Section D

- This section has five questions
- You must answer only four out of the five questions
- This section is marked out of 100
- The marks achieved in this section account for 50% of your final exam result

Instructions:

Answer four out of the five questions. Question 1 is compulsory. Answer three out of the remaining four questions (Questions 2, 3, 4, 5). All questions carry equal marks. Remember always to explain how you reached your conclusion. Calculators are permitted.

Question 1

(a) A curve F has equation $x^2+y^2=u$ where u is a positive constant. Sketch the curve in the positive quadrant of the (x,y)-plane. Make sure you explain your reasoning.

(6)

(b) A curve C has equation $x^{\frac{3}{5}}y^{\frac{2}{5}}=u$ where u is a positive constant.

Find an expression for $\frac{d^2y}{dx^2}$.

(8)

(c) In the positive quadrant of the (x,y)-plane, sketch the curve $x^{\frac{3}{5}} y^{\frac{2}{5}} = u$ for some positive value of u . Make sure you explain your reasoning.

(6)

(d) Find the solutions to the simultaneous equations

(1) $$\frac{\frac{3}{5} x^{-\frac{2}{5}} y^{\frac{2}{5}}}{\frac{2}{5} x^{\frac{3}{5}} y^{-\frac{3}{5}}} = \frac{p}{s}$$

(2) $p x + s y = m$

where p , s and m are positive constants.

(5)

Question 2

(a) A group of students were questioned for an education survey and $\frac{3}{4}$ were found to attend lectures regularly. When questioned about their exam results at the end of the year, $\frac{5}{8}$ said they were satisfied with their marks and, of those who were satisfied, $\frac{19}{20}$ regularly attended lectures during the year.

Find the probability that a randomly selected member of the group

(i) attended lectures and was satisfied with their result,

(5)

(ii) did not attend lectures and was not satisfied with the result.

(5)

(b) Find the exact solution(s) to the following equations:

(i) $2^x - 2^{x-2} = 48$

(5)

(ii) $\ln(y) + \ln(3) = 2\ln(9)$

(5)

(iii) $e^z + 6e^{-z} = 7$

(5)

Question 3

(a) Amrit inherited a large sum of money that he keeps in a savings account. The yearly interest rate is 5% and he withdraws the interest at the end of each year, just after it is paid. Amrit's only income is the interest he gets on this money and there is a 40% income tax rate.

(i) If Amrit gets £14,400 in interest (after tax), how much money does he have in his savings account?

(3)

(ii) Assume the government changes its tax policy: instead of taxing the returns to savings, wealth itself is taxed. Which tax rate would generate the same yearly revenue for the government, assuming that wealth is taxed immediately after interest payments are realised? Assuming Amrit continues to withdraw the amount of money every year that leaves his wealth constant – are there any changes to his income due to the change in the tax policy?

(7)

(b) Sarah wants to buy a car for £28,000 and has made the following plan to finance it: Every month, she deposits an amount Y into a savings account that has a monthly interest rate of 7%. She intends to choose Y such that she will have the necessary £28,000 in exactly 2 years.

(i) Clearly explain why the monthly deposits will satisfy the following equation:

$$28{,}000 = Y + Y(1.07) + Y(1.07)^2 + \ldots + Y(1.07)^{23}$$

(5)

(ii) Calculate the value of Y to two decimal places.

(5)

(iii) If Sarah were to increase the monthly deposits to £700, after how many months would she have enough money to buy the car?

(5)

Question 4

The production of quantity x (for $x \geq 0$) of a widget generates a revenue given by $R(x) = px$ and is associated with the following cost $C(x)$: $C(x) = \begin{cases} \frac{x^2}{2} - 3x + 6 & x > 0 \\ 0 & x = 0 \end{cases}$

(a) For $x > 0$, find the profit function $\pi(x) = R(x) - C(x)$ and the value of x that maximises profits. Explain your reason for choosing this value carefully.

(7)

(b) For $x \geq 0$, explain carefully why it is the case that:

If $p > \sqrt{12} - 3$, then the maximum value of $\pi(x)$ occurs when $x = p + 3$;

if $0 \leq p < \sqrt{12} - 3$, then the maximum value of $\pi(x)$ is zero.

(13)

(c) For $p = \sqrt{12} - 3$, find the value of x that maximises $\pi(x)$ and give your reason for choosing this value of x.

(5)

Question 5

(a) Consider the function $g(x)=ax^2-4x+b$, $a, b \in \mathbb{R}$.

(i) For what values of a and b does $g(x)$ have a stationary point?

(4)

(ii) For what values of a and b does it have a maximum?

(2)

(iii) For what values of a and b does it have a minimum?

(2)

(b) Consider the function $f(x)=\ln[(2-e^x)^5]$

(i)Find the first and

(4)

(ii) second derivative of $f(x)$. Simplify your answers as far as possible.

(2)

(c) Consider the function $f(x)=\exp\left[-\frac{(x-5)^2}{10}\right]$

(i) For what values of x is $f(x)$ positive? For what values of x is it negative?

(2)

(ii) For what values of x is $f(x)$ increasing? For what values of x is it decreasing?

(5)

(iii) Find and classify any stationary points of $f(x)$

(4)