This section is marked out of 100. The marks achieved account for 50% of your final exam result.

Instructions to candidates: Answer FOUR questions. Question 1 is compulsory. Choose 3 questions out of the remaining 4. All questions carry equal marks. Remember always to explain your reasoning. Calculators are permitted.

Question 1

A profit maximising firm chooses the amount of output, $q \geq 0$. Profits are defined as:
$$\pi(q) = R(q) - C(q)$$
where revenues, $R(q)$, are given by:
$$R(q) = pq$$
And costs, $C(q)$, are:
$$C(q) = \frac{3}{4}q^2 + 5q$$

(a) Assume that the price p is constant. Explain why, if $p > 5$, it is optimal for the firm to produce a positive quantity given by:
$$q = \frac{2}{3}(p - 5) \qquad \text{(eq1)}$$

(5)

(b) Explain why, if $0 < p < 5$, it is optimal for the firm not to produce.

(5)

(c) Assume that the price depends on the quantity produced in the following way:
$$p = 45 - \frac{1}{2}q \qquad \text{(eq2)}$$
Solve eq1 and eq2 simultaneously to find equilibrium price, p, and quantity q.

(5)

(d) Now assume that the firm realizes that there are no other firms in the industry and that they can affect the market price by choosing the quantity produced strategically. The revenue function becomes $R(q) = p(q)q$, where $p(q) = 45 - \frac{1}{2}q$. Find the new quantity q that maximizes profits $\pi(q)$.

(5)

(e) Compute the firm's profit, $\pi(q)$,in (c) and (d) and explain why they are different.

(5)

Question 2

(a)   Find the exact solution(s) to the following equations:

(i) $7^{x+5} = 1$

(5)

(ii) $\ln(x) + \ln(x - 2) = \ln(9 - 2x)$

(5)

(b)   Find the values of x for which the following inequality holds:

$$\frac{3}{x^2 - 5x + 6} + \frac{4 - x}{3 - x} > \frac{6 - x}{2 - x}$$

(15)

(a)   Amy has just won the lottery and decides to deposit part of the money on a savings account that pays 8% yearly interest rate. Her objective is to have £2,000,000 on the account after 4 years how much does she need to deposit?

(5)

(b)   Sam wants to buy a motorbike and he can choose between the following payment plans

1. Pay £1700 at the beginning of each year for 5 years
2. Pay £X at the beginning of each month for 2 years

(i)   If the implied yearly interest rate is 7%, what is the value of the motorbike today?

(5)

(ii)   Compute the monthly interest rate that characterizes the second payment plan, assuming that its annualized interest rate is still 7%.

(5)

(iii)   Explain why the monthly payment of the second plan, X, satisfies the following equation:

$$V = X + X(1+r)^{-1} + X(1+r)^{-2} + \ldots + X(1+r)^{-23}$$

Where V is the value of the bike today as computed in (i) and r is the monthly interest rate computed in (ii)

(5)

(iv)   Find X

(5)

Question 4

Consider the function $f(x) = 2x - 3x^3$.

(a)   Find and classify any stationary points.

(6)

(b)   Draw the graph of $y = f(x)$.

(7)

(c)   Consider the region where $y \geq 0$ and $x \geq 0$ and compute the area delimited by the line $y = x$ and the graph $y = f(x)$.

(12)

Question 5

(a) A curve has equation $y^{\frac{1}{2}}x^{\frac{1}{2}} = u$ where u is a positive constant. Sketch the curve in the positive quadrant of the (x,y)-plane. Make sure you explain your reasoning.

(8)

(b) Find an expression for $\frac{d^2y}{dx^2}$

(5)

(c) Find the solution to the following system of equations

$$\begin{cases} \alpha \log(x) + \beta \log(y) = u \\ \dfrac{\alpha x^{-1}}{\beta y^{-1}} = \dfrac{p_x}{p_y} \end{cases}$$

Where α, β, u, p_x and p_y are positive constants

(6)

(d) Find the solution to the following system of equations

$$\begin{cases} p_x x + p_y y = m \\ \dfrac{\alpha x^{-1}}{\beta y^{-1}} = \dfrac{p_x}{p_y} \end{cases}$$

Where α, β, u, p_x and p_y are positive constants

(6)