Discrepancies among TU Games Solutions: A Probabilistic Approach *

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Abstract

In voting theory, many efforts have been undertaken to compute the probability that different voting rules lead to different outcomes under different assumptions on the likelihood of the voting situations. We here intend to propose a similar analysis for linear solutions of cooperative TU games, that is the semivalues and the least square values. First, under an assumption which somehow mimics the IC hypothesis used in voting theory, we compute the likelihood that one semivalue (or least square values) gives more worth to voter i than voter j, while another semivalue (or least square value) leads to the opposite ranking. Secondly, we explore the discrepancies between the Shapley-Shubick index and the Banzhaf index for composite weighted voting games. In conclusion, we suggest that several other models could be used in order to compute the likelihood of discrepancies among semivalues.
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1 Introduction

The literature on cooperative games with transferable utility focuses on how the worth of the grand coalition should be split among the players. It rarely addresses the issue of ranking the players on the basis of the worth that they are able to attain in each coalition. Nevertheless, there exist contexts where this is a relevant problem. For example, consider that $v(S)$ is the number of goals or points scored by a team S of players in a sport like rugby, basket, hockey, etc. The ranking of a player i in the team should be evaluated from the different $v(S)$, $S \ni i$. Similarly, the productivity of an employee can be evaluated from the production of the different coalitions of workers he belongs to. If the objective of the manager is to fire the less productive employee, or to promote the most productive one, he only needs a ranking of the workers. For another example, one could think of an European research network of universities financed by the European Union. The commission in Brussels may wish to know which partners have more cooperation with others universities, in order to favor it for the next programmes. Again, an exact imputation of the worth is not necessary; the problem is to identify the good (or bad) player(s). Then, an important question is to know whether different criteria could lead to very different rankings.

Saari and Sieberg [21] and Laruelle and Merlin [15] studied this issue for two classes of linear solutions to TU games, the least square values and the semivalues. The most famous member of these families is, of course, the Shapley value. Their main results state that any two different least square values (or semivalues) could lead to completely different rankings. In fact, the results they obtain are very similar to those obtained in voting theory: when n candidates are in contention, Saari [20] prove that $(n-1)$ linearly independent scoring rules could lead to $(n-1)$ different rankings of the players (with possibly $n-1$ different winners and losers).

The same issues arise with voting games, though in a more subtle way, when one wants to evaluate the power of the players involved in a decision scheme. In fact, both Saari and Sieberg and Laruelle and Merlin showed that the ranking of the players induced by the Shapley Shubick index and the Banzhaf index are always similar for the class of weighted voting games. However, Straffin [24] displayed a case wherein the two indices significatively diverge. In 1971, the Victoria conference proposed a new voting scheme to approve amendments to the Canadian Constitution. A proposal would have to be approved by Ontario, Quebec, two of the four maritime provinces (New Brunswick, Nova Scotia, Prince Edward Island and Newfoundland), and either British Columbia and one prairie province (Alberta, Saskatchewan and Manitoba) or all the three prairie provinces. We will refer to this game

Table 1: Power indices for the 1971 Victoria scheme

Province	1980 Population	Shapley-Shubick	Banzhaf
Ontario	35.53	31.55	21.78
Quebec	26.52	31.55	21.78
British Columbia	11.32	12.50	16.34
Alberta	9.22	4.17	5.45
Manitoba	4.23	4.17	5.45
Saskatchewan	3.99	4.17	5.45
Nova Scotia	3.49	2.98	5.94
New Brunswick	2.87	2.98	5.94
Newfoundland	2.34	2.98	5.94
Prince Edward Island	0.51	2.98	5.94

as the 'Canadian gama' in the sequel. This procedures can be described as the composition of three different weighted games. In the first game, Ontario and Quebec have one vote, the other provinces zero, and a threshold of two votes must be reached for a decision to be accepted. In the second game, all the players except the maritime provinces, which have one ballot each, have no vote ; A threshold of two must be reached for a decision to be considered. The last part of the constitutional amendment scheme can be described as a weighted game wherein British Columbia gets two mandates, the three prairie provinces obtains one mandate each, and the other province have no voice. Thus, the decision is eligible if a total of three mandates supports it. Of course, a decision is accepted in the composite game if it is supported by a wining coalition in all the three games. Table 1 (reproduced from Straffin [24]) displays the measures of the power of the different provinces according to the Shapley-Shubick and Banzhaf indices. The most striking result is that, according to the Banzhaf index, the maritime provinces seems to enjoy more influence in the decision process than the three prairie states, in contradiction with the evaluation obtained with the Shapley Shubick power index! Thus, as soon as the decision schemes is described by the composition of two or more different games, it seems that there is a room for the most two famous power indices to strongly disagree in their conclusion.

In social choice theory, many efforts have been undertaken to describe the discrepancies among the voting rules and to compute the probability that different voting rules lead to different outcomes. For example, Gehrlein and Fishburn [13] computed the probability that two different scoring rules select the same winner in a three-candidate election for a large electorate. They used the Impartial Culture (IC) probabilistic hypothesis to model the behavior of the voters: Each voter is equally likely to pick any of the 6 possible strict preference orderings. Using the same assumptions, Merlin, Tataru and Valognes [16] have extended these results, and have evaluated the likelihood that all the voting rules (the scoring rules, the Condorcet voting rules, and the scoring runoff procedures) select the same winner in a three candidate election[1]. In the first part of this paper, we intend to propose a similar analysis for the linear solutions of cooperative TU games, that is the semivalues and the least square values. Under an assumption which somehow mimics the IC hypothesis used in voting theory, we compute the likelihood that one semivalue (or least square value) gives more worth to voter i than voter j, while another semivalue (or least square value) leads to the opposite ranking.

Though the idea of computing the probability of a paradox is quite new in game theory, we have to report two recent contributions for the case of simple games. Van Deemen and Rusinowska [27] studied thirteen distributions of the seats in the Second Chamber of the Dutch parliament by the mean of five power indices, including the Shapley-Shubik index, the normalized Banzhaf index and the Penrose Banzhaf index. Subsequently, they search for the occurrence of the paradox of redistribution, the paradox of new members, and the paradox of large size for each power index. Chua Ueng and Huang [5] studied the class of plurality weighted games: A coalition is winning if the sum of the weights of its players is larger than the weights gathered by any other coalition. The space of the games is then identified with the space of the weights: If the sum of the weights equals one, for a n-player game, it is the n-dimensional unit simplex. Thus, the probability that an index does not satisfy a given property is given by the measure of a volume in the simplex. One has to notice that these contributions only consider weighted games. The second part of the paper extends the model of Chua, Ueng and Huang to evaluate the discrepancies between the Shapley-Shubick and Banzhaf indices for composite weighted games. We wrote a computer programme that generates randomly the weights of the composite weighted game, computes the order of the players according to both indices, and check whether these two rankings disagree. Thus, we are able not only to generate easily many examples of conflicts between Shapley-Shubick and Banzhaf (quite an improvement from the only example yet provided by the game theory literature!) but also to give estimate the likelihood of discrepancies according to the probability model we use to generate the weights.

The rest of the paper is organized as follows. In section 2, we first recall the game theoretical background, and introduce the definitions of semivalues

[1]For more on this literature, see the survey articles by Gehrlein [11, 12].

and least square values. We present in detail the results by Saari and Seiberg [21] and Laruelle and Merlin [15]. The first probability model is introduced in section 3. We choose to use a very simple model, which shares some characteristics with the IC hypothesis: Each voter brings (almost) the same worth to all the coalitions he belongs to. Thus, there is no systematic bias in favor of one player. The main results and the computations are displayed in section 4. In particular, we have been able to compute the probability that the Shapley value and the least square prenucleolus rank differently two players. In section 5, we present in details the composite weighted games. We then present two models that can be used to generate randomly these games. One of them turns out to be the model suggested by Chua, Ueng and Huang; The other is directly inspired by the Maximal Culture model suggested by Gehrlein and Lepelley [14] for the computation of the voting paradoxes in social choice theory. Our figures shows that the likelihood of discrepancies increases rapidly with the number of players involved in the game. In the final discussion (section 6), we first present the limitations of our analysis. We then suggest that several other models could be used to compute the probabilities of discrepancies among semivalues and power indices. Open issues and technical problems that could be solved in further studies are also mentioned.

2 Game Theoretical Background

2.1 Games and their properties

A cooperative transferable utility (TU) game is a pair (N, v), where $N = \{1, \ldots, n\}$ denotes the set of players and v is a function which assigns a real number to each non-empty subset (or coalition) of N, with $v(\emptyset) = 0$. The number of players in coalition S is denoted by s. When N is clear from the context we refer to game (N, v) as game v. We denote by $\mathcal{G}^n$ the class of all possible n-player games. Games may have certain nice and reasonable properties in some contexts. A game is non-negative if $v(S) \geq 0$, $\forall S \subseteq N$. The monotony condition requires that $v(S) \leq v(T)$ whenever $S \subseteq T$. A game is superadditive if $v(T) + v(S) \leq v(T \cup S)$ for all coalitions S and T such that $S \cap T = \emptyset$. It is convex whenever for all i and all S and T such that $S \subseteq T \subseteq N \setminus \{i\}$, $v(S \cup i) - v(S) \leq v(T \cup i) - v(T)$.

A *(0-1)-game* is a game in which the function v only takes the values 0 and 1. It is a *simple* game if it is not identically 0, and monotonic. Thus, a simple game is often described by its set of winning coalitions $\mathcal{W}$: $S \in \mathcal{W} \Leftrightarrow v(S) = 1$. We denote by $\mathcal{SG}^n$ the set of all the n-player simple games. A weighted game $G(\gamma, w)$ is defined by a vector of weights, $w = (w_1, \ldots, w_i, \ldots, w_n)$ and a threshold $\gamma < \sum_{i \in N} w_i$, such that $v(S) = 1$ whenever $\sum_{i \in S} w_i > \gamma$, and $v(S) = 0$ otherwise. We denote by $\mathcal{WG}^n$ the set of all n-player weighted games. Let $G(\gamma^1, w^1)$, $G(\gamma^2, w^2)$, $\ldots$ $G(\gamma^k, w^k)$ be a collection of k weighted games. Then, the composite weighted game:

$$G = G(\gamma^1, w^1) \wedge G(\gamma^2, w^2) \wedge \ldots \wedge G(\gamma^k, w^k)$$

is the simple game defined by:

$$v(S) = 1 \Leftrightarrow \sum_{i \in S} w_i^t > \gamma^t \ \forall t = 1, \ldots k.$$

We denote by $k - \mathcal{WG}^n$ the family of composite weighted games with k components. Felsenthal and Machover [8, p 27–28] propose to define the family of composite game more broadly. However, the definition we give here is sufficient for the purpose of this paper. Also notice that a specific k-weighted composite game can sometimes be rewritten as a k'-weighted composite game, with $k' \leq k$. Taylor and Zwicker [26, p 35] define the dimension of a voting game to be the minimal number k such as it can be represented as the meet of k voting rule. For example, Felsenthal and Machover [9] have been able to prove that one of the decision scheme of the European Union, presented in the official text as the conjunction of three weighted voting games, could be represented more simple by the composition of only two weighted voting games.

2.2 Linear solutions for TU games

A solution to a TU game is a function $\psi : \mathcal{G}^n \to \mathbb{R}^n$. The vector $\psi(v) = (\psi_1(v), \ldots, \psi_n(v))$ assigns to each voter a measure of his importance or worth in the game. A solution is efficient if $\sum_{i \in N} \psi_i(v) = v(N)$ for all games; it is then called a value. When designing a solution for a game, one may require that it satisfies the linearity axiom.

Definition 1 *A solution to a game, ψ, is linear iff $\forall v, w \in \mathcal{G}^n$, and $\forall a > 0,\ b > 0$, $\psi(av + bw) = a\psi(v) + b\psi(w)$.*

Two main classes of solutions satisfy this axiom: The semivalues and the least square values.

Definition 2 *A solution Φ to a cooperative TU game v is a semivalue if and only if Φ is given by, $\forall i = 1, \ldots n$,:*

$$\Phi_i(v) = \sum_{S \subseteq N,\, S \ni i} p_s \ \left[v(S) - v(S \setminus \{i\})\right], \tag{1}$$

where $\sum_{s=1}^{n} \binom{n-1}{s-1} p_s = 1$ *and* $p_s \geq 0$.

Definition 3 *A solution* Ψ *to a cooperative TU game* v *is a least square value if and only if* Ψ *is given by,* $\forall i = 1, \ldots, n,$ *:*

$$\Psi_i(v) = \frac{v(N)}{n} + \frac{1}{\alpha n}\left[n \sum_{S \neq N,\, i \in S} m(s)v(S) - \sum_{j=1}^{n} \sum_{S \neq N,\, j \in S} m(s)v(S) \right] \tag{2}$$

where $\alpha = \sum_{s=1}^{n-1} m(s) \binom{n-2}{s-1}$ *and* $m = (m(1), \ldots, m(n-1)) \in \mathbb{R}_+^{n-1}$.

Dubey et alii [7] proposed a characterization of the semivalues based upon linearity; Ruiz et alii [19] also used the linearity axiom to characterize the least square values. These two families mainly differ on the following properties: The semivalues are not generally efficient, while the least square value may assign negative values to some players. Only one solution belongs to both families, the Shapley value [23], denoted by ϕ afterwards, and characterized by $p_s = \frac{(s-1)!(n-s)!}{n!}$ or $m(s) = ((n-1)\binom{n-2}{s-1})^{-1}$.

Nevertheless, there is a close relationship between the two classes. More precisely, Ruiz et alii [19] showed that the orthogonal projection of a semivalue on the efficient plane is a least square value[2]. If $\Phi(v)$ is a semivalue, the orthogonal projection of it on the efficient plane which is given by

$$\bar{\Phi}(v) = \Phi(v) + \frac{1}{n}\left(v(N) - \sum_{i \in N} \Phi_i(v) \right)$$

is a least square value. Furthermore, if $\mathbf{p} = (p_1, \ldots, p_n)$ is the vector associated to the semivalue Φ, then the weight function such that $\bar{\Phi} = \Psi$ is given (up to a positive proportionality factor) by

$$m(s) = p_s + p_{s+1} \text{ for all } 1 \leq s \leq n-1 \tag{3}$$

Without loss of generality, this induces the normalization $\alpha = 1$ for the least square values. Thus, we can associate to each semivalue a least square value. This is in particular true for the following famous semivalues:

1. The Banzhaf semivalue (Banzhaf [1], Coleman [6]) is characterized by

$$p_s = \frac{1}{2^{n-1}}.$$

[2] The converse is not true: we cannot associate a semivalue to each least square value. Also notice that different semivalues can have the same projection in the class of LSVs.

It will be denoted by β. The associated least square value is the least square prenucleolus (Ruiz et alii [18]), which is characterized by

$$m(s) = \frac{1}{2^{n-2}}.$$

Notice that this normalization is different from the one used to defined the normalized Banzhaf index[3], where the Banzhaf values are divided by their sum (this guaranties non negativity).

2. The dictatorial semivalue only takes into consideration the worth of the singletons. Its additive and efficient projection is the center of imputation.

$$m(s) = p_s = \begin{cases} 1 \text{ if } s = 1 \\ 0 \text{ otherwise} \end{cases}$$

3. On the contrary, the marginal semivalue only considers coalitions of size $n-1$.

$$p_s = \begin{cases} 1 \text{ if } s = n \\ 0 \text{ otherwise} \end{cases}$$

The separable cost remaining benefit solution (see for example Young [29]) is the orthogonal projection of the marginal index.

$$m(s) = \begin{cases} 1 \text{ if } s = n-1 \\ 0 \text{ otherwise} \end{cases}$$

2.3 Discrepancies among solutions for TU games

The linear solutions assign to each player a measure of their "power", "importance" or "worth" in the cooperative TU game. Nevertheless, it is quite known that two solutions may estimate quite differently the worth of a player. The 1971 proposal for the Canadian constitutional amendment scheme is a good example of what can go wrong: The numerical estimations of the power as well as the induced ordering of the players can be significatively different according to the game theoretical solution concept we use.

We will here only use an ordinal approach, that is, only consider the way the different linear solutions rank the players according to the value they give to them. In fact, any solution induces a ranking of the players, named a power ranking. These power rankings belongs to $\mathcal{R}(N)$, the set of all the preorderings on $N = \{1, \ldots, n\}$ (the transitive and complete binary relations on N). A first result is that, although a semivalue and its normalization on

[3] Recall that a solution is called a power index when it is applied to simple games.

the efficient plane may award different values to the same player, the power rankings are preserved; from equations (1),(2) and (3), we derive:

$$\begin{aligned}\Phi_i(v) - \Phi_j(v) &= \sum_{s=1}^{n-1} m(s) \left[\sum_{|S|=s,\, i\in S,\, j\notin S} v(S) - \sum_{|S|=s,\, j\in S,\, i\notin S} v(S) \right] \\ &= \Psi_i(v) - \Psi_j(v). \end{aligned} \tag{4}$$

Thus, we will only consider the class of least square values, the results for semivalues being similar (moreover, some least square values are not the additive normalization of any semivalue). Generally, the semivalues are normalized by dividing the Φ_i's by their sum[4]. Notice that, by doing so, we also keep the power ranking. So, theorems in terms of power rankings for least square values are also valid for the classical normalized version of the semivalues.

Possible discrepancies in power rankings in the class of least square values have been extensively described by Saari and Seiberg [21] and Laruelle and Merlin [15]. Their main theorem is the following one:

Theorem 1 *Take $(n-1)$ linearly independent vectors $m_1, \ldots, m_t, \ldots, m_{n-1}$, defining $(n-1)$ different least square values. Choose randomly $(n-1)$ power rankings R_t, $t = 1, \ldots, n-1$ from the set of the possible preorderings $\mathcal{R}(N)$. Then there exist games v (possibly non negative, monotonic, superadditive or convex) such that the power ranking obtained for the game v with the least square value m_t is exactly R_t.*

A consequence of this theorem is that any two different least square values may lead, for some games, to opposite rankings of the players. This result is also true if we consider simultaneously the ranking induced by the Shapley value and the Banzhaf semivalue. Nevertheless, as in voting theory, results like Theorem 1 does not tell us whether examples of strong discrepancies among least square values are just rare events, or, on the contrary, betrays a generalized chaotic behavior. To answer this question, and complete the qualitative results displayed in Theorem 1 with quantitative estimations, we will here compute the likelihood of discrepancies among two or more semivalues in some particular contexts.

[4]The resulting values are efficient and positive, but the linearity axiom is no longer satisfied.

3 The model of almost symmetrical players

The techniques and the arguments we will use are derived from the ones of voting theory. Like the scoring rules, the least square values can be viewed as linear mapping from one space onto an image space. With mild assumptions, Saari showed that most of the voting rules are linear mappings from the set of rational points x in the $m!$ unit simplex (in a m candidate contest, each coordinate x_t gives the fraction of the voters equipped with the type t preference among the $m!$ possible linear orderings) into $\mathbb{R}^m$, as each candidate obtains a final score. Similarly, in cooperative game theory, linear solutions associate to a point in $\mathbb{R}^{n-2}$ an point in $\mathbb{R}^n$.

For any least square value Ψ, the hyperplane $\Psi_i(v) = \Psi_j(v)$ divides $\mathcal{G}^n$ in two half spaces: on one side, the LSV estimates that i has more worth than j, and on the other side, that j has more worth than i. For one LSV, there are $((n-1)n)/2$ such hyperplanes, which divide $\mathcal{G}^n$ in $n!$ regions, corresponding to the $n!$ possible strict power rankings (rankings with ties are located on the hyperplanes and at the intersection of several hyperplanes). Thus, two different LSVs differ in the way they partition $\mathcal{G}^n$. To evaluate the discrepancies between two semivalues Ψ and Φ on the ranking of i and j, one has to examine the regions defined by $\Psi_i(v) > \Psi_j(v)$ and $\Phi_i(v) < \Phi_j(v)$: This a cone in $\mathcal{G}^n$. So, the question resumes to evaluate the volume between the two hypeplanes given a measure on the space of games.

It is true that several different assumptions can be made on the likelihood of a specific game in $\mathcal{G}^n$, or one may be only interested in some subclasses of games, like the weighted games. The model we propose here is inspired by the impartial culture assumption in voting theory: In this model, each voter has a uniform probability to pick any of the possible preference type, and the distribution of the voting situations follow a multivariate normal distribution as the number of voters tends to infinity (for more on this model and its application in voting theory, see the recent surveys by Gehrlein [11, 12]). In particular, there is no bias in favor of a particular candidate under IC. Also notice that in a m-candidate elections, the mass of the points are gatehred around the profile $x = (1/m!, \ldots, \ldots)$, where each preference type is equally represented, as the number of players increases. The counterpart in game theory will be games where the players have almost the same worth: All the coalitions of the same size lead to the same worth for its members, up to an error term. Thus the measure on $\mathcal{G}^n$ will here be gathered around a point in $\mathcal{G}^n$, which corresponds to a symmetric game (possibly non negative, additive, superadditive or even convex).

More precisely, the model of almost symmetrical players assumes that all the coalitions of the same size s have almost the same value, a_s. Nevertheless, the worth of the coalitions of the same size may be slightly different, due to an error term $\epsilon(S)$. Thus:

$$v(S) = a_s + \epsilon(S)$$

The idea is that all the players have the same capacities; the coalition size is the main factor which influences the worth. Nevertheless, some noise may affect the worth of a coalition; this is the $\epsilon(S)$ term. We shall assume that $E(\epsilon(S)) = 0$, and that the $\epsilon(S)$ are independent and identically distributed. The simplest case would be to assume that all the $\epsilon(S)$ follow the same normal distribution $\mathcal{N}(0, \sigma)$, but other distributions such that $Prob(v(S) = a_s + t) = prob(v(S) = a_s - t)\ \forall s = 1, \dots n-1$ may be used.

The random variable is the $(2^n - 2)$-dimensional vector $V = (v(s))$; $v(\emptyset)$ and $v(N)$ are omitted as they have no impact on equation (4), which governs the ranking between player i and player j for a given least square values - they can be adjusted as desired. As each $v(S)$ follows the same normal law, the distribution of the cooperative games is radially symmetric around the point

$$A = (\underbrace{(a_1, \dots, a_1)}_{n\ times}, \underbrace{(a_2, \dots a_2)}_{n(n-1)/2\ times}, \dots \underbrace{(a_s, \dots, a_s)}_{n!/(s!(n-s)!)\ times}, \dots, \underbrace{(a_{n-1}, \dots, a_{n-1})}_{n\ times}).$$

Thus, the normal distribution case is just a particular example of a more general class of distribution where the mass the symmetrically distributed around a single point. In this case the probability of any event is equal to the angle between the two (or more) hyperplanes that describe it. The model is similar to the one described by van Newenhizen [28] in voting theory: Under the IC assumption and more generally for any probability distribution which is radially symmetric around a center point, the probability computations are reduced to the measure of a cone. Her geometric technique has been developed since then by Saari and Tataru [22], Tataru and Merlin [25] and Merlin, Tataru and Valognes [16].

4 Probability of discrepancies among semi-values and least square values

4.1 The three-player case

In the three-player case, the vectors are of the form:

$$V = (v(1), v(2), v(3), v(12), v(13), v(23)) \in \mathbb{R}^6.$$

The distribution of the V's is radially symmetric around a point $A = (a_1, a_1, a_1, a_2, a_2, a_2)$. The least square values are uniquely described by the vector $m = (m(1), m(2))$. With the normalization $\alpha = 1$, we can describe the family of least square values by $m_p = (1-p, p)$, $p \in [0, 1]$. Also notice that, for 3 players, the Shapley and Banzhaf orderings coincide. $p = (1/6, 1/3, 1/6)$ for Shapley-Shubick, which to $m = (1/2, 1/2)$. $p = (1/8, 1/8, 1/8)$ for Banzhaf, which leads to $m = (1/4, 1/4)$ or $m = (1/2, 1/2)$ if we normalize the sum of the weights to one.

4.1.1 Different rankings of two players

We first evaluate the probability that two least square values disagree on the ranking of two players.

Theorem 2 *Consider two LSVs Ψ^p and Ψ^q, respectively defined by m_p and m_q. Then, the probability that $\Psi^p(v)$ and $\Psi^q(v)$ rank differently the players i and j for almost symmetrical games, $P(p, q)$ is given by:*

$$P(p, q) = \frac{1}{\pi} \arccos \left(\frac{1 - p - q + pq}{\sqrt{(1 - 2p + 2p^2)(1 - 2q + 2q^2)}} \right) \tag{5}$$

Proof of Theorem 2. Wlog, consider the players 1 and 2. The equation $\Psi_1^p(v) = \Psi_2^p(v)$ is given by:

$$(1 - p)v(1) + (p - 1)v(2) + 0v(3) + 0v(12) + pv(13) - pv(23) = 0$$

Its normal vector pointing towards the games where the value for player 1 is greater than the value for player 2 is $N_1 = (1-p, p-1, 0, 0, p, -p)$. Similarly, we define for Ψ^q the hyperplane separating the games where player 1 has more worth than player 2, and its normal vector pointing towards games where player 2's value is greater is denoted by N_2.

$$(1 - q)v(1) + (q - 1)v(2) + 0v(3) + 0v(12) + pv(13) - pv(23) = 0$$

$$N_2 = (q - 1, 1 - q, 0, 0, -q, q)$$

The two hyperplanes does not point exactly in the same direction if $p \neq q$. So, we have to compute the volume between these two hyperplanes. As the probability distribution is radially symmetric around A, $P(p, q)$ is twice the angle between the two hyperplanes, α, divided by 2π. We can get this angle by computing the dot product between the normal vectors N_1 and $-N_2$.

$$||N_1|| = \sqrt{2 - 4p + 4p^2}$$

$$||N_2|| = \sqrt{2 - 4q + 4q^2}$$
$$-N_1.N_2 = 2 - 2p - 2q + 4pq$$
$$\cos(\alpha) = \frac{-N_1.N_2}{||N_1||.||N_2||}$$

and, after simplifications,

$$P(p,q) = \frac{2\alpha}{2\pi} = \frac{1}{\pi} \arccos\left(\frac{1 - p - q + 2pq}{\sqrt{(1 - 2p + 2p^2)(1 - 2q + 2q^2)}} \right).$$

QED.

From equation (5), we can derive several interesting facts. First, $P(p,q) = P(1-p, 1-q)$; thus the marginal and the dictatorial cases are symmetric, $P(0,q) = P(1, 1-q)$. Moreover, the probability $P(0,q)$ is increasing in q. It is easy to check that $P(0,1) = \frac{1}{2}$; there is no link between the ranking of two players by the marginal and the dictatorial semi values (for a generalization of this statement, see section 4.3.1.). Another interesting measure is obtained with $p = \frac{1}{2}$. Then, we estimate the probability that any LSV gives a ranking different from the Shapley Banzhaf ordering. With $p = \frac{1}{2}$, we get:

$$P(\frac{1}{2}, q) = \frac{1}{\pi} \arccos\left(\frac{1}{\sqrt{2 - 4q + 4q^2}} \right).$$

On Figure 1, we can see that the values are symmetric around 1/2, with $P(\frac{1}{2}, 0) = P(\frac{1}{2}, 1) = 1/4$.

4.1.2 Opposite Power Rankings

It is quite easy to deduce from the previous figures that the dictatorial and the marginal semivalues are completely independent under these assumptions; They will select the same strict ordering with probability 1/6, and select opposite rankings with probability 1/6 too.

Thus, we explore in detail the probability of agreement and disagreement with the Shapley Banzhaf value.

Theorem 3 *Consider a LSV Ψ^p defined by m_p. Then, the probability that for almost symmetrical games, the rankings for the Banzhaf Shapley value and Ψ^p are the same for $n = 3$, is given, for $p \geq \frac{1}{2}$ by:*

$$A(\frac{1}{2}, p) = 1 + \frac{3}{\pi^2} \int_{\frac{1}{2}}^{p} dvol(C_1)\, dp$$

with

$$\frac{dvol(C_1)}{dp} = \left(\frac{\arccos\left(\frac{-1}{2\sqrt{2-4p+4p^2}}\right)(2p-1)}{(1-2p+2p^2)\sqrt{7-16p+16p^2}} - \frac{\arccos\left(\frac{-1}{\sqrt{2-4p+4p^2}}\right)}{(1-2p+2p^2)} \right)$$

Moreover, $A(\frac{1}{2},p) = A(\frac{1}{2},1-p)$ *and we get* $A(\frac{1}{2},0) = A(\frac{1}{2},1) = 0.443032$.

Theorem 4 *Consider a LSV* Ψ^p *defined by* m_p. *Then, the probability that for almost symmetrical games the ranking given by* Ψ^p *is the exact opposite of the Shapley Banzhaf ranking , is given, for* $p \geq \frac{1}{2}$ *by:*

$$D(\frac{1}{2},p) = \frac{3}{\pi^2}\int_{\frac{1}{2}}^{p} dvol(C_2)\,dp$$

with

$$\frac{dvol(C_2)}{dp} = \left(\frac{-\arccos\left(\frac{1}{2\sqrt{2-4p+4p^2}}\right)(2p-1)}{(1-2p+2p^2)\sqrt{7-16p+16p^2}} + \frac{\arccos\left(\frac{1}{\sqrt{2-4p+4p^2}}\right)}{(1-2p+2p^2)} \right)$$

Moreover, $D(\frac{1}{2},p) = D(\frac{1}{2},1-p)$ *and we get* $D(\frac{1}{2},0) = D(\frac{1}{2},1) = 0.038110$.

Proof of Theorem 3. Wlog, consider the ordering $1 \succ 2 \succ 3$. The inequality $\Psi^p_1(v) > \Psi^p_2(v)$ is given by:

$$(1-p)v(1) + (p-1)v(2) + 0v(3) + 0v(12) + pv(13) - pv(23) > 0 \qquad (6)$$

Its normal vector pointing towards the games where the value for player 1 is greater than the value for player 2 is $N_1 = (1-p, p-1, 0, 0, p, -p)$. For the case $q = \frac{1}{2}$, we obtain

$$\frac{1}{2}v(1) + \frac{1}{2}v(2) + 0v(3) + 0v(12) + \frac{1}{2}v(13) - \frac{1}{2}v(23) > 0 \qquad (7)$$

$$B_1 = (\frac{1}{2}, -\frac{1}{2}, 0, 0, -\frac{1}{2}, \frac{1}{2})$$

Similarly, we define for Ψ^p the hyperplane separating the games where player 2 has more worth than player 3, and its normal vector N_3 pointing towards games where player 2's value is greater.

$$0v(1) + (1-p)v(2) + (p-1)v(3) + pv(12) - pv(13)0v(23) > 0 \qquad (8)$$

$$N_3 = (0, 1-p, p-1, p, -p, 0)$$

For $q = \frac{1}{2}$, we get:

$$0v(1) + \frac{1}{2}v(2) - \frac{1}{2}v(3) + \frac{1}{2}v(12) - \frac{1}{2}v(13) + 0v(23) > 0 \qquad (9)$$

$$B_3 = (0, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, 0)$$

The probability $A(\frac{1}{2}, p)$ is the volume of the cone defined by equations (6) to (9). As the probability distribution is radially symmetric around A, this problem reduces to compute the area of a 3-dimensional spherical simplex on the surface of unit sphere in $\mathbb{R}^4$. To perform this area, we use the Schlăfli's formula which gives the differential volumes of spherical r-simplices (see Saari and Tataru [22], Tataru and Merlin[25]).

$$dvol_r(C_1) = \frac{1}{(r-1)} \sum_{6 \leq j < k \leq 9} vol_{r-2}(S_j \cap S_k) d\alpha_{jk}; \quad vol_0 = 1/6 \qquad (10)$$

where α_{jk} is the dihedral angle formed by the facets S_j, S_k of C_1. To apply this to our problem, remark that C_1 is defined by the inequalities (6), (7), (8) and (9). The dihedral angles between the different facets are easily computed to obtain:

$$\begin{array}{lllllll}
\alpha_{68} & = & \arccos\left(\frac{-N_1 N_3}{||N_2||.||N_3||}\right) & = & \alpha_{79} & = & \pi/3 \\
\alpha_{67} & = & \arccos\left(\frac{-N_1 B_1}{||N_1||.||B_1||}\right) & = & \alpha_{89} & = & \arccos\left(\frac{-1}{\sqrt{2-4p+4p^2}}\right) \\
\alpha_{69} & = & \arccos\left(\frac{-N_1 N_3}{||N_1||.||B_3||}\right) & = & \alpha_{78} & = & \arccos\left(\frac{1}{2\sqrt{2-4p+4p^2}}\right)
\end{array}$$

which implies that:

$$\begin{array}{lllll}
d\alpha_{68} & = & d\alpha_{79} & = & 0 \\
d\alpha_{67} & = & d\alpha_{89} & = & -\frac{1}{1-2p+2p^2} dp \\
d\alpha_{69} & = & d\alpha_{78} & = & \frac{2p-1}{(1-2p+2p^2)\sqrt{7-16p+16p^2}} dp
\end{array}$$

The vectors $v_1 = (1, 1, 1, 0, 0, 0)$ and $v_2 = (0, 0, 0, 1, 1, 1)$ form an orthogonal basis for a subspace orthogonal to C_1. We shall use this information to calculate the vertices of the intersection of the cone with the four-dimensional subspace orthogonal to v_1 and v_2. For example, finding the vertex $p_{678} = S_6 \cap S_7 \cap S_8$ reduces to solving the system:

$$\begin{cases}
(1-p)v(1) + (p-1)v(2) + 0v(3) + 0v(12) + pv(13) - pv(23) = 0 \\
\frac{1}{2}v(1) + \frac{1}{2}v(2) + 0v(3) + 0v(12) + \frac{1}{2}v(13) - \frac{1}{2}v(23) = 0 \\
0v(1) + (1-p)v(2) + (p-1)v(3) + pv(12) - pv(13)0v(23) = 0 \\
0v(1) + \frac{1}{2}v(2) - \frac{1}{2}v(3) + \frac{1}{2}v(12) - \frac{1}{2}v(13) + 0v(23) > 0
\end{cases}$$

By solving similar linear systems, we obtain the vertices of the cone C_1 to be equal to:

$$\begin{aligned}
p_{678} &= (p, p, -2p, 2p-2, -p+1, -p+1)\\
p_{679} &= (-1, -1, 2, 2, -1, -1)\\
p_{689} &= (2p, -p, -p, p-1, p-1, 2-2p)\\
p_{789} &= (-2, 1, 1, 1, 1, -2)
\end{aligned}$$

Also we compute:

$$\begin{aligned}
vol(S_6 \cap S_7) &= vol(S_8 \cap S_9)\\
&= (\widehat{p_{678}, p_{679}}) = \arccos\left(\frac{-1}{\sqrt{2-4\,p+4\,p^2}}\right)\\
vol(S_7 \cap S_8) &= vol(S_6 \cap S_9)\\
&= (\widehat{p_{789}, p_{678}}) = \arccos\left(\frac{-1}{2\sqrt{2-4\,p+4\,p^2}}\right)
\end{aligned}$$

Using (10), with $r = 3$:

$$dvol(C_1) = vol(S_6 \cap S_7)d\alpha_{67} + vol(S_7 \cap S_8)d\alpha_{78}.$$

Hence,

$$\frac{dvol(C_1)}{dp} = \left(\frac{\arccos\left(\frac{-1}{2\sqrt{2-4\,p+4\,p^2}}\right)(2\,p-1)}{(1-2\,p+2\,p^2)\sqrt{7-16\,p+16\,p^2}} - \frac{\arccos\left(\frac{-1}{\sqrt{2-4\,p+4\,p^2}}\right)}{(1-2\,p+2\,p^2)}\right)$$

As the surface of the unit sphere in $\mathbb{R}^4$ is $\omega_4 = 2\pi^2$, the probability that the LSV $m(p)$ and the Shapley-Banzhaf value gives the ranking $1 \succ 2 \succ 3$ is:

$$A'(\frac{1}{2}, p) = 1/6 + \frac{1}{2\pi^2}\int_{\frac{1}{2}}^{p} dvol(C_1)\, dp$$

The values of $A'(\frac{1}{2}, p)$ are displayed on Figure 2 for $p \geq \frac{1}{2}$. We obtain $A(\frac{1}{2}, p) = 6A'(\frac{1}{2}, p)$, and $A(\frac{1}{2}, p) = 6(1/6 - 0.0928283) = 0.443032$. QED

Proof of Theorem 4. The proof of Theorem 4 is very similar to the proof of Theorem 3. We just have to reverse the signs of inequalities (6) and (8), and perform exactly the same computations. We evaluate the volume of a new cone, C_2, again using the Schlăfli's formula.

The dihedral angles between the different facets are easily computed to obtain:

$$\begin{array}{rcccl}\alpha_{68} & = & \alpha_{79} & = & \pi/3 \\ \alpha_{67} & = & \alpha_{89} & = & \arccos\left(\frac{1}{\sqrt{2-4p+4p^2}}\right) \\ \alpha_{69} & = & \alpha_{78} & = & \arccos\left(\frac{-1}{2\sqrt{2-4p+4p^2}}\right)\end{array}$$

which implies that:

$$\begin{array}{rcccl}d\alpha_{68} & = & d\alpha_{79} & = & 0 \\ d\alpha_{67} & = & d\alpha_{89} & = & \frac{1}{1-2\,p+2\,p^2}dp \\ d\alpha_{69} & = & d\alpha_{78} & = & \frac{-2\,p+1}{(1-2\,p+2\,p^2)\sqrt{7-16\,p+16\,p^2}}dp\end{array}$$

We obtain the vertices of the cone C_2 to be equal to:

$$\begin{array}{l}p_{678} = (p, p, -2p, 2p-2, -p+1, -p+1) \\ p_{679} = (1, 1, -2, -2, 1, 1) \\ p_{689} = (2p, -p, -p, p-1, p-1, 2-2p) \\ p_{789} = (2, -1, -1, -1, -1, 2)\end{array}$$

Also we compute

$$\begin{array}{rcl}vol(S_6 \cap S_7) & = & vol(S_8 \cap S_9) \\ & = & (\widehat{p_{678}, p_{679}}) = \arccos\left(\frac{1}{\sqrt{2-4\,p+4\,p^2}}\right) \\ vol(S_7 \cap S_8) & = & vol(S_6 \cap S_9) \\ & = & (\widehat{p_{789}, p_{678}}) = \arccos\left(1/2\,\frac{1}{\sqrt{2-4\,p+4\,p^2}}\right)\end{array}$$

Using (10):

$$dvol(C_2) = vol(S_6 \cap S_7)d\alpha_{67} + vol(S_7 \cap S_8)d\alpha_{78}.$$

Hence,

$$\frac{dvol(C_2)}{dp} = \left(\frac{-\arccos\left(\frac{1}{2\sqrt{2-4\,p+4\,p^2}}\right)(-1+2\,p)}{(1-2\,p+2\,p^2)\sqrt{7-16\,p+16\,p^2}} + \frac{\arccos\left(\frac{1}{\sqrt{2-4\,p+4\,p^2}}\right)}{1-2\,p+2\,p^2}\right)$$

The probability that the LSV ranking is $3 \succ 2 \succ 1$ while the Shapley-Banzhaf value ranking is $1 \succ 2 \succ 3$ is:

$$D'(\frac{1}{2}, p) = \frac{1}{2\pi^2}\int_{\frac{1}{2}}^{p} dvol(C_2)\,dp$$

as $vol_0 = 0$. We obtain $D(\frac{1}{2}, p) = 6A'(\frac{1}{2}, p)$, and $D(\frac{1}{2}, p) = 6(0.00635167) = 0.038110$. The values of $D(\frac{1}{2}, p)$ are displayed on Figure 3 for $p \geq \frac{1}{2}$.QED

4.2 Some results for n players

4.2.1 Elementary Least Square Values

With the normalization $\alpha = 1$, all the vectors $m(s)$ which define the class of the least square values lie in a simplex uniquely characterized by its vertices.

Definition 4 *Let e^k be a vector in $\mathbb{R}_+^{n-1}$ such that:*

$$e^k(s) = \begin{cases} \binom{n-2}{s-1}^{-1} \text{ if } s = k \\ 0 \text{ otherwise} \end{cases} \tag{11}$$

A least square value is called elementary if it is defined by one of the $n-1$ vector e^k. It is denoted by $\Psi^{e^k}(v)$.

Note that the center of imputations is the elementary least square value Ψ^{e^1}, while the equal allocation of nonseparable value corresponds to $\Psi^{e^{n-1}}$. It is easy to see that the family $\{\Psi^{e^k}(v)\}_{k=1,\ldots,n-1}$ of elementary least square values form a base of the class of least square values (see Laruelle and Merlin for a proof [15]).

In section 4.1, we have seen that the results for the $\Psi^{e^1}(v)$ and $\Psi^{e^2}(v)$ are completely independent for $n = 3$. This had to be expected, as each elementary LSV only takes into account the coalitions of a given size. The same argument runs for $n \geq 3$: The probability that two elementary least square values give exactly the same ranking is $(1/n!)$, and they select the same top player with probability $1/n$. We can also notice that all the least square values give the same ranking if and only if all the elementary least square values select the same ranking. As a byproduct, the probability that all the least square values (and all the semivalues) choose exactly the same strict ranking of the players is $(1/n!)^{n-2}$.

4.2.2 Banzhaf versus Shapley

For n players, the most interesting probability to compute is the likelihood that the two most famous semivalues, the Shapley value and the Banzhaf semivalue, lead to different rankings for a pair of players.

Theorem 5 *Let $Prob(\phi \neq \beta, n)$ be the probability that the Shapley value and the Banzhaf semivalue rank differently the players i and j in the power ranking for almost symmetric games of n players. Thus,*

$$Prob(\phi \neq \beta, n) = \frac{1}{\pi} \arccos\left(\frac{(n-1)}{\sqrt{2(n-2)Z(n-2)}}\right)$$

with $Z(n-2) = \sum_{t=1}^{n-2} \binom{n-2}{t}^{-1}$.

Proof of Theorem 5 . Without loss of generality, the reasoning is done for the LSVs, and we consider players 1 and 2. The Banzhaf weights are given by $m(s) = \frac{1}{2^{n-2}}$ for all $s = 1, \ldots, n-1$. The equation of the hyperplane $\beta_1 - \beta_2 = 0$ is such that the coefficients are 0 for all the coordinates where 1 and 2 appear simultaneously or do not appear, $+\frac{1}{2^{n-2}}$ if 1 appears but not player 2, and $-\frac{1}{2^{n-2}}$ otherwise. There are 2^{n-2} coalitions without player 1 and 2, so, there are 2^{n-1} coalitions with 1 or 2, but not both. Thus, if we denote by B the normal vector pointing towards the games where player 1 has more power:

$$\begin{aligned} ||B||^2 &= 2^{n-1}\left(\frac{1}{2^{n-2}}\right)^2 && (12)\\ &= 2^{3-n} && (13) \end{aligned}$$

The Shapley weights are given by $m(s) = ((n-1)\binom{n-2}{s-1})^{-1}$. Again, the coordinates of the hyperplane $\phi_1 - \phi_2 = 0$ obey to the same rule as the ones of the Banzhaf hyperplane. So, there are 2^{n-2} coalitions without 1 and 2, and $\binom{n-2}{t}$, $t = 0, \ldots, n-2$ coalitions of size t with player 1 and without player 2. Thus, if we denote by Sh the normal vector pointing towards the games where the Shapley value of 1 is greater than the Shapley value of 2, we get:

$$\begin{aligned} ||Sh||^2 &= 2\sum_{t=0}^{n-2}\binom{n-2}{t}(m(t+1))^2 && (14)\\ &= 2\sum_{t=0}^{n-2}\binom{n-2}{t}\left(\frac{1}{n-1}\binom{n-2}{t}^{-1}\right)^2 && (15)\\ &= \frac{2}{(n-1)^2}\sum_{t=0}^{n-2}\binom{n-2}{t}^{-1} && (16) \end{aligned}$$

Let us denote $\sum_{t=0}^{n-2}\binom{n-2}{t}^{-1}$ by $Z(n-2)$. It is the sum of the inverse of the entries in the Pascal triangle. One may check that it is decreasing from $n = 6$ and tends to 2 as n grows.

Thus, the angle α between the Shapley and the Banzhaf hyperplanes is given by:

$$\alpha = \arccos\left(\frac{B.Sh}{||B||\,||Sh||}\right)$$

To compute the dot product $Sh.B$, first notice that the t^{th} coordinates in both vectors have always the same sign. There are still 2^{n-2} coalitions with player 1 and without player 2, and $\binom{n-2}{t}$ of them are of size $t+1$. Thus,

$$\begin{aligned} B.Sh &= 2\sum_{t=0}^{n-2}\binom{n-2}{t}m(t+1)\frac{1}{2^{n-2}} & (17)\\ &= \frac{2}{2^{n-2}}\sum_{t=0}^{n-2}\binom{n-2}{t}\frac{1}{n-1}\binom{n-2}{t}^{-1} & (18)\\ &= \frac{2}{2^{n-2}}\sum_{t=0}^{n-2}\frac{1}{n-1} & (19)\\ &= 2^{3-n} & (20)\end{aligned}$$

In turns, we end with:

$$\cos(\alpha) = \frac{(n-1)}{\sqrt{2^{(n-2)}Z(n-2)}}$$

and

$$Prob(\phi \neq \beta, n) = \frac{2\arccos(\alpha)}{2\pi} = \frac{1}{\pi}\arccos\left(\frac{(n-1)}{\sqrt{2^{(n-2)}Z(n-2)}}\right).$$

QED

As $Z(n-2)$ converges to 2 when n goes to infinity, we can state that the probability tends to $\frac{1}{2}$ has the number of players grows, which basically means that the two values become completely independent. Nevertheless, for small values of n, the figures are significatively lower than $\frac{1}{2}$, as shown in Table 1.

5 The discrepancies between Banzhaf and Shapley orderings for composite weighted games

5.1 Composite games, linear games and discrepancies

One of the conclusion to draw from the previous section is that we can to produce rather easily TU-games for which two LSVs radically differ. However, this result depends on the fact that we can pick the $v(S)$ rather freely around the points $A = ((a_1, \ldots, a_1), (a_2, \ldots a_2), \ldots, (a_{n-1}, \ldots, a_{n-1}))$. It is impossible to use this trick for the class of voting games as $v(S) = 0$ or 1; moreover, the monotony condition has to be met. In fact, the following theorem is easy to prove for the class of weighted games:

Table 2: The values of $P(\phi \neq \beta, n)$

n	$Z(n-2)$	$P(\phi_{ij} \neq \beta_{ij})$
4	$\frac{5}{2}$	0.10242
5	$\frac{8}{3}$	0.16666
6	$\frac{8}{3}$	0.22251
7	$\frac{13}{5}$	0.27149
8	$\frac{151}{60}$	0.31403
9	$\frac{256}{105}$	0.34913
10	$\frac{83}{35}$	0.38097
11	$\frac{146}{63}$	0.40624
12	$\frac{1433}{630}$	0.42681
∞	2^+	0.5^-

Theorem 6 (Laruelle and Merlin [15], Saari and Sieberg [21]) *Consider a weighted game $G(\gamma, w) \in \mathcal{SG}^n$. If $w_i > w_j$, then $\Psi_i(v) \geq \Psi_j(v)$ for all the least square values.*

In fact, the player with a higher weight is always more "desirable": if $w_i > w_j$, $v(S \cup i) \geq v(S \cup j)$ $\forall S \not\ni i, j$. Thus, i will never strictly be rank below j in the power ordering, though we cannot rule out the possibility that we obtain a complete indifference ranking $1 \sim 2 \sim .... \sim n$ with one least square value, and the linear ordering $1 \succ 2 \succ \ldots \succ n$ with another one. In fact, the idea that one player is always more desirable than another in order to build a winning coalition as been first used by Isbell (see Taylor and Zwicker for the details [26]).

Definition 5 *Assume that $G = (N, \mathcal{W})$ is a simple game. Then the individual desirability relation for the game G is the binary relation $\leq_D$ on N defined by*

$$i \leq_D j \; iff \; \forall S \subseteq N \setminus \{i, j\}, \; if \; S \cup \{i\} \in \mathcal{W}, \; then \; S \cup \{j\} \in \mathcal{W}$$

The games for which the desirability relation is a weak ordering on the set of players N are called *linear* games (for a precise definition, see Taylor and Zwicker [26]). It is quite obvious that as long as the individual desirability relation $\leq_D$ for the game G is a weak ordering, the least square values cannot reverse it. Thus, if we want to explore the situation where Banzhaf index and

Shapley-Shubick index may rank differently some players, we must consider a class of non-linear games. Notice that the Canadian game is not linear and its desirability relation is incomplete. To check it consider the coalition $S = \{Ontario, Quebec, NewBrunswick, BritishColumbia, Alberta\}$ and the coalition $S' = \{Ontario, Quebec, NewBrunswick, NovaScotia, BritishColumbia\}$. We get:

$$\begin{array}{lcl} S \cup \{PrinceEdwardIsland\} \in \mathcal{W} & \textit{and} & S \cup \{Saskatchewan\} \notin \mathcal{W} \\ S' \cup \{PrinceEdwardIsland\} \notin \mathcal{W} & \textit{and} & S' \cup \{Saskatchewan\} \in \mathcal{W} \end{array}$$

We can neither say that Saskatchewan is more desirable than Prince Edward Island, nor the contrary.

Thus, our objective in this section is to understand to which extend the family of composite weighted game lead to discrepancies between the Shapley-Shubick and the Banzhaf orderings. We focuss our attention to the family of composite weighted games with two components, $2 - \mathcal{WG}^n$. To compute the likelihood, we developed a computer programme which:

1. First generate two n-player weighted games randomly according to the same process. The weights are normalized such as $\sum_{i=1}^{n} w_i = 1$.

$$\begin{array}{llllll} G^1 & \gamma^1 = 0.5 & w_1^1 & w_2^1 & \dots & w_n^1 \\ G^2 & \gamma^2 \in]0,1[& w_1^2 & w_2^2 & \dots & w_n^2 \end{array}$$

2. Next check whether the composite game $G = G^1 \wedge G^2$ is linear or not. If the game is linear, we already know that the Banzhaf and Shapley Shubick orderings will be similar

3. If not, the programme compute the Banzhaf-Coleman and Shapley-Shubick indices to see whether there is a discrepancy between the two rankings.

Each step calls for some comments:

Step 1. First, one may argue that the only example of a strong discrepancy between the two indices concerns the conjunction of three weighted games and we don't know a priori whether such a strange behavior can be observed in the class $2 - \mathcal{WG}^n$. However, in the Canadian case, Quebec and Ontario play no rôle in the paradox, as they belong to all the winning coalition. We can display the same paradox with the 8-player game

$$\begin{array}{llllllllllll} G^1: & \gamma^2 = 3, & w^2 = & 0 & 0 & 0 & 0 & 2 & 1 & 1 & 1 \\ G^2: & \gamma^1 = 2, & w^1 = & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{array}$$

Table 3: The Canadian game: number of coalitions wherein a player is pivotal according to the cardinality of S.

s =	Prairie	BC	Maritime
1,2,3	0	0	0
4	9	18	6
5	12	30	10
6	3	15	5
7	0	3	1
8	0	0	0
β_i	24/128	66/128	22/128
ϕ_i	78/840	264/840	88/840

The first four players are the maritime provinces, the last three are the prairie provinces and player 5 is British Columbia. Table 3 shows the number of time a player is pivotal (that is $V(S) - v(S \setminus \{i\}) = 1$) according to the number of player in a winning coalition S.

It becomes clear that the source of the different rankings lies in the very different distribution of the pivotal positions according to the number of player in the winning coalitions. One can now easily create a whole range of different measures of the power where the ranking for the maritime and prairie provinces disagree. It is even possible to obtain the ranking $BC \succ Maritime \sim Prairie$ by choosing adequately the coefficient p_s of our new power index. The semivalue $p_1 = p_8 = 0.03275$, $p_2 = p_7 = 0.01025$, $p_3 = p_7 = 0.00925$ and $p_4 = p_5 = 0.00575$, which exhibits a nice symmetry, is one of the numerous examples that would lead to this conclusion.

Secondly, notice that we set the value of γ^1 to 0.5 in order to ensure that a coalition and its complement cannot be simultaneously winning. On the other hand, the second threshold can be chosen freely: We will run different simulations from $\gamma^2 = 0.1$ to $\gamma^2 = 0.9$ with an increment of 0.05 in order to examine the influence of this parameter on the likelihood of the paradox.

Step 2. In generating $G^1 \wedge G^2$, we may end up with a composite game that can be rewritten as a traditional weighted game. Unfortunately, there is no easy test (and in particular no fast algorithm) to check whether we really work with a member of $2 - \mathcal{W}^n$ or not. Thus, checking whether a composite game is linear or not is a first test to know whether the game could lead to different results according to the index. Moreover, Taylor and Zwicker have provided us with a way to check for linearity which is easily implementable for an algorithmic point of view.

Theorem 7 (Taylor and Zwicker, [26], p 88-90) *Suppose G is a simple game in $\mathcal{SG}^n$. A game G is said to be swap robust if a one-for-one exchange between two winning coalitions can never render both losing.Then the following assertion are equivalent:*

- *G is a swap-robust game.*
- *G is a linear game.*

Step 3. In the last step, we have to describe precisely what we consider to be a discrepancy between the Shapley Shubick and the Banzhaf indices. We consider that a result is paradoxical each time $\phi_i > \phi_j$ while $\beta_j > \beta_i$ for at least one pair of player. To check for discrepancies, we first represent ordering R_1 with a $n \times n$ matrix B^1: $b^1_{pq} = 0$ if player p cannot is not ranked at position q in the weak ordering R_1, and $b_{pq} = 1$ if there a way to break the ties in the weak ordering that assigns it rank q. Similarly, the weak ordering R_2 is described by a $n \times n$ matrix B^2. Next, we check whether each row in the sum matrix $B^1 + B^2$ contains only 2's. If yes, the two orderings are identical. If the matrices are not identical but each row contains at least one 2, then it possible to find a common ordering between the to ranking by breaking ties in the appropriate way. We consider here that there is a mild or weak discrepancy between the two orderings. At last, if we can prove that there is no value 2 in $B^1 + B^2$ for at least one player, we are sure that the two orderings disagree on the ranking of this player. To give an example, consider $R_1 = 1 \succ 2 \sim 3 \succ 4 \succ 5$, $R_2 = 1 \sim 2 \succ 3 \succ 4 \sim 5$ and $R_3 = 2 \succ 1 \sim 3 \succ 4 \sim 5$. R_1 and R_2 are not strongly different, as it is possible to break the ties to find a compromise, here, the linear ordering $1 \succ 2 \succ 3 \succ 4 \succ 5$. A similar compromise, $2 \succ 1 \succ 3 \succ 4 \sim 5$ exists between R_1 and R_3. However, R_1 and R_3 strongly disagree, as they cannot find a common ranking for players 1 and 2.

$$B^1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},\ B^2 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix},$$

$$B^3 = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix},\ B^1 + B^2 = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix},$$

$$B^2 + B^3 = \begin{pmatrix} 1 & 2 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 & 2 \end{pmatrix},\ B^3 + B^1 = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

On the top of being simple, this method has also another advantage: we can specify the number of player ν for which a common ranking does not exist and get a rough measure of the extend of the discrepancy[5]. The range of this 'disagreement index' is between 0 (perfect agreement or weak disagreement) to n. One can check that in the Canadian example, the value of the index is 3.

The next step is now to describe the results. We will distinguish between the two different model we used in order to generate randomly the composite games.

5.2 The likelihood of disagreement under the IAC assumption

In social choice theory, the Impartial Culture Assumption is quite often used to compute theoretically the likelihood of voting paradox. It asserts that all the distribution of the votes among the different preference types are equally likely. The natural counterpart of this assumption for weighted game is that all the distribution of the weight $(w_1, w_2, \ldots, w_n)$ in the unit simplex are equally likely. This is exactly the assumption Chua, Ueng and Huang used to obtain their results [5]. It is not that easy to generate a distribution of the weights which is uniform in the unit simplex. However, the following algorithm can be used. First, n numbers A_j are drawn randomly between 0 and d, with d large . Then the numbers are ordered from the smallest to the greatest:

$$A_1 < A_2 < A_3 \ldots A_{n-1} < A_n$$

Then, we compute $w_1 = A_1$, $w_2 = A_2 - A_1$, $w_3 = A_3 - A_2$, etc. and we normalize the sum of the weights to one. The final distribution of the the weights $(w_1, w_2, \ldots, w_n)$.

Table 4 displays the results we obtain as a function of n and γ^2 with 500,000 draws in each case. Although we could not run computer simulation for more than 11 players, several interesting facts can already be noted. First,

[5] A more precise measure would be to compute the Kemeny distance between two orderings, that is number of pairs for which the two orderings disagree.

the probability of the paradoxes is always a 'bell shape' curve on the γ^2 axis, with a maximum for $\gamma^2 = 0.5$ or $\gamma^2 = 0.55$. It seems that the peak value tends to shift to $\gamma^2 = 0.55$ as n increases, but general conclusion cannot be drawn yet. This phenomenon can be explained by the fact that a low value of γ^2 implies that the second game only slightly modifies the first one, as almost all the condition will met the threshold. For high values of γ^2, the composite game progressively shift towards an unanimity game whose characteristics are governed by the second game. At both extremes, one game seems to play the major role, the second one only introducing nuances.

However, the most striking result is the rapid increase of the value of the likelihood for any γ^2 as n increases. Almost 60% percent of the games in $2-\mathcal{WG}^{11}$ already exhibit a strong discrepancy between the Banzhaf and Shapley-Shubick index ! Although the structure of simple games puts more constraints on the game than the model of almost symmetrical players, we have been able to provide a floodgate of paradoxical results with our computer programme. This result clearly show that the Canadian game is not that peculiar: Such contradictory results have to be expected in many composite games. We feel that it would be of interest to study precisely real examples of composite weighted games, such as bicameral political systems or complex shareholder design, to seek of other possible occurrences. However, notice that a key point that could explain our striking figures is that the weights of a player are drawn independently in both game. To illustrate this fact, consider the following 10 player game that have been singled out by the algorithm (the sum of the weights has been normalized to 100).

$$\begin{array}{lllrrrrrrrrrrl} G^1: & \gamma^1 = 50, & w^1 = (& 1.8 & 2.4 & 2.1 & 0.5 & 0.5 & 5.3 & 4.3 & 11.8 & 27.1 & 44.1) \\ G^2: & \gamma^2 = 50, & w^2 = (& 23.4 & 10.5 & 10.5 & 0.5 & 4.2 & 1.7 & 10.7 & 4.9 & 16.7 & 16.9) \end{array}$$

Shapley-Shubick ordering for the game $G = G^1 \wedge G^2$ is

$$10 \succ 1 \succ 9 \succ 7 \succ 8 \succ 3 \succ 2 \succ 6 \succ 5 \succ 4$$

while the Banzhaf ordering is

$$10 \succ 1 \succ 9 \succ 7 \succ 3 \succ 2 \succ 8 \succ 6 \succ 5 \succ 4.$$

We obtain a value of $\nu = 3$ as we cannot find a common ranking for players 2, 3, and 8. The problem is that player 8 can be ranked either fifth or seventh. However, this example clearly shows that the weight of a player from a game to the other are completely unrelated. Most of the examples we can provide distillate the same flavor.

Table 4: The likelihood of strong discrepancies between Shapley Shubick and Banzhaf Coleman, IAC model, 500,000 draws.

γ^2	n= 7	n= 8	n= 9	n= 10	n= 11
0.10	ϵ	0.3	0.8	1.8	2.8
0.15	0.2	1.2	2.4	4.8	6.9
0.20	0.5	2.7	5.1	9.1	12.1
0.25	0.9	4.7	8.6	14.2	18,1
0.30	1.5	7.3	12.7	20.1	24.7
0.35	2.3	10.2	17.6	26.7	33.1
0.40	3.1	13.6	23.2	33.8	40.3
0.45	4.0	17.0	29.1	41.5	48.9
0.50	4.5	18.3	32.6	47.0	56.0
0.55	4.3	16.8	31.5	47.2	57.2
0.60	3.8	15.6	30.1	46.2	57.0
0.65	3.4	14.0	28.1	44.3	55.3
0.70	1.7	10.3	22.2	37.7	46.6
0.75	0.6	6.2	15.2	27.8	39.1
0.80	0.2	2.6	8.0	16.4	25.7
0.85	ϵ	0.6	2.7	6.6	11.8
0.9	ϵ	0.1	0.4	1.2	2.7

ϵ: a few occurences found, with a probability lower than 0.1%.

5.3 The likelihood of disagreement under the MC assumption

The maximal culture is another possible way to generate a distribution on the simplex of weights. Again, it has first used in social theory to compute theoretically the likelihood of voting paradoxes (see Gehrlein [11, 12]). The algorithm that generates the weights is slightly different from the one used in IAC. First, A_i numbers, $i = 1, \dots n$ are drawn randomly between 0 and d large. Next, we set

$$w_i = \frac{A_i}{\sum_{i=1}^{n} A_i}$$

The characteristic of the MC model is that the probability of distributions close to $(1/n, 1/n, \dots, 1/n)$ is higher than with the IAC assumption. To understand it, notice that it is quite difficult to obtain a distribution witch gives 90% of the weights to a player with MC. However, this result is equally likely as any other with IAC.

Table 5 displays the results we obtain as a function of n and γ^2 with 500,000 draws of non linear games in each case. Compared to the IAC case, we can observe similar patterns like the bell shape and the increase of the likelihood with n. However, the magnitude of the phenomenon is slightly less important. This is clearly due to the fact that the dispersion of the weights is less pronounced with the MC model. A game where a player obtains 44,1% of the vote share, as in the previous example, may be quite unlikely with MC. Also notice that the peak is always reached for $\gamma^2 = 0.6$; we have bo explanation for this phenomenon.

6 Discussion

To our knowledge, the first authors who tried to build a bridge between the literature on linear solutions to cooperative games and scoring rules are Calvo, Garcia and Gutierrez [2]. They associate to each profile of strict preferences a specific game, the game of the alternatives, where $v(S)$ is the number of candidate who rank the candidate in S a the top of their preference. Then, they prove that there is a one to one mapping between the scoring rules and the least square values, such that the ranking of the alternatives for a given profile with a scoring rule is exactly the ranking of the players in the corresponding game of the alternative with the associated LSV. However, though it would be possible to derive from a probability distribution on the preference profiles (e.g. IC, IAC, MC) a probability measure on the games of the alternatives, this class is not every appealing (the games

Table 5: The likelihood of strong discrepancies between Shapley Shubick and Banzhaf Coleman, MC model, 500,000 draws.

q	N= 7	N= 8	N= 9	N= 10	N= 11	N= 12
0.10	ϵ	ϵ	ϵ	ϵ	ϵ	ϵ
0.15	ϵ	ϵ	ϵ	0.3	0.2	0.3
0.20	0.2	0.2	0.1	0.9	0.9	1.1
0.25	0.5	0.6	1.0	1.9	2.9	3.6
0.35	1.4	1.1	2.2	3.7	5.1	5.5
0.40	2.5	2.4	4.8	8.1	10.3	12.0
0.50	7.2	10.1	17.6	25.4	31.2	38.8
0.55	9.1	9.5	21.2	31.6	38.1	41.3
0.60	9.9	12.0	19.1	33.9	39.8	48.2
0.65	4.6	10.7	13.9	26.5	34.4	41.5
0.70	0.6	6.1	9.3	16.4	22.2	26.6
0.75	ϵ	1.6	3.8	7.1	10.0	12.2
0.80	ϵ	0.2	1.0	2.0	3.1	4.0
0.85	ϵ	ϵ	0.1	0.3	0.4	0.7

cannot be monotonic!) to deserve a thoroughly study. Clearly, our study, as the recent papers by Saari and Sieberg [21] and Laruelle and Merlin [15] is inspired by these ideas, but we try to develop the analysis for more interesting classes of games. In these context, we explored whether it is possible to adapt probability models used for the analysis of voting rules to the analysis of discrepancies among TU game solutions. From our point of view, the answer is positive, though the models we use suffer from several limitations.

Concerning the model of almost symmetrical games, we first notice that it is very specific: All the players are similar, and there is no relationship to expect among the rankings given by the different elementary least square values. For a given player, the values $v(S) - v(S \setminus i)$ are totally independent. The model of almost symmetrical voters can be viewed as an extreme case, where there is only noise. Thus, this assumption probably gives the higher bound for the likelihood of discrepancies. A positive aspect comes from the relationships between the Shapley and Banzhaf rankings: For a small number of players, they tend to rank them in the same way. Although we have not been able to perform the computations, it seems unlikely that the two semivalues lead to opposite rankings. Thus, the probability that the Shapley winner is ranked last by the Banzhaf semivalue could be relatively low for small values of n. Secondly, the complexity of the computations makes it difficult to obtain general statements. The same drawback occurs in social choice theory: We can hardly compute a probability for a problem if we need more than five inequalities to describe it. For complex problems, the only possibility is to rely on computer simulations.

Nevertheless, further researches could explore an alternative assumption on the likelihood of the different games, which would mimics IAC. If we set $v(\emptyset) = 0$, $v(N) = 1$ and $v(S) \in [0,1], \forall S \in N$, the space of all games is identify with the hypercube $C^n = [0,1]^{2^n-2}$. It is then reasonable that all the games in the hypercube are equally likely and that the measure is zero outside the hypercube. This assumption is the natural counterpart of the impartial anonymous culture of voting theory (see Gerhlein [11, 12]): All the voting situations are equally likely to occur. We could next refine the analyzes to more interesting games in C^n, by restricting ourselves only to monotonic, superadditive games, or convex games. Again some techniques from Social Choice theory could be used in order to perform the computations, but they would not allow us to treat general cases: The computations could be undertaken only for a small number of players and a limited number of inequalities. However, the recent papers by Chua and Huang [4] and Cervone, Gehrlein and Zwicker [3] propose new tools that could be useful. The results could also draw a more subtle picture compared to the figures obtained with the almost symmetrical player model.

In our opinion, one of the main contribution of this paper for the class of simple games has been to demonstrate that it is relatively easy to create composite games where the conclusions drawn from the Banzhaf and Shapley-Shubick indices disagree. And we explored only a specific class of composite games. Still, much work remains to be done. To some extent, we relied on brute force by using computer simulation; another route would be to characterize precisely the situation where the two indices disagree. This would clearly enhance our comprehension of the phenomenon, and probably help us to understand which power index is a better measure of this proteus concept that we call 'power'.

Let us also mention that the kind of study we presented here could be extended in two directions, to study the relationships with other non linear solution to TU games, such as the core and the nucleolus, or, for simple games, with other power indices, and to compute the probability that a given solution does not satisfy other desirable axioms, in the line of Chua, Ueng and Huenf [4] and van Deemen and Rusinowska [27]. Thus, we believe that the kind of research we started here could be developed and would add to the growing literature on the probability of paradoxes for solutions of cooperative games.

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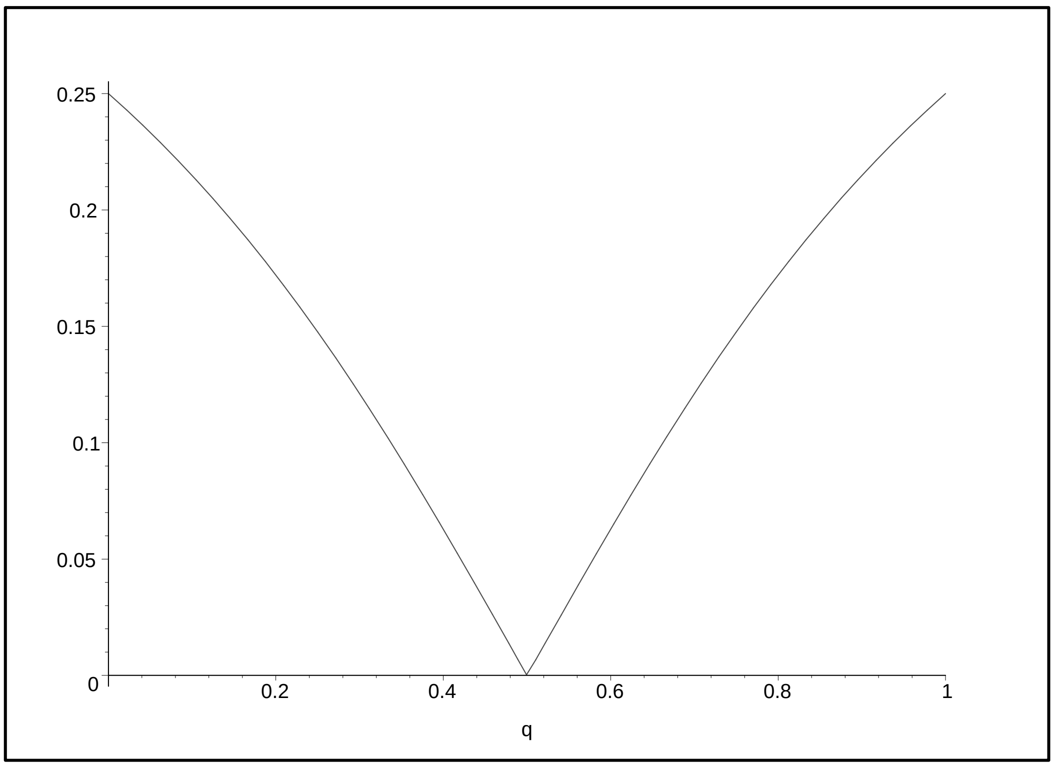

Figure 1: Conflicts with Banzhaf-Shapley on a pair

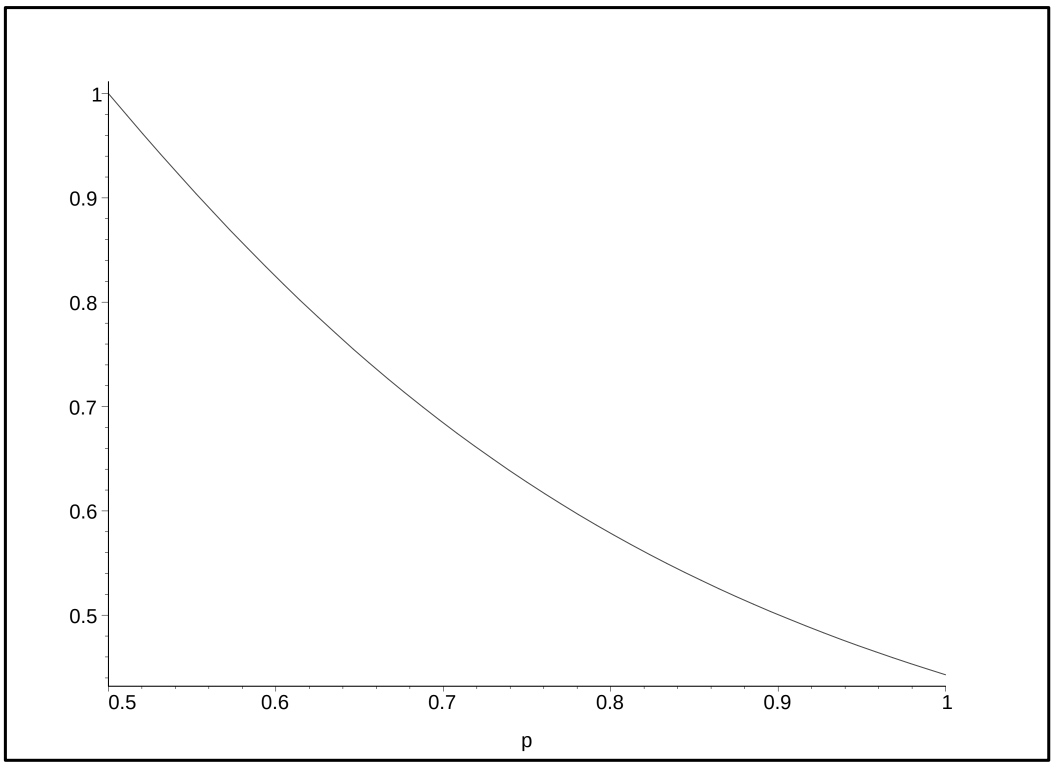

Figure 2: Agreement with the Banzhaf-Shapley ranking

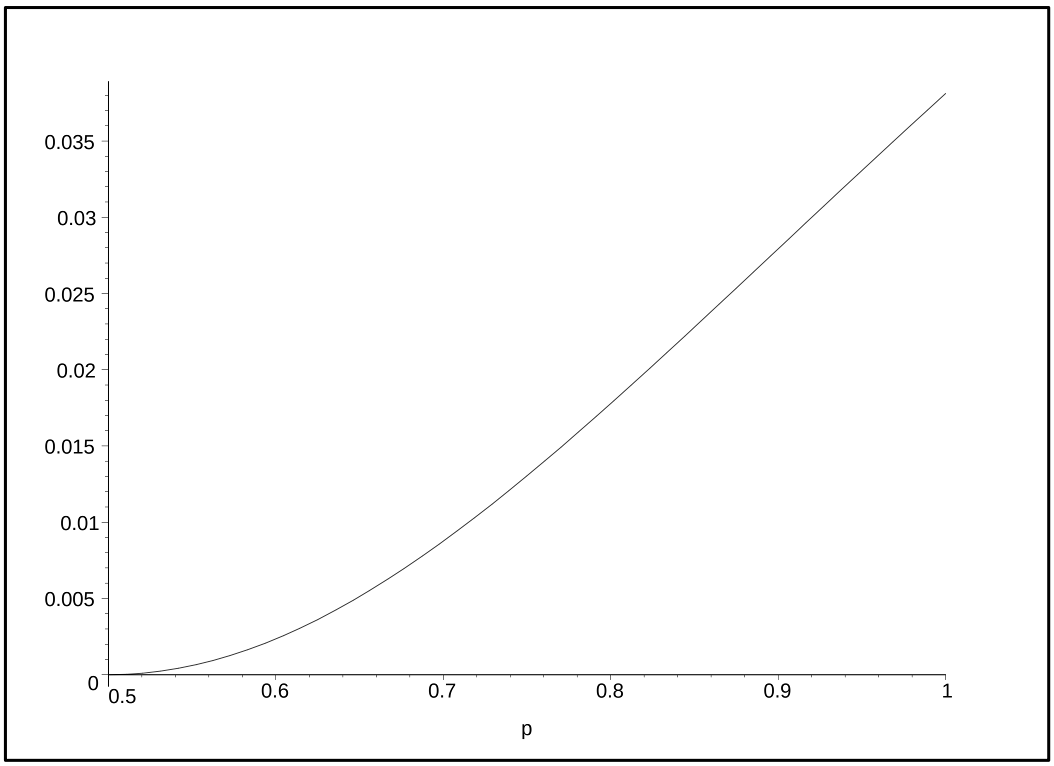

Figure 3: The probability of opposite rankings