{m,k}(x) = x^{1-n/2} J_{\frac{n}{2}-1+m}\left(j_{\frac{n}{2}-1+m,k}\,\frac{x}{\bar{x}}\right) \;\text{ for } k = 1,2,\ldots \text{ and}$$
$$\lambda_{m,k} = -N\,\frac{\left(j_{\frac{n}{2}-1+m,k}\right)^2}{2\,n} \;\text{ for } m = 0,1,\ldots, \text{ and } k = 1,2,\ldots$$

where $C_m^{\frac{n}{2}-1}(\cdot)$ denote the Gegenbauer polynomials, and where $J_{\frac{n}{2}-1+m}(\cdot)$ denote the Bessel function of the first kind, $j_{\nu,k}$ denote the ordered zeros of the Bessel function of the first kind $J_\nu(\cdot)$ with index ν.

Note that the expressions for the eigenfunctions are only valid only for $n > 2$. For $n = 2$ the expression take a different special form, which we skip to save space. The expressions for the eigenvalues are valid for $n \geq 2$.

We remind the reader how the Gegenbauer polynomial and Bessel function, which form an orthogonal base, are defined. The Gegenbauer polynomial $C_m^{\frac{n}{2}-1}(w)$ is given by:

$$C_m^{\frac{n}{2}-1}(w) = \sum_{k=0}^{\lfloor m/2 \rfloor} (-1)^k \frac{\Gamma(m-k+\frac{n}{2}-1)}{\Gamma(\frac{n}{2}-1)k!(m-2k)!}(2w)^{m-2k} \tag{38}$$

For a fixed n, the polynomials are orthogonal on with respect to the weighting function $(1-w^2)^{\frac{n}{2}-1-\frac{1}{2}}$ so that:[10]

$$\int_{-1}^{1} C_m^{(\frac{n}{2}-1)}(w)\, C_j^{(\frac{n}{2}-1)}(w)\, (1-w^2)^{\frac{n}{2}-1-\frac{1}{2}}\, dw = 0 \text{ for } m \neq j \tag{39}$$

and for $m = j$ we get

$$\int_{-1}^{1} \left[C_m^{(\frac{n}{2}-1)}(w)\right]^2 (1-w^2)^{\frac{n}{2}-1-\frac{1}{2}}\, dw = \frac{\pi\, 2^{1-2(\frac{n}{2}-1)}\Gamma(m+2(\frac{n}{2}-1))}{m!(m+\frac{n}{2}-1)[\Gamma(\frac{n}{2}-1)]^2} \tag{40}$$

The Bessel function of the first kind is given by :

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+\nu+1)} \left(\frac{x}{2}\right)^{2k+\nu} \tag{41}$$

For a given ν, the following functions are orthogonal, using the weighting function x^{n-1} so that:[11]

$$\begin{aligned}
&\int_0^{\bar{x}} \left[x^{1-\frac{n}{2}} J_\nu\left(j_{\nu,k}\frac{x}{\bar{x}}\right)\right] \left[x^{1-\frac{n}{2}} J_\nu\left(j_{\nu,s}\frac{x}{\bar{x}}\right)\right] x^{n-1}\, dx \\
&= \int_0^{\bar{x}} J_\nu\left(j_{\nu,k}\frac{x}{\bar{x}}\right) J_\nu\left(j_{\nu,s}\frac{x}{\bar{x}}\right) x\, dx = 0 \text{ if } k \neq s \in \{1,2,3,\dots\} \text{ and} \\
&\int_0^{\bar{x}} \left[x^{1-\frac{n}{2}} J_\nu\left(j_{\nu,k}\frac{x}{\bar{x}}\right)\right]^2 x^{n-1}\, dx = \bar{x}^2 \int_0^{\bar{x}} \frac{x}{\bar{x}} \left[J_\nu\left(j_{\nu,k}\frac{x}{\bar{x}}\right)\right]^2 \frac{dx}{\bar{x}}
\end{aligned} \tag{42}$$

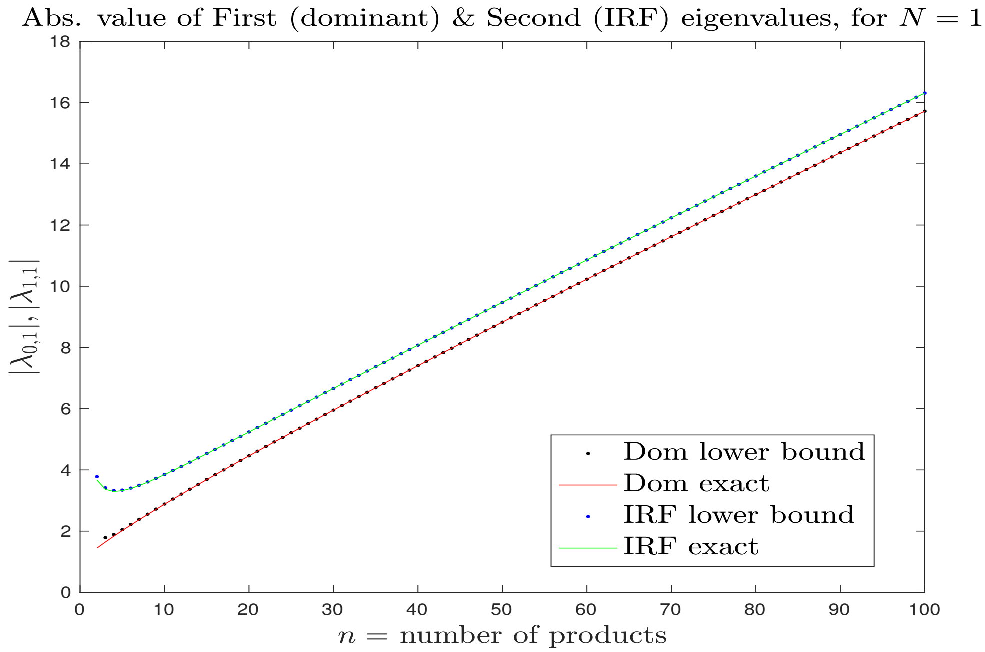
$$= \frac{1}{2}\left(\bar{x}\, J_{\nu+1}\left(j_{\nu,k}\right)\right)^2 \text{ for all } k \in \{1,2,3\dots\} \tag{43}$$

[10] By this we mean that we define the inner product between functions a, b from $[-1,1]$ to $\mathbb{R}$ as : $\langle a, b\rangle = \int_{-1}^{1} a(w)b(w)\left(1-w^2\right)^{\frac{n}{2}-1-\frac{1}{2}} dw$.

[11] By this we mean that we define the inner product between functions a, b from $[0, \bar{x}]$ to $\mathbb{R}$ as: $\langle a, b\rangle = \int_0^{\bar{x}} a(x)b(x)x^{n-1}dx$.

where $j_{\nu,k}$ and $j_{\nu,s}$ are two zeros of $J_\nu(\cdot)$.

Figure 9: Eigenvalues for multiproduct model



Kepping fixed $N = 1$ for all n

Derivation of IRF. Thus we have

$$G(t) \equiv \int_0^{\bar{x}} \int_{-1}^{1} \mathcal{G}(f)(x, w, t)\, p(x, w; 0) dw\, dx$$

As in Section 3, we can write this expected value as:

$$\begin{aligned}
Y(t) &= \int_0^{\bar{x}} \int_{-1}^{1} \mathcal{G} \left(\sum_{k=1}^{\infty} \sum_{m=0}^{\infty} b_{m,k}[f]\, \varphi_{k,m} \right) (x, w, t)\ p(x, w; 0) dw\, dx \\
&= \int_0^{\bar{x}} \int_{-1}^{1} \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} b_{m,k}[f]\, \mathcal{G}\left(\varphi_{k,m}\right)(x, w, t)\ p(x, w; 0) dw\, dx \\
&= \int_0^{\bar{x}} \int_{-1}^{1} \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} b_{m,k}[f]\, e^{\lambda_{m,k}\, t}\, \varphi_{m,k}(x, w)\, p(x, w; 0) dw\, dx \\
&= \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} e^{\lambda_{m,k}\, t}\, b_{m,k}[f] \int_0^{\bar{x}} \int_{-1}^{1} \varphi_{m,k}(x, w)\, p(x, w; 0) dw\, dx
\end{aligned}$$

Then we get:

$$G(t) = \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} e^{\lambda_{m,k}\, t}\, b_{m,k}[f]\, b_{m,k}\left[p(\cdot,0)/\omega\right] \langle \varphi_{m,k}, \varphi_{m,k} \rangle$$

Inner product. We let $\omega(w,x) = x^{1-n}\,(1-w^2)^{\frac{n-3}{2}}$. The inner product of functions a, b from $[0,\bar{x}] \times [-1,1]$ to $\mathbb{R}$ is defied as

$$\langle a, b \rangle = \int_0^{\bar{x}} \int_{-1}^{1} a(x,w)\, b(x,w)\, x^{1-n}\,(1-w^2)^{\frac{n-3}{2}}\, dw\, dx$$

The term $\langle \varphi_{m,k}, \varphi_{m,k} \rangle$ is given by the product of equation (40) and equation (43) found above. Indeed since the polynomials are orthogonal we have:

$$\begin{aligned} b_{m,k}[f] = \frac{\langle f, \varphi_{m,k}\rangle}{\langle \varphi_{m,k}, \varphi_{m,k}\rangle} &= \frac{\int_0^{\bar{x}} \left[\int_{-1}^{1} f(x,w) h_m(w)(1-w^2)^{\frac{n-3}{2}} dw\right] g_{m,k}(x)\, x^{n-1}\, dx}{\left[\int_{-1}^{1} \left(h_m(w)\right)^2 (1-w^2)^{\frac{n-3}{2}} dw\right] \left[\int_0^{\bar{x}} \left(g_{m,k}(x)\right)^2 x^{n-1}\, dx\right]} \\ &= \frac{\int_0^{\bar{x}} \left[\int_{-1}^{1} f(x,w) C_m^{\frac{n}{2}-1}(w)(1-w^2)^{\frac{n-3}{2}} dw\right] J_{m+\frac{n}{2}-1}\left(j_{m+\frac{n}{2}-1,k}\, \frac{x}{\bar{x}}\right) x^{\frac{n}{2}} dx}{\left[\int_{-1}^{1} \left(C_m^{\frac{n}{2}-1}(w)\right)^2 (1-w^2)^{\frac{n-3}{2}} dw\right] \left[\int_0^{\bar{x}} \left(J_{m+\frac{n}{2}-1}\left(j_{m+\frac{n}{2}-1,k}\, \frac{x}{\bar{x}}\right)\right)^2 x\, dx\right]} \end{aligned}$$

Invariant Distribution. After the change in variables we have:

$$\bar{h}(w) = \frac{1}{Beta\left(\frac{n-1}{2}, \frac{1}{2}\right)}\ \left(1-w^2\right)^{(n-3)/2} \quad \text{for} \quad w \in (-1\,,\,1) \tag{44}$$

$$\bar{g}(x) = x\ (\bar{x})^{-n} \left(\frac{2n}{n-2}\right) \left[\,\bar{x}^{n-2} - x^{n-2}\right] \qquad \text{for } x \in [0,\bar{x}] \tag{45}$$

Initial distribution after a small monetary shock.

$$\begin{aligned} &p(w,x;0) = \bar{h}(w(\delta))\bar{g}(x(\delta)) = \bar{h}(w)\bar{g}(x) + \bar{p}'(w,x;0)\delta + o(\delta) \text{ with} \\ &\bar{p}'(w,x;0) = \bar{g}(x)\bar{h}'(w)w'(0) + \bar{h}(w)\bar{g}'(x)x'(0) \end{aligned}$$

where:

$$\begin{aligned} &\frac{\partial}{\partial \delta} x(\delta)|_{\delta=0} = x'(0) = \sqrt{n}\, w \quad \text{and} \quad \frac{\partial}{\partial \delta}\bar{h}(w(\delta))|_{\delta=0} = \bar{h}'(w)w'(0) \\ &\frac{\partial}{\partial \delta} w(\delta)|_{\delta=0} = w'(0) = \frac{\sqrt{n}\,(1-w^2)}{x} \quad \text{and} \quad \frac{\partial}{\partial \delta}\bar{g}(x(\delta))|_{\delta=0} = \bar{g}'(x)x'(0) \end{aligned}$$

Proposition 19. The expressions for $\bar{p}'(x,w;0)$ and the coefficients $b_{1,k}(n)$ for the impulse response of output are given by:

$$
\begin{aligned}
\bar{p}'(w,x;0) &= \bar{g}(x)\bar{h}'(w)w'(0) + \bar{h}(w)\bar{g}'(x)x'(0) \\
&= \frac{w\left(1-w^2\right)^{(n-3)/2}}{Beta\left(\frac{n-1}{2},\frac{1}{2}\right)}\sqrt{n}\left(\frac{2n}{n-2}\right)\frac{\left[(4-n)\bar{x}^{n-2}-(4+n)x^{n-2}\right]}{\bar{x}^n}
\end{aligned}
$$

and the coefficients for the impulse response $b_{1,k}(n) = b_{1,k}[f]\, b_{1,k}[\bar{p}'(\cdot,0)/\omega]\,\langle\varphi_{1,k},\varphi_{1,k}\rangle$ are given by

$$
\begin{aligned}
b_{1,k}(n) = -\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}\frac{2n}{(n-2)\, j_{\frac{n}{2},k}\, J_{\frac{n}{2}+1}\left(j_{\frac{n}{2},k}\right)}&\left[(4-n)\left(\frac{2^{1-\frac{n}{2}}}{\Gamma(\frac{n}{2})\left(j_{\frac{n}{2},k}\right)^{2-\frac{n}{2}}}-\frac{J_{\frac{n}{2}-1}\left(j_{\frac{n}{2},k}\right)}{j_{\frac{n}{2},k}}\right)\right. \\
&\left.-(4+n)2^{-1-\frac{n}{2}}(j_{\frac{n}{2},k})^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right){}_1\tilde{F}_2\left(\frac{n}{2};1+\frac{n}{2},1+\frac{n}{2};-\frac{(j_{\frac{n}{2},k})^2}{4}\right)\right]
\end{aligned}
$$

where ${}_1\tilde{F}_2(a_1;b_1,b_2;z)$ is the regularized generalized hypergeometric function, i.e. it is defined as ${}_1\tilde{F}_2(a_1;b_1,b_2;z) = {}_1F_2(a_1;b_1,b_2;z)/\left(\Gamma(b_1)\Gamma(b_2)\right)$ where ${}_1F_2$ is the generalized hypergeometric function and $j_{\frac{n}{2},k}$ is the k^{th} ordered zero of the Bessel function $J_{\frac{n}{2}}(\cdot)$.

Note that, as our notation emphasizes, the coefficients $b_j(n)$ depends only on the number of products.