Fair and Efficient Representation of the Citizens in a Federal Union

M.R. Feix *, D. Lepelley †, V. Merlin ‡ and J.L. Rouet §

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Abstract

All the federal unions, like the United States of America or the European Union face the issue of finding the right allocation of seats to their member states. There has been a lot of debates in the United States of America since two centuries in order to find the right mechanism to round off the number of representatives per state proportionally to their populations. This problem is now well documented (see Balinski and Young [1]). Another definition of a good apportionment method derives from game theory: Penrose [17] suggested that a fair apportionment method should give the same power to every voter. When the power is measured with the Banzhaf Coleman index, this lead to adopt to square root law in the limit: the number of seats allocated to a state should be proportional to the square root of its population. We propose here another criteria: an apportionment of the seats among the states is efficient if the probability of electing candidate who receives less than 50% of the votes in a two candidate competition over the whole union is minimized. To answer this question, we consider mainly two theoretical probability models derived from social choice literature. In the first model (Impartial Culture, IC), each voter in each states select his candidate randomly with probability one half. In the second model (Impartial Anonymous

*SUBATECH, Ecole des Mines de Nantes, La chanterie, 4 rue A Kastler, BP 20722, F-44307 Nantes cedex 3, France

†Université de Caen. Department of Economics, Université de Caen, Esplanade de la paix, 14032 Caen Cedex, FRANCE

‡CNRS and Université de Caen. MRSH bureau 230, Université de Caen, Esplanade de la Paix, 14032 Caen cedex, France

§Laboratoire de Mathématique, Applications et Physique Mathématique -UMR6628, Université d'Orléans, UFR des Sciences, F-45067 Orléans Cedex 2, France

Culture, IAC), the percentage of votes a candidate receives in a state is drawn from $[0, 1]$ with a uniform probability distribution. Using computer simulations, we suggest that the number of seats should be apportioned with a square root law in the first case, and in proportional way in the second case. These results are consistent with the Penrose law for the IC model.
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1 Introduction

2 Assumptions

2.1 The voting model

Consider a finite set $N = \{1, \ldots, i, \ldots, M\}$ of states (or regions, districts, etc.) which have to take decisions altogether in a political union. We assume that m_i voters live in state i. Two parties, A and B[1], compete in all the states; the winner in state i is the party who obtains a majority of voters on his side (abstention is not allowed). Each state is represented by a_i mandates in the union, and the winner in state i gets all the mandates. Thus, the position that is officially adopted by the union is the one which obtains a majority of mandates at the federal level.

The choice of the number of mandates per state can obey to different logics. One can argue that all the members of the union are equal, and that a_i should be one for all the states, even if they have very different population sizes. Another possibility is to allocate the mandates in proportion to the populations. Ideally we could assume that $a_i = m_i$ but most of the time a_i is supposed to be a relatively small integer. As already mentioned, this rounding off problem is very well documented due to the American case. As the number of representatives per state must be proportional to the population, it happened many times that a state tried to impose in Supreme Court a rounding method that could favor it. This gave a strong incentive for scholar to study the different rounding methods; see Balinski and Young [1]. Several other options exist between the egalitarian and the proportional case. For example the number of electors per states in the electoral college in the US is the number of representatives plus two, the number senators. In the European Union, the number of mandates at the disposal of a country

[1] In fact we do not need institutionalized political parties for this model ; A and B could just be two proposals at stake.

in the council of minister is roughly proportional to the square roots of its inhabitants [3, 9]. Though this may be quite unusual, we could also imagine and justify apportionment rules that attributes mandates in proportion to the exponential of m_i, or the square of m_i. In the extreme case, we get the dictatorship of the biggest state[2].

We adopt here the following normative criteria to evaluate the different apportionment methods : *an apportionment method is said to be efficient if it minimizes the probability that a decision is taken with a majority of mandates at the federal level though it is supported by a minority of voters over the whole union.* Unfortunately, such strange political situations are not only theoretical objects; they often happens, a well known case being the election of George W. Bush against Al. Gore in 2000 (for other examples in US, United Kingdom and France, see [?]). More precisely, we will try to identify which parameter α minimizes the probability of the paradox if we allocate the seats according to the law $a_i = m_i^\alpha$. We answer to this question under a veil of ignorance : we assume that each party are equally likely to win, and we simulate the votes without any reference to a precise political context under different probability models. Though it maybe possible to specify some parameters and the distribution of the votes with applied works in some specified context, we do not consider this possibility here.

2.2 Probability models

There are several ways to model theoretically the behavior of the citizens. We assume that, in every country the behavior is described by the same probability rule. The four model we use throughout the paper are borrowed from social choice theory; the two first are standard, while the other are new.

- The Impartial Culture assumption, IC. Each voter selects a party with equal probability. When m_i is sufficiently large, the distribution of the votes follows a normal law. In each state, the excess of ballots for A or B is then given by

  $$\epsilon_i \sqrt{m_i}$$

  where ϵ_i is drawn randomly according to the Gauss distribution:

  $$(2\pi)^{-1/2} \exp(-\varepsilon^2/2).$$

[2] One could think for example of a military alliance: the number of seats is proportional to his military efforts, that is his military budget or the number of army divisions. Due to increasing returns to scale, the biggest state is able to maintain a very high level of expenses. The same reasoning could hold in a research consortium.

The popular over the whole union is given by

$$\operatorname{sgn}\left(\sum_i \varepsilon_i \sqrt{m_i}\right), \tag{2.1}$$

while the decision taken by the representatives is given by

$$\operatorname{sgn}\left(\sum_i a_i \operatorname{sgn}(\varepsilon_i)\right) \tag{2.2}$$

The sgn function is defined by

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{si} \quad x > 0 \\ -1 & \text{si} \quad x < 0 \end{cases} \tag{2.3}$$

By convention, a one value(respectively minus one value) results in the selection of candidate A (respectively B).

- The Impartial Culture Assumption, IAC. This assumption considers that every distribution of the votes between the two candidates is equally likely to occur. Thus, ϵ_i is now drawn from the distribution $f(\epsilon) = 1/2$ if $-1 < \epsilon < 1$ and 0 otherwise. The excess of ballots in favor of candidate A in state i is given by $\epsilon_i m_i$ and we have to compare

$$\operatorname{sgn}(\sum_i \varepsilon_i m_i) \tag{2.4}$$

for the popular vote with the vote of the representatives

$$\operatorname{sgn}(\sum_i a_i \operatorname{sgn}(\varepsilon_i)) \tag{2.5}$$

- The Rescaled Impartial Anonymous Culture assumption, RIAC. This assumption is a modified version of the IAC model. In most elections, extreme values such as only 10% of a vote in favor of a candidate or 80% support are unlikely. The range of variation for a candidate is more likely to lie between 40% support to 60% support. Thus, ϵ_i is now drawn from the distribution $f(\epsilon) = 1/2\Delta$ if $-\Delta < \epsilon < \Delta$ and 0 otherwise, with $\Delta \in]0, 1]$.

- The Unanimous Culture UC. It may of interest to consider the extreme case where all the voters of a country decide to support the same

Voter 1	A	$\underline{A}$	$\underline{A}$	A	B	$\underline{B}$	$\underline{B}$	B
Voter 2	A	$\underline{A}$	B	$\underline{B}$	$\underline{A}$	A	$\underline{B}$	B
Voter 3	A	B	$\underline{A}$	$\underline{B}$	$\underline{A}$	$\underline{B}$	A	B
IC	1/8	1/8	1/8	1/8	1/8	1/8	1/8	1/8
IAC	1/4	1/12	1/12	1/12	1/12	1/12	1/12	1/4
$RIAC$	0	1/6	1/6	1/6	1/6	1/6	1/6	0
UC	1/2	0	0	0	0	0	0	1/2

Table 1: The 8 voting configurations for 3 voters and the associated probabilities according to IC, IAC, $RIAC$ and UC.

candidate. $f(\epsilon) = 1$ or -1 with probability 1/2. The excess of ballot in favor of candidate A is $\pm m_1$ and we have to compare

$$sgn(\sum_i m_i \,\mathrm{sgn}(\varepsilon_i)) \tag{2.6}$$

for the popular vote with the federal vote

$$\mathrm{sgn}(\sum_i a_i \,\mathrm{sgn}(\varepsilon_i)) \tag{2.7}$$

As an illustration, we display in Table 1 the probability of the different voting situations for three voters. Notice that all the model we introduced assume that there is no bias in favor of A or B; both are treated equally [3]. Thus, the differences among the models do not really come from the fact that the ϵ values are drawn from different distributions, but rather from the scaling factors: $\sqrt{m_i}$ for IC, m_i for IAC and Δm_i for RIAC.

3 The Study of the m_i^α Rule.

Many studies have compared the different merits of the IC and IAC assumptions for the computation of voting paradoxes (see, for example, Berg and Lepelley [4], Gehrlein [11, 12], Cervone, Gehrlein and Zwicker [6]), an the impact of choosing one model or the other on the magnitude of the likelihood. However, the papers on the probability of electing a minority candidate (see

[3]In a previous paper [8], we studied the influence of a bias on the likelihood of electing a minority candidate when all the states have the same population. This is the BRIAC assumption (Biased and Rescaled Impartial Anonymous culture).

Table 2: The values of m_3.

$m_2 \downarrow$ $m_1 \rightarrow$	0.500	0.450	0.400	0.350	0.333
0.250	0.250	−−	−−	−−	−−
0.300	0.200	0.250	0.300	−−	−−
0.333	0.167	0.217	0.266	0.316	0.333
0.350	0.150	0.200	0.250	0.200	−−
0.400	0.100	0.150	0.200	−−	−−
0.450	0.050	0.100	−−	−−	−−
0.500	0	−−	−−	−−	−−

Table 3: The probability $P_{IC}^3(m, \infty)$.

$m_2 \downarrow$ $m_1 \rightarrow$	0.500	0.450	0.400	0.350	0.333
0.250	0.166	−−	−−	−−	−−
0.300	0.168	0.164	0.163	−−	−−
0.333	0.170	0.166	0.163	0.162	0.162
0.350	0.171	0.167	0.164	0.162	−−
0.400	0.179	0.171	0.167	−−	−−
0.450	0.194	0.179	−−	−−	−−
0.500	0.25	−−	−−	−−	−−

Galam [10], Feix, Lepelley, Merlin and Feix [8],) only consider equal population states. In these cases, the sizes m_i play no role; these models only compare the impact of two different drawing (Guassian or uniform).

Our objective is to study the impact of the distribution of the population and the impact of the apportionment rules.

3.1 Exact values for three states

Without loss of generality, we assume in the 3 states that the distribution of the population is given by the vector $m = (m_1, m_2, m_3)$, with $\sum_{i=1}^{3} m_i = 1$, $m_1 \geq m_2 \geq m_3$. Similarly, we assume that the distribution of the representatives is given by $a = (a_1, a_2, a_3)$. For a majority game, we can easily prove that any vectors $a = (a_1, a_2, a_3)$ can be identified with one of these three possible majority games :

- Case 1. $a_1 = 1, a_2 = a_3 = 0$ and state 1 is a dictator

Table 4: The probability $P^3_{IAC}(m,\infty)$.

$m_2 \downarrow \ m_1 \rightarrow$	0.500	0.450	0.400	0.350	0.333
0.250	0.166	$--$	$--$	$--$	$--$
0.300	0.172	0.151	0.134	$--$	$--$
0.333	0.181	0.157	0.137	0.126	0.125
0.350	0.186	0.161	0.140	0.128	$--$
0.400	0.204	0.179	0.156	$--$	$--$
0.450	0.226	0.201	$--$	$--$	$--$
0.500	0.25$\star$	$--$	$--$	$--$	$--$

$\star$: *limit probability.*

- Case 2. $a_1 = a_2 = 1$, $a_3 = 0$. Player 3 has no power and in case of opposite of the two sates, no decision is taken.
- Case 3. $a_1 = a_2 = a_3 = 1$. All the state have the same power.

Thus, we focuss on the third case, the most interesting one. The differences among the population sizes reveal the impact of inequality of population on the likelihood of the paradox.

Proposition 1 *Let $P^3_{IAC}(m,\infty)$ be the likelihood of the majority paradox for three states of large population under IAC for the distribution m. Then:*

$$P^3_{IAC}(m,\infty) = \frac{m_1^3+m_2^3+m_3^3}{24m_1m_2m_3} \frac{-(m_1-m_2)^3-(m_1-m_3)^3-(m_2-m_3)^3}{24m_1m_2m_3}$$

.

Proof: see Appendix 1.

Proposition 2 *Let $P^3_{IC}(m,\infty)$ be the likelihood of the majority paradox for three states of large population under ICC for the distribution m. Then:*

$$P^3_{IC}(m,\infty) = \frac{\arccos\left(\sqrt{m_1}\right)+\arccos\left(\sqrt{m_2}\right)+\arccos\left(\sqrt{m_3}\right)}{\pi} - 0.75$$

Proof: see Appendix 2.

In Table 2 we present several distributions of the vector $m = (m_1, m_2, m_3)$. The corresponding values for $P^3_{IAC}(m,\infty)$ and $P^3_{IAC}(m,\infty)$ are displayed on Table 3 and 4n, respectively. First, our finding are consistent with the equal population case ($m = (1/3, 1/3, 1/3)$ studied previously by Feix, Lepelley, Merlin and Rouet [8]. One can notice that in both cases, the main factor that explain a high value for the paradox is a small value for m_3. In the IC case, when m_3 is lower than 0.1, the probabilities rises from 16.2% to 17.9% and more (up to 25% when $m_3 \to 0$). In the IAC, the rise is more spectacular. The value is already 17.1% for $m_3 = 0.15$, and above 20% for $m_3 = 0.1$.

For $m > 3$, the number of cases becomes too important to give a complete enumeration (see Bison, Bonnet, Lepelley [2]). So, we turn on to the search for an optimal allocation rule.

3.2 Computer simulations

For the general case, we focuss our study on the search on the best apportionment rule. More precisely, we assume that $a_i = m_i^{\beta}$ for the IC case and $a_i = m_i^{\alpha}$ for the IAC case. We then run several computer simulation for different values of the vector m in order to find the optimal values of α and β. Before going to the simulation, notice the following facts. Denote by $\bar{m}_i$ the expression $\sqrt{m_i}$ in equation 2.1. With this new variable, it becomes identical to equation 2.4. Similarly, if we assume that $2\beta = \alpha$, equations 2.2 and 2.5 are the same. Thus, we can state that, to some extent, the results with the IAC assumption are "transposable" to the IC case for populations raised to the square and a distribution of the mandates such as $\beta = 2\alpha$. The meaning of "transposable" will become clear with the computer simulation. However, notice that one difference remains: on one hand the ϵ_i are drawn from a normal law, and on the other from a uniform law.

Figure 1 and 2 display the results of a Monte Carlo simulation for IC and IAC with five states and $m = (1, 1.4, 1.8, 2.3, 3.2)$ (or the square of these values for the IC case). The objective is to perform an integral on a five dimensional space. The curve are varying by plateaus, depending on the underlying discrete power structure of the majority game. For the IAC case, we encounter 6 plateaus, corresponding to the majority games with weight $a = (1, 1, 1, 1, 1)$, $a = (2, 2, 1, 1, 1)$, $a = (3, 2, 2, 1, 1)$, $a = (4, 2, 2, 1, 1)$, $a = (5, 2, 2, 2, 2)$ and $a = (1, 0, 0, 0, 0)$. The optimal value, which lead to a probability of about 16%, is obtained for values of α in between 0.9 and 1.3. The pure federal case leads to a paradox in about 19% of the simulations, and the dictatorial case in about 24%.

The picture for the IC case is very similar if we assume $m'_i = m_i^2$ and $\beta = 2\alpha$. We encounter the six plateaus, corresponding to the same six different voting games. However, the magnitudes are different. We start with a value of 21.5% in the federal, next obtain the a minimum a bit lower than

20% for β in between 0.45 and 0.65 approximately, and then progressively go up to 25.5% for the dictatorial case.

In figure 3, we test the optimality of the rule $\alpha = 1$ by keeping $a_i = 1$ for four states, and let the free a_i vary. For example, the highest value of the paradox is obtained with $a = (1, 1.4^{\alpha}$, 1.8, 2.3, 3.2) and the dictatorial case ($\alpha = 4$). This of course does not exhaust all the possible cases.

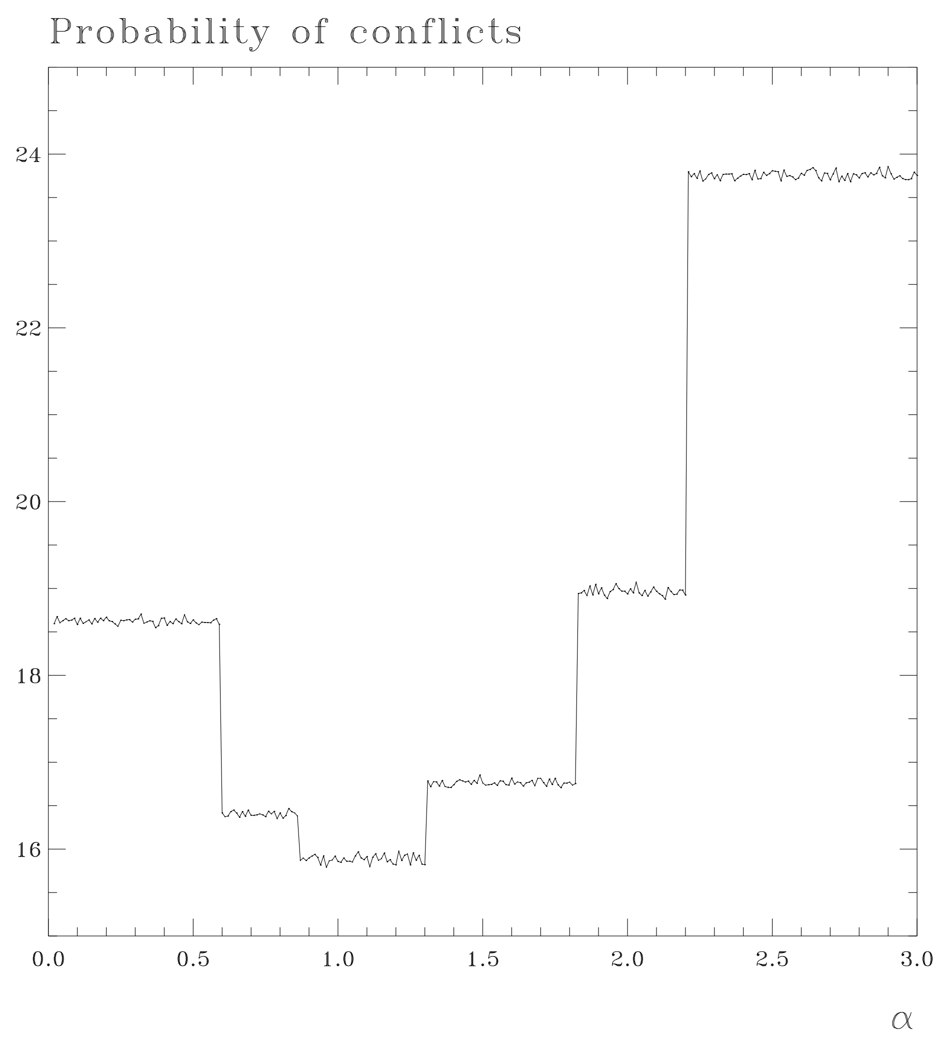

Figure 1: : likelihood of conflicts between the popular vote and the vote by states, estimated after 1 000 000 draws for IAC model and $m_i = 1, 1.4, 1.8, 2.3, 3.2$.

Figure 4 and 5 present a much more complicated situation with 20 states. Due to the numerous number of possible majority weighted games with 20

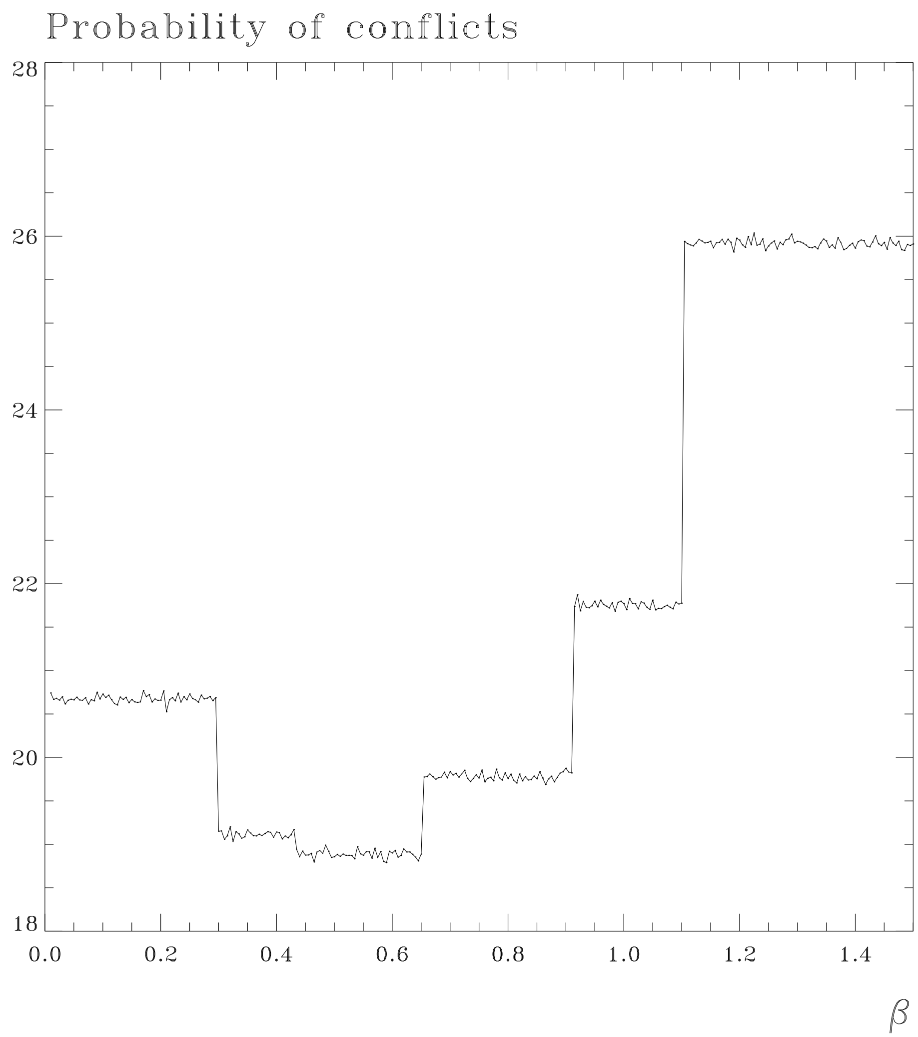

Figure 2: : likelihood of conflicts between the popular vote and the vote by states, estimated after 1 000 000 draws for IC model and $m_i = 1, 1.96, 3.24, 5.29, 10.24$.

players, the curves become almost continuous. The pattern of the two curves are similar : first a plateau around the federal case, then a decline till the optimal value ($\alpha = 1$ in the IAc case, $\beta = 0.5$ in the IC model) and then a regular increase. Notice that we do not reach the dictatorial case in the two simulation.

To conclude, these computer simulation and some other that we do not display lmmake us believe that the following conjecture is true:

CONJECTURE. For N states characterized by the populations m_i voting

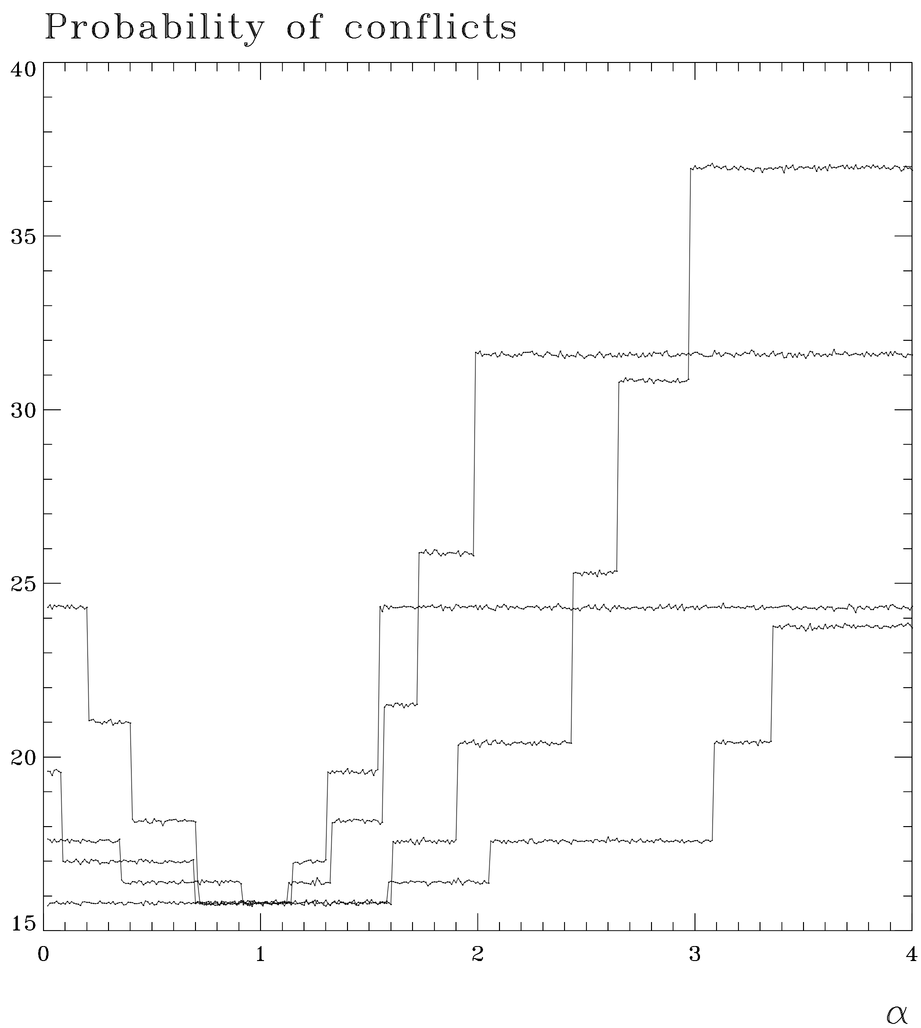

Figure 3: : Testing the optimality of the $a_i = m_i$ for IAC model and the 5 state case, estimated after 1 000 000 draws.

under the IC model (respectively AIC), the number of representatives a_i for state i should vary as $m_i^{1/2}$ (respectively m_i) in order to minimize the likelihood of conflicts. (All the votes being taken through majority votes).

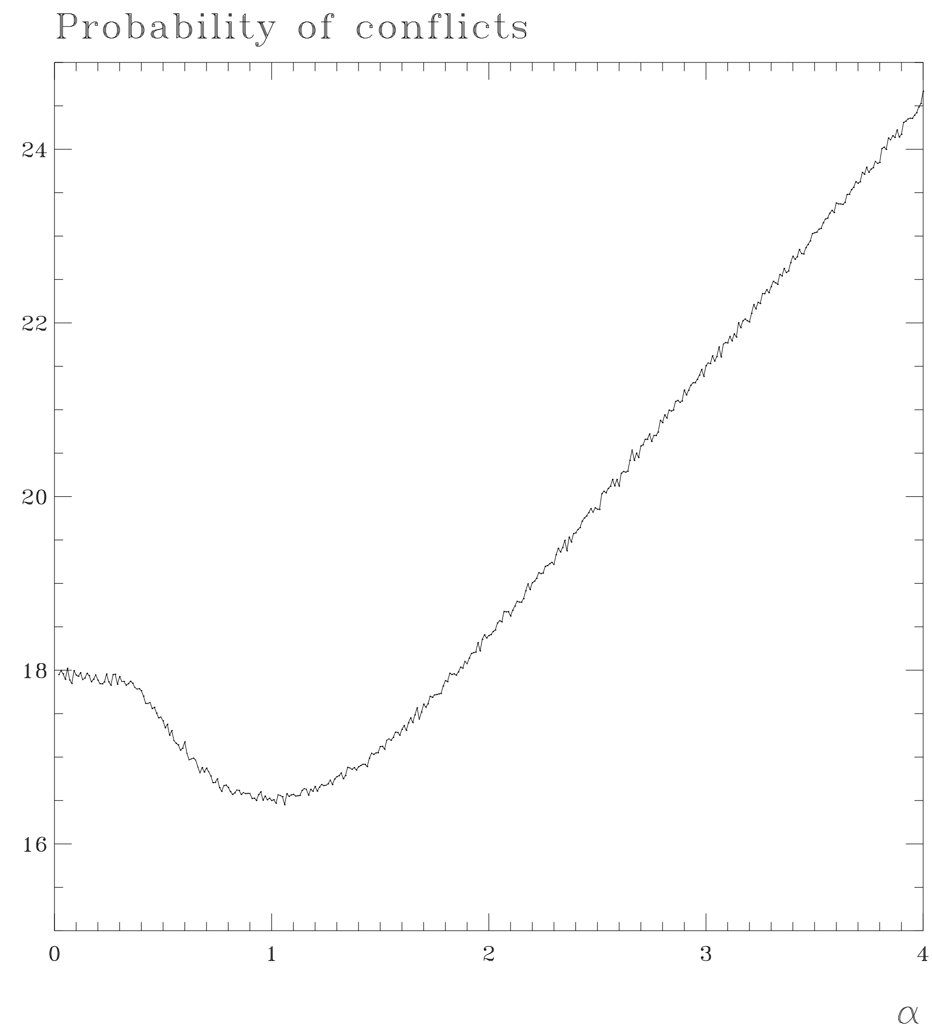

Figure 4: : likelihood of conflicts between the popular vote and the vote by states, estimated after 1 000 000 draws for IC model and m_i= 1, 1.31, 1.53, 1.77, 2.12, 2.54, 2.97, 3.15, 3.39, 3.44, 3.75, 3.93, 4.16, 4.21, 4.34, 4.85, 5.54, 5.72, 5.99.

3.3 An application to the Nice Treaty and European Union.

4 Probability Models and Equal Power

4.1 The Penrose law.

4.2 Equal Power under the IAC Assumption.

4.3 Behavior in the limit

5 Conclusion: A Rejoinder

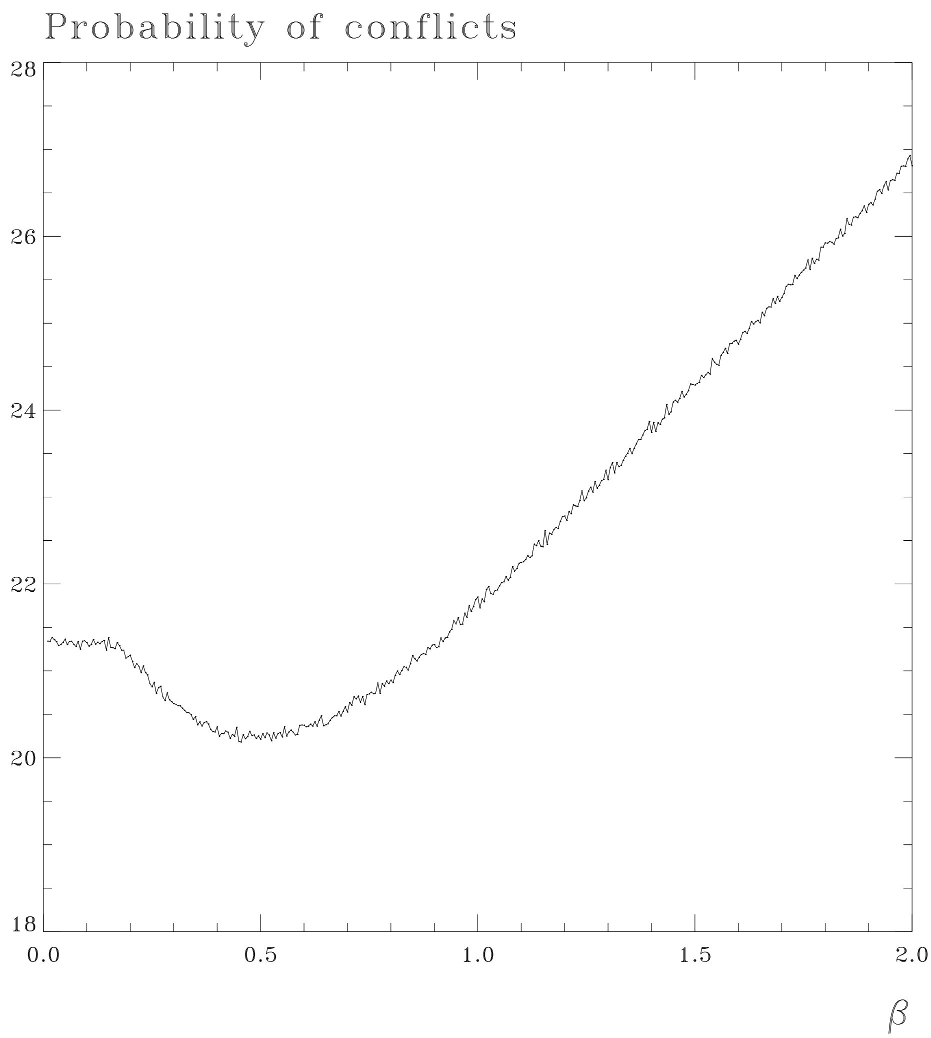

Figure 5: : likelihood of conflicts between the popular vote and the vote by states, estimated after 1 000 000 draws for IC model and $m_i = 1$, 1.71, 2.34, 3.13, 4.49, 5.81, 6.45, 8.82, 9.92, 11.49, 11.83, 14.06, 15.44, 17.31, 17.72, 18.84, 23.52, 30.69,32.72, 35.88.

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