WORKING PAPERS

Measuring HDI – The Old, the New and the Elegant: Implications for multidimensional development and social inclusiveness

Srijit Mishra and Hippu Salk Kristle Nathan

ASIA RESEARCH CENTRE WORKING PAPER 63

Measuring HDI – The Old, the New and the Elegant: Implications for multidimensional development and social inclusiveness

September 2014

Dr Srijit Mishra was the Subir Chowdhury Fellow 2013-14 at the Asia Research Centre, LSE.

Dr Hippu Salk Kristle Nathan National Institute of Advanced Studies, Bangalore.

For further information, please contact:

Asia Research Centre (ARC)
London School of Economics & Political Science
Houghton Street
London
WC2A 2AE
United Kingdom

E-mail: arc@lse.ac.uk
www.lse.ac.uk/asiaResearchCentre

ASIA RESEARCH CENTRE WORKING PAPER 63

Measuring HDI – the Old, the New and the Elegant:
Implications for Multidimensional Development and Social Inclusiveness

Srijit Mishra, Hippu Salk Kristle Nathan

September 2014

Abstract

The Human Development Index (HDI) *is calculated using normalized indicators from three dimensions- health, education, and standard of living (or income). This paper evaluates three aggregation methods of computing* HDI *using a set of axioms. The old measure of* HDI *taking a linear average of the three dimensions satisfies* monotonicity, anonymity, *and* normalization (*or* MAN) *axioms. The current geometric mean approach additionally satisfies the axiom of* uniformity, *which penalizes unbalanced or skewed development across dimensions. We propose an alternative measure, where* HDI *is the additive inverse of the distance from the ideal. This measure, in addition to the above-mentioned axioms, also satisfies* shortfall sensitivity *(the emphasis on the neglected dimension should be at least in proportion to the shortfall) and* hiatus sensitivity to level *(higher overall attainment must simultaneously lead to reduction in gap across dimensions). These axioms make an acronym* MANUSH *and its anagram is* HUMANS. *Using Minkowski distance function we also give an* α*-class of measures, special cases of which turn out to be the old linear averaging method* (α=1) *and our proposed displaced ideal measure* (α=2) *and when* α≥2 *then the* MANUSH *axioms turn out to be both necessary and sufficient. From the perspective of HDI indicating direction of future progress: α=1 can be identified with translation invariance (equal attainment across dimensions in future, independent of historical antecedents), α→∞ can be identified with a Rawlsian leximin ordering, and α=2 will be an intermediary position between the two that satisfies shortfall sensitivity weakly.*

Key words: Displaced ideal, Hiatus sensitivity to level, MANUSH, Minkowski distance, Shortfall sensitivity, Uniform development

JEL codes: D63, I31, O15

Acknowledgements: SM is based at the Indira Gandhi Institute of Development Research (IGIDR), Mumbai and HN is based at the National Institute of Advanced Studies (NIAS), Bengaluru. A part of this revision was carried out when SM was visiting the London School of Economics and Political Science (LSE) as Subir Chowdhury Quality and Economics Fellow during January-April 2014. The authors have benefited from insightful comments by anonymous peer reviewers and by discussions with and comments from Dilip Ahuja, Pradipta Chaudhury, Om Damani, Meghnad Desai, James Foster, Anjan Mukherji, Prasanta Pattanaik, Rathin Roy, S Subramanian, Yongsheng Xu and from presentations made at Coventry University (EFA Research Seminar Series, Coventry Business School); IGIDR, Mumbai; London School of Economics and Political Science (LSE) (CASE Social Exclusion Seminars, STICERD); NIAS, Bengaluru; National Institute of Public Finance and Policy (NIPFP), New Delhi; Oxford Poverty and Human Development Initiative (OPHI), University of Oxford; Ravenshaw University, Cuttack; University of Liverpool (Development Studies Seminar Series under DRIVE); University of Manchester (Development Economics Seminar Series, IDPM); and Xavier Institute of Management, Bhubaneswar. An earlier version has come out as working paper at IGIDR. Usual disclaimers apply.

1. Introduction

In the human development paradigm the emphasis is on human beings as *ends* in themselves and not so much as *means* of development.[1] Further, the ends are in multiple dimensions. It is in this context that Mahbubul Haq, the founder of *Human Development Reports* (HDRs),[2] considers one-dimensionality as the most serious drawback of the income-based measures. This led to the birth of the Human Development Index (HDI), see Haq (1995, chapter 4). The measurement of HDI has evolved over time and has contributed significantly to policy discourse.[3]

The calculation of HDI involves three dimensions–health (h), education (e), and the ability to achieve a decent standard of living, represented by income (y). The performances of each country in these three dimensions are normalized such that $0 \leq h,e,y \leq 1$,[4] and then aggregated to get the composite HDI. Prior to 2010, linear averaging (LA) across three dimensions was used as an aggregation method to obtain HDI, $(h+e+y)/3$; we denote this as HDI_{LA}. In 2010, this aggregation method was changed to the geometric mean (GM), $(h \times e \times y)^{1/3}$. We denote this as HDI_{GM}. In this paper, we propose an alternative aggregation method, which is the additive inverse of the distance from the ideal.[5] Following Zeleny (1982), we refer to this as the displaced ideal (DI) method and denote this as HDI_{DI}.[6]

As a first step, this paper evaluates the above-mentioned three aggregation methods. While evaluating these, it does not look into the rationale behind the choice of the three

[1] For discussions on this see Streeten *et al.* (1981), Sen (1989, 1997, 1999 and 2000), Desai (1991), Streeten (1994), and Haq (1995), among others.

[2] The human development report is being published annually since 1990 and serves as a cornerstone in terms of philosophy as well as an approach of the United Nations Development Programme (UNDP).

[3] For discussions on birth, evolution, and critique of measurement methodology of HDI and its policy discourse, see Anand and Sen, 1993, 1995, 1997, 2000, and 2003; Haq, 1995; Lüchters and Menkhoff, 1996; Hicks, 1997; Noorbakhsh, 1998; Sen, 2000; Fukuda-Parr *et al*., 2003; Jehan, 2003; Raworth and Stewart, 2003; Ranis *et al*., 2006; Grimm *et al*., 2008; UNDP, 2010; and Klugman *et al*. (2011) among others.

[4] The normalization used: Index=(actual-minimum)/(maximum-minimum).

[5] The ideal corresponds to the maximum values for all the three dimensions as posited by UNDP for HDI calculation. It is in this sense that ideal indicates complete attainment. We use distance in the Euclidean sense.

[6] Chakravarty and Majumder (2008) suggest the use of shortfalls in targets, normalized over current deprivations, while evaluating the progress of Millennium Development Goals.

dimensions and how they are measured, scaled, weighed, or normalized.[7] These are important issues, but beyond the scope of this current exercise. Rather, we take these as given or common for all the aggregation methods and then evaluate the methods using a set of axioms, namely, *monotonicity, anonymity, normalization, uniformity, shortfall sensitivity* and *hiatus sensitivity to level* with the acronym MANUSH.[8]

In the second step, we propose a class of measures, $\mathscr{H}_\alpha$, based on the Minkowski distance function. Both HDI_{LA} ($=\mathscr{H}_1$) and HDI_{DI} ($=\mathscr{H}_2$) turn out to be special cases of this class of measures. We also show that MANUSH axioms are necessary and sufficient for $\mathscr{H}_\alpha$ when $\alpha \geq 2$.

Use of Minkowski distance function in the context of human development is not new. Prior to 2010, the *Human Development Reports* used Minkowski distance function across different dimensions of deprivations to calculate Human Poverty Indices (HPI-1 and HPI-2), see Anand and Sen (1997). Subramanian (2006) has also used the Minkowski distance function to the Foster *et al*. (1984) class of poverty measures.

There have been attempts to make the HDI measure sensitive to inequality across individuals or sub-groups of population in each dimension, see, for instance Hicks (1997) and Chatterjee (2005). Others, as also the focus of this paper with an emphasis on aggregation, have been concerned about addressing inequality across dimensions. Chakravarty (2003) has proposed a generalized HDI, $H_\delta =((h^\delta+e^\delta+y^\delta)/3)$;, $\delta\in(0,1]$; $dH_\delta /d\delta<0$ and at its lower bound of $\delta=1$ we get $H_\delta=\text{HDI}_{\text{LA}}$. Based on Atkinson's index, Foster *et al*. (2005) propose a class of measure, $H_\varepsilon=((h^{1-\varepsilon}+e^{1-\varepsilon}+y^{1-\varepsilon})/3)^{1/(1-\varepsilon)}$ for $\varepsilon\geq 0$, $\varepsilon\neq 1$ and $H_\varepsilon=(h\times e\times y)^{1/3}$ for $\varepsilon=1$ such that $H_\varepsilon=\text{HDI}_{\text{LA}}$ at $\varepsilon=0$, $H_\varepsilon=\text{HDI}_{\text{GM}}$ at $\varepsilon=1$, and H_ε takes the form of a harmonic mean at $\varepsilon=2$; also note, $H_\delta=(H_\varepsilon)^{1-\varepsilon}$ as $\varepsilon=1-\delta$ when $\varepsilon\in[0,1)$. Besides addressing inequalities across dimensions,

[7] See Wolff et al (2011) discuss the role of data error observed from subsequent updating , formula revision and the role of threshold in categorizing a country's development status.

[8] Incidentally MANUSH (or *manus*) means human being in some of the South Asians languages such as Assamese, Bengali, Marathi, and Sanskrit among others and also has as its anagram HUMANS.

this measure also uses the ε parameter to address inequalities within each dimension, i.e., across individuals/sub-groups.[9] In this sense, H_ε is known as 'general mean of general means'. Seth (2009) proposes a class of measures in the same form as H_ε with the difference that the two parameters of inequality aversion (between and within dimensions) can be different. In terms of MANUSH property, the proposed class of $\mathscr{H}_\alpha$ measures have an advantage over H_δ and H_ε from the perspective of shortfall sensitivity and hiatus sensitivity to level.

The three different aggregation methods are discussed in section 2. The MANUSH axioms are elaborated in section 3. On the basis of these axioms, the three methods of aggregation, HDI_{LA}, HDI_{GM} (or H_ε when ε=1) and HDI_{DI}, are compared in section 4. In section 5, $\mathscr{H}_\alpha$ class of measures are proposed and their relationships with the MANUSH axioms are explored and we also examine whether H_δ and H_ε (when $\varepsilon \geq 0$, $\varepsilon \neq 1$) can satisfy these axioms. Concluding remarks are given in section 6.

2 The three methods of aggregation

2.1 Linear Averaging

The LA method applied to any set of parameters has an underlying assumption that the parameters are perfectly substitutable. The perfect substitutability assumption means that a differential improvement (or increment) in one indicator at any value can be neutralized by an equal differential decline (or decrement) in another indicator at any other value. This assumption is understandable when used in the case of same parameters like finding the average height of students in a class, or, when production of rice in different plots of land are added to compute yield per unit of land. Thus, LA essentially makes the thinking one dimensional wherein different dimensions are treated as same or similar parameters, which in

[9] The values of h, e, and y, used in H_ε are not simple averages of attainments but values adjusted for inequality across population in the respective dimensions. The inequality adjusted human development index (IHDI) introduced in the 2010 HDR (UNDP, 2010) is based on this measure (see, Alkire and Foster, 2010).

principle are perfectly substitutable. By using LA in the construction of HDI, it is assumed that health, education, and income are perfectly substitutable. Mathematically,

$$HDI_{LA}=1/3(h+e+y). \quad (1)$$

In the three dimensional space (h, e, y), one will have inclined triangular iso-HDI_{LA} planes indicating same HDI_{LA} values. The corresponding locus in two dimension will be 45^0 inclined (or backward hatched) lines. For presentation convenience and without loss of generality, the iso-HDI_{LA} plot for a two-dimensional space of health and education has been given in Figure 1.

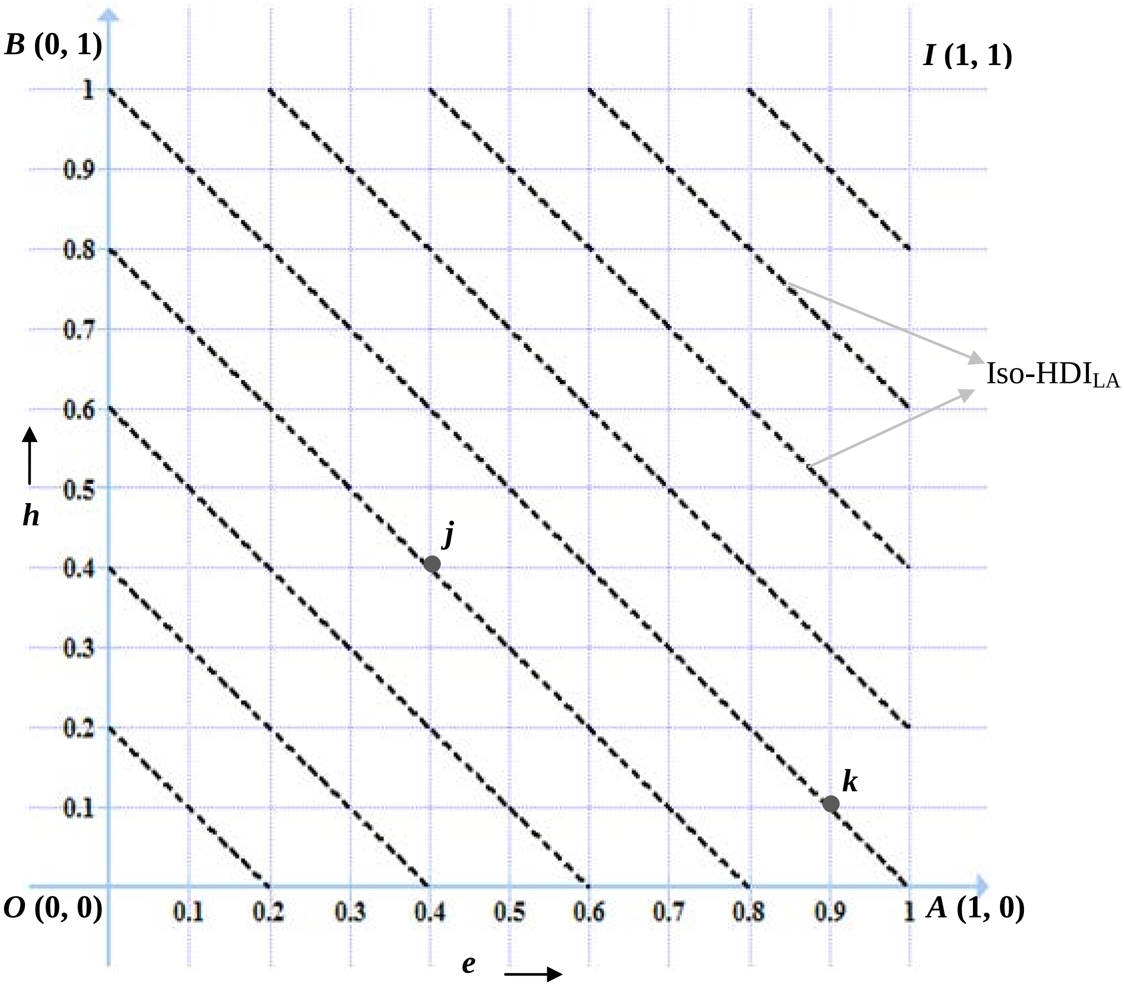


Figure 1. *Iso-HDI_{LA} in a two-dimensional space*

Figure 1 shows HDI space *OAIB* with origin, *O* (0, 0), where education, *e*, and health, *h*, are at their minima, and ideal, *I* (1, 1), where both the indicators are at their maxima.[10] Any random country will occupy a point in the space *OAIB*. The locus of the points having same HDI_{LA} measure is indicated through the iso-HDI_{LA} lines. It is seen that *j* (0.4,0.4) is lower than *k* (0.9,0.1) in terms of HDI_{LA}.

2.2 Geometric mean

The LA method of aggregation which implies perfect substitutability was criticized in the literature for being inappropriate (Desai, 1991; Hopkins, 1991; Palazzi and Lauri, 1998; Sagar and Najam, 1998; Raworth and Stewart, 2003, Herrero *et al.*, 2010a). Perfect substitutability means, "that no matter how bad the health state is, it can be compensated with further education or additional income, *at a constant rate*, which is not very natural" (Herrero *et al.*, 2010a: 4). According to Sagar and Najam (1998: 251), masking of trade-offs between various dimensions suggests that "a reductionist view of human development is completely contrary to the UNDP's own definition." Acknowledging this limitation, in the 20th anniversary edition of human development report (UNDP, 2010), the aggregation method shifted to geometric mean (GM). Mathematically,

$$HDI_{GM}=(h \times e \times y)^{1/3} \qquad (2)$$

Geometric mean does not allow for perfect substitutability, but gives higher importance to the dimension having lower performance, and penalizes unbalanced development (Gidwitz *et al.*, 2010; Herrero *et al.*, 2010b; Kovacevic and Aguña, 2010).

In the three dimensional space (*h*, *e*, *y*), one will have hyperbolic iso-HDI_{GM} surfaces indicating same HDI_{GM} values, the corresponding loci in two dimension will be rectangular hyperbola lines in the positive quadrant. For presentation convenience and without loss of

[10] In a three-dimensional HDI space, ideal, *I*, implies maximum attainment in all the dimensions (*h*=1, *e*=1, *y*=1). Noorbakhsh (1998) had used the concept of ideal for the country with maximum standardized score and suggested calculating a distance from the ideal. This would be in line with the annual maximum/minimum used in the measure of HDI then. Subsequently, as indicated in Dutta e*t al*. (1997) and Panigrahi and Sivaramkrishna (2002), the global maximum/minimum has been used in each dimension.

generality, the iso-HDI_{GM} plot for a two-dimensional space of health and education has been given in Figure 2.

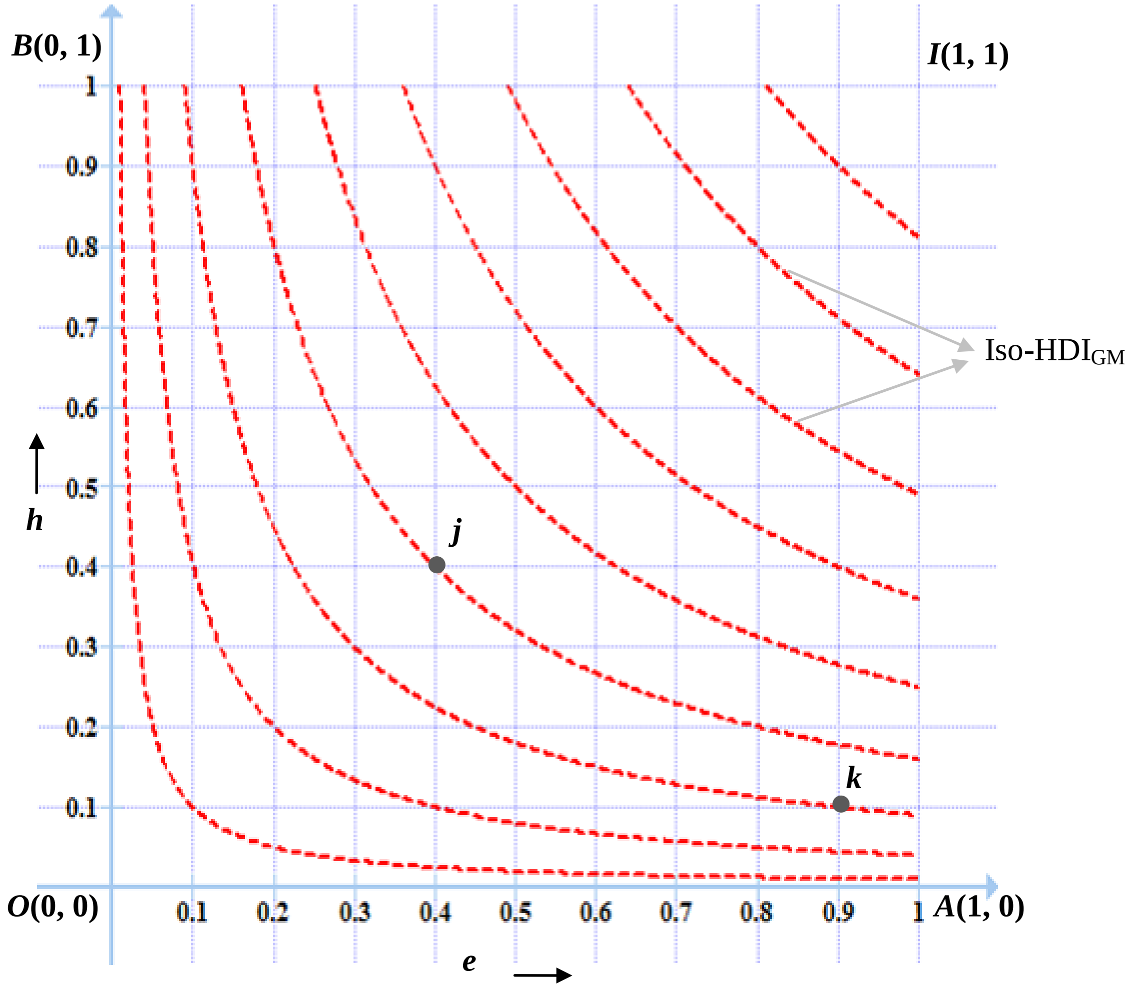


Figure 2. *Iso-HDI_{GM} in a two-dimensional space*

Figure 2 shows the HDI space *OAIB* where *O* and *I* represent origin and ideal, respectively, as in Figure 1. The locus of the points having same HDI_{GM} measure is indicated through the iso-HDI_{GM} lines. Unlike the case of linear average, *j* (0.4,0.4) is higher than *k* (0.9,0.1) in terms of HDI_{GM}.

2.3 Displaced Ideal

The DI method is based the on the concept that a better system should have less distance from ideal (Zeleny 1982).[11] Additive inverse of the normalized Euclidean distance from the ideal gives

$$HDI_{DI} = 1 - (\sqrt{((1-h)^2+(1-e)^2+(1-y)^2)}/\sqrt{3}) \quad (3)$$

where $\sqrt{((1-h)^2+(1-e)^2+(1-y)^2)}$ is the Euclidean distance from the ideal. Dividing the same with $\sqrt{3}$ normalizes it in the three-dimensional space. Thus, for country *j*, the lower the distance from ideal, the higher is HDI_{DI}.

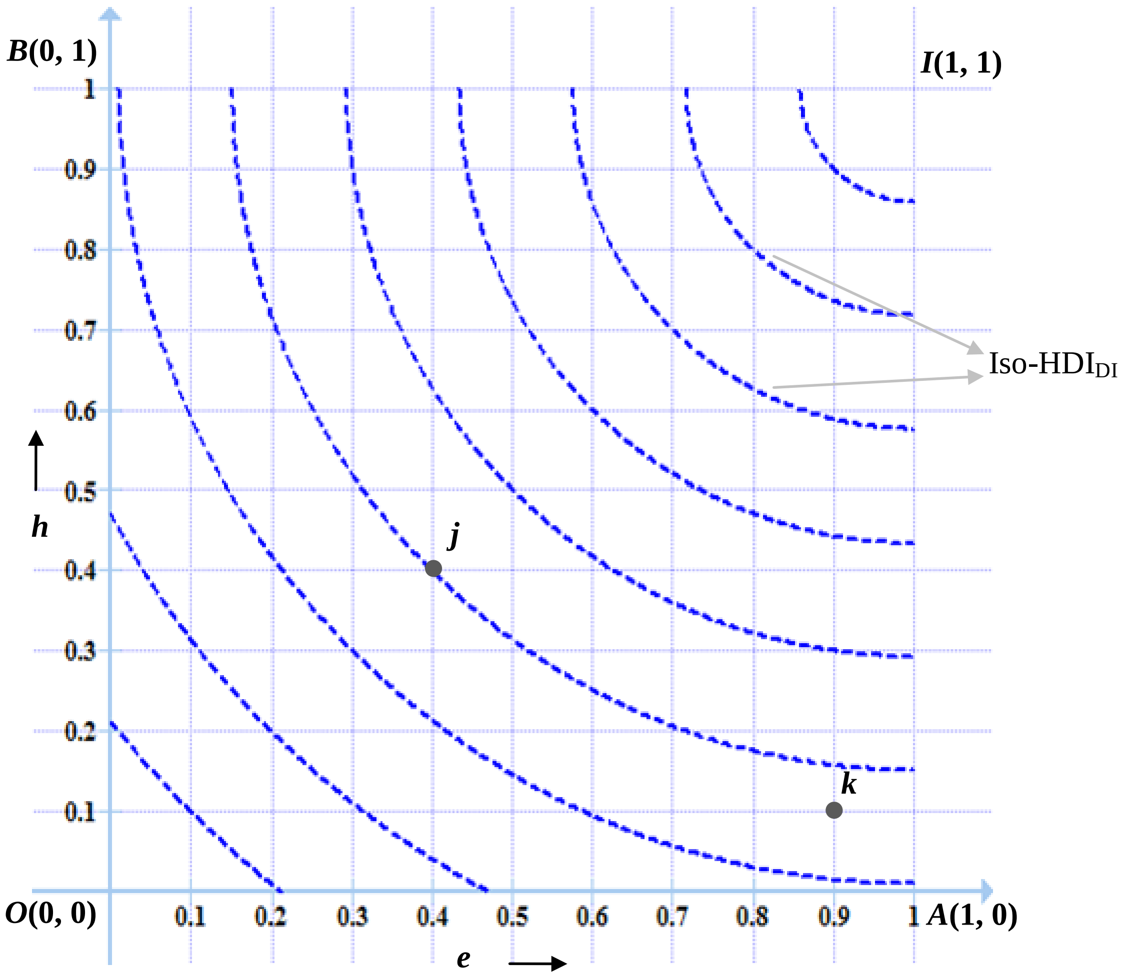


Figure 3. *Iso-HDI_{DI} in a two-dimensional space*

[11] As discussed earlier, full attainment indicates maximum in each dimension (h=1, e=1, y=1) and depends on how each of these are computed. We reiterate that these computations are important, but consider them as given for the current exercise.

In the three-dimensional space, iso-HDI_{DI} surfaces indicating same HDI_{DI} values will be concentric quarter spheres with their centre at ideal. The corresponding locus in two dimensions will be concentric quarter circles. For presentation convenience and without loss of generality, the iso-HDI_{DI} plot for a two-dimensional space of health and education has been given in Figure 3. The two points, *j* and *k*, representing two countries are the same as in Figures 1 and 2. The ranks between *j* and *k*, as in the case of geometric mean, have reversed when compared with the linear averaging method.

3. The MANUSH axioms

This section presents a number of intuitive properties that a measure of HDI should satisfy. They are as follows.

Monotonicity (Axiom M): A measure of HDI should be greater (lower) if the index value in one dimension is greater (lower) with indices value remaining constant in all the other dimensions. With two countries *j* and *k*, this would mean that if indices value remain the same in two dimensions (say, health and education such that $h_j=h_k$ and $e_j=e_k$) and different in the third dimension of income, $y_j \neq y_k$, then $HDI_j \gtreqless HDI_k$ iff $y_j \gtreqless y_k$.

Anonymity (Axiom A): A measure of HDI should be indifferent to swapping of values across dimensions. With two countries *j* and *k*, this would mean that $HDI_j=HDI_k$ if values are interchanged across two dimensions (say, health and education such that $h_j=e_k$ and $h_k=e_j$) and remains the same in the third dimension of income, $y_j=y_k$. This axiom implies a symmetry condition. This is not to be interpreted to indicate that one dimension can be replaced or substituted by another.[12]

Normalization (Axiom N): A measure of HDI should have a minimum and a maximum, HDI $\in$ [0,1]. At its minimum, HDI=0. This indicates minimum development in all the three dimensions (h=0, e=0, y=0). At its maximum, HDI=1. This indicates maximum

[12] When dimensions of HDI have different weights, the swapping has to be appropriately weight-adjusted.

attainment in all the dimensions (h=1, e=1, y=1). Alternatively, in a three-dimensional Cartesian space, the two positions refer to the origin, O, and ideal, I, respectively.[13]

Uniformity (Axiom U): A measure of HDI should be such that for a given mean of indices value across dimensions, μ, a greater (lower) dispersion across dimensions, σ, should indicate a lower (greater) value. For two countries j and k, if $\mu_j=\mu_k$ and $\sigma_j \gtreqless \sigma_k$ then $\text{HDI}_j \lesseqgtr \text{HDI}_k$. This is in line with the notion of human development that each dimension is intrinsic (Sen 1999); and hence they cannot be complete substitutes to each other. So, this axiom rewards balanced or uniform development across dimensions.[14]

Shortfall sensitivity (Axiom S): A measure of HDI should be such that it must indicate that the future emphasis on the worse-off dimension should be at least in proportion to the shortfall. For instance, in a country if the three dimensions of HDI have values as h=0.2, e=0.6, and y=0.8 (indicating that shortfalls are 0.8, 0.4, and 0.2, respectively; note that shortfall in h is twice that of e and four times that of y) then the future emphasis on health is at least twice more than that of education, which is at least twice more than that of income. Thus, without loss of generality, if $h<e<y$ then shortfall sensitivity can be defined as

$$\frac{dh}{de} \geq \frac{1-h}{1-e}; \quad \frac{de}{dy} \geq \frac{1-e}{1-y}$$

Or (4)

$$\frac{dh}{1-h} \geq \frac{de}{1-e} \geq \frac{dy}{1-y}$$

[13] The origin and ideal would depend how each of the indices are measured, scaled, weighed, and normalized. However, as indicated earlier, these are given to us. We also admit that the way we use the axiom of normalization is similar to the inequality literature, but in the HDI literature by Chakravarty (2003), Foster *et al.* (2005), and Seth (2009) it means that if all dimensions have a common value then the HDI value will be equal to this common value and it is implicit in this that the HDI value will lie between zero and unity representing origin and ideal respectively.

[14] Uniformity axiom should not be confused with zero substitutability across dimensions. Suppose, from a uniform value in all dimensions, there is an increase in one dimension with all other dimensions remaining constant. Zero substitutability would not consider this as an improvement in HDI; whereas uniformity axiom simply says that any improvement in HDI from a uniform position will be maximized when the increase is shared equally by all dimensions.

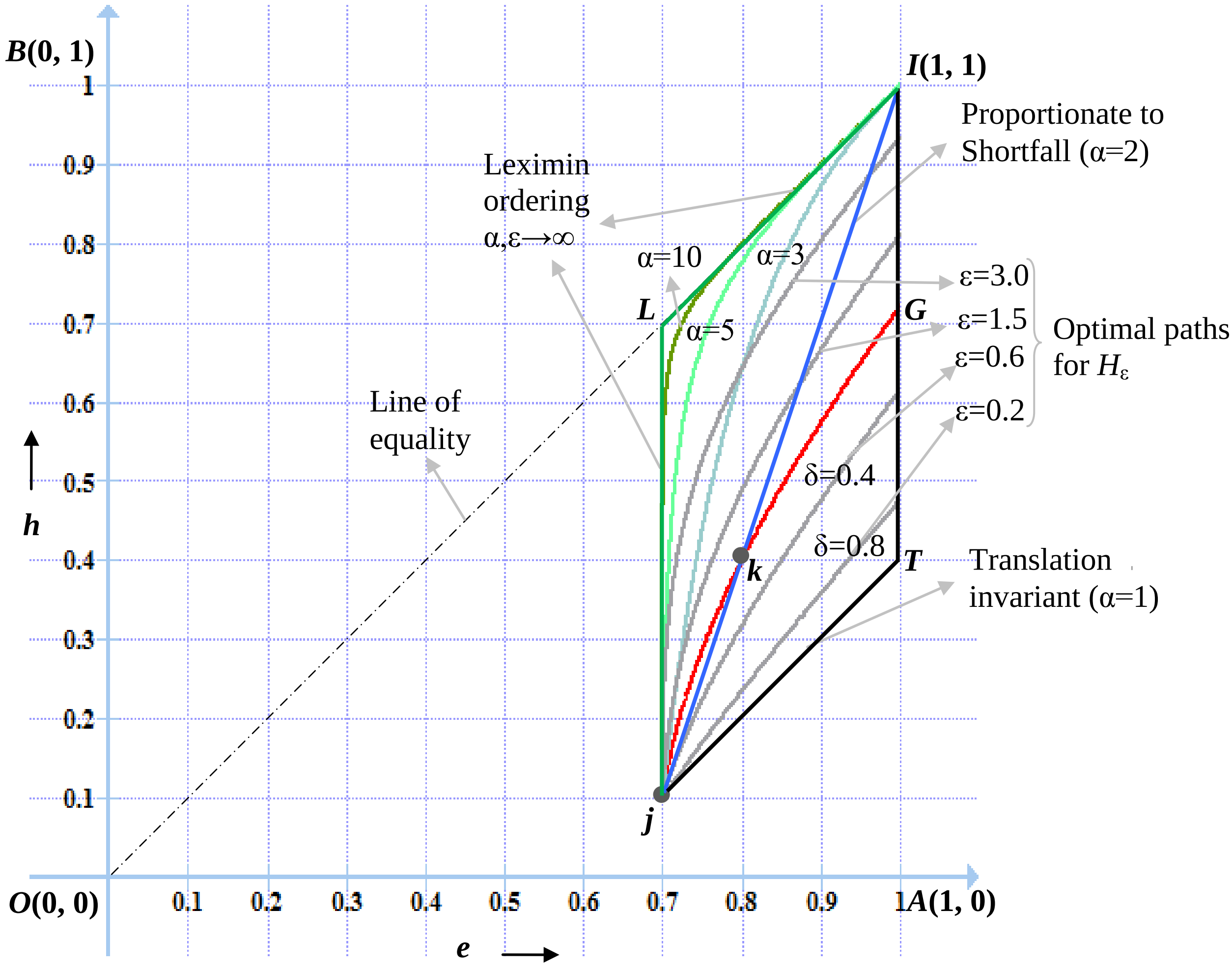
*Sho*rtfall sensitivity is not satisfied when the inequality is reversed in equation (4). It is weakly satisfied when equality holds in (4), i.e. the rate of *change is in proportion to the shortfall*. It will be strongly satisfied if the inequality holds in (4). An exacting situation of this is to give the entire emphasis to the most neglected dimension till it becomes equal to the dimension that is ordered just above it. And then the entire emphasis will be shared equally across both these dimensions till they reach to the dimension that is ordered above them, and then all the three dimensions will get equal emphasis. Under this, $dh/de=dh/dy=\infty$ till $h=e$; then $dh/de=1$ and $dh/dy=\infty$ till $h=e=y$; then onwards $dh/de=1$ and $dh/dy=1$. This is leximin ordering that can be considered equivalent to the Rawlsian scenario.

For country *j* in Figure 4, with a two-dimensional space of health and education, shortfall sensitivity is feasible in the area *jIL*. The line *jI* denotes equal proportion to shortfall case whereas the movement from *j* to *I* along the line segments *jL* and *LI* indicate the leximin ordering case.

Hiatus sensitivity to level (Axiom H): A measure of HDI should be such that the same gap (or hiatus) across dimensions should be considered worse off as the attainment increases.[15] For a given gap, *g*, of indices values across dimensions a measure of HDI should be such that its deviation from its uniform development situation (i.e., when all the dimensions have equal values) will be greater (lower) for a greater (lower) μ. By gap, *g*, we refer to the situation when deviations for each dimension from the mean, μ, are equal. Thus, *g* for two situations *j* and *k* would mean that $\mu_j-h_j=\mu_k-h_k$, $\mu_j-e_j=\mu_k-e_k$, and $\mu_j-y_j=\mu_k-y_k$ and all these can be indicated by stating that $g_j=g_k$. Now we reiterate the axiom. If $g_j=g_k$ and $\mu_j\gtreqless\mu_k$ then $(\mu_j\text{-HDI}_j)\gtreqless(\mu_k\text{-HDI}_k)$. This is in line with development with equity. For any development constituting more than one dimension, higher overall attainment must

[15] This is similar to the level sensitivity axiom in the context of group differential (Mishra and Subramanian, 2006; Mishra, 2008; Nathan and Mishra, 2013).

simultaneously lead to a reduction in gap across dimensions. It supports the view that "concern with inequality increases as a society gets prosperous since the society can 'afford' to be inequality conscious" (Sen, 1997: 36).



Note: For the given position *j*, the line *jT* indicates the translation invariant case which corresponds to the optimal path under HDI_{LA} (or $\mathcal{H}_{\alpha=1}$ or $H_{\delta=1}$ or $H_{\varepsilon=0}$). The area *jIL* satisfies shortfall sensitivity where *jI* indicates the proportion to shortfall case which corresponds to the optimal path under HDI_{DI} (or $\mathcal{H}_{\alpha=2}$). The optimal paths under $\mathcal{H}_\alpha$ for α=3, 5 and 10 are also given. The lines *jL* (which is vertical) and *LI* (which falls on the line of equality, *OI*) indicate the Rawlsian leximin ordering corresponding to the path under $\mathcal{H}_{\alpha\to\infty}$ (or $H_{\varepsilon\to\infty}$). The optimal path under HDI_{GM} (or $H_{\delta\to 0}$ or $H_{\varepsilon=1}$) is indicated by *jG*; it intersects *jI* at *k* indicating that the path fails to satisfy shortfall sensitivity beyond the point *k*. The optimal paths for H_δ; $\delta\in(0,1]$ is equivalent to the geometric mean case in a limiting sense when δ→0 and as δ increases it is below the geometric mean and when δ=1 it is equivalent to the translation invariance case identified with linear average. For H_ε, the optimal paths begin with the translation invariance case at ε=0 and then is equivalent to the geometric mean case when ε=1 and keeps further increasing For illustration, we have given optimal paths for H_ε when ε=0.2, 0,6, 1.5, and 3 (note the similarity with the paths for H_δ at δ=0.4, and 0.8 as δ=1-ε when ε<1). All these paths are either below *jI* or intersect the *jI* line like that for the geometric mean indicating that they fail shortfall sensitivity. However, we need to mention that for an extreme limiting scenario when ε→∞ the optimal path for H_ε is equivalent to the Rawlsian leximin ordering. This is equivalent to α→∞, but we need to reiterate that in $\mathcal{H}\alpha$ shortfall sensitivity is satisfied for α≥2 and not just in the extreme limiting sense.

Figure 4. *Shortfall sensitivity*

The above set of axioms, namely, monotonocity, anonymity, normalization, uniformity, shortfall sensitivity, and hiatus sensitivity to level are collectively referred to, by us, with the acronym of MANUSH.

4. Axiomatic Comparison among LA, GM, and DI methods

The three methods of calculating HDI, viz., LA, GM, and DI, satisfy the axioms of monotonicity (with an exception condition for GM), anonymity, and normalization. Further, the GM and DI methods satisfy the axiom of uniformity. The axioms of shortfall sensitivity and hiatus sensitivity to level are satisfied by the DI method alone. Let us elaborate.

Monotonicity: This axiom is satisfied for all the three methods with an exception condition for $\mathrm{HDI}_{\mathrm{GM}}$. For two countries j and k if the value in one dimension is higher for one, with the other dimensions being the same, say, $h_j>h_k$, while $e_j=e_k$, and $y_j=y_k$, then equations (1), (2), and (3) show $\mathrm{HDI}_{\mathrm{LA}j}>\mathrm{HDI}_{\mathrm{LA}k}$, $\mathrm{HDI}_{\mathrm{GM}j}>\mathrm{HDI}_{\mathrm{GM}k}$, and $\mathrm{HDI}_{\mathrm{DI}j}>\mathrm{HDI}_{\mathrm{DI}k}$, respectively. However, this fails for $\mathrm{HDI}_{\mathrm{GM}}$ for obvious reason, when one of the unchanging dimensions is at zero.

Anonymity: The three methods of aggregation satisfy anonymity. From the mathematical expressions of these three methods, (1), (2), and (3) one can find $\mathrm{HDI}_{\mathrm{LA}}$, $\mathrm{HDI}_{\mathrm{GM}}$, and $\mathrm{HDI}_{\mathrm{DI}}$ are symmetric in h, e, and y. Hence, HDI under these methods does not change to swapping of values across dimensions.

Normalization: In all the three methods, the countries are bounded by the minimum, $\mathrm{HDI}_{\mathrm{LA}}=\mathrm{HDI}_{\mathrm{GM}}=\mathrm{HDI}_{\mathrm{DI}}=0$ at the origin, O (h=0,e=0,y=0); and the maximum, $\mathrm{HDI}_{\mathrm{LA}}=\mathrm{HDI}_{\mathrm{GM}}=\mathrm{HDI}_{\mathrm{DI}}=1$ at the ideal I (h=1,e=1,y=1). Hence, they satisfy normalization.

Uniformity: Both the GM and DI methods satisfy this, while LA fails. For two countries j and k, if $\mu_j=\mu_k$ and $\sigma_j>\sigma_k$ then $\mathrm{HDI}_{\mathrm{GM}j}<\mathrm{HDI}_{\mathrm{GM}k}$ and $\mathrm{HDI}_{\mathrm{DI}j}<\mathrm{HDI}_{\mathrm{DI}k}$, but

HDI_{LAj}=HDI_{LAk}. The GM and DI methods satisfy this as HDI_{GM} and HDI_{DI} increase at a decreasing rate with respect to *h*, *e*, and *y*. In other words, iso-HDI_{GM} and iso-HDI_{DI} are convex to the origin. This axiom along with anonymity implies that for a given mean, HDI_{GM} and HDI_{DI} are maximized when all the three dimensions have equal values (h=e=y).

The LA method, for a given mean, is independent of the deviation from the mean. This makes HDI_{LA} perfectly substitutable, which is "one of the most serious criticisms of the linear aggregation formula" (UNDP, 2010: 216). Linear averaging also enables HDILA to be subgroup decomposable. However, it should not be seen as an advantage as this implies that the subgroup having lower HDI values can be perfectly substitutable by subgroups having higher HDI values. Additionally, our proposed method can also be made subgroup decomposable by considering the *s* subgroup's share of contribution as $n_s v_s / \sum n_s v_s$, where n_s is the subgroup's population share and v_s is the subgroup's value of HDI computed independently. Now, let us state the following proposition.

Proposition 1: A measure of HDI cannot satisfy perfect substitutability and uniformity simultaneously.

Proof: A measure of HDI satisfying perfect substitutability would not change for a given μ even if σ changes. On the contrary, a measure of HDI satisfying uniformity demands the measure to have a lower (higher) value as σ increases (decreases) with μ remaining constant.

Shortfall sensitivity: In order to determine whether a measure of HDI is shortfall sensitive or not, one needs to find what sort of future emphases the measure would suggest. This can be checked by finding the emphases along the optimal path for future progress as indicated by the measure. This path is one where a given increment in HDI can be achieved with a minimal movement. This corresponds to minimizing the Euclidean distance between the current and incremental positions for a given increment in HDI.

For the LA method, the optimal paths are perpendicular to the iso-$\mathrm{HDI}_{\mathrm{LA}}$ lines. This will imply same increment in all dimensions (for the position *j* in Figure 4, movement along the line *jT* indicates this). It is nothing but the translation invariant case ($dh/de=dh/dy=1$ or $dh=de=dy$). This way, under LA, the future emphasis across dimensions is equal and independent of the differences between them in their current attainment. It does not impose greater emphasis on the dimensions that have been hitherto neglected. Hence, it does not satisfy the shortfall sensitivity axiom.

For the GM method, the equation for the optimal path is derived in Appendix 1. The path turns out to be such that the emphases across dimensions are in proportion to the multiplicative inverse of attainment ($hdh=ede=ydy$ implying $dh/de=e/h$, $dh/dy=y/h$ and $de/dy=y/e$).[16] For instance, if $h=0.1$ and $e=0.7$ (refers to point *j* in Figure 4) then the GM method indicates that the emphasis on health, to begin with, should be seven times more than on education. However, as the attainment increases the emphases would change in proportion to changing shortfalls. This results in the optimal path being *jG* in Figure 4. The *kG* segment of the path is outside the area *jIL*, which indicates that beyond *k* (at *k* the optimal path for GM intersects the proportionate to shortfall line *jI*) it does not impose greater emphasis on the dimensions that are neglected. Hence, GM fails shortfall sensitivity.

For the DI method, the optimal paths are the lines joining initial position and the ideal (see line *jI* in Figure 4). Here, the emphases across dimensions turn out to be in proportion to the shortfall throughout the path $\frac{dh}{1-h}=\frac{de}{1-e}=\frac{dy}{1-y}$ implying $dh/de=(1-h)/(1-e)$, $dh/dy=(1-h)/(1-y)$ and $de/dy=(1-e)/(1-y)$). Hence, the shortfall sensitivity axiom is satisfied under DI. To be precise, it satisfies shortfall sensitivity weakly. The proof is given in Appendix 2.

[16] Multiplicative inverse of attainment is not same as shortfall, which is additive inverse to attainment.

Hiatus sensitivity to level: The LA method fails to satisfy this axiom, as there is no deviation of HDI_{LA} values from the uniform development situation at all levels. For a given gap, the deviation of HDI from its uniform development situation is a decreasing function of mean for the GM method while it is an increasing function of mean for the DI method. The proof is given in Appendix 3. This means that GM fails whereas DI satisfies hiatus sensitivity to level. The GM method not satisfying this axiom also means that it penalizes greater proportionate deviation of the given gap from uniform development when average attainment increases. This means that the proportionate deviation for a given gap is higher at a lower level of average attainment. This gives us the following proposition.

Proposition 2: A measure of HDI cannot satisfy hiatus sensitivity to level and also penalize greater proportionate deviation of a given gap from uniform development together.

From the above discussion the following results emerge. The HDI_{LA} method satisfies the axioms of MAN (monotonicity, anonymity, normalization). In addition to these axioms, the axiom of uniformity is also satisfied by HDI_{GM}, which also has an exception condition for monotonicity. The HDI_{DI} method satisfies all the aforementioned axioms including shortfall sensitivity (weakly) and hiatus sensitivity to level. Based on this, we state the following proposition.

Proposition 3: There exists a human development index measure HDI_{DI} that satisfies the MANUSH axioms– monotonicity, anonymity, normalization, uniformity, shortfall sensitivity (weakly) and hiatus sensitivity to level.

Thus, HDI_{DI} measure has some axiomatic advantages over the current HDI_{GM} measure. Nevertheless one must mention that an advantage of the GM method is that the ranking of countries are scale independent to changes in the maximum value for each variable, which is used for normalizing the dimension-specific indicator. However this advantage would not come in the way of our proposed method if one followed the pre-2010 practice of fixing the maximum, in a normative sense, as a goalpost.

The use of an open-ended maximum, amenable under the GM method, also raises some concerns. First, the 1980-2010 observations showed that the maximum income was for United Arab Emirates (UAE) in 1980, which no country has ever reached; UAE too has not been able to reach this again in the period under consideration. Thus, that observation was a historical-accident and may not indicate a scenario that others ought to emulate and attain in the near future. Second, the change in defining maximum meant that compared to 2009, the computations in 2010 had the maximum for the income dimension increased by about two-and-a-half times (per capita gross national income at purchasing power parity US$ terms in 2005 prices increased from 40,000 to more than 100,000) indicating that countries having per capita income more than 40,000 US$ will now be able to add the excess income as attainments to their valuation of HDI and this will favour the very high income countries. A related third concern is that with this shift the shortfall for income has increased and thus increments from income have become more important relative to other dimensions. Fourth, a changing maximum in an advantaged dimension would mean further neglect of a neglected dimension. A rightward shift of the ideal point (h=1, e=1, y=1) extends the optimal path for GM (along jG in Figure 4) to reach G, thereby postponing the move along the vertical segment GI to focus on the neglected dimension. Finally, while conceding that the HDI calculation compared across countries have ordinal relevance, there is merit in an analysis of trends for a specific country or a group of countries over time, as has been carried out by Nathan and Mishra (2010) and UNDP (2010). It is here that our proposed axiomatic advantages gain further importance. It goes without saying that such an analysis should be complemented with an understanding of the state of affairs in health, education and standard of living.

5. The HDI$_\alpha$ Class of Measures

Now, let us define an α-class of HDI measures,

$$\mathscr{H}_\alpha = 1 - D_{\alpha I,w} \tag{5}$$

where

$$D_{\alpha I,w} = \left(\frac{\sum (w_i (1 - x_i))^\alpha}{\sum (w_i)^\alpha} \right)^{\frac{1}{\alpha}} ; i = 1, \ldots, n, \quad \alpha = [1, \infty) \tag{6}$$

is the normalized Minkowski distance function of order α calculated on the basis of the shortfalls from the ideal, I, where x_i refers to the normalized indices for n dimensions (i=1,2…n) such that at the origin $x_i=0\ \forall i$ and at the ideal $x_i=1\ \forall i$, and w_i refers to the weights assigned to each dimension. For equal weights, (6) reduces to

$$D_{\alpha I} = \left(\frac{1}{n} \sum (1 - x_i)^\alpha \right)^{\frac{1}{\alpha}} ; i = 1, \ldots, n,\ \alpha = [1, \infty).^{17} \tag{7}$$

In the $\mathscr{H}_\alpha$ class of measures, the linear average and displaced ideal methods indicated in (1) and (3) respectively turn out to be special cases. This is suggested in the following proposition.

Proposition 4: There exists an α-class of human development index measures, $\mathscr{H}_\alpha$, such that for n=3, $\mathscr{H}_1=\text{HDI}_{\text{LA}}$ and $\mathscr{H}_2=\text{HDI}_{\text{DI}}$.

Proof: Substituting (7) in (5) one gets (1) and (3) for α=1 and α=2, respectively.

Another special case is $\mathscr{H}_\infty$, where the human development index measure reduces to the lowest-valued dimension. This corresponds to a situation where the iso-HDI lines can be depicted through right-angled lines. Thus, as α increases from unity to infinity we move from

[17] Similar to $D_{\alpha I}$, one can have $D_{\alpha O}$, which are α-class of Minkowski distance measures on the basis of attainment from the origin. At α=1, $D_{1O}=1-D_{1I}=\text{HDI}_{\text{LA}}=\mathscr{H}_1$, but for α>1 the identity breaks down and $D_{\alpha O}+D_{\alpha I}>1$. The $D_{\alpha\bullet}$ formulae are similar to the Atkinson's index when ε<0.

a measure that allows for perfect substitutability to one that allows no substitution across dimensions (Figure 5).[18]

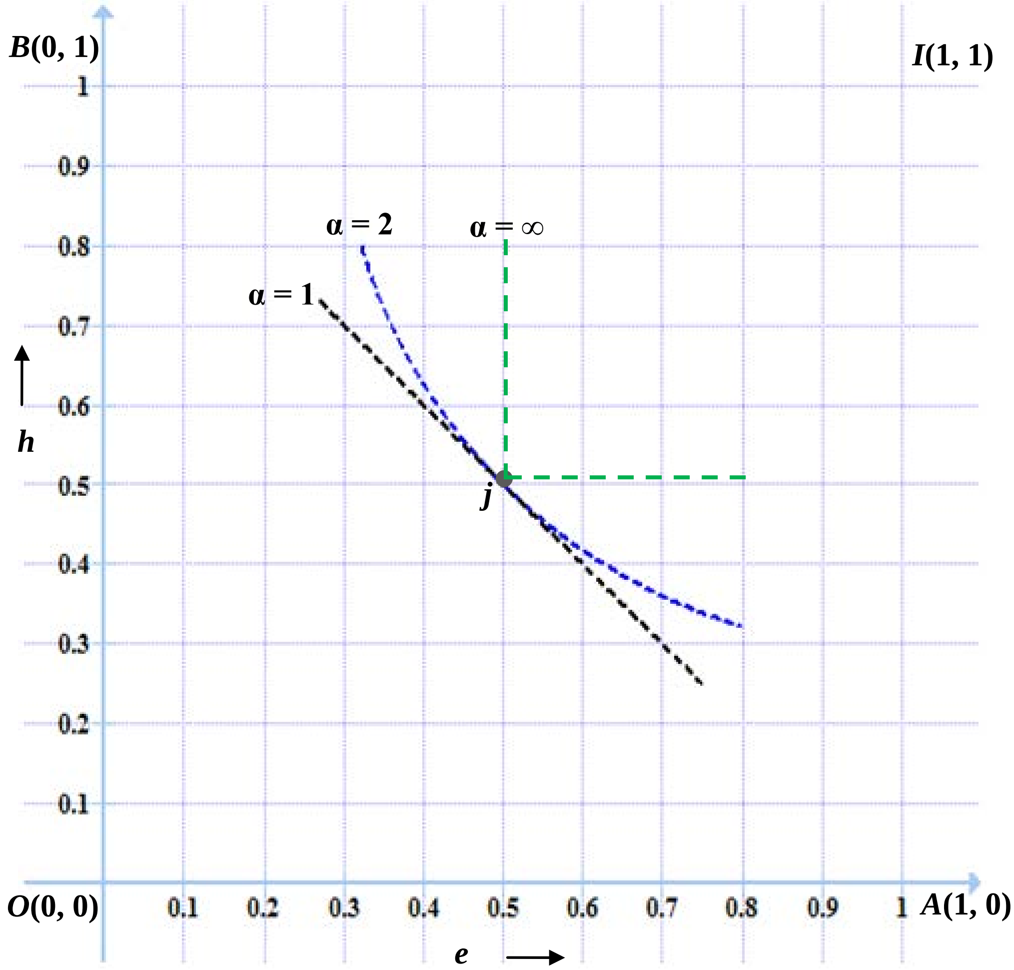


Figure 5. *Class of Measures*

In addition, our class of $\mathcal{H}_\alpha$ measures satisfy the MANUSH axioms. We state that in the following lemma.

Lemma 1: There exists an α-class of human development index measures $\mathcal{H}_\alpha$ such that for $\alpha \geq 2$ the MANUSH axioms are satisfied.

---

[18] The similarity of $\mathcal{H}_\alpha$ class of measures with constant elasticity of substitution (CES) functions is obvious, also see Rao (2011).

Proof: The proof for the six axioms is as follows. $\mathscr{H}_\alpha$ satisfies montonicity, $d\mathscr{H}_\alpha/dx_i>0$ $\forall$ i. It is evident from equations (5) and (7) that $\mathscr{H}_\alpha$ remains the same if the values of x_i and $x_{i'}$ are swapped ($i\neq i'$). Appropriate adjustments can also be made when weights are not equal. Thus, $\mathscr{H}_\alpha$ satisfies anonymity. $\mathscr{H}_\alpha$ satisfies normalization, $\mathscr{H}_\alpha \in[0,1]$. $\mathscr{H}_\alpha$ satisfies uniformity, $d\mathscr{H}_\alpha/dx_i>0$ and $d^2\mathscr{H}_\alpha/d(x_i)^2<0\forall i$. Shortfall sensitivity is not satisfied when $\alpha<2$, it is weakly satisfied for $\alpha=2$, and strongly satisfied for $\alpha>2$ (a formal proof is given in Appendix 2). $\mathscr{H}_\alpha$ satisfies hiatus sensitivity to level for $\alpha\geq 2$ (see Appendix 3 for proof).

The optimal paths, measured in Euclidean distance, of $\mathscr{H}_\alpha$ for $\alpha=1,2$, and ∞ are given in Figure 4 indicating cases of translation invariance, proportionate to shortfall, and leximin ordering, respectively. For values of $\alpha\in(2,\infty)$ the path will be within *jIL* and concave to the line segment *jI*: some sample optimal paths for $\alpha=3,5$, and 10 are given in Figure 4. The choice of α (or substitution across dimensions) is intertwined with the choice of shortfall sensitivity and $\alpha=2$ refers to an intermediary position of weak shortfall sensitivity that lies between translation invariance ($\alpha=1$) and the Rawlsian leximin ordering ($\alpha\rightarrow\infty$).

The existing class of measures such as H_δ (Chakravarthy, 2003) and H_ε (Foster *et al*., 2005), which is also similar to Seth (2009) with regard to aggregation across dimensions, satisfy monotonicity, anonymity, normalization, and uniformity, but they fail to satisfy shortfall sensitivity and hiatus sensitivity to level. Shortfall sensitivity is indicative of the measure providing a direction for future progress that gives emphasis on the dimensions in proportion to their shortfall. The optimal paths indicating a direction for future progress for H_δ and H_ε are given for different values of δ and ε, respectively, in Figure 4.[19] Both do not satisfy shortfall sensitivity as they reach the ideal relatively earlier for the dimension that is doing relatively better (except for the limiting case when $\varepsilon\rightarrow\infty$). For H_δ, the optimal path is

[19] The formulae for the optimal paths of H_δ and H_ε in a two-dimensional situation of $h<e$ are $dh/de=(e/h)^{1-\delta}$; $\delta=(0,1]$ and $dh/de=(e/h)^{\varepsilon}$; $\varepsilon\geqslant 0$, $\varepsilon\neq 1$, respectively. The two formulae are equivalent in the relevant domain as $\varepsilon=1-\delta$.

equivalent to the geometric mean case in a limiting sense when $\delta \rightarrow 0$ and as δ increases it is below the geometric mean and when $\delta=1$ it is equivalent to the translation invariance case identified with linear average. For H_{ε}, the optimal path is equivalent to H_{δ} in the relevant domain as $\varepsilon=1-\delta$. It coincides with translation invariance case at $\varepsilon=0$ and geometric mean case at $\varepsilon=1$. As ε increases the relative emphasis on the neglected dimension increases, but it fails shortfall sensitivity for all finite values of ε (as they are either below the *jI* line or intersect it like the geometric mean, Figure 4) except for the extreme limiting scenario when $\varepsilon \rightarrow \infty$ where it is equivalent to the Rawlsian leximin ordering.[20]

Hiatus sensitivity to level imposes a greater inequity consciousness across dimensions as attainment increases. In other words, for a given gap, the shortfall in H_{δ} and H_{ε} from their corresponding uniform development will be lower for a greater mean, i.e., for $g_j=g_k$ and $\mu_j>\mu_k$ we get $(\mu^{\delta}{}_{j}-H_{\delta j})<(\mu^{\delta}{}_{k}-H_{\delta k})$ and $(\mu_j-H_{\varepsilon j})<(\mu_k-H_{\varepsilon k})$.[21]

While the focus of the current paper has been across dimensions, we have a remark on inequality aversion within dimension, as suggested by Foster et al (2005) and Seth (2009) among others. We feel that the MANUSH axioms would be important across groups either for the specific indicator as also for the aggregated HDI measure. In fact, the discussion of translation invariance and Rawlsian leximin ordering are important constructs in the context of inequality across groups and the discussion of hiatus sensitivity to level has also been borrowed from literature on group differential for a specific indicator. Thus, in equation (7) when *n* refers to individuals or population sub-groups then they can address inequality within dimensions. The satisfaction of monotonicity axiom would also imply that they also satisfy sub-group consistency in the Foster *et al*. (2005) sense, ie., if the value for any sub-group increases/decreases while there are no changes in the other sub-groups then the overall

[20] The latter is equivalent to $\alpha \rightarrow \infty$, but we need to reiterate that in $\mathcal{H}_{\alpha}$ shortfall sensitivity is satisfied for $\alpha \geq 2$ and not just in the extreme limiting sense.

[21] In fact, Chakravarthy (2003: 104) also points out that H_{δ} "will attach greater weight to achievement differences at lower level of attainment." Thus, confirming our observation that it fails hiatus sensitivity to level.

inequality adjusted indicator value will also increase/decrease. Further, as we have also indicated earlier, the measure can be made decomposable with some adjustments.

Now, suppose we have an alternative measure of human development index, $\mathcal{M}(h,e,y)$, that also satisfies the MANUSH axioms then it implies the following. $\mathcal{M}$ is an increasing function across dimensions (monotonicity). The function associated with $\mathcal{M}$ will be symmetric across dimensions (anonymity). There will be bounds to $\mathcal{M}$ such that it lies between zero and unity (normalization). The function increases at a decreasing rate, iso-HDI curves from $\mathcal{M}$ should be convex to the origin (uniformity). $\mathcal{M}$ satisfies equation (4), $\frac{dh}{1-h} \geq \frac{de}{1-e} \geq \frac{dy}{1-y}$ (shortfall sensitivity). And, for a given gap an increment in $\mathcal{M}$ from its corresponding uniform development will be greater for a greater mean (hiatus sensitivity to level). Note that in defining shortfall sensitivity the notions of monotonicity, anonymity and normalization are implicit. Now, if by using equation (4) that defines shortfall sensitivity we obtain an optimal path and this happens to be equivalent to one that we obtained by using $\mathcal{H}_\alpha$ then one can conclude that $\mathcal{M}$ is the same and it would also satisfy uniformity and hiatus sensitivity to level. With this we give a lemma and a theorem.

Lemma 2: There exists a class of measures $\mathcal{M}$ such that it has a one-to-one correspondence with $\mathcal{H}_\alpha$; $\alpha \geq 2$.

Proof: See Appendix 4.

Theorem 1: MANUSH is necessary and sufficient for $\mathcal{H}_\alpha$; $\alpha \geq 2$.

Proof: Lemmas 1 and 2.

Thus, MAN axioms get satisfied for $\mathcal{H}_1$, whereas the MANUSH axioms are necessary and sufficient for $\mathcal{H}_\alpha$; $\alpha \geq 2$. The proposed displaced ideal method $\mathcal{H}_2$ is necessary and sufficient for MANUSH with weak shortfall sensitivity, whereas $\mathcal{H}_\alpha$; $\alpha > 2$ is necessary and sufficient for

MANUSH with strong shortfall sensitivity. In fact, one can state that shortfall sensitivity increases as α increases such that as $\alpha \rightarrow \infty$ the satisfaction of shortfall sensitivity can be identified with a Rawlsian leximin ordering. In targeting and policy intervention for specific situations, shortfall sensitivity may be appropriately increased. For instance, when human immunodeficiency virus/acquired immunodeficiency syndrome (HIV/AIDS) epidemic led to substantive reductions in life expectation in many Sub-Saharan countries it required a much greater emphasis on improving health than just limiting the emphasis to its proportionate shortfall.

6. Conclusions

This exercise evaluated three methods of aggregation across dimensions for measuring human development index through a set of intuitive axiomatic properties. The linear averaging method satisfied the axioms of monotonicty, anonymity, and normalization (or MAN axioms). The geometric mean method, in addition to these three axioms (excluding monotonicity when one of the dimensions continues to have a value of zero), also satisfied the axiom of uniformity (or MANU axioms). The displaced ideal method (additive inverse of the distance from the ideal) satisfied the above-mentioned four axioms as also the axioms of shortfall sensitivity and hiatus sensitivity to level (or MANUSH axioms).

While contextualizing with the existing class of measures like H_δ (Chakravarthy, 2003) and H_ε (Foster *et al*., 2005), we also propose an α-class of measures where α=1 and α=2 turned out to be the linear averaging method and the displaced ideal method, respectively. Further, for the class of measures $\alpha \geq 2$, the MANUSH axioms are both necessary and sufficient. And, the higher is the values of α, the greater is the shortfall sensitivity such that α=1 refers to a translation invariant case that gives no premium to historical antecedents and $\alpha=\infty$ can be identified with the Rawlsian leximin ordering; in between these two lies the intermediary condition of $\alpha=2$ that satisfies shortfall sensitivity weakly.

Further, our proposed class of measures can be used in different contexts. It can also consider the dimensions as subgroups. Under such an interpretation, the related discussions with leximin ordering and translation invariance (linear averaging method) are extreme positions. The geometric mean method is an improvement over the linear average, but it would still keep convergence at bay. Hence, hear also we suggest that a proportionate to shortfall approach be considered as an intermediary position. Of course, we are aware that implementation at the ground level might be different from this measurement exercise, but nevertheless, this will facilitate our understanding.

The word MANUSH means human in many South Asian Languages such as Assamese, Bengali, Marathi, and Sanskrit among others. Besides, MANUSH is an anagram of HUMANS. Thus, we propose the axiom of MANUSH or HUMANS for a human development index.

Appendices

Appendix 1

If the given initial position is (h_1, e_1, y_1) and the next incremental position is a variable point (h_2, e_2, y_2) such that ΔHDI_{GM} is constant, c_1, for all such points,

$$c_1 = (h_2 e_2 y_2)^{1/3} - (h_1 e_1 y_1)^{1/3} \Rightarrow h_2 e_2 y_2 = c_2 \tag{A1}$$

where, $c_2 = \left(c_1 + (h_1 e_1 y_1)^{1/3}\right)^3$. The optimal path corresponds to the incremental position where the distance between the two is least. The distance, θ, to be minimized,

$$\theta^2 = (h_2 - h_1)^2 + (e_2 - e_1)^2 + (y_2 - y_1)^2 \tag{A2}$$

Substituting y_2 from (A1) in (A2) and then applying the minimization conditions,

$$\frac{d(\theta^2)}{dh_2} = 2(h_2 - h_1) + 2\left(\frac{c_2}{h_2 e_2} - y_2\right)\frac{c_2}{e_2}\left(-\frac{1}{h_2^2}\right) = 0 \Rightarrow h_2(h_2 - h_1) = \frac{c_2}{h_2 e_2}\left(\frac{c_2}{h_2 e_2} - y_1\right) \tag{A3}$$

$$\frac{d(\theta^2)}{de_2} = 2(e_2 - e_1) + 2\left(\frac{c_2}{h_2 e_2} - y_1\right)\frac{c_2}{h_2}\left(-\frac{1}{e_2^2}\right) = 0 \Rightarrow e_2(e_2 - e_1) = \frac{c_2}{h_2 e_2}\left(\frac{c_2}{h_2 e_2} - y_1\right) \tag{A4}$$

From, (A3) and (A4);

$$h_2(h_2 - h_1) = e_2(e_2 - e_1) \tag{A5}$$

Similarly, proceeding with h and y;

$$h_2(h_2 - h_1) = y_2(y_2 - y_1) \tag{A6}$$

Thus;

$$h_2(h_2 - h_1) = e_2(e_2 - e_1) = y_2(y_2 - y_1) \tag{A7}$$

The equation for optimal path can be determined by considering infinitesimally small increment and then integrating. From first equation in (A7),

$$\int h\,dh = \int e\,de \Rightarrow h^2 = e^2 + c_3 \tag{A8}$$

The given initial position (h_1, e_1, y_1) will be on the optimal path; so $c_3 = (h_1^2 - e_1^2)$. So,

$$h^2 = e^2 + h_1^2 - e_1^2 \tag{A9}$$

Similarly proceeding with the second equation in (A7)

$$h^2 = y^2 + h_1^2 - y_1^2 \tag{A10}$$

From (A9 and A10)

$$e^2 = y^2 + e_1^2 - y_1^2 \tag{A11}$$

Appendix 2

If the given initial position is (h_1,e_1,y_1) and the next incremental position is a variable point (h_2,e_2,y_2) such that $\Delta \mathrm{HDI}_\alpha$ is constant, c_4, for all such points,

$$c_4 = 1-\left(\frac{(1-h_2)^\alpha + (1-e_2)^\alpha + (1-y_2)^\alpha}{3}\right)^{1/\alpha} - 1 + \left(\frac{(1-h_1)^\alpha + (1-e_1)^\alpha + (1-y_1)^\alpha}{3}\right)^{1/\alpha} \quad \text{(A12)}$$

Expressing y_2 in terms of h_2 and e_2 and simplifying,

$$y_2 = 1-\left(3(c_5-c_4)^\alpha - (1-h_2)^\alpha - (1-e_2)^\alpha\right)^{1/\alpha} \quad \text{(A13)}$$

where, $c_5 = \left(\frac{(1-h_1)^\alpha + (1-e_1)^\alpha + (1-y_1)^\alpha}{3}\right)^{1/\alpha}$

The optimal path corresponds to the incremental position where the distance between the two is least. The distance, θ to be minimized,

$$\theta^2 = (h_2-h_1)^2 + (e_2-e_1)^2 + (y_2-y_1)^2 \quad \text{(A14)}$$

Substituting y_2 from (A13) in (A14) and applying the minimization conditions,

$$\frac{d(\theta^2)}{dh_2} = 2(h_2-h_1) + 2\left((1-y_1) - \left(3(c_5-c_4)^\alpha - (1-h_2)^\alpha - (1-e_2)^\alpha\right)^{1/\alpha}\right)$$

$$\left(\frac{-1}{\alpha}\right)\left(3(c_5-c_4)^\alpha - (1-h_2)^\alpha - (1-e_1)^\alpha\right)^{\frac{1-\alpha}{\alpha}}(-\alpha)(1-h_2)^{\alpha-1}(-1) = 0$$

$$\Rightarrow \frac{(h_2-h_1)}{(1-h_2)^{\alpha-1}} = (y_2-y_1)(1-y_2)^{1-\alpha} \quad \text{(A15)}$$

$$\frac{d(\theta^2)}{de_2} = 2(e_2-e_1) + 2\left((1-y_1) - \left(3(c_5-c_4)^\alpha - (1-h_2)^\alpha - (1-e_2)^\alpha\right)^{1/\alpha}\right)$$

$$\left(\frac{-1}{\alpha}\right)\left(3(c_5-c_4)^\alpha - (1-h_2)^\alpha - (1-e_2)^\alpha\right)^{\frac{1-\alpha}{\alpha}}(-\alpha)(1-e_2)^{\alpha-1}(-1) = 0$$

$$\Rightarrow \frac{(e_2-e_1)}{(1-e_2)^{\alpha-1}} = (y_2-y_1)(1-y_2)^{1-\alpha} \quad \text{(A16)}$$

From (A15) and (A16)

$$\frac{(h_2-h_1)}{(1-h_2)^{\alpha-1}} = \frac{(e_2-e_1)}{(1-e_2)^{\alpha-1}} \quad \text{(A17)}$$

Similarly, proceeding with *h* and *y*;

$$\frac{(h_2-h_1)}{(1-h_2)^{\alpha-1}}=\frac{(y_2-y_1)}{(1-y_2)^{\alpha-1}} \tag{A18}$$

Thus,

$$\frac{(h_2-h_1)}{(1-h_2)^{\alpha-1}}=\frac{(e_2-e_1)}{(1-e_2)^{\alpha-1}}=\frac{(y_2-y_1)}{(1-y_2)^{\alpha-1}} \tag{A19}$$

The equation for optimal path can be determined by considering infinitesimally small increment and then integrating. From first equation in (A19),

$$\frac{dh}{de}=\left(\frac{1-h}{1-e}\right)^{\alpha-1} \tag{A20}$$

For α=1, dh/de=1. This corresponds to $\mathrm{HDI_{LA}}$ case. Applying the condition that the given initial position (h_1,e_1,y_1) will be on the optimal path one can get that the optimal path coincides with translation invariance case (Figure 4). For α=2, $dh/de=(1-h)/(1-e)$; this implies

$$\frac{dh}{1-h}=\frac{de}{1-e} \tag{A21}$$

Integrating,

$$\int\frac{dh}{1-h}=\int\frac{de}{1-e}\Rightarrow \ln(1-h)=\ln(1-e)+c_6\Rightarrow 1-h=c_7(1-e) \tag{A22}$$

where, c_6 and c_7 are constants. The given initial position (h_1,e_1,y_1) will be on the optimal path; so $c_7=(1-h_1)/(1-e_1)$. Thus,

$$h=\left(\frac{1-h_1}{1-e_1}\right)e+\frac{h_1-e_1}{1-e_1} \tag{A23}$$

This shows the proportion to shortfall case (Figure 4).

For α>2,

$$\frac{dh}{(1-h)^{\alpha-1}}=\frac{de}{(1-e)^{\alpha-1}} \tag{A24}$$

Integrating,

$$\int\frac{dh}{(1-h)^{\alpha-1}}=\int\frac{de}{(1-e)^{\alpha-1}}\Rightarrow\frac{(1-h)^{2-\alpha}}{2-\alpha}=\frac{(1-e)^{2-\alpha}}{2-\alpha}+c_8 \tag{A25}$$

where, c_8 is constant. The given initial position (h_1, e_1,y_1) will be on the optimal path; so $c_8=((1-h_1)^{2-\alpha}-(1-e_1)^{2-\alpha})/(2-\alpha)$. Substituting c_8 in (A25) and simplifying,

$$h=1-\left((1-e)^{2-\alpha}+(1-h_1)^{2-\alpha}-(1-e_1)^{2-\alpha}\right)^{\frac{1}{2-\alpha}} \tag{A26}$$

The optimal paths for α=3, 5, and 10 are based on (A26) (see Figure 4).

Proceeding with the other equation in (A19), $\frac{dh}{dy}=\left(\frac{1-h}{1-y}\right)^{\alpha-1}$. For α=1, *dh*/*dy*=1; thus for

HDI_{LA},

$$\frac{dh}{de}=\frac{dh}{dy}=1 \tag{A27}$$

For α=2, *dh*/*dy*=(1-*h*)/(1-*y*); thus for HDI_{DI},

$$\frac{dh}{1-h}=\frac{de}{1-e}=\frac{dy}{1-y} \tag{A28}$$

Similarly for α>2,

$$\frac{dh}{(1-h)^{\alpha-1}}=\frac{de}{(1-e)^{\alpha-1}}=\frac{dy}{(1-y)^{\alpha-1}} \tag{A29}$$

Appendix 3

For positions (*h*, *e*, *y*) having same gap from its respective mean, μ (μ=(*h*+*e*+*y*)/3), such that *h*=μ+c_9, *e*=μ+c_{10}, and *y*=μ+c_{11}, where c_9, c_{10}, and c_{11} are constants (given). Let *V* be the deviation of HDI under GM method from the uniform development situation; *V* is given as,

$$V=\mu-(hey)^{1/3}=\mu-((\mu+c_9)(\mu+c_{10})(\mu+c_{11}))^{1/3} \tag{A30}$$

Differentiating *V* with respect to μ,

$$\frac{dV}{d\mu}=1-\frac{1}{3}((\mu+c_9)(\mu+c_{10})(\mu+c_{11}))^{-2/3}$$
$$((\mu+c_9)(\mu+c_{10})+(\mu+c_{10})(\mu+c_{11})+(\mu+c_{11})(\mu+c_9)) \tag{A31}$$

Simplifying,

$$\frac{dV}{d\mu}=1-\frac{1}{3}(hey)^{1/3}\frac{(he+ey+yh)}{hey}=1-\frac{geometric\ mean\ (h,e,y)}{harmonic\ mean\ (h,e,y)}=1-\frac{GM}{HM} \tag{A32}$$

Since, GM≥HM, *dV*/*d*μ≤0, the equality holds good at the line of equality, i.e. when there is no deviation. Equation (A32) proves, under GM method, deviation is a decreasing function of μ.

Next, let V_1 be the deviation of HDI under DI method from the uniform development situation; V_1 is given as,

$$V_1=\mu-\left(1-\left(\frac{(1-h)^{\alpha}+(1-e)^{\alpha}+(1-y)^{\alpha}}{3}\right)^{1/\alpha}\right) \tag{A33}$$

Replacing h=μ+c_9, e=μ+c_{10}, and y=μ+c_{11}; and differentiating V_1 with respect to μ and simplifying,

$$\frac{dV_1}{d\mu} = 1 - \frac{(1-h)^{\alpha-1} + (1-e)^{\alpha-1} + (1-y)^{\alpha-1}}{3^{\frac{1}{\alpha}}\left((1-h)^{\alpha} + (1-e)^{\alpha} + (1-y)^{\alpha}\right)^{\frac{\alpha-1}{\alpha}}} \qquad \text{(A34)}$$

Comparing the numerator and denominator of the second term in (A34) one can state that $dV_1/d\mu \geq 0$, but equality holds good at the line of equality, i.e. when there is no deviation across dimensions. Thus, under DI method, deviation is an increasing function of μ.

Appendix 4

For a measure, $\mathcal{M}(h, e, y)$, without loss of generality if one assumes $h<e<y$, the measure to satisfy MANUSH, must satisfy shortfall sensitivity, i.e, condition (4), $\frac{dh}{1-h} \geq \frac{de}{1-e} \geq \frac{dy}{1-y}$.

This implies the following relations

$$\frac{dh}{de} \geq \left(\frac{1-h}{1-e}\right) \quad ; \quad \frac{dh}{dy} \geq \left(\frac{1-h}{1-y}\right) \qquad \text{(A35)}$$

Noting that (1-h)>(1-e) and rewriting the first inequality in (A56),

$$\frac{dh}{de} = \left(\frac{1-h}{1-e}\right)^{\beta} \quad ; \quad \beta \geq 1 \qquad \text{(A36)}$$

The conditions for optimal path for $\mathcal{H}_\alpha$ given in equation (A20) is $\frac{dh}{de} = \left(\frac{1-h}{1-e}\right)^{\alpha-1}$. Equation (A36) is same as (A20) for α≥2; this shows that the optimal paths for $\mathcal{M}$ coincide with the optimal paths $\mathcal{H}_\alpha$; α≥2. This can happen only when $\mathcal{M}$ and $\mathcal{H}_\alpha$; α≥2 have one-to-one correspondence.

References

Alkire, S., and J. Foster. 2010. "Designing the Inequality-Adjusted Human Development Index (HDI)." OPHI Working Paper 37, Oxford Poverty & Human Development Initiative, Oxford.

Anand, S., and A. K. Sen. 1993. "Human Development Index: Methodology and Measurement." Human Development Report Office Occasional Paper 12, UNDP, New York.

Anand, S., and A. K. Sen. 1995. "Gender Inequality in Human Development: Theories and Measurement." Human Development Report Office Occasional Paper 19, UNDP. New York.

Anand, S., and A. K. Sen. 1997. "Concepts of Human Development and Poverty: A Multidimensional Perspective." Human Development Papers, United Nations Publications, New York.

Anand, S., and A. K. Sen. 2000. "The Income Component of the Human Development Index." *Journal of Human Development* 1(1): 83–106.

Anand, S., and A. K. Sen. 2003. "Human Development Index: Methodology and Measurement." In *Readings in Human Development*, edited by S. Fukuda-Parr and A. K. Shiva Kumar, 138–151. New York: Oxford University Press.

Chakravarty, S. R. 2003. "A Generalized Human Development Index." *Review of Development Economics* 7(1): 99-114.

Chakravarty, S. R., and A. Majumder. 2008. "Millennium Development Goals: Measuring Progress towards their Achievement." *Journal of Human Development* 9(1): 109–129.

Chatterjee, S. K. 2005. "Measurement of Human Development: an alternative approach." *Journal of Human Development* 6(1): 31–44.

Desai, M. 1991. “Human Developments – Concepts and measurement.” *European Economic Review* 35(2−3): 350–357.

Dutta, B., M. Panda, and W. Wadhwa. 1997. “Human Development in India.” In *Measurement of Inequality and Poverty*, edited by S. Subramanian, 329–357. New Delhi: Oxford University Press.

Fakuda-Parr, S., K. Raworth, and A. K. Shiva Kumar. 2003 “Using the HDI for Policy Analysis.” In *Readings in Human Development*, edited by S. Fukuda-Parr and A. K. Shiva Kumar, 177–187. New York: Oxford University Press.

Foster, J. E., L. F. Calva, and M. Székely 2005. “Measuring the Distribution of Human Development: methodology and an application to Mexico.” *Journal of Human Development* 6(1):5–25.

Foster, J. E., J. Greer, and E. Thorbecke. 1984. “A Class of Decomposable Poverty Measures.” *Econometrica* 52(3): 761–766.

Gidwitz, Z., M. P. Heger, J. Pineda, and F. Rodríguez. 2010. ‘Understanding Performance in Human Development: A Cross-National Study.’ *Human Development Research Paper* 2010/42.

Grimm, H., K. Harttgen, S. Klasen, and M. Misselhorn. 2008. “A Human Development Index by Income Groups.” *World Development* 36(12): 2527–2546.

Haq, M. 1995. *Reflections of Human Development*. New York: Oxford University Press.

Herrero, C., R. Martínez, and A. Villar. 2010a. “Multidimensional Social Evaluation-An Application to the Measurement of Human Development.” *Review of Income and* Wealth 56(3): 483–497.

Herrero, C., R. Martínez, and A. Villar. 2010b. “Improving the Measurement of Human Development.” *Human Development Research Paper* 2010/12.

Hicks, D. 1997. “The Inequality-Adjusted Human Development Index: A Constructive Proposal.” *World Development* 25(8): 1283–1298.

Hopkins, M. 1991. “Human development revisited: A new UNDP report.” *World Development* 19(10): 1469–1473.

Jahan, S. 2003. “Evolution of the Human Development Index.” In *Readings in Human Development*, edited by S. Fukuda-Parr and A. K. Shiva Kumar, 152–163. New York: Oxford University Press.

Klugman, J., F. R. Rodriguez, and H.-J. Choi. 2011. “The HDI 2010: New Controversies, Old Critiques.” *Human Development Research Paper* 2011/01.

Kovacevic, M., and C. G. Aguña. 2010. “Uncertainty and Sensitivity Analysis of the Human Development Index.” *Human Development Research Paper*, 2010/47.

Lüchters, G., and L. Menkhoff. 1996. “Human Development as Statistical Artefact.” *World Development* 24(8): 1385–1392.

Mishra, S. 2008. “On Measuring Group Differentials Displayed by Socio-economic Indicators: An Extension.” *Applied Economics Letters*, 15(12): 935–938.

Mishra, U.S., and S. Subramanian. 2006. “On Measuring Group Differentials Displayed by Socio-economic Indicators.” *Applied Economics Letters* 13(8): 519–521.

Nathan, H. S. K., and S. Mishra. 2010 “Progress in Human Development – Are we on the Right Path?” *International Journal of Economic Policy in Emerging Economies*, 3(3): 199–221.

Nathan, H. S. K., and S. Mishra. (2013) “Group Differential for Attainment and Failure Indicators”, *Journal of International Development* DOI: 10.1002/jid.2898.

Noorbakhsh, F. 1998. “A Modified Human Development Index.” *World Development* 26(3): 517–528.

Palazzi, P., and A. Lauri. 1998. “The Human Development Index: Suggested Corrections.” *Banca Nazionale del Lavoro Quarterly Review* 51(205): 193–221.

Panigrahi, R., and S. Sivramkrishna S. 2002. “An Adjusted Human Development Index: Robust Country Rankings with Respect to the Choice of Fixed Maximum and Minimum Indicator Values.” *Journal of Human Development,* 3(2): 301–311.

Ranis, G., F. Stewart., and E. Samman. 2006. “Human Development: Beyond the Human Development Index.” *Journal of Human Development* 7(3): 323–358.

Rao, T.V.S.R. 2011. “Contemporary Relevance and Ongoing Controversies Related to the CES Production Function.” *Journal of Quantitative Economics* 9(2): 36-57.

Raworth, K., and D. Stewart. 2003. “Critiques of the Human Development Index: A Review.” In *Readings in Human Development*, edited by S. Fukuda-Parr and A. K. Shiva Kumar, 164–176. New York: Oxford University Press.

Sagar A. D., and A. Najam. 1998. “The Human Development Index: A Critical Review.” *Ecological Economics*, 25(3): 249–264.

Sen, A. K. 1989. “Development as Capability Expansion.” *Journal of Development Planning* 19: 41–58.

Sen, A. K. 1997. “Human Capital and Human Capability.” *World Development* 25(12): 1959–1961.

Sen, A. K. 1999. *Development as Freedom*. Oxford: Oxford University Press.

Sen A. K. 2000. “A Decade of Human Development.” *Journal of Human Development* 1(1): 17–23.

Seth, S. 2009. “Inequality, Interactions, and Human Development.” *Journal of Human Development and Capabilities* 10 (3): 375-396.

Streeten, P., J. S. Burki, M. U. Haq, N. Hicks, and F. Stewart. 1981. *First Things First: Meeting Basic Human Needs in Developing Countries*. New York: Oxford University Press.

Streeten, P. 1994 "Human development: means and ends." *The American Economic Review* (Papers and Proceedings) 84(2): 232–237.

Subramanian, S. 2006. "A Re-scaled Version of the Foster-Greer-Thorbecke Poverty Indices based on an Association with the Minkowski Distance Function." In *Rights, Deprivation, and Disparity: Essays in Concepts and Measurement*, Collected Essays Series, New Delhi: Oxford University Press.

United Nations Development Program. 2010. *Human Development Report 2010, The Real Wealth of Nations: Pathways to Human Development*. New York: Oxford University Press.

Wolff, H., H. Chong and M. Auffhammer. 2011. "Classification, Detection and Consequences of Data Error: Evidence from the Human Development Index." The Economic Journal, 121 (553), 843–870.

Zeleny, M. A. 1982. *Multiple Criteria Decision Making*. New York: McGraw-Hill Book Company.