Measuring Influence under Probability Models with Dependent Votes

Claus Beisbart

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Abstract

How can one measure the influence of a voter when the voters do not cast independent votes? In this paper, a well-known notion of voting power is extended to a hierarchy of powers. Using powers of different ranks, the influence of an individual can be described in a very precise way.

1 The problem

We want to measure the influence that a voter has a posteriori. The simplest idea is: Calculate the probability that that voter is critical. Machover (2007) objects to this measure as follows. Consider a probability distribution under which the votes are correlated in such a way that voting profiles under which some voter is critical have probability zero. Clearly, on the simple measure, the power of each is zero. But Machover arguse that this is counterintuitive. So the simple measure does not really measure influence.

His example goes as follows:

Example 1.1 *Consider simple majority voting with 5 votes. There 32 voting profiles possible. There are ten profiles with the structure (2 yes-votes, 3 no-votes) ((2,3) for short) and ten profiles of the structure (3,2). Suppose that (2,3) and (3,2)-profiles have probability zero and that the other profiles have a probability of 1/12 each. Consider an arbitrary voter. The probability of being critical is zero, because all profiles under which a voter is critical have zero probability. But intuitively it seems strange to assign zero power to everybody.*

The problem is: Can we do better? In this paper we follow a hint by Machover (2007), p. 4 (see also the questions in the email), saying that power might be too individualistic. We start with considering combinations of voters (Sec. 2) and come back to single voters in Sec. 3.

2 Voting power of a group

Let us consider an arbitrary subset of the voters. Call this group G. We can assign power to the group in the following way. Voting power of the group is the probability that the group is critical. Criticality is defined as follows:

Definition 2.1 *Let $\mathcal{W}$ be a simple voting game with assembly N. Let be G a subset of the voters N. Consider a specific coalition S.*

1. *G is critical within S, iff S is a winning coalition ($S \in \mathcal{W}$) and $S \setminus G$ is not a winning coalition ($S \setminus G \notin \mathcal{W}$).*[1]
2. *G is critical outside S, iff S is not a winning coalition ($S \notin \mathcal{W}$) and $S \cup G$ is a winning coalition ($S \cup G \in \mathcal{W}$).*
3. *G is critical wrt S, iff S is critical within S or critical outside S.*

It can be shown that G is critical, iff $S \cup G \in \mathcal{W}$ and $S \setminus G \notin \mathcal{W}$.

According to our definition, G is critical, iff the following is true: There is some way in which the members of the group could have voted differently such that the outcome of the election would have been different.[2]

Let us briefly reconsider Example 1.1. Consider the voting profile where 1 votes yes, and the others vote no (i.e. $S = \{1\}$). The group $G \equiv \{2, 3\}$ is critical wrt S, because if 2 and 3 would both have voted differently, the outcome of the election would have been different.

Assume now that an arbitrary probability model over the coalitions S is given.

Definition 2.2 *Let $\mathcal{W}$ be a simple voting game. The power of the group $G \subset N$ is the probability for a coalition S wrt which G is critical.*

This does not exactly parallel the usual definition of voting power, because, in Def. 3.2.2 in Felsenthal & Machover (1998), p. 39 voting power is defined on the base of the number of coalitions within which a voter is critical. It then turns out (Theorem 3.2.4 in Felsenthal & Machover 1998, p. 40) that Banzhaf voting power is the probability that a voter is critical; the probability that she is critical within, given that she votes yes; or the probability that she is critical outside, given that she votes no. Under more general probability models, these equalities do not hold any more. Here, we decided to pick the notion of criticality wrt in order to define voting power of a group. The idea is just that it is equally important, if a group is critical within or critical outside.

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[1]Here $A \setminus B$ denotes the following set: $\{a \in A | a \notin B\}$.

[2]A similar suggestion was made by Beisbart (2007) for voters who have several votes. Note, that paraphrasing criticality of groups in this way assumes that monotonicity holds true. If it doesn't, then we would stick to the colloquial description of criticality and change the definition into: A group G is critical wrt S, iff there are subsets $G', G'' \subset G$ such $(S \setminus G) \cup G' \in \mathcal{W}$ and $(S \setminus G) \cup G'' \notin \mathcal{W}$.

Remark: If the probability model on the profiles is represented as a joint probability distribution $p(\lambda_1, ..., \lambda_n)$, where the v_i denote the votes of the voters, then the power of $G = \{1, 2, ..., g\}$ only depends on the function:

$$\sum_{\lambda_1} ... \sum_{\lambda_g} p(\lambda_1, ..., \lambda_n) \ . \tag{1}$$

where the sums extend over "yes" and "no", each.

Let us reconsider Example 1.1 and $G = \{2, 3\}$. Under the probability model specified above, the power of this group is .5, because the group is critical outside $\{1, 0, 0, 0, 0\}$, $\{0, 0, 0, 1, 0\}$ and $\{1, 0, 0, 0, 1\}$. Also, G is critical within $\{1, 1, 1, 1, 0\}$, $\{1, 1, 1, 0, 1\}$ and $\{0, 1, 1, 1, 1\}$. This example shows that, even under the Example 1.1, a group of voters can have non-zero power.

3 Voting powers of different ranks

So far, we have only defined a notion of power for groups. On this base, we can again become more individualistically.

For a brief motivation, consider Example 1.1 again. Voter 1 could ask: Under the given probability model, is it ever the case that I plus another voter can switch the vote together as a group? If the answer is no, she can also ask: Is it ever the case that I plus two other voters can switch the vote together as a group? And so on. Questions of this type can become very important. Suppose that next week there is a vote. Voter 1 knows the votes that the other voters will cast. Suppose that, under the voting rule that has been adopted, the outcome does not please 1. So voter 1 can ask: Is there another voter such that: if I could convince her to follow suit, the outcome of the vote would be reversed? More generally: How many voters would I have to convince to follow suit in order to secure the vote?

This motivates the introduction of a hierarchy of quantities. Let us start with an extension of the notion of criticality.

Definition 3.1 *Let $\mathcal{W}$ be a simple voting game and $i \in N$ a voter. i is critical of rank $\alpha \in \mathbb{N}_+$ wrt S, iff there is a group G with the following properties:*

1. *$i \in G$;*

2. *G is critical wrt S.*

3. *$|G| = \alpha$.*

That is, i is critical of rank α wrt S, iff i is member of a group with α members that is critical wrt S.

Obviously, we could also define criticality rank α within/outside, but for our present purposes this is not important. Note, that the ordinary notion of criticality coincides with criticality of rank 1.

Note also that, if i is critical of rank α wrt S, then also i is critical of rank $(\alpha + 1)$ wrt S.

In order to render the notion of higher-rank powers more familiar, we consider Example 1.1 again. Consider the profile $(1, 0, 0, 0, 0)$. Each voter 1–4 is critical of rank 2 wrt this profile, whereas 1 is only critical of rank 3.

Definition 3.2 *Let $\mathcal{W}$ be a simple voting game as before. The power of rank $\alpha \in \mathbb{N}$ of voter i, call it $\beta_i^{\prime,\alpha}$, is the probability for a voting profile wrt which i is critical of rank α.*

From what we have said so far, it follows immediately that $\beta_i^{\prime,\alpha+1} \geq \beta_i^{\prime,\alpha}$.

Obviously, the $\beta_i^{\prime,\alpha}$s reach 1, as α increases. Thus, voting power of very high ranks is not interesting.

How can we deal with the Example 1.1? The problem was, that the voters seem to have power, but the probability of being critical was zero for every voter. Using our extended notions of power, we would reply as follows: In some sense, we are afraid, the voters do not have power in this example, because they do not have power of rank 1. But they do have power of rank 2, and so on. In this weaker sense, they do indeed have power.

Let us note that we could have defined the higher-rank powers differently using a different notion of higher-rank criticality.

Definition 3.3 *Let $\mathcal{W}$ be a simple voting game and $i \in N$ a voter. i is v-critical of rank $\alpha \in \mathbb{N}$ wrt S, iff every group G with the following properties:*

1. *$i \in G$;*

2. *$|G| = \alpha$*

is critical wrt S.

Thus, the idea is as follows: i is v-critical of rank α wrt S, if it could reverse the vote with any other combination of $(\alpha - 1)$ voters.

However, we think that the notions of higher-rank criticality is more important than the notion of higher-rank v-criticality, because what really matters from the point of view of one voter is the question, whether there is some way to reverse the vote with the help of $(\alpha - 1)$ voters; and not whether this is relatively easy (i.e. whether any $(\alpha - 1)$ voters would do).

For calculating higher-rank powers, the following lemma is useful.

Lemma 3.1 *Let $\mathcal{W}$ be a simple voting game and let $S \subset N$. Suppose that group G with $|G| = \alpha$ is critical wrt S. Then every voter is critical of rank $(\alpha + 1)$ wrt S.*

The proof of the lemma is trivial.

Let us now examine whether our proposal is in accordance with intuitions. For this, let us examine a variation of Example 1.1. Let us assume that the probabilities are as given below except that the probability for $(1, 0, 0, 0, 0)$ is $1/12 + \epsilon$ and the probability of $(0, 1, 0, 0, 0)$ is $1/12 - \epsilon$ for some $1/12 \geq \epsilon > 0$. On our notion of power, 2 gains power of rank 2, and 1 loses power of rank 2 (the other powers of rank 2 do not change). This seems to be consistent with intuitions, because on the voting profile $(1, 0, 0, 0, 0)$, voter 2 is closer to making a difference than 1 is.

Now this hierarchy of powers with different ranks is complicated. Is there a way to get simpler notions? Obviously, it does not make sense to add up the powers of different ranks – as far as we can see, the result is not a meaningful quantity.

But one can at least say the following thing: Voter i is more powerful than voter j, if $\beta_i^{\prime,\alpha} \geq \beta_j^{\prime,\alpha}$ for all α and $\beta_i^{\prime,\alpha} > \beta_j^{\prime,\alpha}$ for at least one α. This comparative notion of "powerful" induces a pre-ordering of the voters.

We can now pose the following question. Is it ever possible that a voter 1 has more voting power of rank 1 than voter 2, but less voting power of rank 2? The answer is yes, as the following example shows. Note, however, that the probability distribution is very peculiar in this example.

Example 3.1 *Consider the weighted voting game* $[5; 3, 2, 1, 1, 1]$ *(for the notation see Felsenthal & Machover 1998, Def. 2.3.14 on p. 29 f.). Call the voter with weight 3, 1; the voter with weight 2, 2 and so on. Consider the set* $S_1 = \{1, 3, 4\}$. *Voters 1, 3 and 4 are critical wrt S, but 2 is not. Consider now* $S_2 = \emptyset$. *1 and 2 are critical of rank 2 wrt S, but 3, 4 and 5 are not so. Assume now that* $p(S_1) = \epsilon$ *and that* $p(S_2) = 1 - \epsilon$. *Then we have inter alia:* $\beta_3^{\prime,1} = \epsilon$ *and* $\beta_2^{\prime,1} = 0$, *whereas* $\beta_3^{\prime,2} = 1 - \epsilon$ *and* $\beta_2^{\prime,2} = 1$. *So for* $\epsilon > 0$, *we have* $\beta_3^{\prime,1} > \beta_2^{\prime,1}$, *but* $\beta_3^{\prime,2} < \beta_2^{\prime,2}$.

The hope is, however, that under many real probability models, the powers of the different ranks impose a complete ordering on the voters.

References

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