Two types of the Banzhaf-Coleman index in games with a priori unions

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Abstract: The paper presented concerns a voting body divided into subsets (parties). We consider the Banzhaf-Coleman index of an individual counted as a product of his index inside the party in the majority voting game and the index of the party in the weighted voting game and the modified Banzhaf-Coleman index of the game with precoalitions. Both indices are not normalised and have probabilistic interpretation. We also examine changes of values of both indices caused by changes in the structure of precoalitions.

Introduction

In the paper we investigate how to measure the power of individuals in a voting body possibly divided into some parties by means of the absolute Banzhaf-Coleman index. There are two possible ways to do so. The first is the application of the modified Banzhaf-Coleman index introduced by Owen [6]. The second is applying the composite game (Felsenthal, Machover [2]) and counting the index of an individual as a product of the index of his party and his index inside the party. We argue that these two approaches can be applied depending on whether we can assume that all party members vote in the same way (that is whether the party demands discipline in voting).

We begin with describing the formal model and both kinds of indices and the conditions under which we apply each of them. In the sequel we present some examples of applications for a voting body composed of 100 members with various divisions into parties (blocs).

Model

Let $N = \{1, 2 \ldots n\}$ denote the set of voters (or seats). We consider a decision-making situation in which the voting body is supposed to make a decision (to pass or to reject a proposal) by means of a voting rule. We assume that voters who do not vote for a proposal (do not vote "yes") vote against it and there is no possibility of abstention. The voting rule specifies whether the set of voters who accepted the proposal forms a winning coalition or not. Formally we have 2^n possible coalitions (vote configurations) $S \subseteq N$. The voting rule if then defined by the set of winning coalitions W. Usually it is assumed that

- $\varnothing \notin W$,
- $N \in W$,
- if $S \in W$ then $N - S \notin W$,
- if $S \in W$ and $S \subset T$ then $T \in W$.

The voting rule is equivalently given by a simple voting game v_W as follows

$$v_W(S) = \begin{cases} 1 & \text{if } S \in W \\ 0 & \text{if } S \notin W \end{cases}$$

for each $S \subseteq N$.

The Banzhaf-Coleman index of a voter j in this framework is the probability of a voter to be decisive assuming that all voting configurations are equally probable, that is

$$\beta_j(W) = \frac{\#\{S \subset N : (j \in S \in W \wedge S - \{j\} \notin W) \vee (j \notin S \notin W \wedge S \cup \{j\} \in W)\}}{2^n} =$$
$$\frac{1}{2^{n-1}} \sum_{\substack{S \subset N \\ j \in S}} v_w(S) - v_W(S - \{j\}).$$

In real world voting bodies the situation is more complicated since the voters are divided into some blocs (parties) ex ante, which may constrain the actual voting behaviour. This situation can be described by games with a priori unions (precoalitions) introduced by Owen [5]. Let $T = (T_1, T_2, ..., T_m)$ be a partition of a set N into subsets which are nonempty, pairwise disjoint and $\bigcup_{i=1}^{m} T_i = N$. The sets T_i are called precoalitions (a priori unions) and they can be interpreted as parties occupying seats in the voting body (note that some of T_i can be singletons). Let M denote the set of all precoalitions, that is $M = \{1, 2, ..., m\}$. Owen [6] proposed the modification of the Banzhaf-Coleman index – for a voter j in a bloc T_i we have

$$O_j(W, T) = \frac{1}{2^{m+t_i-2}} \sum_{Q \subset M - \{i\}} \sum_{\substack{K \subset T_i \\ j \in K}} v_w(N(Q) \cup K) - v_w(N(Q) \cup (K - \{j\})),$$

where $N(Q) = \bigcup_{p \in Q} T_p$ and t_i denotes the cardinality of T_i.

This index is the ratio of coalitions for which the voter $j \in T_i$ is decisive and no bloc different from T_i can be broken with respect to the total number of such coalitions. Laruelle and Valenciano [3] have given three different probabilistic interpretation of this index. The first is that we assume that all blocs but T_i vote as blocs and the probability of voting for a proposal is equal to ½. On the other hand within the bloc T_i players vote independently with a probability ½ of voting "yes" or "no"). The second interpretation is that we treat all blocs but T_i as single voters, and voters in T_i as independent voters. The third one bases on the two-stage construction of the modified Banzhaf-Coleman index proposed by Owen. First for each bloc T_i we construct new games (which are in general not simple) and then we compute the Banzhaf-Coleman index for each player in T_i. The modified Banzhaf-Coleman index was axiomatised by Albizuri [1].

There is also another possibility of measuring the decisiveness of each voter in the context of games with precoalitions. Suppose that within each party the proposal is accepted or rejected by simple majority voting and then all members of the party vote according to the decision made by previous voting. This is the case of a composite game (see [2]). In this case the Banzhaf-Coleman index of a member of a party T_i is the product of its index in the simple majority voting inside the party and the index of the party T_i treated as a player in the quotient game. In that game the set of players is M, that is players are parties and the set of winning coalitions is $W_T = \{Q \subset M : N(Q) \in W\}$, so for a voter $j \in T_i$ we have

$$C_j(W,T) = \beta_j(W_{t_i}) \cdot \beta_i(W_T),$$

where $W_{t_i} = \{K \subset T_i : \# K \geq \left[\frac{t_i}{2}\right] + 1\}$ and the symbol $[x]$ denotes the largest integer not greater than x, for any real x. In fact this is the Banzhaf-Coleman index in the composite game with the top W_T and the components W_{T_i} for $i = 1,2,..., m$.

Interpretation of the two approaches

The obvious interpretation of the index $C(W,T)$ is that we deal with a situation where all members of each party follow the discipline and vote according to the decision made by internal voting. In this case the power of a voter decomposes into two factors – one is the individual power in the internal voting and second is the power of a party as a whole. The relationships between these two factors where examined in [4].

From this point of view we can interpret the index $O(W,T)$ as an index measuring the power of a member of a party where there is no party whip assuming that in all other parties voters follow the party discipline. It is worth noting that if we compare the situation where some of the sets T_i are singletons with the situation where all singletons are joint into one new party, then the value of the modified Banzhaf-Coleman index for the members of a new party is the same as it was in the previous partition. Formally, suppose that the partition T is of the form $T = (\{j_1\},...,\{j_k\}, T_{k+1},...,T_m)$, where $\# T_i \geq 2$ for $i = k+1,...,m$ and the new partition $\tilde{T} = (\tilde{T}_1, \tilde{T}_2,..., \tilde{T}_{m-k+1})$, where $\tilde{T}_1 = \{j_1,..., j_k\}$ and $\tilde{T}_i = T_{i+k-1}$ for $i = 2,..., m-k+1$. Then

(1) $C_{j_l}(W,T) = O_{j_l}(W,T)$ for $l = 1,..., k$;

(2) $O_{j_l}(W,T)=O_j(W,\widetilde{T})$ for any l = 1,..., k and $j\in\widetilde{T}_1$.

The equality (1) is implied by the fact that if a party T_i is a singleton then the inside power of its member is equal to 1 so $C_j(W,T)=\beta_i(W_T)$ for $j\in T_i$ and it is equal to $O_j(W,T)$ since swings of the player j (or a party composed only of the player j) are exactly the same in both cases.

The equality (2) follows from the observation that when we compute the value of the modified Banzhaf-Coleman index in both case swings of players in singletons are the same as swings of players in the new party $\widetilde{T}_1$. In the first case the number of swings is divided by $2^{m+1-2}=2^{m-1}$ because there are m parties and the cardinality of the singleton is 1. In the second case we divide the number of swings by $2^{m-k+1+k-2}=2^{m-1}$ since the number of parties is equal to $m-k+1$ and the cardinality of the new party $\widetilde{T}_1$ is equal to k.

Therefore we argue to measure the power of members of any party without a party whip by means of the modified Banzhaf-Coleman index. On the other hand we will measure the power of members of disciplined parties applying the index C_j for a modified partition. Assume that the partition $T=(T_1,T_2,...,T_m)$ is such that parties' $T_1,...,T_l$ members follow the party discipline and members of remaining parties vote without discipline. Then, for members of parties $T_1,...,T_l$ we compute the index $C_j(W,\hat{T})$, where $\hat{T}=(T_1,T_2,...,T_l,\{j_1^{l+1}\},...,\{j_{t_{l+1}}^{l+1}\},...,\{j_1^m\},...,\{j_{t_m}^m\})$, that is we treat members of all remaining parties as individuals. The reason is that there is no cause to expect the parties without discipline to act as blocs, any voting configuration of members of parties $T_{l+1},...,T_m$ is possible.

In the rest of the paper we present some examples of applications of our approach for the case of a voting body composed of 100 members.

The case of two parties

Here we consider the case of a voting body with 100 voters divided into two parties with at least two members each and possibly some independent voters. The partition is then $T=(T_1,T_2,\{j_1\},...,\{j_l\})$ where $t_1,t_2\geq 2$ and $t_1+t_2+l=100$. The voting rule is simple majority – we will skip the symbol W while writing the index. We assume that members of both parties follow the party discipline. What we are interested in is how the power of disciplined party members change with respect to the changes of the size of their own party and of the size of the second party. We also compare the situation described above with the one in which members of T_2 vote independently (do not follow the party whip).

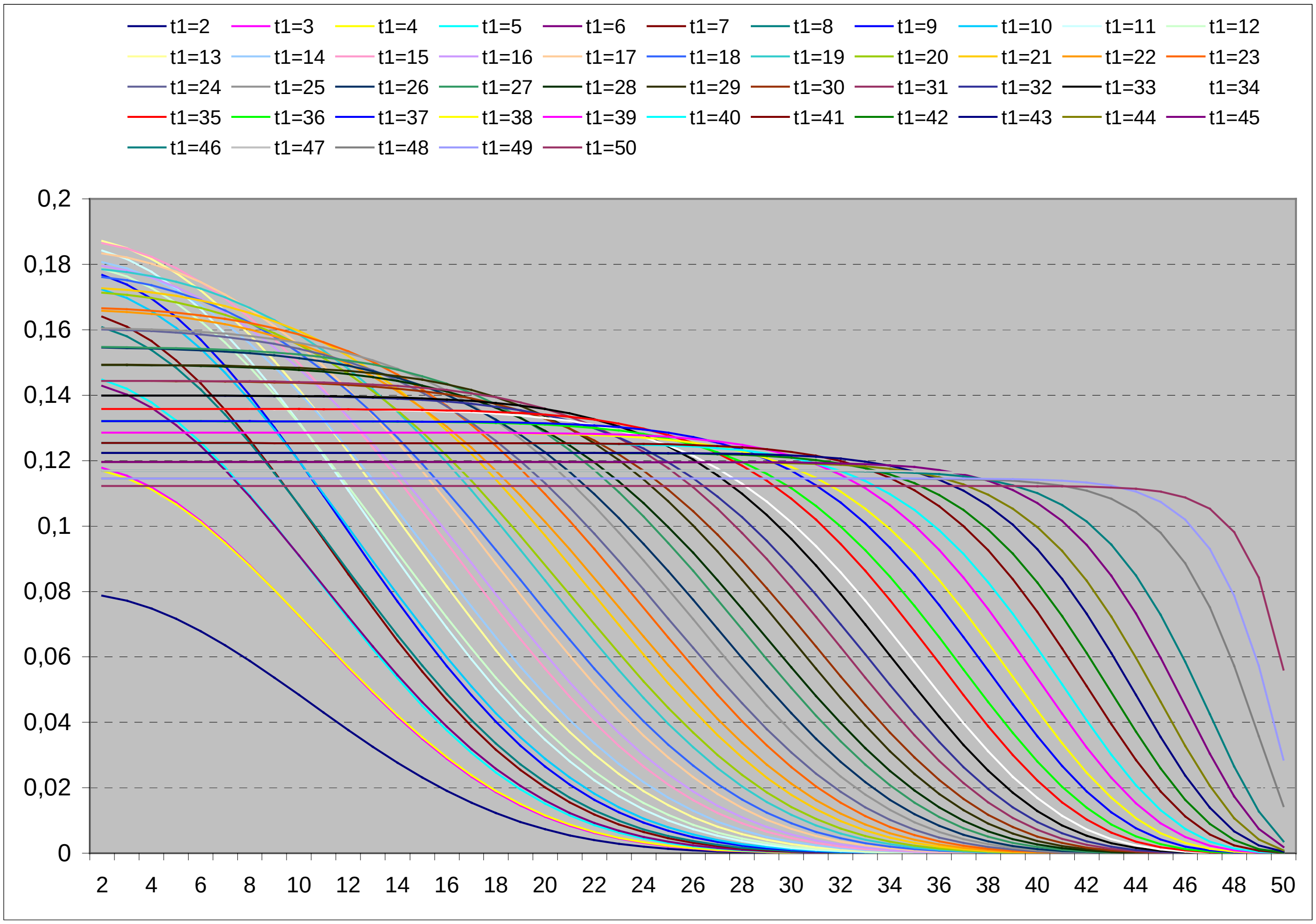
The decisiveness of members of T_1 is then given by

$$C_{T_1}(T)=\frac{1}{2^{100-t_2}}\binom{t_1-1}{[\frac{t_1}{2}]}\left(\sum_{s=\max(51-t_1-t_2,0)}^{\min(50-t_2,l)}\binom{l}{s}+\sum_{s=51-t_1}^{\min(l,50)}\binom{l}{s}\right),$$

(we assume that $t_1,t_2\leq 50$). We will write the subscript T_i instead of j at the symbol of an index since the power of each member of a given party is the same in our case.

We have computed the value of $C_{T_1}(T)$ for every $t_1, t_2 \in \{2,...,50\}$. The results are illustrated at the Figure 1.

Figure 1. The power of a member of T_1 as a function of t_2 for different t_1



For each fixed t_1 the power of a member of the first party is a decreasing function of t_2. The largest value of the index $C_{T_1}(T)$ is achieved for $t_1 = 13$ and $t_2 = 2$. In the table 1 we present the maximal value of the considered index for a fixed t_2 and the value of t_1 at which the maximum is achieved.

Table 1. The maximal value of $C_{T_1}(T)$ for fixed t_2 and the value of t_1 at which the maximum is achieved

t_2	$\max C_{T_1}(T)$	t_1
2	0,187168	13
3	0,184974	13
4	0,182295	15
5	0,178883	15
6	0,174636	17
7	0,170834	17
8	0,16672	19
9	0,162977	19
10	0,159601	21
11	0,156239	23
12	0,153447	23
13	0,15071	25
14	0,148134	25
15	0,145878	27
16	0,143512	27
17	0,141616	29
18	0,139457	29
19	0,137825	31
20	0,135875	31
21	0,134426	33
22	0,132686	33
23	0,131352	35
24	0,129828	35
25	0,128544	37
26	0,127244	37

t_2	$\max C_{T_1}(T)$	t_1
27	0,125952	39
28	0,124881	39
29	0,123525	41
30	0,122689	41
31	0,12152	41
32	0,120614	43
33	0,119737	43
34	0,118593	45
35	0,118013	45
36	0,117129	45
37	0,11626	47
38	0,11576	47
39	0,114947	47
40	0,114203	49
41	0,113895	49
42	0,113336	49
43	0,112329	49
44	0,111398	50
45	0,110521	50
46	0,108767	50
47	0,105258	50
48	0,098241	50
49	0,084206	50
50	0,056138	50

For increasing t_2 the maximal value of $C_{T_1}(T)$ decreases and the value of t_1 at which the maximum is obtained is increasing. This observation is obvious – if the cardinality of the party T_2 increases the maximal possible power of members of T_1 decreases. Moreover the larger is t_2 the larger t_1 has to be in order for $C_{T_1}(T)$ to attain maximum.

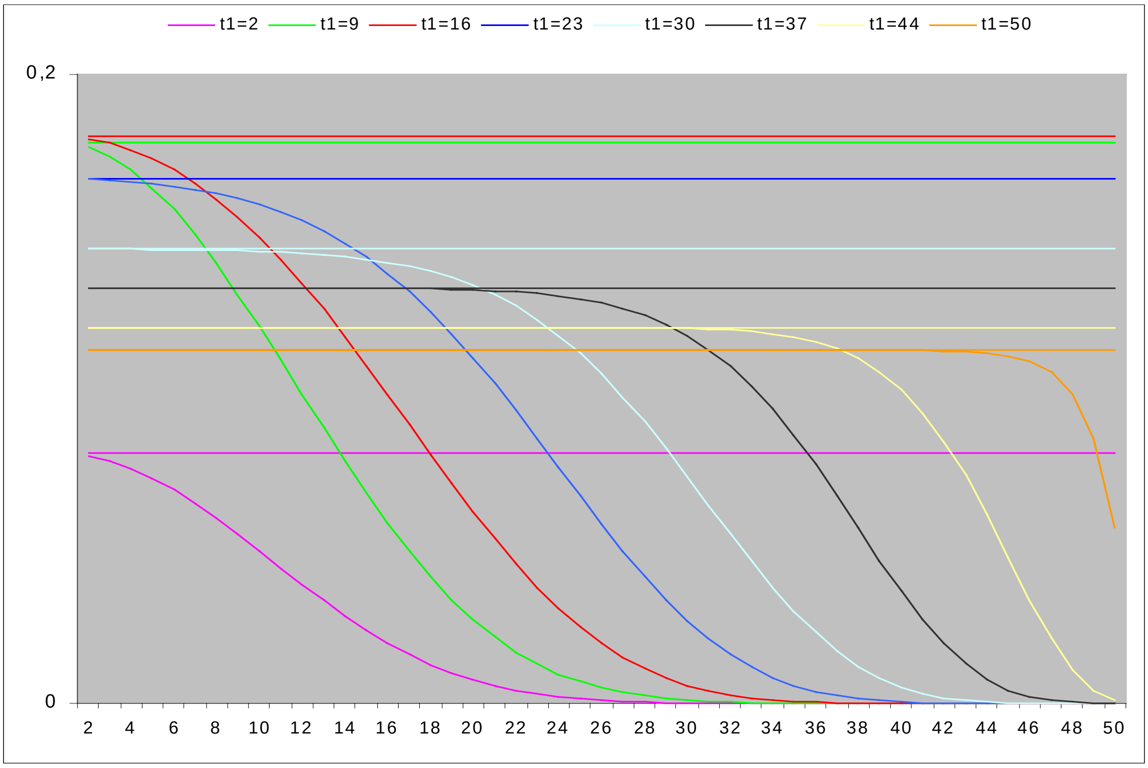
The next question was to compare the situation of members of T_1 in the previous case with one at which there is no party discipline in T_2. In the second case we compute the power of T_1 members using the formula

$$C_{T_1}(\hat{T}) = \frac{1}{2^{99}} \binom{t_1 - 1}{\left[\frac{t_1}{2}\right]} \sum_{s=51-t_1}^{\min(50,l)} \binom{l}{s}$$

for each $t_1 \in \{2,...,50\}$.

As we mentioned before, now we treat members of T_2 as if they were independent individuals, voting "yes" or "no" with the same probability. The conclusions of this comparison are not surprising – it is always better for members of the disciplined party T_1 if their "opponents" – i.e. members of T_2 are not disciplined. The results are presented in the Figure 2.

Figure 2. The comparison of the power of T_1 members for T_2 with and without party whip



At the Figure 2 we present the power of members of T_1 as a function of t_2 for eight different values of t_1 and for the case with and without party discipline in T_2. In case of lack of party discipline in T_2 the power of T_1 members is clearly the constant function of t_2. The two cases for each considered value of t_1 are marked in the same colour. What we can observe is that any horizontal line for a given value of t_1 is always above the respective curve, which means that the lack of party discipline in T_2 is always better for members of T_1. Another interesting observation is that the larger is t_1 the longer the difference between two considered cases is rather small. For example if $t_1 = 2$, then the case without party whip in T_2 is better for members of T_1 for all values of t_2. On the other hand for $t_1 = 44$ the difference between two considered cases becomes significant for $t_2 \geq 30$.
Our interest in this case was concentrated on the decisiveness of party members. The situation of individuals depending on the sizes of two voting blocs was discussed in [4].

The case of three parties

Now we consider a voting body composed of 100 voters with three parties consisting of at least two members. We also admit a possibility of some number of independent voters. The partition is then $T = (T_1, T_2, T_3, \{j_1\}, \ldots, \{j_l\})$ where $t_1, t_2, t_3 \geq 2$, $t_1, t_2, t_3 \leq 50$ and $t_1 + t_2 + t_3 + l = 100$. The voting rule is simple majority as before. We treat all parties as blocs, which mean, that we assume the party discipline in each of them. The decisiveness of members of T_1 is calculated as follows:
if $t_2 + t_3 \geq 51$, then

$$C_{T_1}(T)=\frac{1}{2^{101-t_2-t_3}}\binom{t_1-1}{[\frac{t_1}{2}]}\left(\sum_{s=51-t_1}^{\min(l,50)}\binom{l}{s}+\sum_{s=\max(51-t_1-t_2,0)}^{\min(50-t_2,l)}\binom{l}{s}+\sum_{s=\max(51-t_1-t_3,0)}^{\min(50-t_3,l)}\binom{l}{s}\right),$$

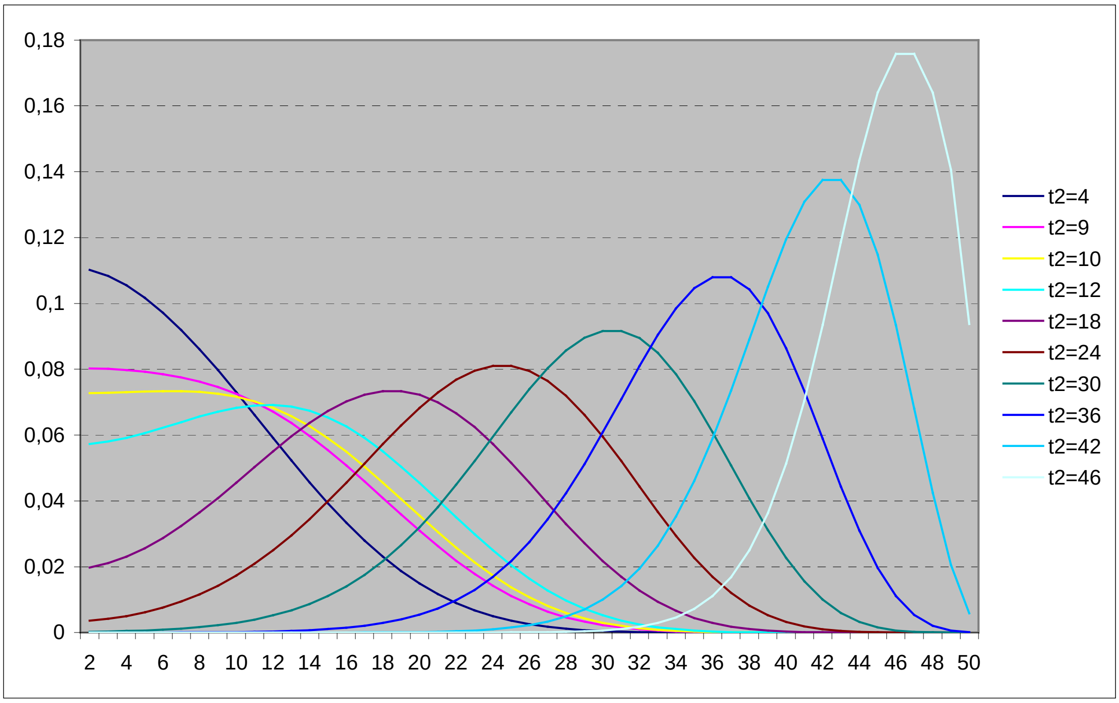
or if $t_2+t_3\leq 50$, then

$$C_{T_1}(T)=\frac{1}{2^{101-t_2-t_3}}\binom{t_1-1}{[\frac{t_1}{2}]}\left(\sum_{s=51-t_1}^{\min(l,50)}\binom{l}{s}+\sum_{s=\max(51-t_1-t_2,0)}^{\min(50-t_2,l)}\binom{l}{s}+\sum_{s=\max(51-t_1-t_3,0)}^{\min(50-t_3,l)}\binom{l}{s}+\sum_{s=\max((51-t_1-t_2-t_3),0)}^{\min(50-t_2-t_3,l)}\binom{l}{s}\right).$$

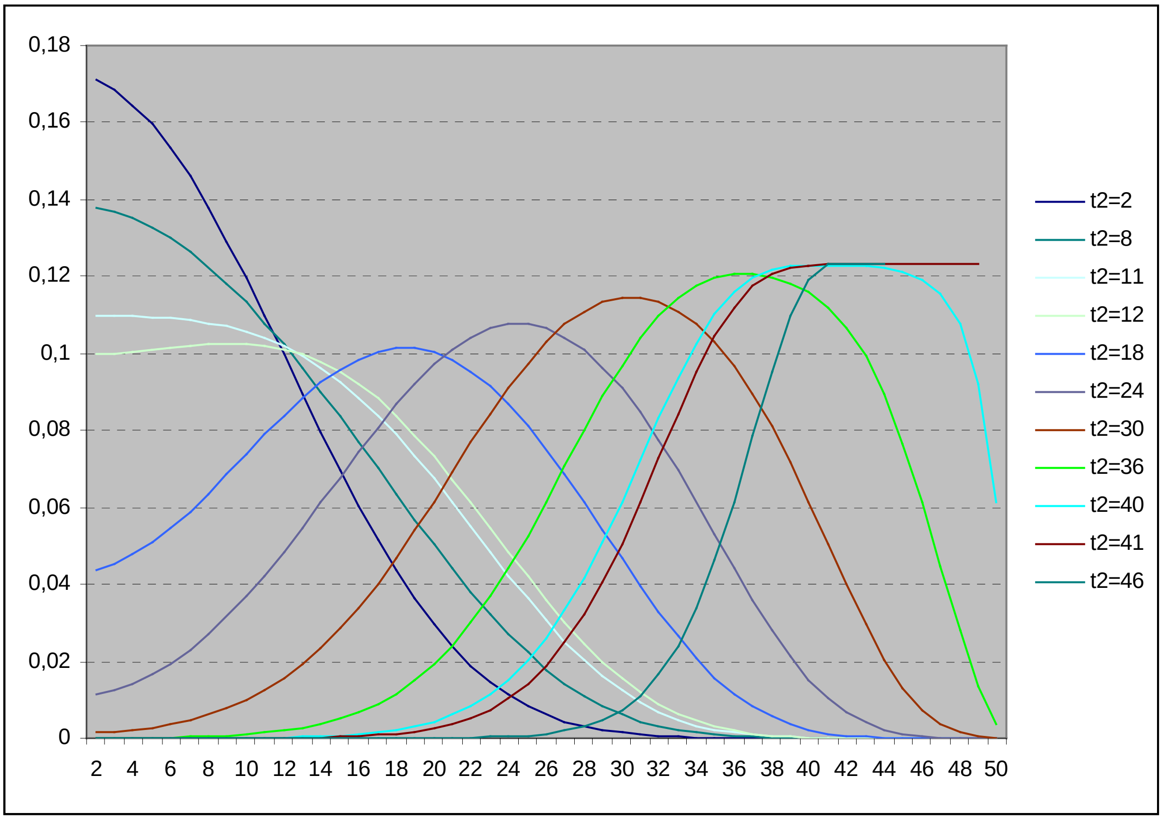
Again, our interest is to examine the power of T_1 members depending on the sizes of T_2 and T_3. First we present results of our calculations for some chosen values of t_1, t_2 at the Figure 3 (a)-(e).

Figure 3. The power of a member of T_1 as a function of t_3 for chosen values of t_1, t_2.

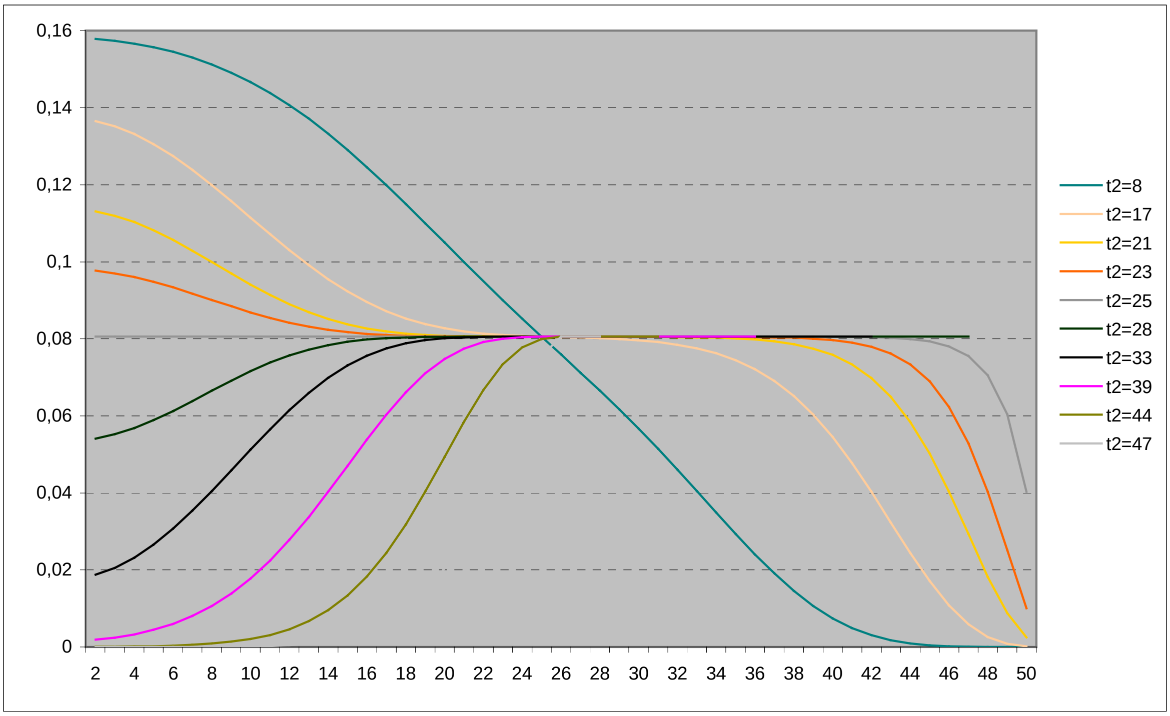
(a) t_1 = 4



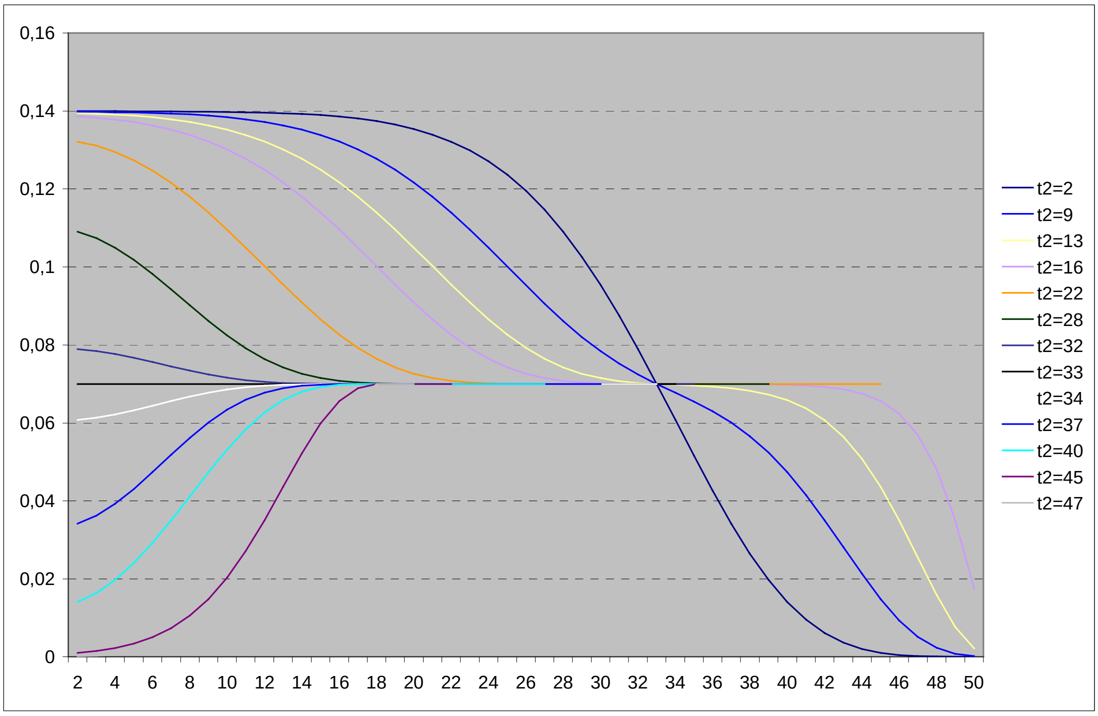
(b) $t_1 = 10$


0,18
0,16
0,14
0,12
0,1
0,08
0,06
0,04
0,02
0
2 4 6 8 10 12 14 16 18 20 22 24 26 28 30 32 34 36 38 40 42 44 46 48 50
t2=2
t2=8
t2=11
t2=12
t2=18
t2=24
t2=30
t2=36
t2=40
t2=41
t2=46

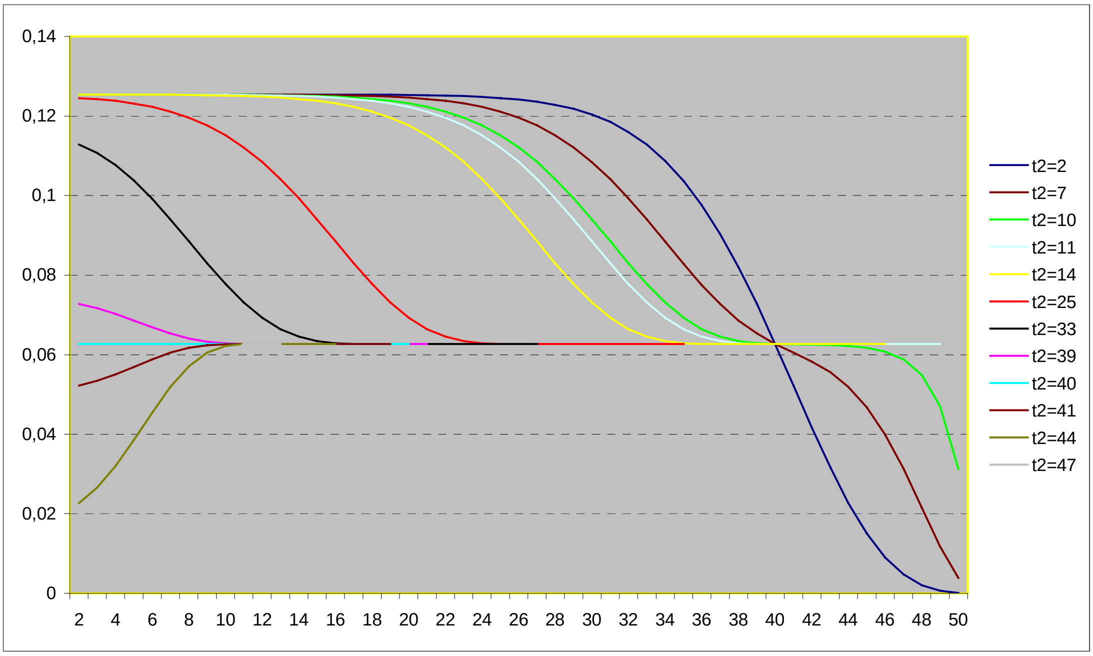
(c) $t_1 = 25$


0,16
0,14
0,12
0,1
0,08
0,06
0,04
0,02
0
2 4 6 8 10 12 14 16 18 20 22 24 26 28 30 32 34 36 38 40 42 44 46 48 50
t2=8
t2=17
t2=21
t2=23
t2=25
t2=28
t2=33
t2=39
t2=44
t2=47

(d) $t_1 = 33$



(e) $t_1 = 40$



First observe that for given values of t_1 and t_2 the cardinality of T_3 is bounded above by $\min(50, 100 - t_1 - t_2)$, so some curves at the Fig. 3 are shorter because t_3 does not obtain the maximal value of 50. Note that for each given value of t_1 there are two distinguished values of t_2. The first is $\widehat{t}_2 = t_1$ or close to t_1 for small t_1. For $t_2 < \widehat{t}_2$ the power of members of T_1 is a decreasing function of t_3. It means that if a second party is less than the first one, then the larger is the third party, the smaller is the power of the first party members. The second distinguished value of t_2 is $\breve{t}_2 = 50 - t_1$. For $\widehat{t}_2 < t_2 \le \widehat{t}_2$ the power of T_1 members first increases with increasing size of T_3, achieves a maximum and then decreases. It means that if the second party is greater than the first one (but not much) then the growing size of the third party causes the increase of power of the first party members until it achieves a maximum. After that the growth of T_3 implies the decrease of the power of T_1 members. And finally for $t_2 > \hat{t}_2$ the power of T_1 members is a nondecreasing function of t_3. Moreover it becomes constant for large t_3.

An interesting case is $t_1 = t_2 = 33$. In this case the power of members of T_1 does not depend on the size of T_3.

Another interesting topic is the power of independent voters depending on their number and on distribution of voters in disciplined parties. Applying the equality (2) we compute the value of $C_{j_k}(\tilde{T})$ for $k = 1,..., l$ as follows

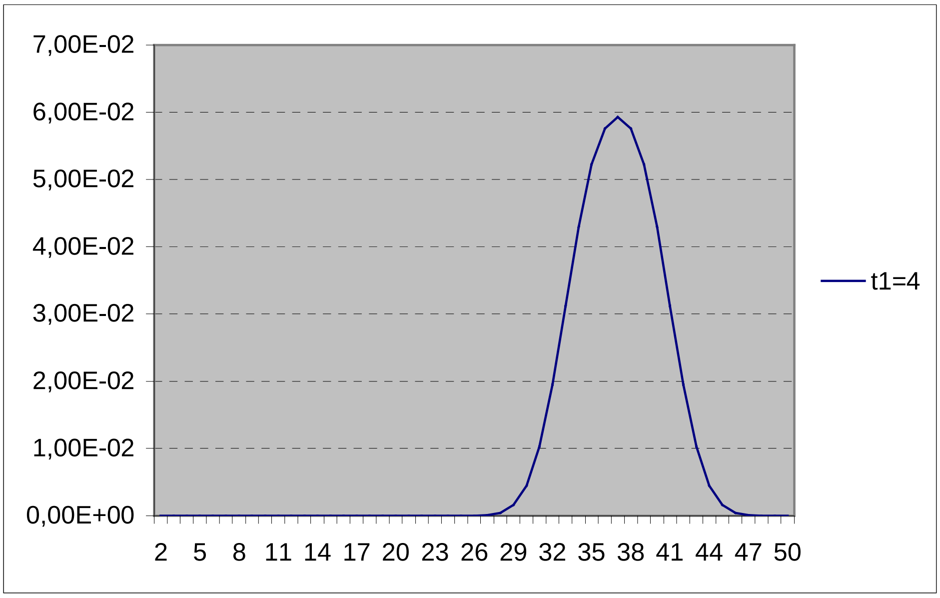
$$C_{j_k}(\tilde{T}) = \frac{1}{2^{l+2}}\left(\binom{l-1}{50} + \binom{l-1}{50-t_1} + \binom{l-1}{50-t_2} + \binom{l-1}{50-t_3} + b_1 + b_2 + b_3 + b_4\right)$$

where $b_1 = \binom{l-1}{50-t_1-t_2}$ for $t_1 + t_2 \le 50$ or $b_1 = 0$ otherwise, $b_2 = \binom{l-1}{50-t_2-t_3}$ for $t_2 + t_3 \le 50$ or $b_2 = 0$ otherwise, $b_3 = \binom{l-1}{50-t_1-t_3}$ for $t_1 + t_3 \le 50$ or $b_3 = 0$ otherwise and finally $b_4 = \binom{l-1}{50-t_1-t_2-t_3}$ for $t_1 + t_2 + t_3 \le 50$ or $b_4 = 0$ otherwise (we assume here that $\binom{n}{k} = 0$ for $n < k$).

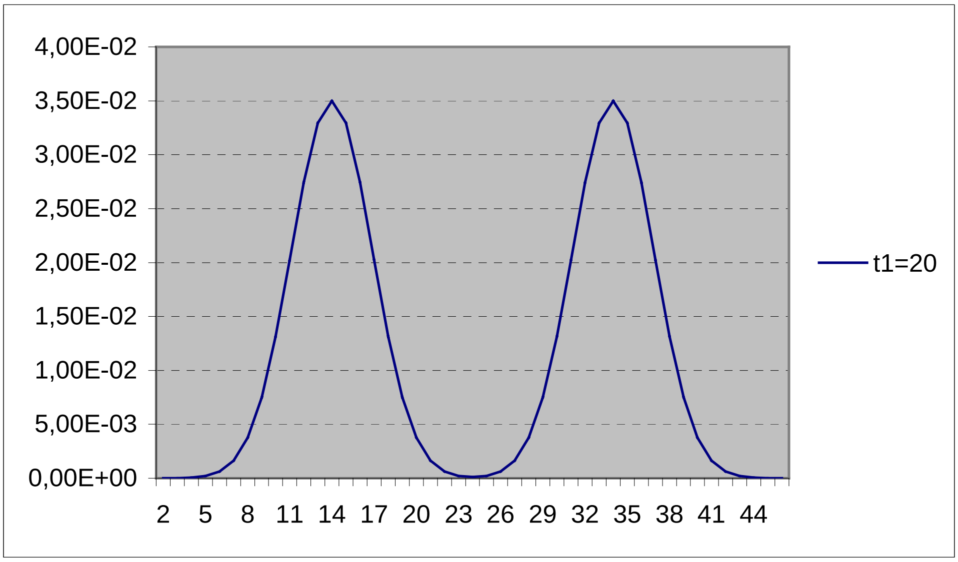
We examine the power of individual voters depending on their number and on the distribution of voters in disciplined parties. We consider the behaviour of the index $C_{j_k}(\tilde{T})$ as a function of t_2 given the values of l and t_1 for $2 \le t_1, t_2, t_3 \le 50$ and $t_1 + t_2 + t_3 + l = 100$. We distinguish three different cases, which are illustrated at the figure 4.

Figure 4.

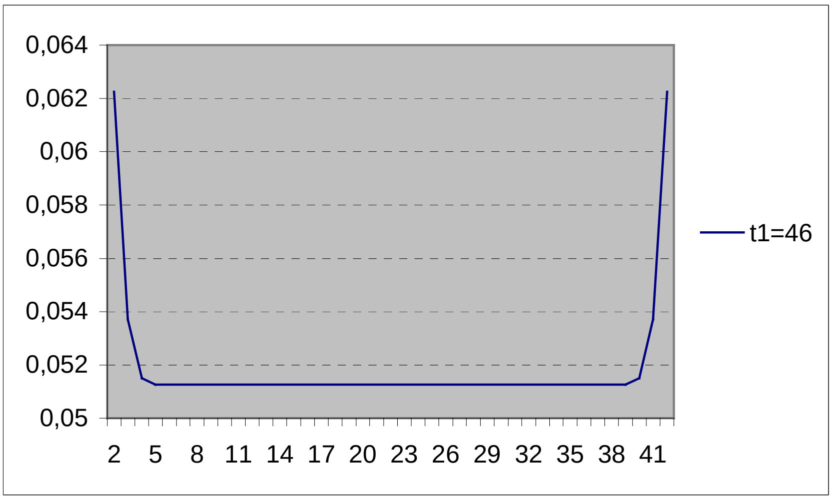
(a) The value of the index $C_{j_k}(\tilde{T})$ as a function of t_2 for $l = 22$, $t_1 = 4$



(b) The value of the index $C_{j_k}(\tilde{T})$ as a function of t_2 for $l = 32$, $t_1 = 20$



(c) The value of the index $C_{j_k}(\tilde{T})$ as a function of t_2 for $l = 10$, $t_1 = 46$



The situation similar to the one illustrated at the Fig. 4 (a), that is the curve with one maximum, occurs in case where $t_1 < 50 - \frac{l}{2}$ and the maximum is obtained at the point where t_2 is equal (or close to) to t_3. The case (b), where we have two points at which the maximum with the same value is achieved, takes place for $t_1 < 50 - \frac{l}{2}$ and both maxima are attained at points for which the largest party's size is equal (or close to) the sum of sizes of remaining parties. The minimal value of $C_{j_k}(\tilde{T})$ in this case occurs when $t_2 = t_3$. Finally, the curves of the shape similar to the one presented at the Fig. 4 (c) belong to the case $t_1 \geq 50 - \frac{l}{2}$.

Concluding remarks

Our main interest in the paper was focused on the behaviour of the decisiveness index of voters who are members of a party following the party discipline or vote independently in spite of being a member of a party (that is do not have to follow the party whip) or, eventually, act in a voting body as independent entities. For members of parties that demand following the party discipline we have computed the "product" index, taking into account the influence of a party member on the decision undertaken inside the party as well as the party power as a whole in the voting body. When computing the above-mentioned index, we treated members of parties without discipline in the same way as individuals not belonging to any party. We concentrated on changes of power of members of a given party depending on changes of structure of remaining parties. More formally, we have settled the size of one party and investigated changes of power of its members implied by changes of size of the second party in case of two parties. In case of three parties we have settled the number of the first and second party members and considered how the power of the first party members depend on changes of size of the third party. It means that we have examined the cases of changing the structure of a voting body in a way that one party gains members from among individual voters. Note that although we focused on computing the value of the index for members of the first party, the results are the same for all remaining parties, since we examined all possible cases of different parties' sizes and the situation of all parties is identical – the model is symmetric.

The power of independent members of a voting body has been computed applying the modified Banzhaf-Coleman index for a modified structure of a voting body. In this case we have investigated the influence of changes in the configuration of sizes of three parties demanding the party discipline on the power of independent voters for a settled number of independent voters.

We restricted our research to the case of two or three parties because it allowed for an illustrative presentation of results. Obviously, the methodology presented here can be applied to examine the power of members of actual voting bodies with an arbitrary structure of parties.

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